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THE H-VECTOR OF THE UNION OF TWO SETS OF POINTS IN THE PROJECTIVE PLANE

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Given two h-vectors, h and h', we study which are the possible h-vectors for the union of two disjoint sets of points in $\mathbb{P}^2$, respectively associated to h and h' and how they can be constructed. We will give some bounds for the resulting h-vector and we will show how to construct the minimal h-vector of the union among all possible ones.

1. Introduction

Let k be an algebraically closed field, $\mathbb{P}^2 = \mathbb{P}^2(k)$ be the projective plane over k and $S = k[x_0, x_1, x_2]$ its homogeneous coordinate ring. Let $Hilb_n(\mathbb{P}^2)$ be the Hilbert scheme of zero-dimensional subschemes of degree n in $\mathbb{P}^2$. It is well known that it is a smooth connected projective variety of dimension $2n$. As the map $I \longrightarrow V(I)$ provides a one to one correspondence between homogeneous radical ideals of height 2 in S and reduced zero-dimensional schemes of $\mathbb{P}^2$, we will refer to a set of points in $\mathbb{P}^2$ also as a reduced zero-dimensional scheme $X \subseteq \mathbb{P}^2$.

If $X \in Hilb_n(\mathbb{P}^2)$ then we will denote by $H_X(i) = \dim_k((S/I_X)_i)$ its Hilbert function and by $h = (h_0, \dots, h_t)$ its h-vector where $h_0 = 1$ and $h_i = \triangle H_X(i) =$

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$H_X(i) - H_X(i-1)\ \forall\ i > 0$. If X is reduced we clearly have $h_0 + \cdots + h_t = \deg(X) =$ number of points in X. Hilbert functions provide a natural stratification of the Hilbert scheme. For any Hilbert function H of a degree n zero-dimensional scheme, let us define a smooth connected subscheme of $Hilb_n(\mathbb{P}^2)$ by

$$\mathbb{G}(H) = \{X \in Hilb_n(\mathbb{P}^2) |\ H_X = H\}.$$

Gotzmann proved that this stratum is smooth, connected and locally closed (see [8]).

In this paper we will concentrate on the h-vectors of reduced zero-dimensional schemes. The classification of all possible h-vectors of a reduced zero-dimensional subscheme of $\mathbb{P}^2$ is well known (see for example [6]). Despite this fact it is not clear what might be the h-vector of the union of two sets of points with given h-vectors h and h'. In particular an interesting question is whether there are some bounds on the h-vector of the union.

In Section 2 we start with some background results concerning the h-vector of the union of two sets of points when one of the sets consists of only one point. We also look at the case when the two sets of points are geometrically linked. Moreover we introduce a partial ordering on the set of the Hilbert functions.

In Section 3 we give some bounds for the h-vector of the union. We also introduce some exclusion criteria which help us to understand which are the possible h-vectors for the union of two given sets of points with fixed h-vectors.

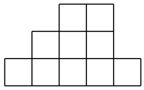
In Section 4, we present an algorithm for obtaining the possible h-vectors for ”the union” of two given h-vectors.

Finally, in Section 5, we prove our main result Proposition: 5.1 . For any two given h-vectors one can construct the unique minimal (with respect to the introduced partial order) h-vector among all admissible h-vectors for the union. This h-vector achieves the bounds given in Section 3.

2. Preliminaries

In order to answer the question of what may be the h-vector of the union of two disjoint sets of points in the projective plane it is natural to see what happens to the h-vector of the union if one of the sets is just a single point or if the sets are ”related” (more precisely linked). We will start in this direction introducing first some definitions.

Remark 2.1. To a given h-vector, $(h_0, h_1, \ldots, h_t)$, we can assign a diagram by drawing columns of h_i boxes for all i. For example if $h = (1,2,3,3,1)$, we will draw the following picture.



Definition 2.2. **(A)** Let A be a homogeneous Artinian graded k-algebra. We define :

$$\tau(A) = \max\{i \mid H_A(i) \neq 0\}.$$

(B) Let $X \subseteq \mathbb{P}^2$ be a zero-dimensional scheme with defining saturated ideal I_X. Then there is a linear form L which is a non-zerodivisor with respect to S/I_X such that $H_{S/(I_X+(L))} = \triangle H_X$. We define as in **(A)**:

$$\tau(X) := \tau(S/(I_X + (L))) = \max\{i \mid \triangle H_X(i) \neq 0\}.$$

Definition 2.3. Let $X, Y \subseteq \mathbb{P}^n$ be two projective subschemes such that no component of X is contained in any component of Y and conversely. Then, X is *geometrically directly linked* to Y by an arithmetically Gorenstein (**aG**) scheme, Z, if $Z = X \cup Y$. This means, in terms of defining ideals, that $I_Z = I_X \cap I_Y$. We also say that X is geometrically **G**-linked to Y. If Z is a complete intersection (**CI**) we say X is geometrically **CI**-linked to Y.

Note that if X and Y are linked by Z, then $\deg(Z) = \deg(X) + \deg(Y)$.

As already mentioned in this section the first question we approach is how does the Hilbert Function of a set of points $X \subseteq \mathbb{P}^2$ change if we add another point which is not in X. It is well known that in this case H_X will increase by one from a certain degree on. We give a short proof of this fact just for convenience. For more details on this and on a related questions see [6].

Lemma 2.4. *Let $X \subseteq \mathbb{P}^2$ be a reduced zero-dimensional scheme and let $P \in \mathbb{P}^2 - X$. Then there exists an integer d such that:*

$$H_{X \cup \{P\}}(i) = \begin{cases} H_X(i), & i < d \\ H_X(i) + 1, & i \geq d. \end{cases}$$

Proof. From the short exact sequence

$$0 \longrightarrow S/I_X \cap I_P \longrightarrow S/I_X \oplus S/I_P \longrightarrow S/(I_X + I_P) \longrightarrow 0,$$

follows that

$$H_{X \cup \{P\}}(i) = H_X(i) + 1 - H_{S/(I_X + I_P)}(i).$$

On the other side $S/I_P \cong k[t]$ and

$$S/(I_X + I_P) \cong \frac{S/I_P}{I_X(S/I_P)} \cong k[t]/I_X k[t].$$

As $k[t]$ is a principal ideal domain (PID), the ideal $I_X k[t]$ is generated by some homogeneous polynomial F of degree d. Therefore we have $I_X k[t] = (t^d)$ and $S/(I_X + I_P) \cong k[t]/(t^d)$. This concludes the proof. □

Remark 2.5. **(A)** From Lemma 2.4 follows, if $h = (h_0, \dots, h_{\tau(X)})$ is the h-vector of X then $d \leq \tau(X) + 1$ (more precisely $b \leq d \leq \tau(X) + 1$, where $b = \mathrm{inDeg}(I_X)$ is the least degree of a generator of I_X (see Theorem 2.10) and the h-vector of $X \cup \{P\}$ is given by

$$h' = \begin{cases} (h_0, \dots, h_{d-1}, h_d + 1, h_{d+1}, \dots, h_{\tau(X)}), & d < \tau(X) + 1 \\ (h_0, \dots, h_{\tau(X)}, 1), & d = \tau(X) + 1 \end{cases}$$

It holds especially

$$\tau(X) + 1 \geq \tau(X \cup \{P\}) \geq \tau(X)$$

(B) Looking at the proof of Lemma 2.4 , it can be easily seen that the statement does not change if we replace the set of points X with any projective variety $V \subseteq \mathbb{P}^2$ such that P does not lie on V.

