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POINCARÉ SERIES OF MONOMIAL RINGS WITH MINIMAL TAYLOR RESOLUTION

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We give a comparison between the Poincaré series of two monomial rings: $R = A/I$ and $R_q = A/I_q$ where I_q is a monomial ideal generated by the q'th power of monomial generators of I. We compute the Poincaré series for a new class of monomial ideals with minimal Taylor resolution. We also discuss the structure a monomial ring with minimal Taylor resolution where the ideal is generated by quadratic monomials.

1. Introduction

Let $A = k[x_1,\dots,x_n]$ be a polynomial ring over a field k and I be an ideal of A. The Poincaré series of $R = A/I$ is the power series

$$P_k^R(z) = \sum_{i\geq 0} \dim_k(\mathrm{Tor}_i^R(k,k))z^i \in \mathbb{Z}[[z]].$$

It is the generating function of the sequence of Betti numbers of a minimal free resolution of k over R. A question that this becomes a rational function was asked by Serre. An affirmative answer was presented by Backelin in [3] when I is a monomial ideal in A and a counter-example was given by Anick in [1, 2] when $I = (x_1^2, x_2^2, x_4^2, x_5^2, x_1x_2, x_4x_5, x_1x_3 + x_3x_4 + x_2x_5) \subset A = k[x_1,\dots,x_5]$.

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In this paper we compute the Poincaré series of some monomial rings. Recall that a monomial ring is the quotient ring $R = A/I$ where I is a monomial ideal in the polynomial ring A. There exists a unique finite set of monomial generators for a monomial ideal I which we denote by $G(I) = \{m_1,\dots,m_t\}$. We also denote the graded maximal ideal of A by $\mathfrak{m}_A = (x_1,\dots,x_n)$, the set containing the first d positive integers by $\mathbb{N}_d$ and by $|S|$ the cardinality of a set S.

We will use polarization of monomial rings introduced by Fröberg in [6] to prove that, for any positive integer q, if $R = A/I$ is a monomial ring such that $I \subseteq \mathfrak{m}_A^2$ and $R_q = A/I_q$ is a monomial ring such that $G(I_q) = \{m^q \mid m \in G(I)\}$, then $P_k^R(z) = P_k^{R_q}(z)$, (Theorem 2.3). This result does not hold if there exists some $x_i \in G(I)$, that is $I \subsetneq \mathfrak{m}_A^2$, as the following example shows.

Example 1.1. Let $R = A/I$ where $I = (x) \subset A = k[x]$. Then $P_k^R(z) = 1$ but, since $R_2 = A/(x^2)$ is a complete intersection, we have $P_k^{R_2}(z) = \frac{1+z}{1-z^2} = \frac{1}{1-z}$.

We will give the minimal generating set of the homology $H(K^R)$ of the Koszul complex K^R of a certain class of monomial rings $R = A/I$ with minimal Taylor resolution (Theorem 3.3). Namely, we consider monomial ideals I such that there exists a total ordering $m_1 \prec m_2 \prec \cdots \prec m_t$ on $G(I)$ and a positive integer $d \leq t$ such that $\gcd(m_i,\dots,m_{i+d-1}) \neq 1$ and $\gcd(m_i, m_{i+d+j}) = 1$ for all $i = 1,\dots,t-d$ and $j > 1$. We use this description to compute the Hilbert series of $H(K^R)$. Then Fröberg in [5] proved that $P_k^R(z)$ is a quotient of the Hilbert series of two associative graded algebras; namely K^R and $H(K^R)$.

It is of some interest to know classes of monomial rings with minimal Taylor resolution which are either a complete intersection or Golod. The reader can see the definition of a Golod ring in e.g. [5]. It is well known that a monomial ring $R = A/I$ is Golod if $G(I)$ has a common factor $\neq 1$. We call such a ring trivially Golod. Since Poincaré series of such rings are already known, one may determine the structure of $R = A/I$ from $P_k^R(z)$. In section 4 we consider monomial rings $R = A/I$ with minimal Taylor resolution when I is either a stable ideal, an ideal with a linear resolution and an ideal generated by quadratic monomials. We study the conditions on $G(I)$ for such a monomial ideal I so that $R = A/I$ becomes a complete intersection or trivially Golod.

