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ON HILBERT'S TYPE OPERATOR NORM INEQUALITIES ON HERZ SPACES

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In this paper some necessary and sufficient conditions are given for the Hilbert's type operators to be bounded on the Herz spaces. The corresponding new operator norm inequalities are obtained.

1. Introduction

Considerable attention has been given to the classical Hilbert operator T defined by

$$T(f,x) = \int_0^\infty \left(\frac{1}{x+y}\right) f(y)\,dy, \tag{1}$$

and to the classical Hilbert inequality

$$\left\{\int_0^\infty \left|\int_0^\infty \frac{f(y)}{x+y}dy\right|^p dx\right\}^{1/p} \leq \frac{\pi}{\sin(\pi/p)}\left\{\int_0^\infty |f(x)|^p\,dx\right\}^{1/p}, \quad \text{for} \quad 1<p<\infty, \tag{2}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible value (Hardy et al. [1]). In view of the mathematical importance and applications, considerable

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attention has also been given to various improvements, refinements and extensions of many inequalities by various authors (see [3-10] and the references cited therein). However, hardly any work was done on inequalities on Herz spaces. It is well-known that Herz spaces play an important role in characterizing the properties of functions and multipliers on classical Hardy spaces. In recent years, considerable attention has been given to the study of the Herz spaces (see [11]).

The aim of this paper is to establish some new inequalities related to the Hilbert's type operator

$$T(f,x) = \int_0^\infty K(x,y)\, f(y)\, dy \tag{3}$$

with the general kernel $K(x,y)$. We obtain some necessary and sufficient conditions for the Hilbert's type operator T to be bounded on the Herz spaces. The corresponding new operator norm inequalities are obtained.

2. Definitions and Main Results

Definition 2.1. Let $\alpha \in \mathbb{R}$, $0 < p$, $q < \infty$, $B_k = \{\mathbf{x} \in \mathbb{R}^n \mid |x| \le 2^k, i = 1,2, \ldots, n\}$, $D_k = B_k \setminus B_{k-1}$, $(k \in Z)$, $\phi_k = \phi_{D_k}$ denote the characteristic function of the set D_k .

(1) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ was defined by Lu et al. [11] as follows:

$$\dot{K}^{\alpha,p}_{q(\mathbb{R}^n)} = \left\{ f \in L^q_{loc}(\mathbb{R}^n - \{0\}) \mid \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty \right\}, \tag{4}$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\phi_k\|_q^p \right\}^{1/p}; \tag{5}$$

(2) The homogeneous Herz type space $\dot{K}_q^{\alpha,p}(\omega)$ is defined by

$$\dot{K}_q^{\alpha,p}(\omega) = \left\{ f \in L^q_{loc}(\mathbb{R}^n - \{0\}) \mid \|f\|_{\dot{K}_q^{\alpha,p}(\omega)} < \infty \right\}, \tag{6}$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\omega)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\phi_k\|_{q,w}^p \right\}^{1/p}. \tag{7}$$

We can define the non-homogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\omega)$ in a similar way, as to Lu et al. [11].

Throughout this paper, we write

$$\|f\|_{p,\omega} = \left(\int_{\mathbb{R}^n} |f(x)|^p \, \omega(x)\, dx\right)^{1/p}; \quad \|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p}.$$

It is easy to see that when $p = q$, we have

$$\dot{K}_p^{\alpha,p}(\mathbb{R}^n) = K_p^{o,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n),$$

and

$$\dot{K}_p^{\alpha/p,p}(\mathbb{R}^n) = K_p^{\alpha/p,p}(\mathbb{R}^n) = L^p\left(|x|^\alpha \, dx\right).$$

Our main result is the following:

Theorem 2.2. *Let* $\alpha \in \mathbb{R}$, $0 < p < \infty$, $\lambda > 0$ *and* $0 < q \le 1$, $\omega(x) = x^{(l-\lambda)q}$. *Let* $K(x,y)$ *be a nonnegative measurable function on* $(0,\infty)\times(0,\infty)$ *which satisfies the following conditions:*

(1) $K(tx,ty) = t^{-\lambda}K(x,y)$ *for all* $t > 0$,

(2) $F(t) = |f(tx)|\, K(1,t)$ *and* $t^{\lambda-1-\frac{1}{q}}K(1,t)$ *are concave functions on* $(0,\infty)$,