(C) If X and Y are two disjoint sets of points in the projective plane then:

$$H_{X \cup Y}(i) = H_X(i) + H_Y(i) - H_{S/(I_X + I_Y)}(i).$$

In particular $S/(I_X + I_Y)$ is an artinian ring.

The h-vectors of geometrically linked sets of points in $\mathbb{P}^2$ are well studied. For example the following is known:

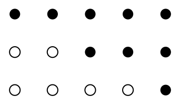
Proposition 2.6. *Let X, Y be two reduced zero-dimensional schemes in $\mathbb{P}^2$ which are geometrically linked over the* **CI** *scheme Z. If X, Y and Z have h-vectors $(1, a_1, \dots, a_{\tau(X)})$, $(1, b_1, \dots, b_{\tau(Y)})$ and $(1, 2, c_2 \dots, c_{\tau(Z)-2}, 2, 1)$. Then*

$$c_i - a_i = b_{(\tau(Z) - i)}.$$

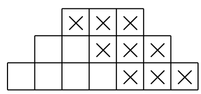
Proof. [2], Theorem 3. □

Remark 2.7. Notice that the last shift in the minimal free resolution of the **CI** ideal I_Z from Proposition 2.6 is $\tau(Z) + 2$.

Example 2.8. Consider the following set of points in $\mathbb{P}^2$:



Denote by X the white dots, by Y the black dots and by Z the set of all points. Then the **CI** Z links X to Y. The defining ideal of Z, I_Z is a **CI** of type $(3,5)$ i.e. the generators of I_Z are of degrees 3 and 5 respectively (one could take for example the first generator to be the product of the three horizontal lines and the second the product of the five vertical lines going through the points). Using the Koszul complex we can easily compute the minimal free resolution of I_Z and thus also the h-vector for Z which is $(1,2,3,3,3,2,1)$. The h-vector of X can also be easily computed and is $(1,2,2,1)$. Now Proposition 2.6 tells us that going to the diagrams of the h-vectors:



where the white squares constitute the h-vector of X, the remaining (crossed) squares (read from right to left) constitutes the h-vector of Y i.e. $(1,2,3,2,1)$.

If our zero-dimensional reduced schemes are geometrically linked, using Proposition 2.6 we can obtain some more information about the ring $S/(I_X+I_Y)$.

Lemma 2.9. *Let X,Y be two reduced zero-dimensional schemes in $\mathbb{P}^2$ which are geometrically linked over the **CI** scheme Z. Then :*

(1) *$S/(I_X+I_Y)$ is Gorenstein;*

(2) $\triangle H_{S/(I_X+I_Y)}(i) = \triangle H_Y(i) - \triangle H_Y(\tau(X\cup Y)-i)\ \forall i \geq 0$*;*

(3) $\tau(S/(I_X+I_Y)) = \tau(X\cup Y)$.

Proof. **(1)** This is a well known fact. It can be easily proved using the mapping cone (see for example [9], Chapter II, section 4) on the sequence

$$0 \longrightarrow S/I_Z \longrightarrow S/I_X \oplus S/I_Y \longrightarrow S/(I_X+I_Y) \longrightarrow 0.$$

(2) Let $(1,a_1,\ldots,a_{\tau(X)})$, $(1,b_1,\ldots,b_{\tau(Y)})$ and $(1,2,c_2\ldots,c_{\tau(X\cup Y)-2},2,1)$ be the h-vectors of X,Y and $Z=X\cup Y$ respectively.
By Proposition 2.6 using the above exact sequence we have:

$$\triangle H_{S/(I_X+I_Y)}(i) = a_i + b_i - c_i = b_i - b_{\tau(X\cup Y)-i},\ \forall i \geq 0.$$

(3) By Remark 2.5 **(A)** we have $\tau(Y) \leq \tau(X \cup Y)$. According to Proposition 2.6 $c_i - a_i = b_{\tau(X\cup Y)-i}$ and thus $b_{\tau(X\cup Y)} = 0$. Therefore $\tau(Y) < \tau(X \cup Y)$.
In **(2)** we have seen $\triangle H_{S/(I_X+I_Y)}(i) = b_i - b_{\tau(X\cup Y)-i}$, $\forall i \geq 0$ so that

$$\triangle H_{S/(I_X+I_Y)}(\tau(X \cup Y)) = b_{\tau(X\cup Y)} - b_0 = -1$$

and

$$H_{S/(I_X+I_Y)}(i) = 0 \; \forall i \geq \tau(X \cup Y) + 1.$$

Therefrom $\tau(S/(I_X + I_Y)) = \tau(X \cup Y)$. □

From the h-vector of a set of points in $\mathbb{P}^2$ using the following theorem due to E. D. Davis (see [1]) one can also obtain information about the geometric properties and the defining ideal of the points.

Theorem 2.10 (Davis)**.** *The h-vector $(h_0, \dots, h_{\tau(X)})$ of a reduced zero-dimensional scheme $X \subset \mathbb{P}^2$ satisfies the following conditions:*

- *$h_d = d+1$, for $d = 0, 1, \dots, b-1$, and $h_b \leq b$;*
- *$h_{d+1} \leq h_d$, for $d \geq b-1$;*
- *If $h_d = h_{d+1} = e$ for some $d \geq b-1$, the generators of I_X of degree at most $d+1$ have a common factor of degree e. This leads to a partition of X into $X_1 \cup X_2$, where X_1 lies on a curve of degree e and X_2 has the h-vector given by $(h_e - e, h_{e+1} - e, \dots, h_{d-1} - e)$.*
  Where $b = \mathrm{inDeg}(I_X)$ is the least degree of a generator of I_X.

Definition 2.11. Let $h = (h_0, \dots, h_{\tau(X)})$ be the h-vector of some reduced zero-dimensional scheme $X \subset \mathbb{P}^2$. If $h_d = h_{d+1} = e$, for some $d \geq b-1$, where $b = \mathrm{inDeg}(I_X)$ we say that the h-vector of X has a *flat of height e*.

Remark 2.12. **(A)** The result of Davis can be used to exclude some h-vectors which cannot be the h-vector of the union of two sets of points with given h-vectors h and h' . For example the h-vector $(1,2,1,1,1)$ cannot be the h-vector of the union of two sets of points with h-vectors $(1,1,1)$ and $(1,1,1)$, simply because it corresponds to a set of points where 5 are on a line and one is out and (1,1,1) correspond to 3 points on a line. A similar argument shows that the h-vector $(1,2,3,3,2,2,1,1,1,1)$ cannot be the h-vector of the the union of two sets of points with h-vectors $(1,2,3,3,1,1)$ and $(1,2,3)$.

