LE MATEMATICHE
Vol. LXVIII (2013) – Fasc. II, pp. 219–225
doi: 10.4418/2013.68.2.16

SOME INEQUALITIES INVOLVING RATIOS AND PRODUCTS OF THE GAMMA FUNCTION

MRIDULA GARG - AJAY SHARMA - PRATIBHA MANOHAR

In this paper we establish some generalized inequalities for the gamma function, using the properties of logarithmically convex/concave functions.

1. Introduction

The **Euler gamma function** $\Gamma(x)$ is defined for $x > 0$ by [2]

$$\Gamma(x) = \int_0^\infty e^x t^{x-1} dt. \tag{1}$$

The **psi or digamma function** (the logarithmic derivative of the gamma function) can be expressed as

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, x > 0. \tag{2}$$

A function $f : D \to \mathbb{R}_+$ is said to be **log-convex** if [5]

$$f[ux + (1-u)y] \leq [f(x)]^u . [f(y)]^{1-u}, 0 < u < 1 \tag{3}$$

Entrato in redazione: 6 dicembre 2012

AMS 2010 Subject Classification: 33B20.
Keywords: Convex/concave log-function, Digamma function, Euler gamma function, Inequalities.

and **log-concave** if

$$f[ux+(1-u)y] \geq [f(x)]^u . [f(y)]^{1-u}, 0<u<1, \tag{4}$$

holds for all $x,y \in D$, where D is a subinterval of the real line $\mathbb{R}$ and $\mathbb{R}_+$ denotes the positive real-axis.

It is well-known that a family of log-convex functions is closed under addition and multiplication of real numbers.

Logarithmically convex (log-convex) functions are of interest in many areas of mathematics and science. They have been found to play an important role in the theory of special functions and mathematical statistics (see [1], [7]).

In Section 2, we establish some generalized inequalities for gamma functions, given in the form of Corollary 2.4, Theorem 2.6 and Theorem 2.8. This research is motivated by the results obtained by Neuman [6]. Some of the techniques used in the subsequent sections are the same as those used in [6].

2. Main Results

Theorem 2.1. *Let us take*

$$\phi \equiv \phi(a,b,c,d,x) = \left[\frac{f(a+bx)}{f(c+dx)}\right]^{\frac{1}{(a-c)+(b-d)x}}, \tag{5}$$

where $a+bx, c+dx \in D$, $(a-c)+(b-d)x \neq 0$ *and* $f:\mathbb{R}\to\mathbb{R}_+$ *is a log-convex function as defined by* (3).
Then the function $\phi(a,b,c,d,x)$ *increases with an increase in either of* a,b,c *or* d.

Proof. Let

$$\Phi = \ln\phi = \frac{\ln f(a+bx) - \ln f(c+dx)}{(a-c)+(b-d)x}. \tag{6}$$

From [4] , [6] we infer that Φ, being first order divided difference of a convex function, is an increasing function in both $a+bx$ and $c+dx$ and thus ϕ has the desired monotonicity property in either of a,b,c or d. □

Remark 2.2. In particular for $b=d=1$, Theorem 2.1 reduces to the one established by Neuman [6].

Corollary 2.3. *The function* $\left[\frac{\Gamma(1+kx)}{\Gamma(1+ky)}\right]^{\frac{1}{(x-y)k}}$, $(kx,ky>-1, x\neq y, k\neq 0)$ *is increasing in both the variables x and y and* $[\Gamma(1+kx)]^{\frac{1}{kx}}$, $(kx>-1, x\neq 0, k\neq 0)$ *is increasing in the variable x.*

Proof. In Theorem 2.1, if we take $f(x)=\Gamma(x)$, $a=c=1$, replace x by k, b by x and d by y, we arrive at the first part of the corollary and the second part follows on taking $y=0$ in the first part. □

Corollary 2.4. *If in Theorem 2.1 we take f to be continuously differentiable function on D, then the following inequalities hold.*

For $x\geq 0$,

$$\{(a-c)+(b-d)x\}\frac{f'(c+dx)}{f(c+dx)}\leq \ln\frac{f(a+bx)}{f(c+dx)} \leq \{(a-c)+(b-d)x\}\frac{f'(a+bx)}{f(a+bx)}, \quad (7)$$

where $a+bx, c+dx\in D$, $(a-c)+(b-d)x\neq 0$
and

$$\{(a-c)+(b-d)x\}\Psi(c+dx)\leq \ln\frac{\Gamma(a+bx)}{\Gamma(c+dx)} \leq \{(a-c)+(b-d)x\}\Psi(a+bx), \quad (8)$$

where $a+bx, c+dx\in\mathbb{R}_+$.

