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AN ASYMPTOTIC FORMULA OF COSINE POWER SUMS

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In the paper, the authors find several accurate approximations of some cosine power sums and present an asymptotic formula for these cosine power sums.

1. Introduction

In [5, 6], the first author presented two open problems concerning the asymptotic behavior of cosine power sums,

$$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^p \left(\frac{k\pi}{n} \right) , \tag{1}$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x.

If the powers p are even, then these cosine power sums can be determined exactly, without approximations.

Theorem 1.1. *Let n and p be two positive integers. Then*

$$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^{2p} \left(\frac{k\pi}{n} \right) = -\frac{1}{2} + \frac{n}{2^{2p+1}} \sum_{k=-\lfloor \frac{p}{n} \rfloor}^{\lfloor \frac{p}{n} \rfloor} \binom{2p}{p+kn} .$$

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This result was recently published in [4]. If p in (1) is odd, such formulas are not known. For this reason, we will give some accurate approximations of the cosine power sums (1), when $p < 2n$. These approximations have simple forms and can be used to quickly compute values of the power sums (1).

2. Cosine power sums estimated by integrals

In this section we show that the cosine power sums (1) can be approximated using Wallis's integral

$$I(p) = \int_0^{\frac{\pi}{2}} \cos^p(x)dx \ .$$

Theorem 2.1. *Let n and p be two positive integers. Then*

$$\frac{n}{\pi}\cdot I(p) - 1 < \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^p\left(\frac{k\pi}{n}\right) < \frac{n}{\pi}\cdot I(p) \ .$$

Proof. To prove the theorem, we take into account that $f(x) = \cos^p(x)$ is a positive and decreasing function on the interval $\left(0, \frac{\pi}{2}\right)$, i.e.,

$$f'(x) = -p\cos^{p-1}(x)\sin(x) < 0 \ , \qquad 0 < x < \frac{\pi}{2} \ .$$

Thus, the left Riemann sum amounts to an underestimation and the right Riemann sum amounts to an overestimation of f on the interval $\left[0, \frac{\pi}{2}\right]$. On the one hand, we can write

$$\frac{\pi}{2n}\sum_{k=1}^{n} \cos^p\left(\frac{k\pi}{2n}\right) < I(p) \tag{2}$$

and

$$\frac{\pi}{2n}\sum_{k=0}^{n-1} \cos^p\left(\frac{k\pi}{2n}\right) > I(p) \ . \tag{3}$$

By (2) and (3), we get

$$\frac{2n}{\pi}\cdot I(p) - 1 < \sum_{k=1}^{n-1} \cos^p\left(\frac{k\pi}{2n}\right) < \frac{2n}{\pi}\cdot I(p) \ . \tag{4}$$

On the other hand, we have

$$\frac{\pi}{2n+1}\sum_{k=1}^{n} \cos^p\left(\frac{k\pi}{2n+1}\right) < I(p) \tag{5}$$

and

$$\begin{aligned} I(p) &< \frac{\pi}{2n+1}\sum_{k=0}^{n-1}\cos^p\left(\frac{k\pi}{2n+1}\right)+\left(\frac{\pi}{2}-\frac{n\pi}{2n+1}\right)\cos^p\left(\frac{n\pi}{2n+1}\right) \\ &< \frac{\pi}{2n+1}\sum_{k=0}^{n}\cos^p\left(\frac{k\pi}{2n+1}\right) . \end{aligned} \tag{6}$$

By (5) and (6), we obtain

$$\frac{2n+1}{\pi}\cdot I(p)-1<\sum_{k=1}^{n}\cos^p\left(\frac{k\pi}{2n+1}\right)<\frac{2n+1}{\pi}\cdot I(p) . \tag{7}$$

According to (4) and (7), the theorem is proved. □

A lower bound and an upper bound of the cosine power sums (1) are given in Theorem 2.1. Thus, the cosine power sums (1) could be approximated with the arithmetic mean of the two boundaries. It is clear that the absolute approximation error is less than $\frac{1}{2}$ without any further condition.

Theorem 2.2. *Let n and p be two positive integers. Then*

$$\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}\cos^p\left(\frac{k\pi}{n}\right)\approx\frac{n}{\pi}\cdot I(p)-\frac{1}{2} .$$

A double factorial $p!!$ can be defined by

$$p!!=\prod_{k=0}^{\lfloor\frac{p-1}{2}\rfloor}(p-2k) ,$$

for any given positive integer p. The formula

$$I(p)=\frac{(p-1)!!}{p!!}\cdot\begin{cases}1, & \text{if } p \text{ is odd,}\\ \frac{\pi}{2}, & \text{otherwise}\end{cases} \tag{8}$$

is well known (see [2, p. 389, eq. 3.621 4, eq. 3.621 5]) and allows us to rewrite Theorem 2.2 as the following corollaries.