(B) Let $X \subseteq \mathbb{P}^2$ be a reduced zero-dimensional scheme with defining ideal $I_X = (F_1, \ldots, F_s)$ and let F be a common factor of degree e for some of the minimal generators of I_X as in Theorem 2.10. Then X is the union of two sets of points X_1 and X_2 where X_1 are the points in X lying on the curve F. Scheme-theoretically the defining saturated ideals of X_1 and X_2 are

$$I_{X_1} = (I_X + (F))^{Sat} \text{ and } I_{X_2} = (I_X \underset{S}{:} F).$$

As in [12], Chapter 1.3 using

$$\frac{I_X + (F)}{(F)} \cong \frac{I_X}{I_X \cap (F)} = \frac{I_X}{F \cdot (I_X \underset{S}{:} F)}$$

we can obtain two exact sequences:

$$\textbf{(1)} \quad 0 \longrightarrow I_{X_2}(-e) \longrightarrow I_X \longrightarrow \frac{I_X + (F)}{(F)} \longrightarrow 0$$

and

$$\textbf{(2)} \quad 0 \longrightarrow S(-e) \longrightarrow I_X + (F) \longrightarrow \frac{I_X + (F)}{(F)} \longrightarrow 0.$$

In order to find out what are the possible h-vectors of the union of two sets of points, we would like to be able to "compare" the h-vectors of zero-dimensional reduced schemes in the projective plane. For this reason we will introduce a partial order on the set

$$\mathcal{H} := \{\triangle H_X \,|\, X \subseteq \mathbb{P}^2 \text{ is a finite set of points}\}.$$

Definition 2.13. Let $\mathbb{H}_1 = (H_X(i))_{i\in\mathbb{N}}$ and $\mathbb{H}_2 = (H_Y(i))_{i\in\mathbb{N}}$ be two Hilbert functions, we will say that $\mathbb{H}_2$ is *more generic* than $\mathbb{H}_1$, and we will write $\mathbb{H}_1 \leq_g \mathbb{H}_2$, if $H_X(i) \leq H_Y(i)$, $\forall\, i \in \mathbb{N}$. We say also in this situation that $\mathbb{H}_1$ is *more special* than $\mathbb{H}_2$.

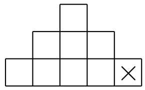
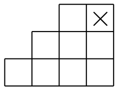
Remark 2.14. It follows immediately from the definition that $\leq_g$ is a partial order. It is not difficult to see, considering for example the Hilbert functions $(1,3,6,7,8,9,10,11,\longrightarrow)$ and $(1,3,5,7,9,10,11,\longrightarrow)$, that it is not a total order.

This partial order induce also a partial order (we use the same notation $\leq_g$) on $\mathcal{H}$. Indeed, if $(h_0,h_1,\ldots,h_s)$ and $(h'_0,h'_1,\ldots,h'_t)$ are the h-vectors of two finite sets of points in $\mathbb{P}^2$. Then:

$$(h_0,h_1,\ldots,h_s) \leq_g (h'_0,h'_1,\ldots,h'_t) \;:\Longleftrightarrow\; \sum_{i=0}^{j} h_i \leq \sum_{i=0}^{j} h'_i,\ \forall\ j = 0,\ldots,s$$

Going to the diagram of an h-vector, this means that if *we move one box from a row to an upper row*, so that the result is admissible (i.e. there is a set of points in $\mathbb{P}^2$ with this h-vector), we get a more generic h-vector. The following example shows that.

Example 2.15. Given the two h-vectors, $h = (1,2,3,2,1)$, $h' = (1,2,3,3)$. We have clearly $h \leq_g h'$. Considering the corresponding diagrams

we see that if we move the crossed box to the upper row, we obtain a more generic h-vector.

Remark 2.16. We have seen in Theorem 2.10 that if we have a set of points $X \subseteq \mathbb{P}^2$ with given h-vector $h = (h_0, \dots, h_t)$ which has a flat of degree $e < b = \operatorname{inDeg}(I_X)$ then some of the points of X will be on a curve of degree e. In general if an h-vector h and an integer d is given, one can consider the set

$$\left\{ \triangle H_Y \;\middle|\; \begin{array}{c} Y \subseteq X \subseteq \mathbb{P}^2,\ X \text{ is a set of points with } \triangle H_X = h \\ Y \text{ lies on a curve of degree d,} \\ \text{which does not contain all of the points of } X \end{array} \right\}.$$

According to the partial order introduced above this set is ordered and it contains a unique maximal element (for a proof see [4], Theorem 3.15).

3. Bounds on the h-vectors of the union of sets of points

Given two h-vectors h and h' we want to see what are the possible h-vectors h'' for the union of two disjoint sets of points in $\mathbb{P}^2$ with the given h-vectors h and h'. In this section we give some bounds that the resulting h-vector h'' has to satisfy.

We will use $\tau(X)$ as defined in 2.2. An equivalent description is

$$\tau(X) = \max\{i \mid h_i \neq 0\} = \min\{i \mid H_X(i) = H_X(i+1)\}.$$

The h-vector h corresponding to X has $\tau(X)+1$ non-zero entries. We say that $\tau(X)+1$ is *the length* of the h-vector. Sometimes we will use $\tau(h)$ instead of $\tau(X)$.

Proposition 3.1 (Subset criterion)**.** *Let $X \subset Y$ be two sets of points in $\mathbb{P}^2$, let h be the h-vector of X and h' be the h-vector of Y. We have then:*

$$h'_i \geq h_i \text{ for all } i = 0, \dots, \tau(X).$$

Proof. By Remark 2.5 **(A)** the h-vector of $X \cup \{P\}$ is the h-vector of X increased in some degree d by one. In particular if $Y = X \cup \{P\}$ then $h_i \leq h_i' \; \forall i \geq 0$.
The thesis easily follows by induction on $|Y|$. □

Remark 3.2. (A) In our graphical way of representing h-vectors this means that the h-vectors of X and Y have to fit into the h-vector of the union $X \cup Y$. For example the h-vectors $(1,2,1)$ and $(1,1,1,1)$ fit into the h-vector $(1,2,2,1,1,1)$ but $(1,2,3,1)$ does not. This is an easy way to see that the h-vector $(1,2,2,1,1,1)$ cannot be obtained as the union of the h-vector $(1,2,3,1)$ and something else.

(B) With the subset criterion, one can easily see that if X is a set of points in $\mathbb{P}^2$ with h-vector $(h_0,\ldots,h_{\tau(X)})$ then at most $\tau(X)+1$ of the points of X can be collinear.

The result of the following Proposition is known but we prove it for completeness.

Proposition 3.3. *Let $X \subseteq \mathbb{P}^2$ be a set of points with h-vector $h = (h_0,\ldots,h_{\tau(X)})$. Then the defining ideal of X, I_X is generated in degree at most $\tau(X)+1$.*

Proof. Let

$$0 \longrightarrow \bigoplus_{i=1}^{t} S(-b_i) \longrightarrow \bigoplus_{i=1}^{t+1} S(-a_i) \longrightarrow I_X \longrightarrow 0$$

be the minimal free resolution of I_X, where $a_1 \leq a_2 \leq \ldots \leq a_{t+1}$ are the degrees of the minimal generators of I_X. For $i = 1,\ldots,t+1$ denote by $g_j = |\{a_i | a_i = j\}|$ the number of the integers a_i equal to j. Let $\triangle h$ be the first difference of the h-vector of X i.e. $\triangle h(i) = h_i - h_{i-1}$ and $w = \min\{i \in \mathbb{N} | \triangle h(i) < 0\}$. According to [11], Theorem 1.2 we have

$$-\triangle^2 h(j) \leq g_j \leq -\triangle h(j) \; \forall j > w$$

As $\triangle h(\tau(X)+1) = -h_{\tau(X)} \neq 0$ and $\triangle h(j) = 0$ for all $j > \tau(X)+1$ the maximal value of j for which g_j could be non zero, or equivalently there exists some a_i equal j, is $\tau(X)+1$.

□

We now want to give a bound on the length of the h-vector of the union of X and Y.

Definition 3.4. A homogeneous element f of the homogeneous coordinate ring S/I_X of X is a *separator* of the point $P \in X$ if $f(P) \neq 0$ and $f(Q) = 0$ for all $Q \in X$, $Q \neq P$.

Proposition 3.5. *For any set of points $X \subset \mathbb{P}^2$, $\tau(X)$ is the least degree for which there are separators for every point $P \in X$.*

Proof. For more details on separators and for the proof see [5]. □

Theorem 3.6. *Given two sets of points X and Y in $\mathbb{P}^2$, we have that*

$$\max\{\tau(X), \tau(Y)\} \leq \tau(X \cup Y) \leq \tau(X) + \tau(Y) + 1.$$

In other words, the length of the h-vector of the union is at most the sum of the lengths of the given h-vectors.

