LE MATEMATICHE
Vol. LXVII (2012) – Fasc. II, pp. 203–216
doi: 10.4418/2012.67.2.15

UNIVALENT HARMONIC FUNCTIONS DEFINED BY SALAGEAN INTEGRAL OPERATOR WITH RESPECT TO SYMMETRIC POINTS

MOHAMED K. AOUF - RABHA M. EL-ASHWAH
ALI SHAMANDY - SHEZA M. EL-DEEB

In this paper, we define and investigate a subclass of univalent harmonic functions defined by Salagean integral operator with respect to symmetric points. We obtain coefficient conditions, extreme points, distortion bounds, convex combinations for this family of harmonic univalent functions.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply-connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply-connected domain we can write

$$f = h + \overline{g}\,, \tag{1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h^{'}(z)| > |g^{'}(z)|$ in D (see [5]).

Denote by S_H the class of functions f of the form (1) that are harmonic univalent and sense-preserving in the unit disc $U = \{z : |z| < 1\}$ for which

Entrato in redazione: 23 marzo 2012
AMS 2010 Subject Classification: 30C45.
Keywords: Harmonic univalent functions, Integral operator, Symmetric points.

$f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k\,,\; g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{2}$$

In 1984 Clunie and Shell-Small [6] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.

For $f = h + \overline{g}$ given by (2), we define the modified integral Salagean operator of f as

$$I^m f(z) = I^m h(z) + (-1)^m \overline{I^m g(z)}\ (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \mathbb{N} = \{1, 2, ...\})\,, \tag{3}$$

where

$$I^m h(z) = z + \sum_{k=2}^{\infty} k^{-m} a_k\, z^k \text{ and } I^m g(z) = \sum_{k=1}^{\infty} k^{-m} b_k\, z^k.$$

The integral operator I^m was introduced by Salagean [10](see also [2], [3] and [7]).

Definition 1.1. For $0 \le \alpha < 1$, $m \in \mathbb{N}_0$ and $z = re^{i\theta} \in U$, let $SH_s(m;\alpha)$ denote the family of harmonic functions f of the form 2 such that

$$\text{Re}\left\{\frac{2I^m f(z)}{I^{m+1} f(z) - I^{m+1} f(-z)}\right\} > \alpha, \tag{4}$$

where $I^m f$ is defined by (3).

We denote by $\overline{SH}_s(m;\alpha)$ the subclasses of harmonic functions $f_m = h + \overline{g}_m$ in $SH_s(m;\alpha)$ such that h and g_m are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k,\; g_m(z) = (-1)^m \sum_{k=1}^{\infty} b_k z^k,\; |b_1| < 1. \tag{5}$$

Remark 1.2. If the co-analytic part of $f = h + \overline{g}$ is zero, then $SH_s(-1;\alpha)$ turns out to be the class $S_s^*(\alpha)$ of starlike functions with respect to symmetric points which was introduced by Sakaguchi [9]. Also, $SH_s(-2,\alpha)$ turns out to be the class $K_s(\alpha)$ of convex functions with respect to symmetric points which was introduced by Das and Singh [8].
Also, we note that $SH_s(-n-1,\alpha) = SH_s(n,\alpha)$ and $\overline{SH_s}(-n-1,\alpha) = \overline{SH_s}(n,\alpha)$ $(n \in \mathbb{N}_0)$, which were studied by AL-Khal and Al-Kharsani [1].

In this paper, we extend the above results to the classes $SH_s(m;\alpha)$ and $\overline{SH}_s(m;\alpha)$. We also obtain coefficient conditions, extreme points, distortion bounds, convolution conditions and convex combinations for $\overline{SH_s}(m;\alpha)$.

2. Coefficient characterization

Unless otherwise mentioned, we assume throughout this paper that $m \in \mathbb{N}_0$, $a_1 = 1$ and $0 \leq \alpha < 1$. We begin with a sufficient condition for functions in the class $SH_s(m;\alpha)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be such that h and g are given by (2). Furthermore, let*

$$\sum_{k=1}^{\infty} \Big\{ (2k-1)^{-m-1}(2k-1-\alpha)|a_{2k-1}| + (2k)^{-m}|a_{2k}| + (2k-1)^{-m-1}(2k-1+\alpha)|b_{2k-1}| + (2k)^{-m}|b_{2k}| \Big\} \leq 2(1-\alpha). \quad (6)$$

Then f is sense-preserving, harmonic univalent in U and $f \in SH_s(m;\alpha)$.

