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SOME FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS IN NON-ARCHIMEDEAN MENGER PROBABILISTIC METRIC SPACES VIA COMMON LIMIT RANGE PROPERTY

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In this paper, we utilize the notion of common limit range property in Non-Archimedean Menger PM-spaces and prove some fixed point theorems for two pairs of weakly compatible mappings. Some illustrative examples are furnished to support our results. As an application to our main result, we present a common fixed point theorem for four finite families of self mappings. Our results improve and extend several known results existing in the literature.

1. Introduction

In 1974, Istrătescu and Crivăt [11] introduced the concept of Non-Archimedean probablistic metric spaces (briefly, N.A. PM-spaces). Istrătescu [8, 9] obtained some fixed point theorems on N.A. Menger PM-spaces and generalized the results of Sehgal and Bharucha-Reid [21] (also see [10, 12]). Further, Hadžić [6] improved the results of Istrătescu [8, 9].

In 1987, Singh and Pant [24] introduced the notion of weakly commuting mappings on N.A. Menger PM-spaces and proved some common fixed

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point theorems. Dimri and Pant [5] studied the application of N.A. Menger PM-spaces to product spaces. Jungck and Rhoades [13, 14] weakened the notion of compatible mappings by introducing weakly compatible mappings and proved common fixed point theorems without any requirement of continuity of the involved mappings. Many mathematicians proved common fixed point theorems in N.A. Menger PM-spaces using different contractive conditions (see [2, 4, 5, 15–17, 22, 23, 26]). In 2002, Aamri and Moutawakil [1] defined the notion of property (E.A) which contained the class of non-compatible mappings. It is observed that the property (E.A) requires the completeness (or closedness) of the subspaces for the existence of the common fixed point. In 2011, Sintunavarat and Kumam [27] defined the notion of common limit range property for a pair of self mappings in fuzzy metric spaces. They showed that common limit range property never requires the closedness of the subspace (also see [28]). Recently, Singh et al. [25] proved a common fixed point theorem for a pair of weakly compatible self mappings in N.A. Menger PM-space employing common limit range property.

In this paper, we extend the notion of common limit range property to two pairs of self mappings in N.A. Menger PM-spaces and prove some fixed point theorems. Some examples are given which demonstrate the validity of our main result. As an application to our main result, we derive a fixed point theorem for four finite families of self mappings which can be utilized to derive common fixed point theorems involving any finite number of mappings.

2. Preliminaries

Definition 2.1. [20] A t-norm $\mathcal{T}$ is a binary operation on the unit interval [0,1] such that for all $a,b,c,d \in [0,1]$ and the following conditions are satisfied:

1. $\mathcal{T}(a,1) = a$;
2. $\mathcal{T}(a,b) = \mathcal{T}(b,a)$;
3. $\mathcal{T}(a,b) \leq \mathcal{T}(c,d)$, whenever $a \leq c$ and $b \leq d$;
4. $\mathcal{T}(a,\mathcal{T}(b,c)) = \mathcal{T}(\mathcal{T}(a,b),c)$.

Definition 2.2. [20] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is said to be a distribution function if it is non-decreasing and left continuous with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$. We shall denote $\Im$ by the set of all distribution functions.

If X is a non-empty set, $\mathcal{F} : X \times X \to \Im$ is called a probabilistic distance on X and $F(x,y)$ is usually denoted by $F_{x,y}$.

Definition 2.3. [9, 11] The ordered pair $(X,\mathcal{F})$ is said to be N.A. PM-space if X is a non-empty set and $\mathcal{F}$ is a probabilistic distance satisfying the following conditions: for all $x,y,z \in X$ and $t,t_1,t_2 > 0$,

1. $F_{x,y}(t) = 1 \Leftrightarrow x = y$;
2. $F_{x,y}(t) = F_{y,x}(t)$;
3. if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$ then $F_{x,z}(\max\{t_1,t_2\}) = 1$.

The ordered triplet $(X,\mathcal{F},\mathcal{T})$ is called a N.A. Menger PM-space if $(X,\mathcal{F})$ is a N.A. PM-space, $\mathcal{T}$ is a t-norm and the following inequality holds:

$$F_{x,z}(\max\{t_1,t_2\}) \geq \mathcal{T}\left(F_{x,y}(t_1),F_{y,z}(t_2)\right),$$

for all $x,y,z \in X$ and $t_1,t_2 > 0$.