Proof. Let P be a point in X, then there exists a separator f of $P \in X$ which has degree $\tau(X)$. If $P \notin Y$ we can find a generator g of I_Y such that $g(P) \neq 0$, it has degree at most $\tau(Y) + 1$. In the case $P \in Y$ we choose g to be a separator of P in Y of degree less than $\tau(Y) + 1$. Now fg is a separator of P in $X \cup Y$ of degree at most $\tau(X) + \tau(Y) + 1$. This can be done for all points in X and Y. Now the theorem follows from Proposition 3.5. The first inequality follows from Remark 2.5. □

Definition 3.7. We define $b(X) = \max\{h_i |\ i = 0, 1, \ldots, \tau(X)\}$ as the *height* of the h-vector associated to X.

Proposition 3.8. *For any set of points $X \subseteq \mathbb{P}^2$, we have that the height of the h-vector is equal to the initial degree of the ideal of X.*

$$b(X) = \text{inDeg}(I_X).$$

Proof. The height of the h-vector is equal to the initial degree of the ideal I_X, as by Davis' Theorem the h-vector grows by one in each degree until it gets the maximum value $b(X)$ and then, it starts to decrease (not necessarily strictly). This means that the Hilbert function of X is equal to the Hilbert function of S up to degree $b(X) - 1$. From degree $b(X)$ on the Hilbert function of S/I_X is strictly smaller than the one of S, this means that I_X has no generators of degree smaller than $b(X)$ and it has at least one generator in degree $b(X)$. □

Theorem 3.9. *Given two sets of points X and Y in $\mathbb{P}^2$, we have the following bounds for the height of the resulting h-vector:*

$$\max\{b(X), b(Y)\} \leq b(X \cup Y) \leq \min\{b(X) + b(Y), b(G)\}$$

where G consists of $\deg(X \cup Y)$ *generic points.*

Proof. We first observe that the height of G is given exactly by

$$b(G) = \max\left\{ i \,\middle|\, \frac{i(i+1)}{2} \leq \deg(X \cup Y) \right\},$$

and this is the h-vector with maximal height that we can have for $\deg(X \cup Y)$ points: so, $b(X \cup Y) \leq b(G)$. Moreover, it is easily proved that the height is also bounded by the sum of the heights: indeed, $I_X \cdot I_Y \subseteq I_X \cap I_Y$, and so $b(X \cup Y) \leq \mathrm{InDeg}(I_X \cdot I_Y) = b(X) + b(Y)$.

The lower bound for the $b(X \cup Y)$ is obtained from the inclusions $I_X \cap I_Y \subseteq I_X, I_Y$. □

Example 3.10. We now show in two simple examples how to exclude h-vectors from the set of the admissible ones for the union of two given h-vectors. The h-vectors $(1,1,1)$ and $(1,1,1,1)$ cannot give the h-vector $(1,2,3,1)$, since the maximum height that we can have is the sum of the heights, i.e. 2. Another example of exclusion is that we cannot achieve $(1,2,1,1,1,1)$ from $(1,2,1)$ and $(1,2)$, since the maximal length of the union is 5.

Another useful exclusion technique uses arguments about the number of points that lie on a curve of certain degree.

Definition 3.11. We define

$$\eta(h,d) := \sum_{i=0}^{\tau(X)} \min\{h_i, d\}.$$

One can think of $\eta(h,d)$ as the sum of the entries of the h-vector h when we cut it off in height d. The following proposition gives a bound on η.

Proposition 3.12. *Let $X \subseteq \mathbb{P}^2$ be a set of points such that n points of X lie on a curve of degree d. Then the h-vector of X is such that*

$$\eta(h,d) \geq n.$$

Proof. Let Y be the subset of X consisting of n points lying on the degree d curve. The curve is an element of I_Y hence by Proposition 3.8, the height of the h-vector is at most d. Moreover, the sum of the components of h-vector of Y is n. Now the result follows from the subset criterion. □

Together with Davis' Theorem, we can use the proposition above to introduce another exclusion criterion for generic h-vectors.

Proposition 3.13. *Let h be the h-vector of X with flat of height r, h' the h-vector of Y with flat of height s and h'' the h-vector of $X \cup Y$. Then we have*

$$\eta(h'', r+s) \geq \eta(h, r) + \eta(h', s).$$

Proof. We know that at least $\eta(h,r)$ points lie on a curve of degree r and another $\eta(h',s)$ points lie on a degree s curve, so these points together lie on a degree $r+s$ curve. □

Remark 3.14. When we apply the previous statement to the case in which both r and s are equal to the maximal height of the h-vector we get the upper bound for the height.

The following exclusion criterion makes use of the flats in the h-vectors. Instead of a flat we can also use the maximal height.

Theorem 3.15. *Given two h-vectors h and h' with a flats of degree d and d' respectively. For the union we can then exclude h-vectors h'' which have a flat of degree $d'' \geq d, d'$ with*

$$\max\left\{\eta(h,d'') + \eta(h',d''+d') - \eta(h',d') + \binom{d'+2}{2} - 2, \eta(h',d'')+\right.$$
$$\left.\eta(h,d''+d) - \eta(h,d) + \binom{d+2}{2} - 2\right\} < \eta(h'',d'') < \eta(h,d) + \eta(h',d').$$

Proof. We want to show that it is not possible to satisfy both inequalities. So we assume the second one holds and then show that the first cannot hold.

By hypothesis, h, h' and h'' have flats respectively of degrees, d, d' and d'', so let C, C' and C'' be the curves given by Davis' Theorem. Because of the second inequality not both curves C and C' can be components of C''. We will now investigate how many points of $X \cup Y$ can lie on C'' given this restriction. If we assume C is a component of C'' then at most $\eta(h,d'')$ points from X lie on C''. At most $\binom{d'+2}{2} - 2$ points in $Y \cap C'$ can lie on C'', since otherwise C' would be a component of C''. The remaining points in Y not on C' have the h-vector which you get from h' by cutting of the last d' lines, so here at most $\eta(h,d''+d) - \eta(h,d)$ can lie on a degree d'' curve. If we add up these numbers we get the first entry of the max, the second one is yield by exchanging the roles of C and C'. □

Example 3.16. Configurations of points with h-vectors $(1,2,3,4,5,2,2,2)$ and $(1,1,1,1,1,1,1,1)$ (8 on a line) cannot give h-vectors with a flat of degree 2 and $17 < \eta(h'',2) < 24$.

4. Linear Configurations and Partitions

Later in this section, we will introduce a construction for the "union" of two h-vectors. In order to do this, we need first to give some tools, namely, we need to define linear configurations, 2-type vectors and pseudo 2-type vectors.

Definition 4.1. A 2-*type vector* is a vector $(d_1, d_2, \ldots, d_t)$, where $0 < d_1 < d_2 < \cdots < d_t$.

To any h-vector, associated to a reduced zero-dimensional subscheme of $\mathbb{P}^2$, corresponds only one 2-type vector. The following theorem explains this correspondence.

Theorem 4.2 ([7], Theorem 2.4 and Theorem 2.5)**.** *Let S_2 denote the collection of Hilbert functions of all reduced zero-dimensional schemes in $\mathbb{P}^2$. Then, there is a 1-1 correspondence between S_2 and the set of* 2*-type vectors. Moreover, let $T = (d_1, d_2, \ldots, d_t)$ be a* 2*-type vector , and H_i the Hilbert function of d_i collinear points. Then, T corresponds to the Hilbert function defined by $H(j) = H_t(j) + \cdots + H_1(j - (t-1))$.*

Once we have the definition of 2-type vector we can define the concept of linear configuration.