Proof. Note that f is sense-preserving in U. Now we show that $f \in SH_s(m;\alpha)$. We only need to show that if (6) holds then the condition (4) is satisfied.

Using the fact that $\operatorname{Re} w > \alpha$ if and only if $|1-\alpha+w| > |1+\alpha-w|$, it suffices to show that

$$\left|2I^m f(z) + (1-\alpha)[I^{m+1}f(z) - I^{m+1}f(-z)]\right| - \left|2I^m f(z) - (1+\alpha)[I^{m+1}f(z) - I^{m+1}f(-z)]\right| > 0 \quad (7)$$

Substituting for $I^m f(z)$ and $I^{m+1}f(z)$ in (7) yields, by using (6) and $0 \leq \alpha < 1$, we obtain

$$\begin{aligned}
&\Bigg|2(2-\alpha)z + \sum_{k=2}^{\infty} k^{-m-1}\left[2k + (1-\alpha)(1-(-1)^k)\right] a_k z^k \\
&\qquad - (-1)^{m+1}\sum_{k=1}^{\infty} k^{-m-1}\left[2k - (1-\alpha)(1-(-1)^k)\right] \overline{b_k z^k}\Bigg| \\
&\qquad - \Bigg|-2\alpha z + \sum_{k=2}^{\infty} k^{-m-1}\left[2k - (1+\alpha)(1-(-1)^k)\right] a_k z^k \\
&\qquad - (-1)^{m+1}\sum_{k=1}^{\infty} k^{-m-1}\left[2k + (1+\alpha)(1-(-1)^k)\right] \overline{b_k z^k}\Bigg| \\
&\qquad = \Bigg|2(2-\alpha)z + 2\sum_{k=2}^{\infty} \Big\{ (2k-2)^{-m} a_{2k-2} z^{2k-2} \\
&\qquad\qquad + (2k-1)^{-m-1}[2k-1-\alpha+1] a_{2k-1} z^{2k-1} \Big\} -
\end{aligned}$$

$$2(-1)^{m+1}\sum_{k=1}^{\infty}\left\{(2k)^{-m}\overline{b_{2k}z^{2k}}+(2k-1)^{-m-1}[2k-1+\alpha-1]\overline{b_{2k-1}z^{2k-1}}\right\}\Bigg|$$

$$-\Bigg|-2\alpha z+2\sum_{k=2}^{\infty}\left\{(2k-2)^{-m}a_{2k-2}z^{2k-2}\right.$$

$$\left.+(2k-1)^{-m-1}[2k-1-\alpha-1]a_{2k-1}z^{2k-1}\right\}-$$

$$2(-1)^{m+1}\sum_{k=1}^{\infty}\left\{(2k)^{-m}\overline{b_{2k}z^{2k}}+(2k-1)^{-m-1}[2k-1+\alpha+1]\overline{b_{2k-1}z^{2k-1}}\right\}\Bigg|$$

$$\geq 4(1-\alpha)|z|-4\sum_{k=2}^{\infty}\left\{(2k-2)^{-m}|a_{2k-2}||z|^{2k-2}\right.$$

$$\left.+(2k-1)^{-m-1}[2k-1-\alpha]|a_{2k-1}||z|^{2k-1}\right\}$$

$$-4\sum_{k=1}^{\infty}\left\{(2k)^{-m}|b_{2k}||z|^{2k}+(2k-1)^{-m-1}[2k-1+\alpha]|b_{2k-1}||z|^{2k-1}\right\}$$

$$\geq 4(1-\alpha)-4\left[\sum_{k=2}^{\infty}\left\{(2k-2)^{-m}|a_{2k-2}|+(2k-1)^{-m-1}[2k-1-\alpha]|a_{2k-1}|\right\}\right.$$

$$\left.+\sum_{k=1}^{\infty}\left\{(2k)^{-m}|b_{2k}|+(2k-1)^{-m-1}[2k-1+\alpha]|b_{2k-1}|\right\}\right].$$

This last expression is non-negative by (6).

