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SOME PROPERTIES OF TWO-FOLD SYMMETRIC ANALYTIC FUNCTIONS

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In this paper, we introduce a new class of two-fold symmetric functions analytic in the unit disc. We prove such results as subordination and superordination properties, convolution properties, distortion theorems, and inequality properties of this new class.

1. Introduction

Let $\mathcal{A}(m)$ denote the class of functions f:

$$f(z) = z + \sum_{k=m+1}^{\infty} a_k z^k, \quad m \in \mathbb{N} = \{1, 2, \dots\}, \tag{1}$$

which are analytic in the open unit disc $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let $\mathcal{H}[a, m+1]$ be the class of analytic functions of the form

$$f(z) = a + a_{m+1} z^{m+1} + a_{m+2} z^{m+2} + \dots, \quad z \in E.$$

If f and g are analytic in E, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$

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in E such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in E, then we have the following equivalence holds, see [4, 5]

$$f(z) \prec g(z) \ \ (z \in E) \Longleftrightarrow f(0) = g(0) \text{ and } f(E) \subset g(E).$$

For function f, $g \in \mathcal{A}(m)$, where f is given by (1) and g is defined by

$$g(z) = z + \sum_{k=m+1}^{\infty} b_k z^k, \quad m \in \mathbb{N} = \{1, 2, \ldots\},$$

then the Hadamard product (or convolution) $f * g$ of the function f and g is defined by

$$(f * g)(z) = z + \sum_{k=m+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

In [8], Sakaguchi defined the class of starlike functions with respect to symmetrical points as follows:

Let $f \in \mathcal{A}$. Then f is said to be starlike with respect to symmetrical points in E if, and only if,

$$Re \, \frac{zf'(z)}{f(z) - f(-z)} > 0, \, z \in E.$$

Obviously, it forms a subclass of close-to-convex functions and hence univalent. Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin, see [8].

Definition 1.1. A function $f \in \mathcal{A}(m)$ is said to be in the class $\mathcal{B}^{\lambda,\mu}(m, A, B)$, if it satisfies the following subordination condition:

$$\begin{aligned}(1-\lambda)&\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \\ &\prec \frac{1+Az}{1+Bz},\end{aligned} \tag{2}$$

where and throughout this paper unless otherwise mention the parameters λ, μ, A and B are constrained as follows:

$$\lambda \in \mathbb{C} : \ Re\,(\mu) > 0 : \ -1 \leq B \leq 1, \, A \neq B, \, A \in \mathbb{R} \text{ and } m \in \mathbb{N},$$

and all powers are understood as principal values.

In this paper, we prove such results as subordination and superordination properties, convolution properties, distortion theorems, and inequality properties of the class $\mathcal{B}^{\lambda,\mu}(m, A, B)$.

For interested readers see the work done by the authors [1, 2, 10–13].

2. Preliminary Results

Definition 2.1. Let $\mathcal{Q}$ be the set of all functions f that are analytic and injective on $\overline{E}\backslash U(f)$, where

$$U(f)=\left\{\zeta\in\partial E:\lim_{z\to\zeta}f(z)=\infty\right\},$$

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial E\backslash U(f)$.

To establish our main results we need the following Lemmas.

Lemma 2.2 (Miller and Mocanu [4, 5])**.** *Let the function $h(z)$ be analytic and convex (univalent) in E with $h(0)=1$. Suppose also that the function $\Phi(z)$ given by*

$$\Phi(z)=1+c_{m+1}z^{m+1}+c_{m+2}z^{m+2}+\ldots$$

is analytic in E,

$$\Phi(z)+\frac{z\,\Phi'(z)}{\gamma}\prec h(z)\ \ (z\in E;\ Re\,\gamma\geq 0;\ \gamma\neq 0), \tag{3}$$

then

$$\Phi(z)\prec\Psi(z)=\frac{\gamma}{(m+1)z^{\frac{\gamma}{m+1}}}\int_0^z t^{\frac{\gamma}{m+1}-1}h(t)dt\prec h(z)\ \ (z\in E),$$

and $\Psi(z)$ is the best dominant of (3).

Lemma 2.3 (Shanmugam et al. [9])**.** *Let $\sigma\in\mathbb{C}$, $\eta\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}$ and let q be a convex univalent function in E with*

$$Re\,\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0;-Re\,\frac{\sigma}{\eta}\right\},\ z\in E.$$

If p is analytic in E and

$$\sigma p(z)+\eta zp'(z)\prec\sigma q(z)+\eta zq'(z), \tag{4}$$

then $p(z)\prec q(z)$, and q is the best dominant of (4).