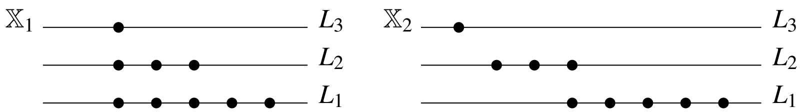
Definition 4.3. Let $T = (d_1, \ldots, d_t)$ be a 2-type vector. Let $L_1, \ldots, L_t$ be t distinct lines in $\mathbb{P}^2$ and $\mathbb{X}_i$ a set of d_i distinct points on L_i, for all $i = 1, \ldots, t$. Moreover, we suppose that, for $i \neq j$, L_i does not contain any point of $\mathbb{X}_j$. Then, $\mathbb{X} = \cup_{i=1}^{t} \mathbb{X}_j$ is called a *linear configuration of type T*.

The following result shows that the Hilbert function associated to a linear configuration of a given type T depends only on the type, and not on the choice of the lines and of the points on them.

Theorem 4.4 ([7], Theorem 2.8)**.** *Let $\mathbb{X}$ be a linear configuration of type T, and let H be the Hilbert function associated to T. Then the Hilbert function of $\mathbb{X}$, $H_{\mathbb{X}}$ is H.*

The theorem above can be visualized in the following example: these two different linear configurations have the same Hilbert function.

Example 4.5. Let us fix the 2-type vector $T = (1, 3, 5)$. Consider the following linear configurations of this given type.



The two reduced zero-dimensional schemes, $\mathbb{X}_1$ and $\mathbb{X}_2$, have the same h-vector since they are associated to the same 2-type vector.

Remark 4.6. Hence, whenever we have such linear configurations associated to 2-type vectors, we can choose any set of distinct of points on each line, and still obtain a linear configuration with that given type, and with that given h-vector associated to the type.
Moreover, given a reduced zero-dimensional scheme which is also a linear configuration, its type is nothing else than a partition of the degree of the scheme constituted by strictly increasing positive integers.

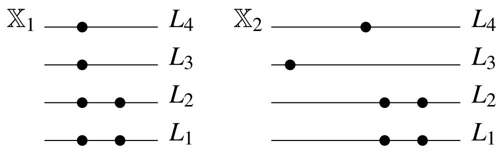
A similar definition of linear configurations can be given for partitions of the degree of the scheme constituted by non-decreasing positive integers.

Definition 4.7. A *pseudo type vector* is a sequence of positive integers $T = (d_1, \dots, d_t)$, where $d_i \leq d_{i+1}\ \forall i$, and if $d_{i-1} = d_i$, then $d_i < d_{i+1}$.
A *pseudo linear configuration* of type T is a set of points $\mathbb{X} = \cup_{i=1}^{t} \mathbb{X}_i$, where $\mathbb{X}_i$ is a set of d_i distinct points on a line L_i. The lines $L_1, \dots, L_t$ are all different, and none of the points of $\mathbb{X}_i$ lies on L_j for $i \neq j$.

Also in this case, an O-sequence can be associated to a pseudo type vector (see [7]), but in general the Hilbert function of a pseudo linear configuration of type $T = (d_1, \dots, d_t)$ is not uniquely determined. It is uniquely determined only if a certain condition on the first difference of the pseudo type vector holds, namely:

$$\textit{between any two zero entries of } \Delta T \textit{ there is at least one entry } > 1. \qquad (1)$$

The Example 3.8 in [7] shows that in the case in which the previous condition does not hold, for instance when $T = (1, 1, 2, 2)$, we cannot choose any set of distinct points in the lines and keep the same Hilbert function.



$\mathbb{X}_1$ and $\mathbb{X}_2$ have not the same h-vector, namely $\mathbb{X}_1$ has h-vector $(1, 2, 2, 1)$, and $\mathbb{X}_2$ has h-vector $(1, 2, 3)$, since they are 6 generic points.

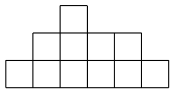
Let $(h_0, \dots, h_t)$ be the h-vector of a reduced zero-dimensional scheme in $\mathbb{P}^2$. To this given h-vector we can associate a monomial ideal I such that the standard graded k-algebra S/I has the given h-vector. We do this as follows:

1. From the given h-vector $(h_0, \dots, h_t)$ we pass to its geometric representation by drawing h_i boxes for all i.

2. By Davis's Theorem we have that $h_i = i+1$ for $i = 0,\dots,b-1$ and $h_i \geq h_{i+1}$ $\forall i \geq b-1$. Denote by d_i the number of squares in the i-th row, for $i = 1,\dots,b$. Notice that the vector $D = (d_1,\dots,d_b)$ is a 2-type vector.

3. Let I be the ideal generated by $x^{d_b}, x^{d_{b-1}}y,\dots,y^b$. Then the k-algebra S/I has h-vector $(h_0,h_1,\dots,h_t)$. Note also that the ideal I is a lex-segment ideal i.e. if a monomial $M \in I$ then every larger monomial (with respect to the lexicographical ordering) of the same degree is also in I. In this way to a given h-vector of a reduced zero-dimensional scheme in $\mathbb{P}^2$ we can assign in a unique way a lex-segment ideal.

Example 4.8. For $h = (1,2,3,2,2,1)$ we have



Therefore $b = 3, d_3 = 6, d_2 = 4, d_1 = 1$ and the corresponding ideal is

$$I = (x^6, x^4y, xy^2, y^3)$$

To the ideal $I = (x^{d_b}, x^{d_{b-1}}y,\dots,y^b)$ we can assign a set of points in $\mathbb{P}^2$, whose defining ideal has the same h-vector. This set of points is a linear configuration of type D. In order to get it we first choose two sets of distinct elements in k, $\{\alpha_1,\dots,\alpha_{d_b}\}$ and $\{\beta_1,\dots,\beta_b\}$, and then replace every generator x^iy^j, of I by $(x-\alpha_1 z)\dots(x-\alpha_i z)(y-\beta_1 z)\dots(y-\beta_j z)$. The ideal that we obtain in this way is the defining ideal of the following set of points in the projective plane:

d_b points with coordinates $(\alpha_i : \beta_1 : 1)$, $i = 1,\dots,d_b$;

d_{b-1} points with coordinates $(\alpha_i : \beta_2 : 1)$, $i = 1,\dots,d_{b-1}$;

$\vdots$

d_1 points with coordinates $(\alpha_i : \beta_b : 1)$, $i = 1,\dots,d_1$.

In the special case where $\alpha_1 = 0,\dots,\alpha_{d_b} = d_b - 1$ and $\beta_1 = 0,\dots,\beta_b = b-1$ we get

d_b points with coordinates $(i : 0 : 1)$, $i = 0,\dots,d_b - 1$;

d_{b-1} points with coordinates $(i : 1 : 1)$, $i = 0,\dots,d_{b-1} - 1$;

$\vdots$

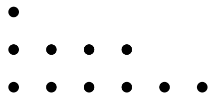
d_1 points with coordinates $(i : b-1 : 1)$, $i = 0,\dots,d_1 - 1$.

We will call this the *standard linear configuration of type D*.
Note that by this construction every box in the diagram of a given h-vector corresponds to a point in the projective plane.