The harmonic univalent functions

$$f(z)=z+\sum_{k=2}^{\infty}\left\{\frac{1-\alpha}{(2k-2)^{-m}}X_{2k-2}z^{2k-2}+\frac{1-\alpha}{(2k-1)^{-m-1}(2k-1-\alpha)}X_{2k-1}z^{2k-1}\right\}$$
$$+\sum_{k=1}^{\infty}\left\{\frac{1-\alpha}{(2k)^{-m}}\overline{Y_{2k}z^{2k}}+\frac{1-\alpha}{(2k-1)^{-m-1}(2k-1+\alpha)}\overline{Y_{2k-1}z^{2k-1}}\right\}, \quad (8)$$

where $\sum_{k=2}^{\infty}(|X_{2k-2}|+|X_{2k-1}|)+\sum_{k=1}^{\infty}(|Y_{2k}|+|Y_{2k-1}|)=1$, show that the coefficient bound given by (6) is sharp. The functions of the form (8) are in $SH_s(m;\alpha)$ because

$$\sum_{k=2}^{\infty}\left\{\frac{(2k-2)^{-m}}{1-\alpha}|a_{2k-2}|+\frac{(2k-1)^{-m-1}(2k-1-\alpha)}{1-\alpha}|a_{2k-1}|\right\}$$
$$+\sum_{k=1}^{\infty}\left\{\frac{(2k)^{-m}}{1-\alpha}|b_{2k}|+\frac{(2k-1)^{-m-1}(2k-1+\alpha)}{1-\alpha}|b_{2k-1}|\right\}$$

$$= \sum_{k=2}^{\infty} \left(|X_{2k-2}| + |X_{2k-1}|\right) + \sum_{k=1}^{\infty} \left(|Y_{2k}| + |Y_{2k-1}|\right) = 1.$$

This completes the proof. □

Putting $m = -n-1$ $(n \in \mathbb{N}_0)$ in Theorem 2.1, we obtain the following corollary, correcting the results obtained by AL-Khal and Al-Kharsani [1, Theorem 2.1].

Corollary 2.2. *Let $f = h + \overline{g}$ with h and g given by (2). Furthermore, let*

$$\sum_{k=1}^{\infty} \Big\{ (2k-1)^n (2k-1-\alpha) |a_{2k-1}| + (2k)^{n+1} |a_{2k}| + (2k-1)^n (2k-1+\alpha) |b_{2k-1}| + (2k)^{n+1} |b_{2k}| \Big\} \leq 2(1-\alpha).$$

Then f is sense-preserving, harmonic univalent in U and $f \in SH_s(n,\alpha)$.

In the following theorem, it is shown that the condition (6) is also necessary for functions $f_m = h + \overline{g}_m$, where h and g_m are of the form (5).

Theorem 2.3. *Let $f_m = h + \overline{g}_m$ be such that h and $\overline{g}_m$ are given by (5). Then $f_m \in \overline{SH_s}(m;\alpha)$ if and only if*

$$\sum_{k=1}^{\infty} \Big\{ (2k)^{-m} |a_{2k}| + (2k-1)^{-m-1} (2k-1-\alpha) |a_{2k-1}| + (2k)^{-m} |b_{2k}| + (2k-1)^{-m-1} (2k-1+\alpha) |b_{2k-1}| \Big\} \leq 2(1-\alpha). \quad (9)$$

Proof. Since $\overline{SH_s}(m;\alpha) \subset SH_s(m;\alpha)$, we only need to prove the "only if" part of the theorem. To this end, for functions f_m of the form (5), we notice that the condition $\operatorname{Re}\left\{ \dfrac{2I^m f_m(z)}{I^{m+1} f_m(z) - I^{m+1} f_m(-z)} \right\} > \alpha$ is equivalent to

$$\operatorname{Re}\left\{ \frac{2(1-\alpha)z - \sum_{k=2}^{\infty} k^{-m-1}\left[2k-\alpha(1-(-1)^k)\right] a_k z^k - (-1)^{2(m+1)} \sum_{k=1}^{\infty} k^{-m-1}\left[2k+\alpha(1-(-1)^k)\right] \overline{b_k z^k}}{2z - \sum_{k=2}^{\infty} k^{-m-1}(1-(-1)^k) a_k z^k + (-1)^{2(m+1)} \sum_{k=1}^{\infty} k^{-m-1}(1-(-1)^k) \overline{b_k z^k}} \right\} > 0$$