2. Ideals generated by Powers of generators

For a monomial $m \in A$, let $\mathrm{Supp}(m) = \{j \mid x_j \text{ divides } m\}$. Given a monomial ideal $I \subset A$, if $j \notin \bigcup_{m\in G(I)} \mathrm{Supp}(m)$, then we have $R = A/I \cong A'/(I \cap A') \otimes_k k[x_j]$ where $A' = k[x_1,\dots,\hat{x}_j,\dots,x_n]$. Since $P_k^{k[x_j]}(z) = 1+z$ and the Poincaré series of a tensor product of two algebras is the product of Poincaré series of the algebras, it follows that $P_k^R(z) = (1+z)P_k^{A'/(I\cap A')}(z)$. So we may assume, without any loss of generality, that $\bigcup_{m\in G(I)} \mathrm{Supp}(m) = \{1,\ldots,n\}$. Furthermore, let $I \subset A$ be a monomial ideal such that $G(I) = \{x_1,\ldots,x_s,m_1,\ldots,m_t\}$ for some $s \le n$ where $\deg(m_i) > 1$ for each $i = 1,\ldots,t$. Since $\cup_{i=1}^{t} \mathrm{Supp}(m_i) = \{s+1,\ldots,n\}$, it follows that $R \cong k \otimes_k R' \cong R'$ where $R' = k[x_{s+1},\ldots,x_n]/(m_1,\ldots,m_t)$. Hence $P_k^R(z) = P_k^{R'}(z)$. So we shall consider monomial rings $R = A/I$ such that $\deg(m) > 1$ for each $m \in G(I)$, that is $I \subseteq \mathfrak{m}_A^2$.

The following Proposition is due to Fröberg in [6], pp. 30-32. We put it for a quick reference since it plays an important role in proving Theorem 2.3.

Proposition 2.1. *Let $R = \boldsymbol{k}[x_1,\ldots,x_n]/I$ be a monomial ring. There exists a monomial ring $S = \boldsymbol{k}[y_1,\ldots,y_N]/I'$ such that $R = S/(f_1,\ldots,f_{N-n})$, where $f_1,\ldots,f_{N-n}$ is a regular sequence of homogeneous elements of degree one. Moreover,*

1. *R is complete intersection if and only if S is complete intersection.*
2. *R is Golod if and only if S is Golod.*
3. $P_{\boldsymbol{k}}^R(z) = \frac{P_{\boldsymbol{k}}^S(z)}{(1+z)^{N-n}}$.

Recall that a squarefree monomial ring is the quotient of a polynomial ring by a squarefree monomial ideal. We shall explain how to get the squarefree monomial ring S from R. Let

$$I(x_i) = \max\{\alpha \in \mathbb{Z}_{\ge 0} \mid x_i^{\alpha} \text{ divides some } m \in G(I)\} \tag{1}$$

If each $I(x_i) = 1$, then R is squarefree. If there exists some i with $I(x_i) > 1$, we introduce new variables and replace each monomial $m \in G(I)$ by a squarefree monomial of degree equal to $\deg(m)$ in $N = \sum_{i=1}^{n} I(x_i)$ new variables, say $y_1,\ldots,y_N$. This set of squarefree monomials generate a monomial ideal I' in $A' = k[y_1,\ldots,y_N]$ and we take $S = A'/I'$.

Example 2.2. Consider the monomial ring $R = k[x_1,x_2,x_3]/(x_1^3,x_2^2x_3,x_1x_2x_3)$. Then $N = 3+2+1 = 6$. Replacing x_1^3 by $y_1y_2y_3$, $x_2^2x_3$ by $y_4y_5y_6$ and $x_1x_2x_3$ by $y_1y_4y_6$ we obtain a squarefree monomial ideal $I' = (y_1y_2y_3, y_4y_5y_6, y_1y_4y_6) \subset A' = k[y_1,\ldots,y_6]$. It is not difficult to see that $R \cong S/(y_1-y_2, y_1-y_3, y_4-y_5)$ where $S = A'/I'$.