Lemma 2.4 ([5])**.** *Let $q(z)$ be convex univalent in E and $k\in\mathbb{C}$. Further assume that Re $k>0$. If*

$$g(z)\in\mathcal{H}[q(0),1]\cap\mathcal{Q},$$

and

$$g(z)+kzq'(z)\prec g(z)+kzg'(z),$$

implies $q(z)\prec g(z)$ and $q(z)$ is the best subordinant.

Lemma 2.5 ([3])**.** *Let F be analytic and convex in E. If $f,g \in \mathcal{A}(1)$ and $f,g \prec F$, then*

$$\lambda f + (1-\lambda)g \prec F \quad (0 \le \lambda \le 1).$$

Lemma 2.6 ([7])**.** *Let*

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$$

be analytic in E and

$$g(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$$

be analytic and convex in E. If $f(z) \prec g(z)$, then

$$|a_k| < |b_1|, \;\; k \in \mathbb{N}.$$

3. Main Results

Theorem 3.1. *Let $f(z) \in \mathcal{B}^{\lambda,\mu}(m,A,B)$ with Re $\lambda > 0$. Then*

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \prec \psi(z) = \frac{\mu}{\lambda(m+1)} \int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu}{\lambda(m+1)}-1} du \prec \frac{1+Az}{1+Bz}, \quad (5)$$

and $\psi(z)$ is the best dominant.

Proof. Set

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} = h(z), \;\; z \in E. \quad (6)$$

Then $h(z)$ is analytic in E with $h(0) = 1$.

Logarithmic differentiation of (5) and simple computations yield

$$(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)} \left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}$$
$$= h(z) + \frac{\lambda}{\mu} z h'(z) \prec \frac{1+Az}{1+Bz}. \quad (7)$$

Applying Lemma 2.2 to (7) with $\gamma = \frac{\mu}{\lambda}$, we have

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \prec \psi(z) = \frac{\mu}{\lambda(m+1)} z^{-\frac{\mu}{\lambda(m+1)}} \int_0^z \frac{1+At}{1+Bt} t^{\frac{\mu}{\lambda(m+1)}-1} dt$$
$$= \frac{\mu}{\lambda(m+1)} \int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu}{\lambda(m+1)}-1} du \prec \frac{1+Az}{1+Bz}, \quad (8)$$

and $\psi(z)$ is the best dominant. This completes the proof. □

Theorem 3.2. *Let $q(z)$ be univalent in E, $\lambda \in \mathbb{C}^*$. Suppose also that $q(z)$ satisfies the following inequality:*

$$Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -Re\left(\frac{\mu}{\lambda}\right)\right\}. \tag{9}$$

If $f \in \mathcal{A}(m)$ satisfies the following subordination:

$$(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \prec q(z)+\frac{\lambda}{\mu}zq'(z), \tag{10}$$

then

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Let the function $h(z)$ be defined by (6). We know that the first part of (7) holds true. Combining (7) and (10), we have

$$h(z)+\frac{\lambda}{\mu}zh'(z) \prec q(z)+\frac{\lambda}{\mu}zq'(z). \tag{11}$$

By using Lemma 2.3 and (11), we easily get the assertion of Theorem 3.2. □

Corollary 3.3. *Let $\lambda \in \mathbb{C}^*$ and $-1 \leq B < A \leq 1$. Suppose also that*

$$Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -Re\left(\frac{\mu}{\lambda}\right)\right\}.$$

If $f \in \mathcal{A}(m)$ satisfies the following subordination:

$$(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}$$
$$\prec \frac{1+Az}{1+Bz}+\lambda\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

If f is subordinate to F, then F is superordinate to f. We now derive the following superordination result for the class $\mathcal{B}^{\lambda,\mu}(m,A,B)$.

Theorem 3.4. *Let q be convex univalent in $E, \lambda \in \mathbb{C}$ with Re $\lambda > 0$. Also let*

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \in \mathcal{H}[q(0),1] \cap \mathcal{Q}$$

and

$$(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)} \left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}$$

be univalent in E. If

$$\begin{aligned} & q(z) + \frac{\lambda}{\mu} z q'(z) \\ & \prec (1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)} \left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}, \end{aligned}$$

then

$$q(z) \prec \left(\frac{f(z)-f(-z)}{2z}\right)^{\mu},$$

and q is the best subordinant.