(3) $K(1,t)$ *has compact support on* $(0,\infty)$,

then the Hilbert's type operator T *defined by (3):* $\dot{K}_q^{\alpha,p}(\omega) \to \dot{K}_q^{a,p}\left(\mathbb{R}_+^1\right)$ *exists as a bounded operator if and only if*

$$\int_0^\infty t^{\lambda-\alpha-1-(1/q)}K(1,t)\, dt < \infty. \tag{8}$$

Moreover, when (8) holds, the operator norm $\|T\|$ *of* T *on* $\dot{K}_q^{\alpha,p}(\omega)$ *satisfies the following inequality:*

$$\int_0^\infty t^{\lambda-\alpha-1-(1/q)}K(1,t)\, dt \le \|T\| \le C(p,q,\alpha)\int_0^\infty t^{\lambda-\alpha-1-(1/q)}K(1,t)\, dt, \tag{9}$$

where

$$C(p,q,\alpha) = \begin{cases} 2^{1/p-1/q-2}q^{-1/p}(1+q)^{1/q}(p+q)^{1/p}\left(1+2^{|\alpha|}\right), & 0 < p \le q \le 1 \\ 2^{1/q-2}(1+q)^{1/q}\left(1+2^{|\alpha|}\right), & 0 < q \le p < 1 \\ 2^{1/q-2/p-1}(1+q)^{1/q}(1+1/p)\left(1+2^{|\alpha|}\right), & 0 < q < 1 \le p < \infty \end{cases}. \tag{10}$$

There are some similar results for $1 < q < \infty$ *and the non-homogeneous Herz spaces. We omit the details here.*

3. Proof of Theorem 2.2

We require the following Lemmas to prove our result.

Lemma 3.1. *Let f be a nonnegative measurable function on $[0,b]$. If $1 \leq p < \infty$, then*

$$\int_0^b f(x)\,dx \leq b^{1-(1/p)} \left\{ \int_0^b f^p(x)\,dx \right\}^{1/p}. \tag{11}$$

Proof. Lemma 3.1 is an immediate consequences of Holder's inequality. □

Lemma 3.2. (See [2]) *Let f be a nonnegative measurable and concave function on $[a,b]$, $0 < \alpha \leq \beta$, then*

$$\left\{ \left(\frac{\beta+1}{b-a} \right) \int_a^b [f(x)]^\beta\,dx \right\}^{1/\beta} \leq \left\{ \left(\frac{\alpha+1}{b-a} \right) \int_a^b [f(x)]^\alpha\,dx \right\}^{1/\alpha}. \tag{12}$$

Setting $a = 0$ and for $\alpha = 1$, $\beta = p \geq 1$, we obtain the Favard inequality:

$$\int_0^b f^p(x)\,dx \leq \left(\frac{2^p}{p+1} \right) b^{1-p} \left(\int_0^b f(x)\,dx \right)^p. \tag{13}$$

When $\alpha = p$, $\beta = 1$, that is, $0 < p \leq 1$, inequality (12) yields

$$\left(\int_0^b f(x)dx \right)^p \leq \left(\frac{p+1}{2^p} \right) b^{p-1} \int_0^b f^p(x)dx. \tag{14}$$

We next write simply $\dot{K}_q^{\alpha,p}\left(\mathbb{R}_+^1\right)$ and $\dot{K}_q^{\alpha,p}(\omega)$ to denote K and $K(\omega)$, respectively.

Proof of Theorem 2.2. Since $K(1,t)$ has compact support on $(0,\infty)$, there exists

$b>0$, such that support $K(1,t)\subset[0,b]$. Using (14) and setting $y=tx$, we obtain

$$\begin{aligned}
&\|(Tf)\phi_k\|_q \\
&=\left\{\int_{D_k}\left|\int_0^\infty K(x,y)f(y)\,dy\right|^q dx\right\}^{1/q} \\
&\le \left\{\int_{D_k}\left(\int_0^b K(1,t)\sup_{x\in(0,\infty)}|f(tx)|\,dt\right)^q x^{(1-\lambda)q}dx\right\}^{1/q} \\
&\le \frac{1}{2}(1+q)^{1/q}b^{1-(1/q)}\left\{\int_0^b\left(\int_{D_k}\left(\sup_{x\in(0,\infty)}|f(tx)|\right)^q x^{(1-\lambda)q}dx\right)\right. \\
&\qquad\qquad\left.\times K^q(1,t)\,dt\right\}^{1/q} \\
&=\frac{1}{2}(1+q)^{1/q}b^{1-(1/q)}\left\{\int_0^b\left(\int_{2^{k-1}t<y\le 2^k t}|f(y)|^q y^{(1-\lambda)q}dy\right)\right. \\
&\qquad\qquad\left.\times t^{(\lambda-1)q-1}K^q(1,t)\,dt\right\}^{1/q}.
\end{aligned}$$