Theorem 2.3. *Let $R = A/I$ be a monomial ring such that $I \subseteq \mathfrak{m}_A^2$, and $R_q = A/I_q$ be a monomial ring such that $G(I_q) = \{m^q \mid m \in G(I)\}$ for some integer $q > 1$. Then $P_k^R(z) = P_k^{R_q}(z)$.*

Proof. We consider two cases to prove the theorem depending on the values of $I(x_i)$ defined in (1).

(a): Let $I(x_i) = 1$ for all $i \in \mathbb{N}_n$. Then I is a squarefree monomial ideal. It suffices to prove that $\mathrm{Tor}_i^{R_q}(k,k) \cong \mathrm{Tor}_i^R(k,k)$ for all $i \geq 0$. Put $A_q = k[x_1^q, \dots, x_n^q]$ and $R' = A_q/A_qI_q$. There exists a natural isomorphism of k-algebras $A \to A_q$ and, since I is a squarefree monomial ideal, this induces an isomorphism of k-algebras $R \to R'$. On the other hand, since $A_qI_q \subset I_q$ we have a ring map $R' \to R_q$ which defines an R'-module structure on R_q. In fact one has $R_q = \oplus_{|\alpha|<q} R'(X^\alpha \bmod I_q)$, where $\alpha \in \mathbb{Z}^n_{\geq 0}$, so R_q is free over R'. Considering k both as an R'-module and an R_q-module, and using the ring map $R' \to R_q$, it follows, by [9, Prop. 3.2.9], that

$$\mathrm{Tor}_i^{R'}(k,k) \cong \mathrm{Tor}_i^{R_q}(k \otimes_{R'} R_q, k)$$

for all $i \geq 0$. Using the projection map $R_q \to R' \cong R$, we obtain

$$k \hookrightarrow k \otimes_R R_q \to k \otimes_R R \cong k.$$

So we have $\mathrm{Tor}_i^{R_q}(k \otimes_R R_q, k) \cong \mathrm{Tor}_i^R(k,k)$.

(b): Let $I(x_i) > 1$ for some i. We use Prop. 2.1 and (a) above to prove $P_k^R(z) = P_k^{R_q}(z)$. Put $N = \sum_{i=1}^n I(x_i)$. By Prop. 2.1 there exists a square free monomial ideal $J \subset B = k[y_1, \dots, y_N]$ with $G(J) = \{M_1, \dots, M_t\}$ and a regular sequence $f_1, \dots, f_{N-n} \in S = B/J$ of degree one such that $R \cong S/(f_1, \dots, f_{N-n})$. It also follows by Prop. 2.1 that

$$P_k^R(z) = \frac{P_k^S(z)}{(1+z)^{N-n}} \tag{2}$$

Now for an integer $q > 1$, let $J_q \subset B$ be a monomial ideal with $G(J_q) = \{M_1^q, \cdots, M_t^q\}$ and put $S_q = B/J_q$. By (a) above we have

$$P_k^S(z) = P_k^{S_q}(z). \tag{3}$$

Since each M_j is a squarefree monomial, $j \in \mathbb{N}_t$, we have $J_q(y_i) = q > 1$ for each $i \in \mathbb{N}_N$. Again by Prop. 2.1 there exists a square free monomial ideal $J' \subset C = k[z_1, \dots, z_{Nq}]$ and a regular sequence $g_1, \dots, g_{Nq-N} \in S' = C/J'$ such that $S_q \cong S'/(g_1, \dots, g_{Nq-N})$ and, moreover,

$$P_k^{S_q}(z) = \frac{P_k^{S'}(z)}{(1+z)^{Nq-N}} \tag{4}$$

Combining (2-4), we obtain $P_k^R(z) = \frac{P_k^{S'}(z)}{(1+z)^{Nq-n}}$.

On the other hand, since $G(I_q) = \{m_1^q, \dots, m_t^q\}$, one has $I_q(x_i) = q \cdot I(x_i) > 1$ for each i. So there exists a square free monomial ideal $I'_q \in C = k[z_1, \dots, z_{Nq}]$ and a regular sequence $h_1, \dots, h_{Nq-n} \in S' = C/I'_q$ such that $R_q \cong S'/(h_1, \dots, h_{Nq-n})$. So $P_k^{R_q}(z) = \frac{P_k^{S'}(z)}{(1+z)^{Nq-n}}$. We thus have $P_k^{R_q}(z) = \frac{P_k^{S'}(z)}{(1+z)^{Nq-n}} = P_k^R(z)$. □