Example 2.4. Let X be any set with at least two elements. If we define $F_{x,x}(t) = 1$ for all $x \in X$, $t > 0$ and

$$F_{x,y}(t) = \begin{cases} 0, & \text{if } t \leq 1; \\ 1, & \text{if } t > 1, \end{cases}$$

where $x,y \in X, x \neq y$, then $(X,\mathcal{F},\mathcal{T})$ is a N.A. Menger PM-space with $\mathcal{T}(a,b) = \min\{a,b\}$ or (ab) for all $a,b \in [0,1]$.

Example 2.5. Let $X = \mathbb{R}$ be the set of real numbers equipped with metric defined by $d(x,y) = |x-y|$ and

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

Then $(X,\mathcal{F},\mathcal{T})$ is a N.A. Menger PM-space with $\mathcal{T}$ as continuous t-norm satisfying $\mathcal{T}(a,b) = \min\{a,b\}$ or (ab) for all $a,b \in [0,1]$.

Definition 2.6. [4] A N.A. Menger PM-space $(X,\mathcal{F},\mathcal{T})$ is said to be of type $(C)_{\mathfrak{g}}$ if there exists a $\mathfrak{g} \in \Omega$ such that

$$\mathfrak{g}(F_{x,z}(t)) \leq \mathfrak{g}(F_{x,y}(t)) + \mathfrak{g}(F_{y,z}(t)),$$

for all $x,y,z \in X$, $t \geq 0$, where $\Omega = \{\mathfrak{g} \mid \mathfrak{g} : [0,1] \to [0,\infty)$ is continuous, strictly decreasing with $\mathfrak{g}(1) = 0$ and $\mathfrak{g}(0) < \infty\}$.

Definition 2.7. [4] A N.A. Menger PM-space $(X,\mathcal{F},\mathcal{T})$ is said to be of type $(D)_{\mathfrak{g}}$ if there exists a $\mathfrak{g} \in \Omega$ such that

$$\mathfrak{g}(\mathcal{T}(t_1,t_2)) \leq \mathfrak{g}(t_1) + \mathfrak{g}(t_2),$$

for all $t_1,t_2 \in [0,1]$.

Remark 2.8. [4] If a N.A. Menger PM-space $(X,\mathcal{F},\mathcal{T})$ is of type $(D)_{\mathfrak{g}}$ then

1. it is of type $(C)_{\mathfrak{g}}$.
2. it is metrizable, where the metric d on X is defined by

$$d(x,y)=\int_0^1 \mathfrak{g}(F_{x,y}(t))dt,$$

for all $x,y\in X$.

Throughout this paper $(X,\mathcal{F},\mathcal{T})$ is a N.A. Menger PM-space of type $(D)_{\mathfrak{g}}$ with a continuous strictly increasing t-norm $\mathcal{T}$.

Let $\phi:[0,\infty)\to[0,\infty)$ be a function satisfying the condition (Φ): ϕ is upper semi-continuous from the right and $\phi(t)<t$ for $t>0$.

Lemma 2.9. [4] *If a function* $\phi:[0,\infty)\to[0,\infty)$ *satisfies the condition* (Φ) *then we have:*

1. *for all* $t\geq 0, \lim_{n\to\infty}\phi^n(t)=0$, *where* $\phi^n(t)$ *is the* n^{th} *iteration of* $\phi(t)$.
2. *If* $\{t_n\}$ *is a non-decreasing sequence of real numbers and* $t_{n+1}\leq\phi(t_n)$ *where* $n=1,2,\dots$ *then* $\lim_{n\to\infty}t_n=0$. *In particular, if* $t\leq\phi(t)$, *for each* $t\geq 0$ *then* $t=0$.

Definition 2.10. A pair (A,S) of self mappings of a N.A. Menger PM-space $(X,\mathcal{F},\mathcal{T})$ is said to satisfy (E.A) property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=z,$$

for some $z\in X$.

Definition 2.11. [19] A pair (A,S) of self mappings of a non-empty set X is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Az=Sz$ for some $z\in X$, then $ASz=SAz$.

It is known that a pair (A,S) of compatible mappings is weakly compatible but converse is not true in general.

Remark 2.12. It is noticed that the notions of weak compatibility and property (E.A) are independent to each other [18, Example 2.2].