Proof. Let the function $h(z)$ be defined by (6). Then

$$\begin{aligned} & q(z) + \frac{\lambda}{\mu} z q'(z) \\ & \prec (1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)} \left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \\ & = h(z) + \frac{\lambda}{\mu} z h'(z). \end{aligned}$$

An application of Lemma 2.4 yields the assertion of Theorem 3.4. □

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.4, we obtain the following corollary.

Corollary 3.5. *Let $q(z)$ be convex univalent in E and $-1 \leq B < A \leq 1$, $\lambda \in \mathbb{C}$ with Re $\lambda > 0$. Also let*

$$0 \neq \left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \in \mathcal{H}[q(0),1] \cap \mathcal{Q},$$

and

$$(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)} \left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}$$

be univalent in E. If

$$\frac{1+Az}{1+Bz}+\lambda\frac{(A-B)z}{(1+Bz)^2}$$
$$\prec(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu},$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu},$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Combining the above results of subordination and superordination, we easily get the following "sandwich-type result".

Corollary 3.6. *Let q_1 be convex univalent and let q_2 be univalent in E, $\lambda\in\mathbb{C}$ with Re $\lambda>0$. Let q_2 satisfy (9). If*

$$0\neq\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\in\mathcal{H}[q(0),1]\cap\mathcal{Q},$$

and

$$(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}$$

is univalent in E, also

$$q_1(z)+\frac{\lambda zq_1'(z)}{\mu}$$
$$\prec(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}$$
$$\prec q_2(z)+\frac{\lambda zq_2'(z)}{\mu},$$

then

$$q_1(z)\prec\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinant and dominant.

Theorem 3.7. *If $\lambda\in\mathbb{C}$, $\mu>0$ and $f(z)\in\mathcal{B}^{0,\mu}(m,1-2\rho,-1)$ $(0\leq\rho<1)$, then $f(z)\in\mathcal{B}^{\lambda,\mu}(m,1-2\rho,-1)$ for $|z|<R$,*

where

$$R = \left(\left(\sqrt{\left(\frac{|\lambda|(m+1)}{\mu} \right)^2 + 1} \right) - \frac{|\lambda|(m+1)}{\mu} \right)^{\frac{1}{m+1}}. \tag{12}$$

The bound R is best possible.

Proof. Set

$$\left(\frac{f(z) - f(-z)}{2z} \right)^{\mu} = (1-\rho)h(z) + \rho, \quad z \in E, \quad 0 \leq \rho < 1. \tag{13}$$

Then, clearly the function $h(z)$ is analytic in E with $h(0) = 1$. Proceeding as an Theorem 3.1, we have

$$\frac{1}{1-\rho} \left\{ (1-\lambda) \left(\frac{f(z)-f(-z)}{2z} \right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)} \left(\frac{f(z)-f(-z)}{2z} \right)^{\mu} - \rho \right\} = h(z) + \frac{\lambda z h'(z)}{\mu}. \tag{14}$$

Using the following well-known estimate, see [6]

$$\left| z h'(z) \right| \leq \frac{2(m+1) r^{m+1} Re\,(h(z))}{(1 - r^{2(m+1)})} \quad (|z| = r < 1)$$

in (14), we obtain that

$$Re\, \frac{1}{1-\rho} \left\{ (1-\lambda) \left(\frac{f(z)-f(-z)}{2z} \right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)} \left(\frac{f(z)-f(-z)}{2z} \right)^{\mu} - \rho \right\} \geq Re\, h(z) \left\{ 1 - \frac{2|\lambda|(m+1) r^{m+1}}{\mu(1 - r^{2(m+1)})} \right\}. \tag{15}$$

Right hand side of (15) is positive, provided that $r < R$, where R is given by (12).

In order to show that the bound R is best possible, we consider the function $f(z) \in \mathcal{A}(m)$ defined by

$$\left(\frac{f(z)-f(-z)}{2z} \right)^{\mu} = (1-\rho) \frac{1+z^{m+1}}{1-z^{m+1}} + \rho, \;\; z \in E, 0 \leq \rho < 1.$$

We note that

$$\frac{1}{1-\rho}\left\{(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}-\rho\right\} = \frac{1+z^{m+1}}{1-z^{m+1}}+\frac{2|\lambda|(m+1)z^{m+1}}{\mu(1-z^{m+1})^2}=0,$$

for $|z| = R$, we conclude that the bound is the best possible and this proves the theorem. □