3. Rings with a Minimal Taylor Resolution

Let $R = A/I$ be a monomial ring with $G(I) = \{m_1, \ldots, m_t\}$ and T be the exterior algebra of a rank t free A-module with a standard basis $e_{i_1,\ldots,i_l}$ for $1 \leq i_1 < \cdots < i_l \leq t$. Consider T as a finite free resolution of R with a differential

$$d(e_{i_1,\ldots,i_l}) = \sum_{j=1}^{l} (-1)^{j-1} \frac{m_{i_1,\ldots,i_l}}{m_{i_1,\ldots,\hat{i_j},\ldots,i_l}} e_{i_1,\ldots,\hat{i_j},\ldots,i_l}$$

where $m_{i_1,\ldots,i_l} = \mathrm{lcm}(m_{i_1}, \ldots, m_{i_l})$. This resolution is called the Taylor resolution of R, see also [4]. It is far from being minimal. But we get a minimal Taylor resolution whenever $m_{i_1,\ldots,i_l} \neq m_{i_1,\ldots,\hat{i_j},\ldots,i_l}$ for all $i_1, \ldots, i_l \in \mathbb{N}_t$, or equivalently, whenever each m_i contains a variable with a maximal power. The reader may refer [5] for other equivalent conditions. It is evident that if a monomial ring $R = A/I$ has a minimal Taylor resolution, then so will the monomial ring $R_q = A/I_q$ where $G(I_q) = \{m^q \mid m \in I\}$ for an integer $q > 0$.

Let $R = A/I$ be a monomial ring. A differential graded, associative and commutative algebra structure for the Taylor resolution of R was given by Gemeda in [7]. Using this algebra structure, Fröberg in [5, Theorem 3] proved that the Poincaré series of $R = A/I$ having a minimal Taylor resolution is the quotient of the Hilbert series of the Koszul complex K^R of R and the Hilbert series of its Homology $H(K^R)$. More precisely,

$$P_k^R(z) = \frac{Hilb(K^R)(z)}{Hilb(H(K^R))(-z,z)} = \frac{(1+z)^n}{Hilb(H(K^R))(-z,z)}$$

where n is the embedding dimension of R and we consider $H(K^R)$ as a bi-graded algebra by a polynomial degree and a total degree. The basis for $H(K^R)$ can be described in terms of representatives $T_1, \ldots, T_n$ in K^R by

$$f_{i_1,\ldots,i_l} = \frac{\mathrm{lcm}(m_{i_1}, \ldots, m_{i_l})}{x_{i_1} \cdots x_{i_l}} T_{i_1} \cdots T_{i_l}$$

for $1 \leq i_1 < \ldots < i_l \leq n$.

Example 3.1. We will describe how to compute $P_k^R(z)$ for a monomial ring $R = A/I$ where $I = (x^2y, y^2z, z^2) \subset A = k[x,y,z]$. Let T_1, T_2, T_3 be the standard generators for K^R. Then

$$f_1 = \frac{x^2y}{x}T_1 = xyT_1 \qquad f_2 = \frac{y^2z}{y}T_2 = yzT_2$$

$$f_3 = \frac{z^2}{z}T_3 = zT_3 \qquad f_{12} = \frac{lcm(x^2y, y^2z)}{xy}T_1T_2 = xyzT_1T_2$$

$$f_{23} = \frac{lcm(y^2z, z^2)}{yz}T_2T_3 = yzT_2T_3 \qquad f_{13} = \frac{lcm(x^2y, z^2)}{xz}T_1T_3 = xyzT_1T_3$$

$$f_{123} = \frac{lcm(x^2y, y^2z, z^2)}{xyz} = xyzT_1T_2T_3$$

We then have $\bar{f}_1\bar{f}_3 = \bar{f}_{13}$ and, otherwise, $\bar{f}_I\bar{f}_J = 0$ for all $I, J \subset \{1,2,3\}$. So we obtain $H(K^R) = k(X_1, X_2, X_3, X_{12}, X_{23}, X_{123}/(X_IX_J \mid$ for all I, J except $I = \{1\}$, and $J = \{3\})$. The Hilbert series, then, becomes

$$Hilb(H(K^R))(X,Y) = 1 + 3XY + 2XY^2 + X^2Y^2 + XY^3$$

and the Poincaré series is

$$P_k^R(z) = \frac{Hilb(K^R)(z)}{Hilb(H(K^R))(-z,z)} = \frac{(1+z)^3}{1-3z^2-2z^3}.$$