Definition 2.13. Two pairs (A,S) and (B,T) of self mappings of a N.A. Menger PM-space $(X,\mathcal{F},\mathcal{T})$ are said to satisfy the common property (E.A), if there exists two sequences $\{x_n\}$, $\{y_n\}$ in X for some z in X such that

$$\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}By_n=\lim_{n\to\infty}Ty_n=z.$$

Inspired by Sintunavarat and Kumam [27], we define the "common limit in the range" property in N.A. Menger PM-space as follows:

Definition 2.14. A pair (A,S) of self mappings of a N.A. Menger PM-space $(X,\mathcal{F},\mathcal{T})$ is said to satisfy the common limit range property with respect to mapping S, denoted by (CLR_S), if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in S(X)$.

Definition 2.15. Two pairs (A,S) and (B,T) of self mappings of a N.A. Menger PM-space $(X,\mathcal{F},\mathcal{T})$ are said to satisfy the common limit range property with respect to mappings S and T, denoted by (CLR_{ST}), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$.

Now, we show examples for two pairs of self mappings (A,S) and (B,T) which are satisfying the (CLR_{ST}) property.

Example 2.16. Let $(X,\mathcal{F},\mathcal{T})$ be a N.A. Menger PM-space, where $X = [1,\infty)$ and metric d is defined as condition (2) of Remark 2.8. Define the self mappings A,B,S and T on X by $A(x) = x+2$, $B(x) = x+1$, $S(x) = 3x$ and $T(x) = \frac{3x}{2}$ for all $x \in X$. Then with sequences $\{x_n\} = \{1+\frac{1}{n}\}_{n\in\mathbb{N}}$ and $\{y_n\} = \{2+\frac{1}{n}\}_{n\in\mathbb{N}}$ in X, we can easily verify that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 3 \in S(X) \cap T(X),$$

which shows that the pairs (A,S) and (B,T) satisfy the (CLR_{ST}) property.

Example 2.17. Let $(X,\mathcal{F},\mathcal{T})$ be a N.A. Menger PM-space, where $X = [0,\infty)$ and metric d is defined as condition (2) of Remark 2.8. Define the self mappings A,B,S and T on X by $A(x) = \frac{x}{3}$, $B(x) = \frac{x}{4}$, $S(x) = \frac{2x}{3}$ and $T(x) = \frac{3x}{4}$ for all $x \in X$. Let the sequences $\{x_n\} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ and $\{x_n\} = \{\frac{2}{n}\}_{n\in\mathbb{N}}$ in X. Since

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 0 \in S(X) \cap T(X),$$

therefore both the pairs (A,S) and (B,T) satisfy the (CLR_{ST}) property.

Definition 2.18. [7] Two families of self mappings $\{A_i\}$ and $\{S_j\}$ are said to be pairwise commuting if:

1. $A_iA_j = A_jA_i$, $i,j \in \{1,2,\ldots,m\}$,
2. $S_kS_l = S_lS_k$, $k,l \in \{1,2,\ldots,n\}$,
3. $A_iS_k = S_kA_i$, $i \in \{1,2,\ldots,m\}$, $k \in \{1,2,\ldots,n\}$.

3. Results

Before proving our main result, we begin with the following observation.

Lemma 3.1. *Let A,B,S and T be four self mappings of a N.A. Menger PM-space $(X,\mathcal{F},\mathcal{T})$, where $\mathcal{T}$ is a continuous t-norm. Suppose that*

1. *the pair (A,S) satisfies the (CLR_S) property $\Big($or the pair (B,T) satisfies the (CLR_T) property$\Big)$,*
2. $A(X) \subset T(X)$ $\Big($*or* $B(X) \subset S(X)\Big)$,
3. $T(X)$ $\Big($*or* $S(X)\Big)$ *is a closed subset of* X,
4. *$\{By_n\}$ converges for every sequence $\{y_n\}$ in X whenever $\{Ty_n\}$ converges $\big($or $\{Ax_n\}$ converges for every sequence $\{x_n\}$ in X whenever $\{Sx_n\}$ converges$\big)$,*
5. $$\mathfrak{g}(F_{Ax,By}(t)) \leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{Sx,Ty}(t)),\mathfrak{g}(F_{Sx,Ax}(t)),\mathfrak{g}(F_{Ty,By}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{Sx,By}(t))+\mathfrak{g}(F_{Ty,Ax}(t))\right)\end{array}\right\}\right), \quad (1)$$
   for all $x,y \in X$, $t>0$, where $\mathfrak{g} \in \Omega$ and ϕ satisfies the condition (Φ).

Then the pairs (A,S) and (B,T) enjoy the (CLR_{ST}) property.