For $x<0$,

$$\{(a-c)+(b-d)x\}\frac{f'(a+bx)}{f(a+bx)}\leq \ln\frac{f(a+bx)}{f(c+dx)} \leq \{(a-c)+(b-d)x\}\frac{f'(c+dx)}{f(c+dx)}, \quad (9)$$

where $a+bx, c+dx\in D$, $(a-c)+(b-d)x\neq 0$ *and*

$$\{(a-c)+(b-d)x\}\Psi(a+bx)\leq \ln\frac{\Gamma(a+bx)}{\Gamma(c+dx)} \leq \{(a-c)+(b-d)x\}\Psi(c+dx), \quad (10)$$

where $a+bx, c+dx\in\mathbb{R}_+$.

Proof. Differentiating (5) partially with respect to ‘b’ and ‘d’ and using Theorem 2.1 i.e. ϕ increases with increase in b and d, we get

$$\{(a-c)+(b-d)x\}^2\frac{\partial\phi}{\partial b} = \phi x\left[\{(a-c)+(b-d)x\}\frac{f'(a+bx)}{f(a+bx)}-\ln\frac{f(a+bx)}{f(c+dx)}\right]\geq 0, \quad (11)$$

and

$$\{(a-c)+(b-d)x\}^2 \frac{\partial\phi}{\partial d}$$
$$= -\phi x \left[\{(a-c)+(b-d)x\} \frac{f'(c+dx)}{f(c+dx)} - \ln \frac{f(a+bx)}{f(c+dx)}\right] \geq 0. \quad (12)$$

In the above inequalities (11) and (12), if we consider both the cases $x > 0$ and $x < 0$, we get the inequality (7) established for $x > 0$ and the inequality (9) for $x < 0$.

Next we prove the double inequality (7) for $x = 0$. For this, we use Theorem 2.1 with $x = 0$ and differentiate equation (5) partially with respect to $'a'$ and $'c'$ to arrive at

$$(a-c)^2 \frac{\partial\phi}{\partial a} = \phi \left[(a-c) \frac{f'(a)}{f(a)} - \ln \frac{f(a)}{f(c)}\right] \geq 0,$$

and

$$(a-c)^2 \frac{\partial\phi}{\partial c} = -\phi \left[(a-c) \frac{f'(c)}{f(c)} - \ln \frac{f(a)}{f(c)}\right] \geq 0,$$

These results combined together give inequality (7) for $x = 0$.

The inequalities (8) and (10) can be established on taking $f(x) = \Gamma(x)$ in (7) and (9) respectively and using the definition (2). ☐

Lemma 2.5. *Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a function and $a_i, x_i > 0$, $1 \leq i \leq n$. If the function $\frac{g(x)}{x}$ is increasing on $\mathbb{R}_+$, then*

$$g\left(\frac{\sigma}{m}\right) \geq \frac{na}{m} \sum_{i=1}^{n} g\left(\frac{a_i x_i}{na}\right), \quad (13)$$

where $\sigma = \sum_{i=1}^{n} a_i x_i$, $m = \sum_{i=1}^{n} a_i$, $a = \max(a_i)$.