A *strictly ordered partition* of a finite totally ordered set $(S, \prec)$ is a sequence $(S_1, \ldots, S_l)$ of non-empty subsets of S such that they form an ordered partition and $\max(S_i) \prec \min(S_{i+1})$ for all $i = 1, \ldots, l-1$. In this case we call l the *length* and $(|S_1|, \ldots, |S_l|)$ the *weight* of the partition. The following is evident.

Proposition 3.2. *Fix positive integers $d \leq t$. For any non-empty subset S of $\mathbb{N}_t$ there exists a strictly ordered partition $(S_1, \ldots, S_l)$ such that:*

1. *Any two consecutive numbers in S_i differ at most by $d-1$.*
2. $\min(S_{j+1}) - \max(S_j) \geq d$ *for each $j = 1, \ldots, l$.*

Theorem 3.3. *Fix a positive integer d. Let $R = A/I$ be a monomial ring with a minimal Taylor resolution. Assume that $|G(I)| = t$ and there exists a total ordering on $G(I)$ such that* $\gcd(m_i, m_{i+1}, \ldots, m_{i+d}) \neq 1$ *and* $\gcd(m_i, m_{i+d+j}) = 1$ *for any $j \geq 0$ and $i = 1, \ldots, t-d$. Consider the collection*

$$\mathcal{B} = \{S \subseteq \mathbb{N}_t \mid \textit{any two consecutive numbers in } S \textit{ differ at most by } d-1\}.$$

We have the following:

1. *The minimal generating set of the homology* $H(K^R)$ *of* K^R *is* $\{X_S \mid S \in \mathcal{B}\}$, *that is* $H(K^R) = k(X_S)_{S\in\mathcal{B}}/J$ *where* J *is the ideal*

$$J = (X_{S_1}\cdots X_{S_l} \mid \text{ each } S_i \in \mathcal{B}, \quad S_i \cup S_j \notin \mathcal{B} \text{ for all } i,j \in \mathbb{N}_l \text{ and } S_i \cap S_j \neq \emptyset \text{ for some } i,j \in \mathbb{N}_l).$$

2. *For any non-empty set* $S \subseteq \mathbb{N}_t$ *there exists a partition* $S_1,\ldots,S_l$ *of* S *such that each* $S_j \in \mathcal{B}$ *and* $S_i \cup S_j \notin \mathcal{B}$. *Put* $m = |S|$. *Then the Hilbert series of* $H(K^R)$ *is*

$$Hilb(H(R^K))(X,Y) = \sum_{(n_1,\ldots,n_l)} f(n_1,\ldots,n_l)X^lY^m. \tag{5}$$

where $f(n_1,\ldots,n_l)$ *is the number of subsets of* N_t *having a strictly ordered partition defined in Prop. 3.2 for a given length* $l \leq t$ *and a weight* $(n_1,\ldots,n_l) \in \mathbb{Z}^l_{>0}$.

Proof. (1): Note that $H(K^R) = k(X_S)_{S\subseteq\mathbb{N}_t}/J$. Now let $(S_1,\ldots,S_l)$ be a strictly ordered partition given in Prop. 3.2 of a set $S \subseteq \mathbb{N}_t$. Then each $S_i \in \mathcal{B}$ and by assumption any two monomials in $G(I)$ indexed by elements of different subpartitions are relatively prime. We then have $\bar{f}_S = \bar{f}_{S_1}\cdots\bar{f}_{S_l}$ and so $X_S = X_{S_1}\cdots X_{S_l}$. It follows that set $\{X_{S_i} \mid S_i \in \mathcal{B}\}$ generates $H(K^R)$. Since each $S_i \in \mathcal{B}$ has a strictly ordered partition of length 1, $\{X_{S_i} \mid S_i \in \mathcal{B}\}$ becomes a minimal generating set. (2): Follows from Prop 3.2. □

Now we give an example.