Proof. If the pair (A,S) satisfies the (CLR_S) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z \tag{2}$$

where $z \in S(X)$. Since $A(X) \subset T(X)$, hence for each $\{x_n\} \subset X$ there corresponds a sequence $\{y_n\} \subset X$ such that $Ax_n = Ty_n$. Therefore, due to closedness of $T(X)$,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z, \tag{3}$$

where $z \in S(X)\cap T(X)$. Thus in all, we have $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$ as $n\to\infty$. By (4), the sequence $\{By_n\}$ converges and in all we need to show that $By_n \to z$ as $n\to\infty$. On using inequality (1) with $x = x_n$, $y = y_n$, we get

$$\mathfrak{g}(F_{Ax_n,By_n}(t)) \leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{Sx_n,Ty_n}(t)),\mathfrak{g}(F_{Sx_n,Ax_n}(t)),\mathfrak{g}(F_{Ty_n,By_n}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{Sx_n,By_n}(t))+\mathfrak{g}(F_{Ty_n,Ax_n}(t))\right)\end{array}\right\}\right),$$

Let $By_n \to l(\neq z)$ as $n\to\infty$. Then, passing to limit as $n\to\infty$, we get

$$\begin{aligned}\mathfrak{g}(F_{z,l}(t)) &\leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{z,z}(t)),\mathfrak{g}(F_{z,z}(t)),\mathfrak{g}(F_{z,l}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{z,l}(t))+\mathfrak{g}(F_{z,z}(t))\right)\end{array}\right\}\right),\\ &= \phi\left(\max\left\{\mathfrak{g}(1),\mathfrak{g}(1),\mathfrak{g}(F_{z,l}(t)),\frac{1}{2}\left(\mathfrak{g}(F_{z,l}(t))+\mathfrak{g}(1)\right)\right\}\right)\\ &= \phi\left(\max\left\{0,0,\mathfrak{g}(F_{z,l}(t)),\frac{1}{2}\left(\mathfrak{g}(F_{z,l}(t))\right)\right\}\right)\\ &= \phi\left(\mathfrak{g}(F_{z,l}(t))\right).\end{aligned}$$

Owing Lemma 2.9, we have $z = l$. Hence the pairs (A,S) and (B,T) share the (CLR_{ST}) property. □

Theorem 3.2. *Let A,B,S and T be four self mappings of a N.A. Menger PM-space $(X,\mathcal{F},\mathcal{T})$, where $\mathcal{T}$ is a continuous t-norm satisfying inequality (1) of Lemma 3.1. Suppose that the pairs (A,S) and (B,T) share the (CLR_{ST}) property, then (A,S) and (B,T) have a coincidence point each. Moreover, A,B,S and T have a unique common fixed point provided both the pairs (A,S) and (B,T) are weakly compatible.*

Proof. Since the pairs (A,S) and (B,T) enjoy the (CLR_{ST}) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z,$$

where $z \in S(X)\cap T(X)$. Since $z \in S(X)$, there exists a point $u \in X$ such that $Su = z$. First we assert that $Au = Su$. On using inequality (1) with $x = u$, $y = y_n$, we get

$$\mathfrak{g}(F_{Au,By_n}(t)) \leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{Su,Ty_n}(t)),\mathfrak{g}(F_{Su,Au}(t)),\mathfrak{g}(F_{Ty_n,By_n}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{Su,By_n}(t))+\mathfrak{g}(F_{Ty_n,Au}(t))\right)\end{array}\right\}\right),$$

which on making $n\to\infty$, reduces to

$$\begin{aligned}\mathfrak{g}(F_{Au,z}(t)) &\leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{z,z}(t)),\mathfrak{g}(F_{z,Au}(t)),\mathfrak{g}(F_{z,z}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{z,z}(t))+\mathfrak{g}(F_{z,Au}(t))\right)\end{array}\right\}\right)\\ &= \phi\left(\max\left\{\mathfrak{g}(1),\mathfrak{g}(F_{z,Au}(t)),\mathfrak{g}(1),\frac{1}{2}\left(\mathfrak{g}(1)+\mathfrak{g}(F_{z,Au}(t))\right)\right\}\right)\\ &= \phi\left(\mathfrak{g}(F_{z,Au}(t))\right).\end{aligned}$$

On employing Lemma 2.9, we get $Au = Su = z$, which shows that u is a coincidence point of the pair (A,S).