Example 4.9. Consider again the h-vector $h = (1,2,3,2,2,1)$. Using the standard linear configuration, we obtain

$$\begin{aligned}
d_3 &= 6 \text{ points with coordinates } (i:0:1),\ i = 0,\dots,5\\
d_2 &= 4 \text{ points with coordinates } (i:1:1),\ i = 0,\dots,3\\
d_1 &= 1\ \text{ point with coordinates } (0:2:1)
\end{aligned}$$

i.e. we obtain the following picture



Remark 4.10. The picture in the previous example can also be seen as the set of all monomials in x, y which are not in the ideal $I = (x^6, x^4y, xy^2, y^3)$. In the following, such a picture will have the same double interpretation.

To find all other possible pseudo linear configurations associated to a given h-vector, different from the standard one, we use the following idea, which comes from a paper of Maggioni and Ragusa, see [11].

Given an h-vector $h = (h_0, h_1, h_2, \dots, h_t)$, associated to a set of points in $\mathbb{P}^2$, let us consider the first difference $\Delta(h)$, and the second difference, $\Delta^2(h)$, let g_i be the number of generators in degree i in an ideal with that given h-vector, and s_i the number of first syzygies in degree i, let $w = \min\{i \mid \Delta^2 h(i) < 0\}$, then according to [11], Theorem 1.2, we have the following bounds:

$$\begin{aligned}
&g_w = -\Delta^2 h(w),\ s_w = 0;\\
&-\Delta^2 h(i) \le g_i \le -\Delta h(i),\ \ s_i = g_i + \Delta^2 h(i),\ \forall\, i > w.
\end{aligned} \tag{2}$$

All the numbers, g_i and s_j, satisfying these bounds are allowed and give a set a possibilities for the generators and the syzygies.

Let $x = \max\{i \mid \Delta h(i) \neq 0\} = \tau(h) + 1 = \text{length}(h)$, let us write the degrees of the generators and the syzygies in the following table

$$\begin{gathered}
\overbrace{w\ w \cdots w}^{g_w\ times}\ \overbrace{w+1 \cdots w+1}^{g_{w+1}\ times} \cdots \overbrace{x \cdots x}^{g_x\ times}\\
\overbrace{w+1\ \cdots w+1}^{s_{w+1}\ times} \cdots\quad \cdots \overbrace{x \cdots x}^{s_x\ times}\,.
\end{gathered} \tag{3}$$

From such a table, we can construct different partitions of the degree of the scheme, i.e. we can obtain all different pseudo type vectors corresponding to the given h-vector. Notice that this way we obtain also different lex-segment monomial ideals, associated to the given h-vector. More precisely once we have made the choice for the g_i's, and thus for the s_i's, we can write the chosen values in form of a table as follows .

$$D = \begin{pmatrix} a_1 & a_2 & \dots & a_{\alpha+1} \\ \cdot & b_1 & \dots & b_\alpha \end{pmatrix}.$$

where $\alpha \le \tau(h)+1 = \text{length}(h)$.

Proposition 4.11. *Starting from the table D, the partition of the degree of the scheme is obtained in the following way:*

$$\overbrace{a_2 - [(b_\alpha - a_{\alpha+1}) + \dots + (b_2 - a_3)], \dots, a_2 - [(b_\alpha - a_{\alpha+1}) + \dots + (b_2 - a_3),]}^{b_1 - a_2 \textit{ times}},$$

$$\dots, \overbrace{a_\alpha - (b_\alpha - a_{\alpha+1}), \dots, a_\alpha - (b_\alpha - a_{\alpha+1})}^{b_{\alpha-1} - a_\alpha \textit{ times}}, \overbrace{a_{\alpha+1}, \dots, a_{\alpha+1}}^{b_\alpha - a_{\alpha+1} \textit{ times}},$$

Proof. Let us introduce $e_i = b_i - a_i$ and $f_j = b_j - a_{j+1}$, for $i, j = 1, \dots, \alpha$. Since $\deg(X) = \sum_{i \le j} e_i f_j$ (see [3], Corollary 3.10) and $\sum_{i=1}^{\alpha} b_i = \sum_{j=1}^{\alpha+1} a_j$, we have that

$$\deg(X) = \sum_{i=1}^{\alpha} f_i(e_1 + \dots + e_i) =$$

$$\sum_{i=1}^{\alpha-1} (b_i - a_{i+1})\,[a_{i+1} - (b_\alpha - a_{\alpha+1}) - \dots - (b_{i+1} - a_{i+2})] + (b_\alpha - a_{\alpha+1})a_{\alpha+1};$$

which is what we wanted to prove.
Note that the elements of the partition are ordered in a non-decreasing way. □

Remark 4.12. By permuting elements in the first row and elements in the second row of D, but maintaining the conditions $b_j > a_{j+1}$ for each j, we will get other partitions of the degree of the scheme, with the same choice of $g_i's$.

Remark 4.13. The partition of the degree of the scheme that we get with this construction have not to be necessarily 2-type vectors or pseudo vectors. For instance, by using Remark 4.12, given the h-vector $(1,1,1)$, we have the following corresponding tables

$$A_1 = \begin{pmatrix} 1 & 3 \\ \cdot & 4 \end{pmatrix}, \; A_2 = \begin{pmatrix} 3 & 1 \\ \cdot & 4 \end{pmatrix}.$$

They correspond to the partitions (3) and $(1,1,1)$ respectively. The first is a 2-type vector but the second is not even a pseudo type vector.

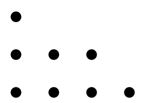
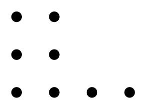
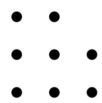
We give now the algorithm for this construction and then an example on what it actually does. It takes as an input a h-vector, and gives as an output all the associated partitions.

Name : $FindPartition$;
Input : $h-vector, \mathbf{h} = (h_0, h_1, \dots, h_t)$;
Output : $all\ the\ partitions\ associated\ to\ \mathbf{h}\ of\ \sum_{i=0}^{t} h_i$;
$h'_0 = 1$
$FOR\ i = 1, \dots, t+1$
  $h'_i = h_i - h_{i-1}$
$END\ FOR$;
$h''_0 = 1$
$FOR\ i = 1, \dots, t+2$
  $h''_i = h'_i - h'_{i-1}$
$END\ FOR$;
$w := min\{i |\ h''i < 0\}$
$g_w := -h''_w; s_w := 0$;
$FOR\ g_{w+1} = max\{0, -h''_{w+1}\}, \dots, -h'_{w+1}; \dots$
$FOR\ g_{t+2} = max\{0, -h''_{t+2}\}, \dots, -h'_{t+2}$;
  $FOR\ j = w+1, \dots, t+2,$
    $s_j = g_j + h''_j,$
  $Write\ what\ we\ get\ in\ 3\ in\ the\ form$
  $\begin{pmatrix} a_1 & a_2 & \dots & a_{\alpha+1} \\ \cdot & b_1 & \dots & b_\alpha \end{pmatrix}.$
    $FOR\ any\ \sigma \in S_{\alpha+1},\ FOR\ any\ \delta \in S_\alpha$
      $IF\ (b_{\delta(1)} > a_{\sigma(2)}) \& \cdots \& (b_{\delta(\alpha)} > a_{\sigma(\alpha+1)}),$
      $compute$
      $(a_{\sigma(2)} - \sum_{i=2}^{\alpha} (b_{\delta(i)} - a_{\sigma(i+1)}))(b_{\delta(1)} - a_{\sigma(2)}\ times), \dots,$
      $\dots, a_{\sigma(\alpha+1)}(b_{\delta(\alpha)} - a_{\sigma(\alpha+1)}\ times),$
    $END\ FOR$;
  $END\ FOR$;
$END\ FOR; \dots; END\ FOR.$

Example 4.14. Let us consider the h-vector $h=(1,2,3,2)$. In this case, $w=3$, $x=4$, and moreover $g_3=2$, $1\le g_4\le 2$, $g_5=0$. Let us choose $g_4=2$, this implies that $s_4=1$, $s_5=2$.
The following degree tables all correspond to this choice:

$$\begin{pmatrix} 3 & 3 & 4 & 4 \\ \cdot & 4 & 5 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 3 & 4 \\ \cdot & 5 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 4 & 3 \\ \cdot & 5 & 5 & 4 \end{pmatrix}.$$

From these tables, we obtain that the following partitions are all associated to the same choice of the g_i's.