Example 3.4. Let $R = A/I$ where

$$I = (x^2yz, y^2zw, z^2wu, w^2u, u^2) \subset A = k[x,y,z,w,u].$$

We want to compute $Hilb(H(K^R))(X,Y)$. Consider the ordering $m_1 = x^2yz \prec m_2 = y^2zw \prec m_3 = z^2wu \prec m_4 = w^2u \prec m_5 = u^2$ where $d = 3$. Then $\mathcal{B}$ consists of all non-empty subsets of $\mathbb{N}_5$ except $\{1,4\},\{1,5\},\{2,5\},\{1,2,5\}$ and $\{1,4,5\}$. That is, $\bar{f}_{14} = \bar{f}_1\bar{f}_4$, $\bar{f}_{15} = \bar{f}_1\bar{f}_5$, $\bar{f}_{25} = \bar{f}_2\bar{f}_5$, $\bar{f}_{125} = \bar{f}_{12}\bar{f}_5$ and $\bar{f}_{145} = \bar{f}_1\bar{f}_{45}$. Therefore, $Hilb(H(K^R))(X,Y) =$

$$= 1 + 5XY + 8XY^2 + 2X^2Y^2 + 8XY^3 + 2X^2Y^3 + 5XY^4 + XY^5.$$

We obtain a formula for Hilbert series of $H(K^R)$ if $d \leq 2$ in Theorem 3.3.

Proposition 3.5. *Keep all the assumptions of Theorem 3.3 for a monomial ring* $R = A/I$.

1. *If $d = 1$, R is a complete intersection. If $d = t$, R is trivially Golod.*

2. *If $d = 2$, the Hilbert series of $H(K^R)$ is*

$$Hilb(H(R^K))(X,Y) = \sum_{(n_1,\dots,n_l)} \binom{t-m+1}{l} X^l Y^m. \tag{6}$$

Proof. (1) is clear, so we prove only (2). If $d = 2$, the strictly ordered partition of a non-empty subset S of $\mathbb{N}_t$ is given by a partition $(S_1,\dots,S_l)$ such that each S_i contains consecutive integers and $\min(S_i) - \max(S_{i-1}) \geq 2$. Now let $S_i = \{a_i, a_i+1, \dots, a_i+n_i-1\}$ where $a_i = \min(S_i)$ for each i. We then obtain the following inequalities:

$$\begin{aligned}
&1 \leq a_1 \\
&a_1 + n_1 - 1 + 2 \leq a_2 \Rightarrow a_1 + n_1 < a_2 \\
&a_2 + n_2 - 1 + 2 \leq a_3 \Rightarrow a_1 + n_1 + n_2 < a_3 \\
&a_3 + n_3 - 1 + 2 \leq a_4 \Rightarrow a_1 + n_1 + n_2 + n_3 < a_4 \\
&\vdots \\
&a_{l-1} + n_{l-1} - 1 + 2 \leq a_l \Rightarrow a_1 + \sum_{i=1}^{l-1} n_i < a_l \\
&a_l + n_l - 1 \leq t.
\end{aligned}$$

This is equivalent to the inequality system $1 \leq a_1 < a_2 - n_1 < a_3 - (n_1+n_2) < \cdots < a_l - (\sum_{i=1}^{l-1} n_i) \leq t - m + 1$. The number of solutions we get for this inequality is $\binom{t-m+1}{l}$. □

Remark 3.6. It is known that for a monomial ring $R = A/I$ with minimal Taylor resolution, the dimension of the m'th homology of K^R is $\binom{|G(I)|}{m}$, see [5]. This value also equals to the sum of the coefficients in $Hilb(H(R^K))(X,Y)$ with terms containing Y^m. It then follows from Prop. 3.5 that

$$\sum_{\substack{(n_1,\dots,n_l) \\ \sum_i n_i = m}} \binom{|G(I)| - m + 1}{l} = \binom{|G(I)|}{m}$$

where $(n_1,\dots,n_l)$ is the weight of a strictly ordered partition defined in Prop. 3.2 for a set $S \subset \mathbb{N}_t$ with $|S| = m$ and $d = 2$. It would be interesting to know if there is any combinatorial reason why these two numbers are the same.