Again, $z \in T(X)$, there exists a point $v \in X$ such that $Tv = z$. We show that $Bv = Tv$. On using inequality (1) with $x = u$, $y = v$, we get

$$
\begin{aligned}
\mathfrak{g}(F_{Au,Bv}(t)) &\leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{Su,Tv}(t)),\mathfrak{g}(F_{Su,Au}(t)),\mathfrak{g}(F_{Tv,Bv}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{Su,Bv}(t))+\mathfrak{g}(F_{Tv,Au}(t))\right)\end{array}\right\}\right)\\
\mathfrak{g}(F_{z,Bv}(t)) &\leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{z,z}(t)),\mathfrak{g}(F_{z,z}(t)),\mathfrak{g}(F_{z,Bv}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{z,Bv}(t))+\mathfrak{g}(F_{z,z}(t))\right)\end{array}\right\}\right)\\
&= \phi\left(\max\left\{\mathfrak{g}(1),\mathfrak{g}(1),\mathfrak{g}(F_{z,Bv}(t)),\frac{1}{2}\left(\mathfrak{g}(F_{z,Bv}(t))+\mathfrak{g}(1)\right)\right\}\right)\\
&= \phi\left(\mathfrak{g}(F_{z,Bv}(t))\right).
\end{aligned}
$$

Appealing to Lemma 2.9, we have $Bv = Tv = z$, which shows that v is a coincidence point of the pair (B,T).

Since the pair (A,S) is weakly compatible and $Au = Su$, hence $Az = ASu = SAu = Sz$. Now, we show that z is a common fixed point of the pair (A,S). Putting $x = z$ and $y = v$ in inequality (1), we have

$$
\begin{aligned}
\mathfrak{g}(F_{Az,Bv}(t)) &\leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{Sz,Tv}(t)),\mathfrak{g}(F_{Sz,Az}(t)),\mathfrak{g}(F_{Tv,Bv}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{Sz,Bv}(t))+\mathfrak{g}(F_{Tv,Az}(t))\right)\end{array}\right\}\right)\\
\mathfrak{g}(F_{Az,z}(t)) &\leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{Az,z}(t)),\mathfrak{g}(F_{Az,Az}(t)),\mathfrak{g}(F_{z,z}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{Az,z}(t))+\mathfrak{g}(F_{z,Az}(t))\right)\end{array}\right\}\right)\\
&= \phi\left(\max\left\{\mathfrak{g}(F_{Az,z}(t)),\mathfrak{g}(1),\mathfrak{g}(1),\frac{1}{2}\left(\mathfrak{g}(F_{Az,z}(t))+\mathfrak{g}(F_{z,Az}(t))\right)\right\}\right)\\
&= \phi\left(\mathfrak{g}(F_{Az,z}(t))\right).
\end{aligned}
$$

In view of Lemma 2.9, we have $Az = z = Sz$ which shows that z is a common fixed point of the pair (A,S).

Also the pair (B,T) is weakly compatible and $Bv = Tv$, then $Bz = BTv = TBv = Tz$. On using inequality (1) with $x = u$, $y = z$, we have

$$
\begin{aligned}
\mathfrak{g}(F_{Au,Bz}(t)) &\leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{Su,Tz}(t)),\mathfrak{g}(F_{Su,Au}(t)),\mathfrak{g}(F_{Tz,Bz}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{Su,Bz}(t))+\mathfrak{g}(F_{Tz,Au}(t))\right)\end{array}\right\}\right)\\
\mathfrak{g}(F_{z,Bz}(t)) &\leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{z,Bz}(t)),\mathfrak{g}(F_{z,z}(t)),\mathfrak{g}(F_{Bz,Bz}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{z,Bz}(t))+\mathfrak{g}(F_{Bz,z}(t))\right)\end{array}\right\}\right)\\
&= \phi\left(\max\left\{\mathfrak{g}(F_{z,Bz}(t)),\mathfrak{g}(1),\mathfrak{g}(1),\frac{1}{2}\left(\mathfrak{g}(F_{z,Bz}(t))+\mathfrak{g}(F_{Bz,z}(t))\right)\right\}\right)\\
&= \phi\left(\mathfrak{g}(F_{z,Bz}(t))\right).
\end{aligned}
$$

On using Lemma 2.9, we have $Bz = z = Tz$ which shows that z is a common fixed point of the pair (B,T) and in all z is a common fixed point of the pairs

(A,S) and (B,T). The uniqueness of common fixed point is an easy consequence of inequality (1) in view of Lemma 2.9. This concludes the proof. $\square$

From the proof of Theorem 3.2, it is asserted that the common limit range property never requires any condition on closedness of the underlying subspaces, continuity of one or more mappings and containment of ranges amongst involved mappings.