which implies that

$$\operatorname{Re}\left\{ \frac{(1-\alpha)z - \sum_{k=2}^{\infty} \Big\{ (2k-2)^{-m} a_{2k-2} z^{2k-2} + (2k-1)^{-m-1} [2k-1-\alpha] a_{2k-1} z^{2k-1} \Big\}}{z - \sum_{k=2}^{\infty} (2k-1)^{-m-1} a_{2k-1} z^{2k-1} + (-1)^{2(m+1)} \sum_{k=1}^{\infty} (2k-1)^{-m-1} \overline{b_{2k-1} z^{2k-1}}} \right.$$

$$\left.-\frac{(-1)^{2(m+1)}\sum\limits_{k=1}^{\infty}\left\{(2k)^{-m}\overline{b_{2k}z^{2k}}+(2k-1)^{-m-1}[2k-1+\alpha]\overline{b_{2k-1}z^{2k-1}}\right\}}{z-\sum\limits_{k=2}^{\infty}(2k-1)^{-m-1}a_{2k-1}z^{2k-1}+(-1)^{2(m+1)}\sum\limits_{k=1}^{\infty}(2k-1)^{-m-1}\overline{b_{2k-1}z^{2k-1}}}\right\}>0 \tag{10}$$

The above required condition (10) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\le z=r<1$, we must have

$$\left\{\frac{(1-\alpha)-\sum\limits_{k=2}^{\infty}\left\{(2k-2)^{-m}a_{2k-2}r^{2k-3}+(2k-1)^{-m-1}[2k-1-\alpha]a_{2k-1}r^{2k-2}\right\}}{1-\sum\limits_{k=2}^{\infty}(2k-1)^{-m-1}a_{2k-1}r^{2k-2}+\sum\limits_{k=1}^{\infty}(2k-1)^{-m-1}b_{2k-1}r^{2k-2}}\right. \tag{11}$$

$$\left.-\frac{\sum\limits_{k=1}^{\infty}\left\{(2k)^{-m}b_{2k}r^{2k-1}+(2k-1)^{-m-1}[2k-1+\alpha]b_{2k-1}r^{2k-2}\right\}}{1-\sum\limits_{k=2}^{\infty}(2k-1)^{-m-1}a_{2k-1}r^{2k-2}+\sum\limits_{k=1}^{\infty}(2k-1)^{-m-1}b_{2k-1}r^{2k-2}}\right\}>0.$$

If the condition (9) does not hold, then the numerator in (11) is negative for r sufficiently close to 1. Hence there exists $z_0=r_0$ in $(0,1)$ for which the quotient in (11) is negative. This contradicts the required condition for $f_m\in\overline{SH_s}(m;\alpha)$ and so the proof of Theorem 2 is completed. □

Putting $m=-n-1$ $(n\in\mathbb{N}_0)$ in Theorem 2, we obtain the following corollary, correcting the result obtained by AL-Khal and Al-Kharsani [1, Theorem 2.2].

Corollary 2.4. *Let $f_n=h+\overline{g}_n$ be given by (5). Then $f_n\in\overline{SH_s}(n,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\left\{(2k-1)^{n}(2k-1-\alpha)|a_{2k-1}|+(2k)^{n+1}|a_{2k}|\right.$$
$$\left.+(2k-1)^{n}(2k-1+\alpha)|b_{2k-1}|+(2k)^{n+1}|b_{2k}|\right\}\le 2(1-\alpha).$$

3. Extreme points and distortion theorem

Our next theorem is on the extreme points of convex hulls of $\overline{SH_s}(m;\alpha)$ denoted by clco $\overline{SH}_s(m;\alpha)$.

Theorem 3.1. *Let $f_m = h + \overline{g}_m$ be such that h and $\overline{g}_m$ are given by (5). Then $f_m \in \overline{SH_s}(m;\alpha)$ if and only if*

$$f_m(z) = \sum_{k=1}^{\infty} \left[\left(X_{2k-1}\, h_{2k-1}(z) + X_{2k}\, h_{2k}(z)\right) + \left(Y_{2k-1} g_{m_{2k-1}}(z) + Y_{2k} g_{m_{2k}}(z)\right) \right], \tag{12}$$

where $h_1(z) = z$,

$$h_{2k-1}(z) = z - \frac{1-\alpha}{(2k-1)^{-m-1}(2k-1-\alpha)}\, z^{2k-1} \quad (k \geq 2),$$