For each $t\in(0,\infty)$, there exists an integer m such that $2^{m-1}<t\le 2^m$. Setting

$$A_{k,m}=\left\{y\in(0,\infty):2^{k+m-1}<y\le 2^{k+m}\right\},$$

we obtain

$$\begin{aligned}
\|(Tf)\phi_k\|_q &\le \frac{(1+q)^{1/q}}{2b^{(1/q)-1}}\left\{\int_0^b\left(\int_{A_{(k-1),m}}|f(y)|^q\omega(y)\,dy\right.\right. \\
&\qquad\left.\left.+\int_{A_{k,m}}|f(y)|^q\omega(y)\,dy\right)t^{(\lambda-1)q-1}K^q(1,t)\,dt\right\}^{1/q} \\
&\le \frac{(1+q)^{1/q}}{2b^{(1/q)-1}}\left\{\int_0^b\left(\left\|f\phi_{(k+m-1)}\right\|_{q,\omega}^q+\|f\phi_{k+m}\|_{q,\omega}^q\right)\right. \\
&\qquad\left.\times t^{(\lambda-1)q-1}K^q(1,t)\,dt\right\}^{1/q}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|(Tf)\|_K \le \frac{(1+q)^{1/q}}{2b^{(1/q)-1}}&\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left[\int_0^b\left(\left\|f\phi_{(k+m-1)}\right\|_{q,\omega}^q+\|f\phi_{k+m}\|_{q,\omega}^q\right)\right.\right. \\
&\qquad\left.\left.\times\, t^{(\lambda-1)q-1}K^q(1,t)\,dt\right]^{p/q}\right\}^{1/p}. \qquad (15)
\end{aligned}$$

Now, we consider three cases:

Case 1. $0 < p \le q \le 1$.

In this case, it follows from (15) and (14) that

$$\begin{aligned}
&\|(Tf)\|_K \\
&\le \frac{(1+q)^{1/q}(1+p/q)^{1/p}}{2^{1+(1/q)}b^{(1/p)-1}} \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left[\int_0^b \left(\|f\phi_{k+m-1}\|_{q,\omega}^p + \|f\phi_{k+m}\|_{q,\omega}^p \right) \right] \right. \\
&\qquad \left. \times\, t^{(\lambda-1-1/q)p} K^p(1,t)\, dt \right\}^{1/p} \\
&\le \frac{(1+q)^{1/q}(1+p/q)^{1/p}}{2^{2+(1/q)-(1/p)}b^{(1/p)-1}} \left\{ \left[\sum_{k=-\infty}^{\infty} 2^{(k+m-1)\alpha p} \|f\phi_{k+m-1}\|_{q,\omega}^p \right. \right. \\
&\qquad \left. \times \int_0^b 2^{-(m-1)\alpha p} t^{(\lambda-1-1/q)p} K^p(1,t)\, dt \right]^{1/p} \\
&+ \left. \left[\sum_{k=-\infty}^{\infty} 2^{(k+m)\alpha p} \|f\phi_{k+m}\|_{q,\omega}^p \int_0^b 2^{-m\alpha p} t^{(\lambda-1-1/q)p} K^p(1,t)\, dt \right]^{1/p} \right\} \\
&\le 2^{1/p-1/q-2}(1+q)^{1/q}(1+p/q)^{1/p} \|f\|_{K(\omega)} \\
&\qquad \times \int_0^b \left(2^{-(m-1)\alpha} + 2^{-m\alpha} \right) t^{\lambda-1-1/q} K(1,t)\, dt \\
&\le 2^{1/p-1/q-2}(1+q)^{1/q}(1+p/q)^{1/p} \left(1 + 2^{|\alpha|} \right) \|f\|_{K(\omega)} \\
&\qquad \times \int_0^\infty t^{\lambda-\alpha-1-1/q} K(1,t)\, dt. \qquad (16)
\end{aligned}$$

Case 2. $0 < q < p < 1$.