Remark 3.3. Theorem 3.2 improves the results of Cho et al. [4], Singh et al. [22, Theorem 3.1, Corollary 3.3] and Singh et al. [23, Theorem 3.1, Corollary 3.1] and generalizes the results of Rao and Ramudu [19, Theorem 14].

Now, we give an example which demonstrates the validity of the hypotheses and degree of generality of our main result over comparable ones from the existing literature.

Example 3.4. Let (X,d) be a metric space with the usual metric d where $X = [1,15)$ and let $(X,\mathcal{F},\mathcal{T})$ be the induced N.A. Menger PM-space with $\mathfrak{g}(t) = 1-t$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x,y \in X$ and $t > 0$. Let A,B,S and T be four mappings from X to itself defined as

$$A(x) = \begin{cases} 1, & \text{if } x \in \{1\} \cup (3,15); \\ 14, & \text{if } x \in (1,3]. \end{cases} \qquad B(x) = \begin{cases} 1, & \text{if } x \in \{1\} \cup (3,15); \\ 5, & \text{if } x \in (1,3]. \end{cases}$$

$$S(x) = \begin{cases} 1, & \text{if } x = 1; \\ 6, & \text{if } x \in (1,3]; \\ \frac{x+1}{4}, & \text{if } x \in (3,15). \end{cases} \qquad T(x) = \begin{cases} 1, & \text{if } x = 1; \\ 9+x, & \text{if } x \in (1,3]; \\ x-2, & \text{if } x \in (3,15). \end{cases}$$

Then we have $A(X) = \{1,14\} \nsubseteq [1,13) = T(X)$ and $B(X) = \{1,5\} \nsubseteq [1,4) \cup \{6\} = S(X)$. Taking the sequences $\{x_n\} = \left\{3+\frac{1}{n}\right\}$, $\{y_n\} = \{1\}$ $\Big($or $\{x_n\} = \{1\}$, $\{y_n\} = \left\{3+\frac{1}{n}\right\}\Big)$, the pairs (A,S) and (B,T) satisfy the(CLR_{ST}) property, that is,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 \in S(X) \cap T(X).$$

By a routine calculation, one can easily verify the inequality (1). Thus, all the conditions of Theorem 3.2 are satisfied and 1 is a unique common fixed point of the pairs (A,S) and (B,T). It is noted that in this example that $S(X)$ and $T(X)$ are not closed subsets of X. Also, all the involved mappings are even discontinuous at their unique common fixed point 1.

Notice that the subspaces $S(X)$ and $T(X)$ are not closed subspaces of X, therefore Theorem 3.1 of Chauhan and Kumar [3] can not be used in the context of this example which establishes the genuineness of our extension.

Theorem 3.5. *Let A,B,S and T be four self mappings of a N.A. Menger PM-space $(X,\mathcal{F},\mathcal{T})$, where $\mathcal{T}$ is a continuous t-norm satisfying all the hypotheses of Lemma 3.1. Then A,B,S and T have a unique common fixed point provided both the pairs (A,S) and (B,T) are weakly compatible.*

Proof. In view of Lemma 3.1, the pairs (A,S) and (B,T) share the (CLR_{ST}) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z,$$

where $z \in S(X)\cap T(X)$. The rest of the proof can be completed on the lines of the proof of Theorem 3.2. □

Here, it is worth noting that the conclusions in Example 3.4 cannot be obtained using Theorem 3.5 as conditions (2) and (3) of Lemma 3.1 are not fulfilled. In what follows, we give another example which creates a situation wherein conclusion can be reached using Theorem 3.5.