$$h_{2k-2}(z) = z - \frac{1-\alpha}{(2k-2)^{-m}}\, z^{2k-2} \qquad (k \geq 2),$$

and

$$g_{m_{2k-1}}(z) = z + (-1)^m \frac{1-\alpha}{(2k-1)^{-m-1}(2k-1+\alpha)}\, \overline{z}^{2k-1} \qquad (k \geq 1),$$

$$g_{m_{2k}}(z) = z + (-1)^m \frac{1-\alpha}{(2k)^{-m}} \overline{z}^{2k} \qquad (k \geq 1),$$

$$X_k \geq 0, Y_k \geq 0, \sum_{k=1}^{\infty} \left(X_{2k-1} + X_{2k} + Y_{2k-1} + Y_{2k}\right) = 1 .$$

In particular, the extreme points of $\overline{SH_s}(m;\alpha)$ are $\{h_{2k-1}\}$, $\{h_{2k-2}\}$, $\{g_{m_{2k-1}}\}$ and $\{g_{m_{2k}}\}$.

Proof. For functions f_m of the form (12), we have

$$\begin{aligned} &f_m(z) \\ &= z - \sum_{k=2}^{\infty} \left\{ \frac{1-\alpha}{(2k-2)^{-m}} X_{2k-2}\, z^{2k-2} + \frac{1-\alpha}{(2k-1)^{-m-1}(2k-1-\alpha)} X_{2k-1}\, z^{2k-1} \right\} \\ &\quad + (-1)^m \sum_{k=1}^{\infty} \left\{ \frac{1-\alpha}{(2k)^{-m}} Y_{2k} \overline{z}^{2k} + \frac{1-\alpha}{(2k-1)^{-m-1}(2k-1+\alpha)} Y_{2k-1}\, \overline{z}^{2k-1} \right\}. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty} \left\{ \tfrac{(2k-2)^{-m}}{1-\alpha} . \left(\tfrac{1-\alpha}{(2k-2)^{-m}} X_{2k-2} \right) \right. \\ &\qquad \left. + \tfrac{(2k-1)^{-m-1}(2k-1-\alpha)}{1-\alpha} . \left(\tfrac{1-\alpha}{(2k-1)^{-m-1}(2k-1-\alpha)} X_{2k-1} \right) \right\} \end{aligned}$$

$$+\sum_{k=1}^{\infty}\left\{\tfrac{(2k)^{-m}}{1-\alpha}\cdot\left(\tfrac{1-\alpha}{(2k)^{-m}}Y_{2k}\right)+\tfrac{(2k-1)^{-m-1}(2k-1+\alpha)}{1-\alpha}\cdot\left(\tfrac{1-\alpha}{(2k-1)^{-m-1}(2k-1+\alpha)}Y_{2k-1}\right)\right\}$$

$$=\sum_{k=2}^{\infty}(X_{2k-2}+X_{2k-1})+\sum_{k=1}^{\infty}(Y_{2k}+Y_{2k-1})=1-X_1\leq 1$$

and so $f_m \in \overline{SH_s}(m;\alpha)$.

Conversely, if $f_m \in clco\,\overline{SH_s}(m;\alpha)$. Setting

$$X_{2k-1}=\frac{(2k-1)^{-m-1}(2k-1-\alpha)}{1-\alpha}a_{2k-1}, X_{2k-2}=\frac{(2k-2)^{-m}}{1-\alpha}a_{2k}\,(k\geq 2),$$

and

$$Y_{2k-1}=\frac{(2k-1)^{-m-1}(2k-1+\alpha)}{1-\alpha}b_{2k-1}, Y_{2k-2}=\frac{(2k)^{-m}}{1-\alpha}b_{2k}\,(k\geq 1).$$

We obtain

$$f_m(z)=\sum_{k=1}^{\infty}\left[(X_{2k-1}\,h_{2k-1}(z)+X_{2k}\,h_{2k}(z))+\left(Y_{2k-1}g_{m_{2k-1}}(z)+Y_{2k}g_{m_{2k}}(z)\right)\right]$$

as required. □

The following theorem gives the distortion bounds for functions in the class $\overline{SH_s}(m;\alpha)$ which yields a covering result for this class.