In this case, it follows from (15) and (11) that

$$\begin{aligned}
&\|(Tf)\|_K \\
&\le \frac{(1+q)^{1/q}}{2b^{(1/p)-1}} \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \int_0^b \left(\|f\phi_{k+m-1}\|_{q,\omega}^q + \|f\phi_{k+m}\|_{q,\omega}^q \right)^{p/q} \right. \\
&\qquad \left. \times t^{(\lambda-1-1/q)p} K^p(1,t)\, dt \right\}^{1/p} \\
&\le \frac{(1+q)^{1/q}}{2^{1/p-1/q+1}b^{(1/p)-1}} \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \int_0^b \left(\|f\phi_{k+m-1}\|_{q,\omega}^p + \|f\phi_{k+m}\|_{q,\omega}^q \right) \right. \\
&\qquad \left. \times\, t^{(\lambda-1-1/q)p} K^p(1,t)\, dt \right\}^{1/p}
\end{aligned}$$

$$\leq \frac{(1+q)^{1/q}}{2^{2-(1/q)}b^{(1/p)-1}}\left\{\left[\sum_{k=-\infty}^{\infty}2^{(k+m-1)\alpha p}\|f\phi_{k+m-1}\|_{q,\omega}^{p}\right.\right.$$

$$\left.\times\int_0^b 2^{-(m-1)\alpha p}t^{(\lambda-1-(1/q))p}K^p(1,t)\,dt\right]^{1/p}$$

$$\left.+\left[\sum_{k=-\infty}^{\infty}2^{(k+m)\alpha p}\|f\phi_{k+m}\|_{q,\omega}^{p}\int_0^b 2^{-m\alpha p}t^{(\lambda-1-1/q)p}K^p(1,t)\,dt\right]^{1/p}\right\}$$

$$\leq 2^{1/q-2}(1+q)^{1/q}\|f\|_{K(\omega)}\int_0^b\left(2^{-(m-1)\alpha}+2^{-m\alpha}\right)t^{\lambda-1-1/q}K(1,t)\,dt$$

$$\leq 2^{1/q-2}(1+q)^{1/q}\left(1+2^{|\alpha|}\right)\|f\|_{K(\omega)}\int_0^\infty t^{\lambda-\alpha-1-(1/q)}K(1,t)\,dt.$$

Case 3. $0<q<1\leq p<\infty$.

It follows from (14) and (15) that

$$\|(Tf)\|_K$$

$$\leq \frac{(1+q)^{1/q}}{2b^{(1/p)-1}}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\int_0^b\left(\|f\phi_{k+m-1}\|_{q,\omega}^{q}+\|f\phi_{k+m}\|_{q,\omega}^{q}\right)^{p/q}\right.$$

$$\left.\times\, t^{(\lambda-1-1/q)p}K^p(1,t)\,dt\right\}^{1/p}$$

$$\leq \frac{(1+q)^{1/q}}{2^{1/p-1/q+1}b^{(1/p)-1}}\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\int_0^b\left(\|f\phi_{k+m-1}\|_{q,\omega}^{p}+\|f\phi_{k+m}\|_{q,\omega}^{p}\right)\right.$$

$$\left.\times\, t^{(\lambda-1-1/q)p}K^p(1,t)\,dt\right\}^{1/p}$$

$$\leq \frac{(1+q)^{1/q}}{2^{1/p-1/q+1}b^{(1/p)-1}}\left\{\left[\sum_{k=-\infty}^{\infty}2^{(k+m-1)\alpha p}\|f\phi_{k+m-1}\|_{q,\omega}^{p}\right.\right.$$

$$\left.\times\int_0^b 2^{-(m-1)\alpha p}t^{(\lambda-1-1/q)p}K^p(1,t)\,dt\right]^{1/p}$$

$$\left.+\left[\sum_{k=-\infty}^{\infty}2^{(k+m)\alpha p}\|f\phi_{k+m}\|_{q,\omega}^{p}\int_0^b 2^{-m\alpha p}t^{(\lambda-1-1/q)p}K^p(1,t)\,dt\right]^{1/p}\right\}$$

$$\begin{aligned}
&\leq 2^{1/q-2/p-1}\,(1+q)^{1/q}\,(1+1/p)\,\|f\|_{K(\omega)} \\
&\qquad\qquad \times \int_0^b \left(2^{-(m-1)\alpha}+2^{-m\alpha}\right) t^{\lambda-1-(1/q)} K(1,t)\,dt \\
&\leq 2^{1/q-2/p-1}\,(1+q)^{1/q}\,(1+1/p)\left(1+2^{|\alpha|}\right)\|f\|_{K(\omega)} \\
&\qquad\qquad \times \int_0^\infty t^{\lambda-\alpha-1-(1/q)} K(1,t)\,dt. \qquad (17)
\end{aligned}$$

In view of above inequalities (16)–(17), we obtain

$$\|T\| \leq C(p,q,\alpha)\int_0^\infty t^{\lambda-\alpha-1-1/q} K(1,t)\,dt, \qquad (18)$$

where $C(p,q,\alpha)$ defined by (10).