Corollary 2.3. *Let n and p be two positive integers. If p is even, then*

$$\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}\cos^p\left(\frac{k\pi}{n}\right)\approx\frac{n}{2}\cdot\frac{(p-1)!!}{p!!}-\frac{1}{2} ,$$

with equality if $p<2n$.

Proof. This follows from Theorem 1.1 and Theorem 2.2 using (8). □

Corollary 2.4. *Let n and p be two positive integers. If p is odd, then*

$$\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \cos^p\left(\frac{k\pi}{n}\right) \approx \frac{n}{\pi}\cdot\frac{(p-1)!!}{p!!} - \frac{1}{2}\ .$$

Proof. The same as above. □

3. Accurate approximations of cosine power sums

When p is even, we have exact formulas for the cosine power sums (1). It is clear that for $p < 2n$ the lower border and the upper border given by Theorem 2.1 can be improved. We do this using the following improved version of Wallis's inequality [1]

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}\ , \tag{9}$$

where $n!!$ denotes the double factorial. The constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible. More details can be found in [1] and a collection of refinements of Wallis's inequality was published in [7, Section 7.4].

Theorem 3.1. *Let n and p be two positive integers such that $p < 2n$. Then*

$$\frac{n}{2\sqrt{\pi\left(\lfloor\frac{p-1}{2}\rfloor+\frac{4}{\pi}\right)}} - \frac{1}{2} < \sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \cos^p\left(\frac{k\pi}{n}\right) < \frac{n}{2\sqrt{\pi\left(\lfloor\frac{p}{2}\rfloor+\frac{1}{4}\right)}} - \frac{1}{2}\ .$$

Proof. To prove the theorem we take into account Corollary 2.3, the double inequality (9) and

$$\cos^{2p+2}(x) < \cos^{2p+1}(x) < \cos^{2p}(x)\ , \qquad 0 < x < \frac{\pi}{2}\ .$$

□

Another improved version of Wallis's inequality is proved in [8],

$$2L(2n) < \frac{(2n-1)!!}{(2n)!!} < 2R(2n)\ , \tag{10}$$

where

$$L(n) = \frac{1}{\sqrt{2e\pi n}}\left(1+\frac{1}{n}\right)^{\frac{1}{2}\left(n-\frac{1}{3n}\right)}$$

and

$$R(n) = \frac{1}{\sqrt{2e\pi n}}\left(1+\frac{1}{n}\right)^{\frac{1}{2}\left(n-\frac{1}{3n+8}\right)} .$$

If $n > 1$, this inequality is better than the inequality (9). For more details, one can refer to [8] and the references therein.

Theorem 3.2. *Let n and p be two positive integers such that $p < 2n$.*

1. *If p is even, then*

$$nL(p) - \frac{1}{2} < \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^p\left(\frac{k\pi}{n}\right) < nR(p) - \frac{1}{2} .$$

2. *If p is odd, then*

$$nL(p+1) - \frac{1}{2} < \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^p\left(\frac{k\pi}{n}\right) < nR(p-1) - \frac{1}{2} .$$

Proof. The same as the proof of Theorem 3.1, replacing the inequality (9) with the inequality (10). □

The asymptotic expansion [3, p. 12] of the central binomial coefficients

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}\left(1 - \frac{1}{64\left(n+\frac{1}{4}\right)^2} + \frac{21}{8192\left(n+\frac{1}{4}\right)^4} - \cdots\right) \tag{11}$$

leads us to accurate approximations for the cosine power sums.

Theorem 3.3. *Let n and p be two positive integers such that $p < 2n$. Then*

$$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \cos^p\left(\frac{k\pi}{n}\right) = \frac{n}{\sqrt{(2p+1)\pi}}\left(1 - \frac{1}{4(2p+1)^2} + O\left(\frac{1}{p^4}\right)\right) - \frac{1}{2} .$$

Proof. Taking into account that

$$\lim_{p\to\infty} \frac{I(p+1)}{I(p)} = 1 \qquad \text{and} \qquad \frac{1}{2^{2p}}\binom{2p}{p} = \frac{(2p-1)!!}{(2p)!!} ,$$

the proof follows easily by Corollary 2.3 and the asymptotic expansion (11). □

Theorem 3.3 improves the formula given in [5]. This formula can be obtained using the following relation:

$$\binom{2p}{p} = \frac{2^{2p}}{\sqrt{\pi p}}\left(1 - \frac{c_p}{p}\right) \qquad \text{where} \qquad \frac{1}{9} < c_p < \frac{1}{8} \qquad \text{for all} \qquad p > 0 .$$

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