Example 3.6. In the setting of Example 3.4, replace the self mappings A,B,S and T by the following, besides retaining the rest:

$$A(x) = \begin{cases} 1, & \text{if } x \in \{1\}\cup(3,15); \\ 10, & \text{if } x \in (1,3]. \end{cases} \qquad B(x) = \begin{cases} 1, & \text{if } x \in \{1\}\cup(3,15); \\ 4, & \text{if } x \in (1,3]. \end{cases}$$

$$S(x) = \begin{cases} 1, & \text{if } x = 1; \\ 4, & \text{if } x \in (1,3]; \\ \frac{x+1}{4}, & \text{if } x \in (3,15). \end{cases} \qquad T(x) = \begin{cases} 1, & \text{if } x = 1; \\ 13, & \text{if } x \in (1,3]; \\ x-2, & \text{if } x \in (3,15). \end{cases}$$

It is noted that $A(X) = \{1,10\} \subset [1,13] = T(X)$ and $B(X) = \{1,4\} \subset [1,4] = S(X)$. Clearly, both the pairs (A,S) and (B,T) satisfy the (CLR_{ST}) property, that is,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 \in S(X)\cap T(X).$$

Also all the conditions of Theorem 3.5 can be easily verified and 1 is a unique common fixed point of the pairs (A,S) and (B,T). Here, it is worth noting that Theorems 3.2 can not be used in the context of this example as $S(X)$ and $T(X)$ are closed subsets of X. Also, all the involved mappings are even discontinuous at their unique common fixed point 1.

By choosing A,B,S and T suitably, we can deduce corollaries involving two as well as three self mappings. The details of possible corollaries are not included here.

Now we utilize the notion of commuting pairwise due to Imdad et al. [7] and extend Theorem 3.2 to six self mappings in N.A. Menger PM-spaces.

Theorem 3.7. *Let A,B,R,S,H and T be six self mappings of a N.A. Menger PM-space $(X,\mathcal{F},\mathcal{T})$, where $\mathcal{T}$ is a continuous. Suppose that*

1. *the pairs (A,SR) and (B,TH) satisfy the $(CLR_{(SR)(TH)})$ property,*
2. $$\mathfrak{g}(F_{Ax,By}(t)) \leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{SRx,THy}(t)), \mathfrak{g}(F_{SRx,Ax}(t)),\\ \mathfrak{g}(F_{THy,By}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{SRx,By}(t)) + \mathfrak{g}(F_{THy,Ax}(t))\right)\end{array}\right\}\right), \tag{4}$$

for all $x,y \in X$, $t > 0$, where $\mathfrak{g} \in \Omega$ and ϕ satisfies the condition (Φ). Then (A,SR) and (B,TH) have a coincidence point each. Moreover, A,B,H,R,S and T have a unique common fixed point provided $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$, $BH = HB$ and $TH = HT$.

Proof. By Theorem 3.2, A,B,SR and TH have a unique common fixed point z in X. We show that z is a unique common fixed point of the self mappings A,B,R,S,H and T. Putting $x = Rz$ and $y = z$ in inequality (4), we have

$$\begin{aligned}
\mathfrak{g}(F_{A(Rz),Bz}(t)) &\leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{SR(Rz),THz}(t)), \mathfrak{g}(F_{SR(Rz),A(Rz)}(t)),\\ \mathfrak{g}(F_{THz,Bz}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{SR(Rz),Bz}(t)) + \mathfrak{g}(F_{THz,A(Rz)}(t))\right)\end{array}\right\}\right)\\
\mathfrak{g}(F_{Rz,z}(t)) &\leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{Rz,z}(t)), \mathfrak{g}(F_{Rz,Rz}(t)), \mathfrak{g}(F_{z,z}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{Rz,z}(t)) + \mathfrak{g}(F_{z,Rz}(t))\right)\end{array}\right\}\right)\\
&= \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{Rz,z}(t)), \mathfrak{g}(1), \mathfrak{g}(1),\\ \frac{1}{2}\left(\mathfrak{g}(F_{Rz,z}(t)) + \mathfrak{g}(F_{z,Rz}(t))\right)\end{array}\right\}\right)\\
&= \phi\left(\mathfrak{g}(F_{Rz,z}(t))\right).
\end{aligned}$$

On using Lemma 2.9, we have $z = Rz$. Hence $S(z) = S(Rz) = z$. Therefore we have $z = Az = Sz = Rz$. Now we assert that z is a common fixed point of B,T and H. To accomplish this, we use inequality (4) with $x = z$, $y = Hz$, we get