To prove the opposite inequality for each $\varepsilon \in (0,1)$, we set,

$$f_\varepsilon(x) = \begin{cases} 0, & 0 < x \leq 1 \\ x^{(\lambda-\alpha-1-1/q-\varepsilon)}, & x > 1, \end{cases}$$

then

$$\|f_\varepsilon \phi_k\|_{q,\omega}^q = \int_{2^{k-1}<x\leq 2^k} x^{(\lambda-\alpha-1-1/q-\varepsilon)q} x^{(1-\lambda)q} dx \;=\; 2^{-k(\alpha+\varepsilon)q} C_0, \qquad (19)$$

where

$$C_0 = \left|\frac{2^{(\alpha+\varepsilon)q}-1}{(\alpha+\varepsilon)q}\right|.$$

It follows that

$$\|f_\varepsilon\|_{K(\omega)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(C_0^{1/q}\, 2^{-k(\alpha+\varepsilon)}\right)^p\right\}^{1/p} = C_0^{1/q}\frac{2^{-\varepsilon}}{\left(1-2^{-p\varepsilon}\right)^{1/p}}. \qquad (20)$$

We note that

$$T(f_\varepsilon,x) = x^{-(\alpha+1/q+\varepsilon)}\int_{x^{-1}}^\infty t^{\lambda-(\alpha+1+1/q+\varepsilon)} K(1,t)\,dt. \qquad (21)$$

For each $\varepsilon \in (0,1)$, there exists a positive integer l such that $2^{l-1} \leq 1/\varepsilon < 2^l$, so that

$$\begin{aligned}
\|Tf_\varepsilon\|_K^p &= \sum_{k=1}^{\infty} 2^{k\alpha p}\left\{\int_{x>1}\left[T\left(f_\varepsilon,x\right)\phi_k\left(x\right)\right]^q dx\right\}^{p/q} \\
&= \sum_{k=1}^{\infty} 2^{k\alpha p}\left\{\int_{2^{k-1}<x\le 2^k} x^{-(\alpha+1/q+\varepsilon)q}\right. \\
&\qquad\qquad \left.\times\left(\int_{x^{-1}}^{\infty} t^{\lambda-(\alpha+1+1/q+\varepsilon)}K(1,t)\,dt\right)^q dx\right\}^{p/q} \\
&\ge \sum_{k=l+1}^{\infty} 2^{k\alpha p}\left(\int_{\varepsilon}^{\infty} t^{\lambda-(\alpha+1+1/q+\varepsilon)}K(1,t)\,dt\right)^p \\
&\qquad\qquad \times\left(\int_{2^{k-1}<x\le 2^k} x^{-(\alpha+1/q+\varepsilon)q}dx\right)^{p/q} \\
&= \left(\int_{\varepsilon}^{\infty} t^{\lambda-(\alpha+1+1/q+\varepsilon)}K(1,t)\,dt\right)^p \sum_{k=l+1}^{\infty} 2^{k\alpha p}\left(C_0\, 2^{-k(\alpha+\varepsilon)q}\right)^{p/q} \\
&= C_0^{p/q}\left(\int_{\varepsilon}^{\infty} t^{\lambda-(\alpha+1+1/q+\varepsilon)}K(1,t)\,dt\right)^p \left(\frac{2^{-\varepsilon p(l+1)}}{1-2^{-\varepsilon p}}\right). \qquad (22)
\end{aligned}$$

Thus,

$$\|T\| \ge \frac{\|Tf_\varepsilon\|_K}{\|f_\varepsilon\|_{K(\omega)}} \ge 2^{-\varepsilon l}\int_{\varepsilon}^{\infty} t^{\lambda-(\alpha+1+1/q+\varepsilon)}K(1,t)\,dt. \qquad (23)$$

In the limit as $\varepsilon \to 0$ in (23), we obtain

$$\|T\| \ge \int_0^{\infty} t^{\lambda-(\alpha+1+1/q)}K(1,t)\,dt.$$

This completes the proof of Theorem 2.2.

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