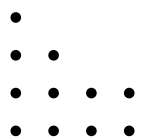




Once we have such a partition, say $\alpha_1,\cdots,\alpha_n$ with $\alpha_j\le\alpha_{j+1}$, we can associate to this a monomial ideal in x,y with the given invariants, i.e. the degrees of the generators, the degrees of the first syzygies and, of course, the h-vector, as described in the example below.

Example 4.15. Given $h=(1,2,3,4,1)$, using the bounds in (2), we have that an ideal, associated to this h-vector, can be generated by 4 elements of degree 4, i.e. $g_4=4$, $g_i=0\ \forall\, i>4$: this implies that there are 2 syzygies of degree 5, and one of degree 6. So, in this case, the table we have chosen is:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \cdot & b_2 & b_3 & b_4 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 4 & 4 \\ \cdot & 5 & 5 & 6 \end{pmatrix}.$$

The degree of this scheme is 11, and, hence, a partition of 11, by Proposition 4.11, is given by: $b_4-a_4=2$ times $a_4=4$, $b_3-a_3=1$ times $a_3-(b_4-a_4)=2$, $b_2-a_2=1$ time $a_2-[(b_4-a_4)+(b_3-a_3)]=1$. So we get the partition $(1,2,4,4)$.



Therefore, the ideal $I=(x^4,x^2y^2,xy^3,y^4)$, associated to the picture above, is a monomial ideal with the chosen degree table and such that $k[x,y]/I$ has the given h-vector.
According to Remark 4.12, also the following degree tables produce partitions of 11:

$$\begin{pmatrix} 4 & 4 & 4 & 4 \\ \cdot & 5 & 6 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 4 & 4 & 4 \\ \cdot & 6 & 5 & 5 \end{pmatrix},$$

namely, $(4,3,3,1)$ and $(4,3,2,2)$.

Remark 4.16. In the previous example, we have seen how to get a monomial ideal from a partition. Chosen a table, by considering the transpose of a partition, we are simply looking at the monomial ideal with exchanged variables x and y, which has still the same table and h-vector.

In this way, we can find all the partitions of the degree of the scheme associated to its h-vector, and so *all monomial ideals I in two variables, x,y, with pure powers, such that $k[x,y]/I$ has the given h-vector* . We now use the theory of 2-type vectors and pseudo type vectors in order to introduce the concept of the sum of two partitions.

Remark 4.17. Given a partition of a number n, i.e. $(c_1,\dots,c_t)$, where $c_i \leq c_{i+1}$ and $c_1+\cdots+c_t = n$, by adding zero entries, we get other partitions of n. Notice that we need to consider them differently in order to define the concept of sum of two partitions.

Definition 4.18. Let $c = (c_1,\dots,c_t)$ and $d = (d_1,\dots,d_v)$ be two partitions of n respectively m. Assume in addition that at least one of those partitions is either a 2-type or a pseudo type vector whose first difference satisfies Condition (1). We say that a partition of $n+m$ is the *sum* of c and d, if it is obtained by ordering the sequence

$$\{c_i + d_j\}_{i=1,\dots,t;j=1,\dots,v}$$

(where each c_i and d_j appear exactly once in the sums) in a non-decreasing way.

Notice that if a vector, $D = (d_1,\dots,d_t)$, with non-decreasing positive integer entries, satisfies Condition (1), it has to be at least a pseudo type vectors: by contradiction, suppose there are three consecutive equal entries, i.e. $d_{i-1} = d_i = d_{i+1}$, for some i, then in ΔD the consecutive entries, $d_i - d_{i-1}$ and $d_{i+1} - d_i$, are both zero, but there isn't any entry between them which is greater than 1.
So *Condition (1) is a sufficient condition for a partition to be 2-type vector or a pseudo type vector*. Hence, we can actually use Theorem 4.4 or the similar result holding in the case of pseudo type vectors.
The following example shows why it is necessary that at least one of the two partitions has to satisfy Condition (1).

Example 4.19. The vectors $h = (1,1,1)$ and $h' = (1,1,1,2)$ are not a pseudo type vectors. If we take their sum, by definition, we can get $(1,1,1,1,1,3)$: it sufficies to sum them as $(1+\mathbf{0},1+\mathbf{0},0+\mathbf{1},0+\mathbf{1},0+\mathbf{1},1+\mathbf{2})$, where with the bold font we indicate the partition associated to h'. This vector is associated to the h-vector $(1,2,2,1,1,1)$, which is not possible for geometric reasons (see the table in Section 5).

The following algorithm gives all the partitions associated to the possible h-vectors of the union.

Name : $hvectorOfTheUnion$;
Input : $h-vectors, \mathbf{h} = (h_0, h_1, \dots, h_t), \mathbf{h}' = (h'_0, h'_1, \dots, h_v)$;
Output : $all\ the\ partitions\ associated\ to\ the\ union$;
$A = FindPartition(\mathbf{h})$;
$A' = FindPartition(\mathbf{h}')$;
$FOR\ P = (p_1, \dots, p_m) \in A\ \&\ Q = (q_1, \dots, q_n) \in A'$
$IF \Delta P\ SATISFIES\ COND.\ 1\ OR\ \Delta Q\ SATISFIES\ COND.\ 1$;
$IF\ n > m,\ DEFINE\ p_{m+1} = \cdots = p_n = 0$;
$IF\ m > n,\ DEFINE\ q_{n+1} = \cdots = q_m = 0$;
$FOR\ all\ \sigma, \delta \in S_{max\{n,m\}}$
$order\ in\ a\ non-decreasing\ way\ the\ set : \{p_{\alpha(i)} + q_{\delta(i)}\}_{i=1,\dots,max\{n,m\}}$
$END\ FOR$;
$FOR\ all\ i = 1, \dots, n\ \&FOR\ all\ j = 1, \dots, m$
$IF\ i + m = j + n$,
$DEFINE\ P_i = (p_1, \dots, p_m, \underbrace{p_{m+1}, \dots, p_{m+i}}_{=0})$
$Q_j = (q_1, \dots, q_n, \underbrace{q_{n+1}, \dots, q_{n+j}}_{=0})$
$FOR\ all\ \sigma, \delta \in S_{i+m}$
$compute\ and\ sort\ \{p_{\sigma(l)} + q_{\delta(l)}\}_{l=1,\dots,i+m}$
$END\ FOR$;
$END\ FOR$;
$END\ FOR$;

We can state here a conjecture on the construction of all the possible h-vectors for the union of two sets of points in $\mathbb{P}^2$.

Conjecture 4.20. Given h and h' two h-vectors, associated to two disjoint reduced zero-dimensional schemes, with degrees, n respectively m, the h-vectors of the monomials ideals associated to the partitions of $n+m$, that are obtained by summing, according to the previous definition, any partition of n associated to h and any partition of m associated to h', are all the possible h-vectors for the union of the two schemes.

The following example explains what can be obtained with this method.