Next if $\frac{g(x)}{x}$ is decreasing on $\mathbb{R}_+$, then we have reverse inequality

$$g\left(\frac{\sigma}{m}\right) \leq \frac{na}{m} \sum_{i=1}^{n} g\left(\frac{a_i x_i}{na}\right). \quad (14)$$

Proof. We have

$$\frac{\sigma}{m} = \frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i} \geq \frac{a_i x_i}{\sum_{i=1}^{n} a_i} \geq \frac{a_i x_i}{na}, \; \forall\, i \in \{1, 2..., n\}.$$

Next, by using the assumption that $\frac{g(x)}{x}$ is increasing on $\mathbb{R}_+$, we get

$$\frac{g\left(\frac{\sigma}{m}\right)}{\frac{\sigma}{m}} \geq \frac{g\left(\frac{a_i x_i}{na}\right)}{\frac{a_i x_i}{na}}, \forall\, i \in \{1,2...,n\}. \tag{15}$$

Using (15), we now write

$$g\left(\frac{\sigma}{m}\right) = \frac{\sigma}{m}\frac{g\left(\frac{\sigma}{m}\right)}{\frac{\sigma}{m}} \geq \frac{1}{m}\sum_{i=1}^{n} a_i x_i \frac{g\left(\frac{a_i x_i}{na}\right)}{\frac{a_i x_i}{na}} \geq \frac{na}{m}\sum_{i=1}^{n} g\left(\frac{a_i x_i}{na}\right). \tag{16}$$

This proves the inequality (13).

The reverse inequality (14) in the case when $\frac{g(x)}{x}$ is decreasing on $\mathbb{R}_+$ can be proved on similar lines. □

Theorem 2.6. *If $a_i, x_i > 0$, $1 \leq i \leq n$, $a = \max_i (a_i)$, $m = \sum_{i=1}^{n} a_i$ then for $r \geq 1$, we have*

$$\frac{\left(\sum_{i=1}^{n} a_i x_i\right)^r}{m^r}\left[\Gamma\left(\frac{\sum_{i=1}^{n} a_i x_i}{m}+1\right)\right]^{\frac{m}{\sum_{i=1}^{n} a_i x_i}} \geq \frac{na}{m}\sum_{i=1}^{n}\left(\frac{a_i x_i}{na}\right)^r\left[\Gamma\left(\frac{a_i x_i}{na}+1\right)\right]^{\frac{na}{a_i x_i}} \tag{17}$$

and for $r \leq 0$

$$\frac{\left(\sum_{i=1}^{n} a_i x_i\right)^r}{m^r}\left[\Gamma\left(\frac{\sum_{i=1}^{n} a_i x_i}{m}+1\right)\right]^{\frac{m}{\sum_{i=1}^{n} a_i x_i}} \leq \frac{na}{m}\sum_{i=1}^{n}\left(\frac{a_i x_i}{na}\right)^r\left[\Gamma\left(\frac{a_i x_i}{na}+1\right)\right]^{\frac{na}{a_i x_i}}. \tag{18}$$

Proof. We know by [7] that the function $\frac{x^r[\Gamma(x+1)]^{\frac{1}{x}}}{x}$, $x > 0$ is strictly increasing for $r \geq 1$ and strictly decreasing for $r \leq 0$.
If we take $g(x) = x^r [\Gamma(x+1)]^{\frac{1}{x}}$, $r \geq 1$ and apply the first part (13) of Lemma 2.5 for $r \geq 1$, we obtain the inequality (17) and for $r \leq 0$, using the second part (14) of Lemma 2.5, we get the inequality (17). □

Remark 2.7. In particular, for $r = 0$, $a_1 = a_2 = ... = a_n = a$, $y_i = \frac{x_i}{n}$, inequality (18) reduces to

$$\left[\Gamma\left(\sum_{i=1}^{n} y_i + 1\right)\right]^{\frac{1}{\sum_{i=1}^{n} y_i}} \leq [\Gamma(y_i+1)]^{\frac{1}{y_i}}$$

which is the same as obtained by Neuman [6].

Theorem 2.8. *For $1 \leq i \leq n$, $a_i > 0$, $x_i > -1$ $(x_i \neq 0)$ the following double inequality holds*

$$\prod_{i=1}^{n}[\Gamma(1+x_i)]^{\frac{a_i\sigma}{x_i m}} \leq \left[\Gamma\left(1+\frac{\sigma}{m}\right)\right]^m \leq \prod_{i=1}^{n}[\Gamma(1+x_i)]^{a_i}, \tag{19}$$

where $m = \sum_{i=1}^{n} a_i$ and $\sigma = \sum_{i=1}^{n} a_i x_i$.