Theorem 3.8. *Let* $0 \le \lambda_1 \le \lambda_2$ *and* $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$. *Then*

$$\mathcal{B}^{\lambda_2,\mu}(m,A_2,B_2) \subset \mathcal{B}^{\lambda_1,\mu}(m,A_1,B_1). \tag{16}$$

Proof. Suppose that $f \in \mathcal{B}^{\lambda_2,\mu}(m,A_2,B_2)$. We know that

$$\left\{(1-\lambda_2)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda_2\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\right\} \prec \frac{1+A_2z}{1+B_2z}.$$

Since $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$, we easily find that

$$\left\{(1-\lambda_2)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda_2\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\right\} \prec \frac{1+A_2z}{1+B_2z} \prec \frac{1+A_1z}{1+B_1z}, \tag{17}$$

that is $f \in \mathcal{B}^{\lambda_2,\mu}(m,A_1,B_1)$.
Thus the assertion (16) holds true for $0 \le \lambda_1 = \lambda_2$. If $\lambda_2 > \lambda_1 \ge 0$, by Theorem 3.1 and (17), we know that $f \in \mathcal{B}^{0,\mu}(m,A_2,B_2)$, that is,

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \prec \frac{1+A_1z}{1+B_1z}. \tag{18}$$

At the same time, we have

$$\left\{(1-\lambda_1)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda_1\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\right\} = \frac{\lambda_1}{\lambda_2}\left[(1-\lambda_2)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda_2\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\right]+ \left(1-\frac{\lambda_1}{\lambda_2}\right)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}. \tag{19}$$

Moreover,

$$0 \le \frac{\lambda_1}{\lambda_2} < 1,$$

and the function $\frac{1+A_1z}{1+B_1z}$, $-1 \le B_1 < A_1 \le 1$, $z \in E$ is analytic and convex in E. Combining (17-19) and Lemma 2.5, we find that

$$\left\{(1-\lambda_1)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda_1 \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\right\} \prec \frac{1+A_1z}{1+B_1z},$$

that is $f \in \mathcal{B}^{\lambda_1,\mu}(m,A_1,B_1)$, which implies that the assertion (16) of Theorem 3.8 holds and this completes the proof. □

Theorem 3.9. *Let* $f \in \mathcal{B}^{\lambda,\mu}(m,A,B)$ *with* $\lambda > 0$ *and* $-1 \le B_1 < A_1 \le 1$. *Then*

$$\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du$$
$$< Re\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} < \frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du. \tag{20}$$

The extremal function of (20) is defined by

$$F_{\lambda,\mu,m,A,B}(z) = 2z\left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{\frac{1}{\mu}}. \tag{21}$$

Proof. Let $f \in \mathcal{B}^{\lambda,\mu}(m,A,B)$ with $\lambda > 0$. From Theorem 3.1, we know that (5) holds, which implies that

$$Re\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} < \sup_{z\in E} Re\left\{\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\mu}{\lambda(m+1)}-1}du\right\}$$
$$\le \left\{\frac{\mu}{\lambda(m+1)}\int_0^1 \sup_{z\in E} Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu}{\lambda(m+1)}-1}du\right\}$$
$$< \frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du, \tag{22}$$

and

$$Re\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} > \inf_{z\in E} Re\left\{\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\mu}{\lambda(m+1)}-1}du\right\}$$
$$\geq \left\{\frac{\mu}{\lambda(m+1)}\int_0^1 \inf_{z\in E} Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu}{\lambda(m+1)}-1}du\right\}$$
$$> \frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du. \tag{23}$$

Combining (22) and (23), we obtain (20). Noting that the function $F_{\lambda,\mu,m,A,B}(z)$ defined by (21) belongs to the class $\mathcal{B}^{\lambda,\mu}(m,A,B)$, we get that inequality (20) is sharp. This completes the proof. □

In view of Theorem 3.9, we have the following distortion theorems for the class $\mathcal{B}^{\lambda,\mu}(m,A,B)$.

Corollary 3.10. *Let $f(z)\in\mathcal{B}^{\lambda,\mu}(m,A,B)$ with $\lambda>0$ and $-1\leq B<A\leq 1$. Then for $|z|=r<1$, we have*

$$2r\left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1-Aur}{1-Bur}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{\frac{1}{\mu}}$$
$$<|f(z)-f(-z)|<2r\left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Aur}{1+Bur}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{\frac{1}{\mu}}. \tag{24}$$

The extremal function of (24) is defined by (21).