Example 4.21. Let $h = (1,2,3,4,1)$ and $h' = (1,2,3,4,5,3)$, choosing the degree tables,

$$\begin{pmatrix} 4 & 4 & 4 & 4 & 5 \\ \cdot & 5 & 5 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 5 & 5 & 5 & 6 & 6 & 6 \\ \cdot & 6 & 6 & 7 & 7 & 7 \end{pmatrix},$$

we obtain the partitions $(1,2,3,5)$ and $(1,2,4,5,6)$. Thus the corresponding configurations of points are:

$$\begin{array}{lllll} \\ \\ \bullet & & & & \\ \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \qquad \begin{array}{lllll} \circ & & & & \\ \circ & \circ & & & \\ \circ & \circ & \circ & & \\ \circ & \circ & \circ & & \\ \circ & \circ & \circ & \circ & \\ \circ & \circ & \circ & \circ & \circ \end{array}$$

By summing these two partitions we get:

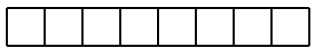
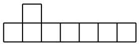
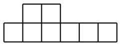
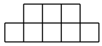
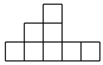
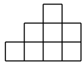
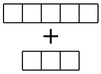
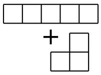
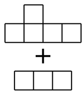
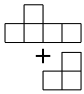
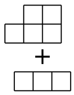
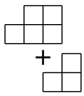
$$\begin{array}{llllllll} \bullet & & & & & & & \\ \circ & \circ & & & & & & \\ \circ & \bullet & \bullet & & & & & \\ \circ & \circ & \circ & \circ & & & & \\ \circ & \circ & \circ & \circ & \circ & & & \\ \circ & \circ & \circ & \bullet & \bullet & \bullet & & \\ \circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

from which we can read the corresponding h-vector $(1,2,3,4,5,6,7,1)$. As expected this is the h-vector corresponding to 29 generic points, obtained as the union of 11 and 18 generic points, which have respectively h-vectors h and h'.

5. Results and conjectures on the h-vectors of the union of two sets of points

In Section 3 we have introduced some tools to exclude possibilities for the h-vector of the union of two sets of points. In Section 4 we have showed a way to construct possible h-vectors. In this section, we show, in an example, how to use these tools practically. Moreover, we prove a theorem that constructs the least generic possible h-vector. Finally, we state our conjecture on the construction of possible h-vectors for the union.

We start with some examples. The following table shows the possible combinations of h-vectors of 5 points and 3 points. Here ”$-$” denotes not possible and ”$\checkmark$” possible.

5+3						
	$\checkmark$	$-^*$	$\checkmark$	$\checkmark$	$-$	$-$
	$-$	$\checkmark$	$\checkmark$	$-^+$	$\checkmark$	$-$
	$-$	$\checkmark$	$-^*$	$\checkmark$	$\checkmark$	$\checkmark$
	$-$	$-$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	$-$	$-$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
	$-$	$-$	$-$	$\checkmark$	$\checkmark$	$\checkmark$

To fill this table we first used the method described in Section 4 and got all the $\checkmark$. The bounds in Section 3 give the areas at the lower left and upper right where the combinations are not possible. Beside that we have 3 more combinations which are not possible. The two marked with * can be excluded using Theorem 3.15. For the one marked with $^+$ we use a similar argumentation: we have 5 points on a line and 3 points which are not on a line. In the result there cannot be more than 5 on a line, so none of the 3 points is on the line of the 5 points. Since the 3 points are not on a line, the 8 points cannot be on a quadric.

The least generic possibility is always easy to get. An informal description is: "Put the diagrams of the h-vectors side by side and then move columns to the left until you get a valid shape according to Davis' Theorem."

In the following proposition, we prove it formally by actually building the most specific h-vector.

Proposition 5.1. *For any two given h-vectors, we can always construct the unique minimum h-vector for the union among all the admissible ones. This h-vector achieves the lower bound for the height and the upper bound for the length.*

Proof. Let $h = (h_0, h_1, \ldots, h_{b-1}, h_b, \ldots, h_r)$ and $h' = (h'_0, h'_1, \ldots, h'_{c-1}, h'_c, \ldots, h'_s)$ be two given h-vectors, X and Y two sets of points with these h-vectors and

$b = b(X)$, $c = b(Y)$. Let $\alpha_i = \left|\{j|\ h_j = i\}\right|$, for $i = 1,\dots,b$ and let $\gamma_i = \left|\{j|\ h'_j = i\}\right|$, for $i = 1,\dots,c$. W.L.O.G. assume that $c \le b$. We claim that the least generic h-vector for the union $X \cup Y$ is:

$$(1,\dots,b-1,\underbrace{b,\dots,b}_{\alpha_b},\underbrace{b-1,\dots,b-1}_{\alpha_{b-1}-1},\dots,\underbrace{c+1,\dots,c+1}_{\alpha_{c+1}-1},\underbrace{c,\dots,c}_{\alpha_c+\gamma_c-1},\dots,\underbrace{1,\dots,1}_{\alpha_1+\gamma_1-1})$$

It is clear that we cannot decrease by one any entry of the previous h-vector until the degree $\alpha_b + \alpha_{b-1} + \alpha_{b-2} + \cdots + \alpha_{c+1} + c + 1$, since until that point we have exactly the same entries of the h-vector with the biggest height. To obtain a smaller h-vector (with respect to the introduced partial order) one has to move one box from one row to a lower row. As we just mentioned, no box from the top $c-d$ rows of the diagram can be moved. It is also not possible to move any box to the lowest row, since the maximal possible length for the h-vector is already achieved . One can try to see if it possible to move one box from one row to a lower row between $b-c$ and 1.

To do this, we first observe that the values of the h-vector from 1 to c occur at least two times each.This means that in the tail of our h-vector diagram the values $i = 1,\dots,c-1$ occur at least once. So by moving one box, we will obtain a flat, and thus we can use Theorem 2.10.

Suppose we have moved the box to a row of degree f, where $1 < f \le c-1$. Then in that degree we will have a flat of length $\alpha_f + \gamma_f$, and, by Davis' Theorem, the following number of points will lie on a curve of degree f:

$$N = (\alpha_1 + 2\alpha_2 + \cdots + f\alpha_b) + (\gamma_1 + 2\gamma_2 + \cdots + f\gamma_c) + 1.$$

Consider the set $Z \subseteq X \cup Y$ consisting of all these points lying on the curve of degree f. Let $Z = X' \cup Y'$, where $X' \subseteq X$ and $Y' \subseteq Y$. The h-vector corresponding to Z must have height equal to f, and since X', Y' lie also on the curve of degree f, their ideals must be generated at most in degree f, so their h-vectors have length at most f. Moreover, by the subset criterion, the h-vectors of X' and Y' have to fit into $(h_0,h_1,\dots,h_{b-1},h_b,\dots,h_r)$ and $(h'_0,h'_1,\dots,h'_{c-1},h'_c,\dots,h'_s)$. respectively. But counting the number of boxes below the degree f in the given h-vectors, we obtain $\sum\limits_{i=0}^{\alpha_1+\cdots\alpha_c-1} \min\{h_i, f\} + \sum\limits_{i=0}^{\gamma_1+\cdots+\gamma_d-1} \min\{h'_i, f\} = N-1$, which is a contradiction. □

In the example at the beginning of the section, we have seen that all possible h-vectors of the union of two sets of points can be obtained with the methods introduced in Section 4. We conjecture that this is the case in general.

Conjecture 5.2. Given two h-vectors, every possible h-vector for the union of two sets of points that have these h-vectors can be constructed with the methods showed in Section 4.

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