$$\begin{aligned}
\mathfrak{g}(F_{Az,B(Hz)}(t)) &\leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{SRz,TH(Hz)}(t)), \mathfrak{g}(F_{SRz,Az}(t)),\\ \mathfrak{g}(F_{TH(Hz),B(Hz)}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{SRz,B(Hz)}(t)) + \mathfrak{g}(F_{TH(Hz),Az}(t))\right)\end{array}\right\}\right)\\
\mathfrak{g}(F_{z,Hz}(t)) &\leq \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{z,Hz}(t)), \mathfrak{g}(F_{z,z}(t)), \mathfrak{g}(F_{Hz,Hz}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{z,Hz}(t)) + \mathfrak{g}(F_{Hz,z}(t))\right)\end{array}\right\}\right)\\
&= \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{z,Hz}(t)), \mathfrak{g}(1), \mathfrak{g}(1),\\ \frac{1}{2}\left(\mathfrak{g}(F_{z,Hz}(t)) + \mathfrak{g}(F_{Hz,z}(t))\right)\end{array}\right\}\right)\\
&= \phi\left(\mathfrak{g}(F_{z,Hz}(t))\right).
\end{aligned}$$

Thus, by Lemma 2.9, we have $z = Hz$. Hence $T(z) = T(Hz) = z$. Therefore z is a common fixed point of self mappings A,B,R,S,H and T. Uniqueness of common fixed point is an easy consequence of inequality (4). □

In view of Theorem 3.7, we can drive a fixed point theorem for four finite families of self mappings.

Corollary 3.8. *Let $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ be four finite families of self mappings of a N.A. Menger PM-space $(X,\mathcal{F},\mathcal{T})$, where $\mathcal{T}$ is a continuous with $A = A_1A_2\ldots A_m, B = B_1B_2\ldots B_n, S = S_1S_2\ldots S_p$ and $T = T_1T_2\ldots T_q$ satisfying inequality (1) of Lemma 3.1 such that the pairs (A,S) and (B,T) share the (CLR_{ST}) property. Then $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ have a unique common fixed point provided the pairs of families $(\{A_i\},\{S_k\})$ and $(\{B_r\},\{T_h\})$ commute pairwise, where $i \in \{1,2,\ldots,m\}, k \in \{1,2,\ldots,p\}, r \in \{1,2,\ldots,n\}$ and $h \in \{1,2,\ldots,q\}$.*

By setting $A_1 = A_2 = \ldots = A_m = A$, $B_1 = B_2 = \ldots = B_p = B$, $S_1 = S_2 = \ldots = S_n = S$ and $T_1 = T_2 = \ldots = T_q = T$ in Corollary 3.8, we deduce the following:

Corollary 3.9. *Let A,B,S and T be self mappings of a N.A. Menger PM-space $(X,\mathcal{F},\mathcal{T})$, where $\mathcal{T}$ is a continuous. Suppose that*

1. *the pairs (A^m,S^p) and (B^n,T^q) satisfy the (CLR_{S^p,T^q}) property, where m,n,p,q are fixed positive integers,*
2. $$\mathfrak{g}(F_{A^mx,B^ny}(t)) \le \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{S^px,T^qy}(t)),\mathfrak{g}(F_{S^px,A^mx}(t)),\\ \mathfrak{g}(F_{T^qy,B^ny}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{S^px,B^ny}(t))+\mathfrak{g}(F_{T^qy,A^mx}(t))\right)\end{array}\right\}\right), \quad (5)$$

for all $x,y \in X$, $t > 0$, $\mathfrak{g} \in \Omega$ where ϕ satisfies the condition (Φ). Then A,B,S and T have a unique common fixed point provided $AS = SA$ and $BT = TB$.

Remark 3.10. The conclusions of Theorems 3.2 remains true if we replace the inequality (1) by one of the following (for all $x,y \in X$, $t > 0$, where $\mathfrak{g} \in \Omega$ and ϕ satisfies the condition (Φ).):

$$\mathfrak{g}(F_{Ax,By}(t)) \le \phi\left(\max \mathfrak{g}(F_{Sx,Ty}(t)),\mathfrak{g}(F_{Sx,Ax}(t)),\mathfrak{g}(F_{Ty,By}(t)),\mathfrak{g}(F_{Sx,By}(t))\right), \quad (6)$$

$$\mathfrak{g}(F_{Ax,By}(t)) \le \phi\left(\max \mathfrak{g}(F_{Sx,Ty}(t)),\mathfrak{g}(F_{Sx,Ax}(t)),\mathfrak{g}(F_{Ty,By}(t))\right). \quad (7)$$

Remark 3.11. The results similar to Theorem 3.2, Theorem 3.5, Theorem 3.7, Corollary 3.8 and Corollary 3.9 can also be outlined in view of inequalities (6)-(7).

Remark 3.12. In view of Remark 3.10, the results improve the results of Khan and Sumitra [16, Theorem 2, Corollary 1].

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