4. Complete Intersection and Trivially Golod Rings

From the algorithm to compute $P_k^R(z)$ given in [5] it follows, for a monomial ring $R = A/I$ with minimal Taylor resolution, that $P_k^R(z) = (1-z)^n/(1-z^2)^t$ if and only if $\gcd(m_i.m_j) = 1$ for all $i \neq j$ where $G(I) = \{m_1, \ldots, m_t\}$, i.e. R is a complete intersection. Furthermore $P_k^R(z) = (1+z)^n/(1-\sum_{i=1}^{t}\binom{t}{i}z^{i+1})$ if $G(I)$ has a common factor $\neq 1$, i.e. R is trivially Golod. This gives Prop. 4.1.

Proposition 4.1. *Let $R = A/I$ and $R_q = A/I_q$ be two monomial rings such that $G(I_q) = \{m^q \mid m \in G(I)\}$ for some integer $q > 0$. Then*

1. *R_q is trivially Golod if and only if R is trivially Golod.*
2. *R_q is complete intersection if and only if R is complete intersection.*

For a monomial m, put $i_0 = \max(\mathrm{Supp}(m))$. Recall that a monomial ideal I is said to be stable if for each monomial $m \in I$ and all $i < i_0$, we have $x_i m/x_{i_0} \in I$. In [8] Okudaira and Takayama proved that such an ideal I has a minimal Taylor resolution if and only if each $m_i \in G(I)$ has the form $m_i = x_i(\prod_{j=1}^{i} x_j^{n_j})$ for $i = 1, \ldots, t$ and for some integers $n_1, \ldots n_r \geq 0$. It follows that R is trivially Golod if $n_1 > 0$; and R is a complete intersection if each $n_i = 0$.

A monomial ideal I with a linear resolution possesses a minimal Taylor resolution if and only if each $m_i \in G(I)$ is of the form $m_i = ux_{j_i}$ for some $j_i \in \mathbb{N}_n$ and a monomial $u \in A$, see [8]. It follows then that R is a complete intersection if $u = 1$, it is trivially Golod if $u \neq 1$.

Theorem 4.2. *Let $R = A/I$ be a monomial ring with a minimal Taylor resolution and each $m \in G(I)$ is a quadratic monomial. Then R is a k-tensor product of a complete intersection and trivially Golod rings.*

Proof. Let $P_1, \ldots, P_r$ be a partition of $G(I)$ such that any two elements of one subpartition have a common factor and elements between any pair of different subpartitions are relatively prime. Put $A_i = k[x]_{x \in \mathrm{Supp}(P_i)}$. Since $\mathrm{Supp}(G(I))$ is partitioned by the collection $\{\mathrm{Supp}(P_i)\}_i$, we have $A = \otimes_{i=1}^{r} A_i$. If each P_i is a singleton, by construction, R is a complete intersection. Since the monomials in each partitions are quadratic, there exists a $j \in \mathbb{N}_n$ such that $\gcd(m)_{m \in P_i} = x_j$. So we obtain a monomial ideal $I_i = (m)_{m \in P_i} \subset A_i$ such that $I = \sum_i I_i$, $R = \otimes A_i/I_i$ and each $R_i = A_i/I_i$ has a minimal Taylor resolution. Moreover, if I_i is principal, then R_i is a complete intersection and otherwise R_i is trivially Golod. □

Example 4.3. Consider the monomial ideal $I = (x_1^2, x_1x_2, x_1x_4, x_3^2, x_5^2) \subset A = k[x_1, \ldots, x_5]$. It is easy to see that $R = A/I$ has a minimal Taylor resolution. We have three partitions $P_1 = \{x_1^2, x_1x_2, x_1x_4\}, P_2 = \{x_3^2\}, P_3 = \{x_5^2\}$ and monomial ideals $I_1 = (x_1^2, x_1x_2, x_1x_4) \subset A_1 = k[x_1, x_2, x_4]$, $I_2 = (x_3^2) \subset A_2 = k[x_3]$ and $I_3 =$

$(x_5^2) \subset A_3 = k[x_5]$. We thus have a trivially Golod ring $R_1 = A_1/I_1$, complete intersections $R_2 = A_2/I_2$ and $R_3 = A_3/I_3$. So $R = R_1 \otimes_k R'$ where $R' = R_2 \otimes_k R_3$ is a complete intersection.

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