By noting that

$$(Re\,(v))^{\frac{1}{2}}\leq Re\,(v^{\frac{1}{2}})\leq |v|^{\frac{1}{2}},\ \ v\in\mathbb{C}; Re\,v\geq 0.$$

From Theorem 3.9, we can easily derive the following result.

Corollary 3.11. *Let $f(z)\in\mathcal{B}^{\lambda,\mu}(m,A,B)$ with $\lambda>0$ and $-1\leq B<A\leq 1$. Then*

$$\left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{\frac{1}{2}}$$
$$<Re\left(\frac{f(z)-f(-z)}{2z}\right)^{\frac{\mu}{2}}<\left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{\frac{1}{2}}.$$

Theorem 3.12. *Let*

$$f(z)=z+\sum_{k=m+1}^{\infty}a_kz^k\in\mathcal{B}^{\lambda,\mu}(m,A,B),\ m\in\mathbb{N}. \tag{25}$$

Then

$$|a_{m+1}|\leq\left|\frac{2(A-B)}{\lambda(m+1)+2\mu}\right|. \tag{26}$$

The inequality (26) is sharp, with the extremal function defined by (21).

Proof. Combining (2) and (25), we have

$$\begin{aligned}&(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\\&=1+[1+\frac{\lambda(m+1)}{2\mu}]\mu a_{m+1}\,z^{m+1}+\cdots\prec\frac{1+Az}{1+Bz}\\&=1+(A-B)z+\dots\end{aligned} \tag{27}$$

An application of Lemma 2.5 to (27) yields

$$\left|[1+\frac{\lambda(m+1)}{2\mu}]\mu a_{m+1}\right|\leq|A-B|. \tag{28}$$

Thus, from (28), we easily arrive at (26) asserted by Theorem 3.12. □

Theorem 3.13. *Let $f(z)\in\mathcal{B}^{\lambda,\mu}(m,A,0)$ with $Re\,\lambda>0, A>0$ and $|\lambda|\left(1+Re\,\frac{\mu}{\lambda(m+1)}\right)>A\mu$. Then*

$$\left|\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}-1\right|<\frac{A\left[|\lambda|\left(m+1+Re\,\frac{\mu}{\lambda}\right)+\mu\right]}{|\lambda|\left[|\lambda|(m+1+Re\,\frac{\mu}{\lambda}-A\mu\right]}.$$

Proof. Let $h(z)$ be defined by (5). It follows from (6) that

$$h(z)+\frac{\lambda zh'(z)}{\mu}=1+Aw(z), \tag{29}$$

where

$$w(z)=\sum_{k=m+1}^{\infty}w_kz^k,\ \ m\in\mathbb{N},$$

is analytic in E with $|w(z)|<1$, $z\in E$. From (29), we can get

$$\begin{aligned}h(z)&=1+A\frac{\mu}{\lambda}\int_0^1 t^{\frac{\mu}{\lambda}-1}w(tz)dt\\&=1+A\frac{\mu}{\lambda}\sum_{k=m+1}^{\infty}\frac{1}{k+\frac{\mu}{\lambda}}w_kz^k.\end{aligned} \tag{30}$$

It follows from (30) that

$$
\begin{aligned}
(zh(z))' &= 1 + A\frac{\mu}{\lambda}\sum_{k=m+1}^{\infty}\frac{k+1}{k+\frac{\mu}{\lambda}}w_k z^k \\
&= 1 + A\frac{\mu}{\lambda}\sum_{k=m+1}^{\infty}\frac{1}{k+\frac{\mu}{\lambda}}w_k z^k \\
&\quad + A\frac{\mu}{\lambda}\left(w(z) - \frac{\mu}{\lambda}\int_0^1 t^{\frac{\mu}{\lambda}-1}w(tz)dt\right). \qquad (31)
\end{aligned}
$$

We now find from (30) and (31) that

$$
zh'(z) = A\frac{\mu}{\lambda}\left(w(z) - \frac{\mu}{\lambda}\int_0^1 t^{\frac{\mu}{\lambda}-1}w(tz)dt\right). \qquad (32)
$$

Combining (30) and (32), we can get

$$
\left|\frac{zh'(z)}{h(z)}\right| < \frac{A\mu\left[|\lambda|\left(m+1+Re\,\frac{\mu}{\lambda}\right)+\mu\right]}{|\lambda|\left[|\lambda|\,(m+1+Re\,\frac{\mu}{\lambda}-A\mu\right]}. \qquad (33)
$$

Thus, from (6) and (33), we easily arrive at the assertion of Theorem 3.13. □

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