Proof. To prove second part of the inequality (19), we use the log-convex property (3) of gamma function,

$f(x) = \Gamma(1+x)$ for $x > -1$ and write

$$\left[\Gamma\left(1+\frac{\sigma}{m}\right)\right]^m = \left[\Gamma\left(1+\frac{\sum_{i=1}^n a_i x_i}{m}\right)\right]^m$$
$$= \left[\Gamma\left\{1+\sum_{i=1}^n\left(\frac{a_i}{m}\right)x_i\right\}\right]^m \leq \prod_{i=1}^n \{\Gamma(1+x_i)\}^{a_i}, \; x_i > -1. \quad (20)$$

Further, to prove the first part of the inequality (19), we use the fact that $f(x) = [\Gamma(1+x)]^{\frac{1}{x}}$ is log-concave for $x > -1$ (see [3]) and write

$$\left[\Gamma\left(1+\frac{\sigma}{m}\right)\right]^m = \left\{\Gamma\left(1+\frac{\sigma}{m}\right)\right\}^{\frac{\sigma}{\sigma/m}} = \left[\Gamma\left(1+\frac{\sum_{i=1}^n a_i x_i}{m}\right)^{\frac{1}{\sum_{i=1}^n a_i x_i/m}}\right]^\sigma$$
$$\geq \left[\prod_{i=1}^n \{\Gamma(1+x_i)\}^{a_i/x_i}\right]^\sigma, x_i > -1, x_i \neq 0. \quad (21)$$

□

Remark 2.9. In particular for $a_1 = a_2 = ... = a_n = 1, \sigma = \sum_{i=1}^n x_i$ in inequality (19) it reduces to

$$\prod_{i=1}^n [\Gamma(1+x_i)]^{\frac{\sigma}{x_i n}} \leq \left[\Gamma\left(1+\frac{\sigma}{n}\right)\right]^n \leq \prod_{i=1}^n [\Gamma(1+x_i)],$$

which is the same as obtained by Neuman [6].

Acknowledgements

We thank the worthy referee for the useful comments.

REFERENCES

[1] B. C. Carlson, *Special Functions of Applied Mathematics*, Academic Press, New York, 1977.

[2] A. Erdelyi - W. Magnus - F. Oberhettinger - F. Tricomi (EDS), *Higher Transcendental Functions, Vol. I*, Mc-Graw-Hill, New York, 1953.

[3] P. J. Grabner - R. Thichy - U. T. Zimmermann, *Inequalities for the gamma function with applications to permanents*, Discrete Math. 154 (1996), 53–62.

[4] Z. Kadelburg - D. Dukic - M. Lukic - I. Matic, *Inequalities of karamata, schur and muirhead, and some applications*, The Teaching of Mathematics VIII (1) (2005), 31–45.

[5] J. F. C. Kingman, *A convexity property of positive matrices*, Quart. J. Math. Oxford 12 (2) (1961), 283–284.

[6] E. Neuman, *Some inequalities for the Gamma function*, Appl. Mat. Comp. 218 (8) (2011), 4349–4352.

[7] F. Qi - CH. P. Chen, *Monotonicity and convexity results for functions involving the Gamma function*, RGMIA Res. Rep. Coll. 6 (4) (2003), Art. 10.

MRIDULA GARG
Department of Mathematics
University of Rajasthan
Jaipur-302004, Rajasthan
India
e-mail: gargmridula@gmail.com

AJAY SHARMA
Department of Mathematics
University of Engineering and Management
Jaipur-303807, Rajasthan
India
e-mail: ajaysince1984@gmail.com

PRATIBHA MANOHAR
Department of Statistics, Mathematics and Computer Science
SKN College of Agriculture
Swami Keshwanand Rajasthan Agriculture University
Jobner-303329, Rajasthan
India
e-mail: prati_manohar@yahoo.co.in