Theorem 3.2. *Let $f_m = h+\overline{g}_m$ be such that h and $\overline{g}_m$ are given by (5) be in the class $\overline{SH_s}(m;\alpha)$. Then for $|z|=r<1$ we have*

$$|f_m(z)|\leq(1+|b_1|)\,r+\frac{1}{2^{-m-1}}\left\{\frac{1-\alpha}{2}-\frac{1+\alpha}{2}|b_1|\right\}r^2,$$

and

$$|f_m(z)|\geq(1-|b_1|)\,r-\frac{1}{2^{-m-1}}\left\{\frac{1-\alpha}{2}-\frac{1+\alpha}{2}|b_1|\right\}r^2$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $f_m(z)\in\overline{SH_s}(m;\alpha)$. Taking the absolute value of f_m we have

$$\begin{aligned}|f_m(z)| &\le (1+|b_1|)\,r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k\\ &\le (1+|b_1|)r+r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|)\\ &= (1+|b_1|)r+\frac{(1-\alpha)}{2^{-m}}\sum_{k=2}^{\infty}\frac{2^{-m}}{1-\alpha}(|a_k|+|b_k|)r^2\\ &\le (1+|b_1|)r+\frac{1}{2^{-m-1}}\left[\frac{1-\alpha}{2}-\frac{1+\alpha}{2}|b_1|\right]r^2.\end{aligned}$$

The bounds given in Theorem 3.2 for functions $f_m = h+\overline{g}_m$ of form (5) also hold for functions of the form (2) if the coefficient condition (6) is satisfied. The upper bound given for $f_m \in \overline{SH_s}(m;\alpha)$ is sharp and the equality occurs for the functions

$$f_m(z) = z+b_1\overline{z}+\frac{1}{2^{-m-1}}\left[\frac{1-\alpha}{2}-\frac{1+\alpha}{2}b_1\right]\overline{z}^2,$$

and

$$f_m(z) = z-b_1\overline{z}-\frac{1}{2^{-m-1}}\left[\frac{1-\alpha}{2}-\frac{1+\alpha}{2}b_1\right]\overline{z}^2$$

show that the bounds given in Theorem 3.2 are sharp. □

The following covering result follows from the left hand inequality in Theorem 3.2.

Corollary 3.3. *Let the functions f_m defined by (5) belong to the class $\overline{SH_s}(m;\alpha)$. Then*

$$\left\{w : |w| < \frac{2^{-m}-1+\alpha}{2^{-m}}-\frac{2^{-m}-1+\alpha}{2^{-m}}|b_1|\right\}\subset f_m(U).$$

4. Convolution and convex combination

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form:

$$f_m(z) = z-\sum_{k=2}^{\infty}a_k z^k+(-1)^m\sum_{k=1}^{\infty}b_k\overline{z}^k,\ |b_1|<1 \tag{13}$$

and

$$F_m(z) = z-\sum_{k=2}^{\infty}A_k z^k+(-1)^m\sum_{k=1}^{\infty}B_k\overline{z}^k\ (A_k\ge 0;\, B_k\ge 0) \tag{14}$$

we define the convolution of two harmonic functions f_m and F_m as

$$(f_m * F_m)(z) = f_m(z) * F_m(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^m \sum_{k=1}^{\infty} b_k B_k \bar{z}^k. \tag{15}$$

Using this definition, we show that the class $\overline{SH_s}(m;\alpha)$ is closed under convolution.

Theorem 4.1. *For $0 \le \beta \le \alpha < 1$, let $f_m \in \overline{SH_s}(m;\alpha)$ and $F_m \in \overline{SH_s}(m;\beta)$. Then $f_m * F_m \in \overline{SH_s}(m;\alpha) \subset \overline{SH_s}(m;\beta)$.*

Proof. Let the function $f_m(z)$ defined by (13) be in the class $\overline{SH_s}(m;\alpha)$ and let the function $F_m(z)$ defined by (14) be in the class $\overline{SH_s}(m;\beta)$. Then the convolution $f_m * F_m$ is given by (15). We wish to show that the coefficients of $f_m * F_m$ satisfy the required condition given in Theorem 2.3. For $F_m \in \overline{SH_s}(m;\beta)$ we note that $0 \le A_k \le 1$ and $0 \le B_k \le 1$. Now, for the convolution function $f_m * F_m$ we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \left\{ \frac{(2k-2)^{-m}}{1-\beta} |a_{2k-2}| A_{2k-2} + \frac{(2k-1)^{-m-1}(2k-1-\beta)}{1-\beta} |a_{2k-1}| A_{2k-1} \right\} \\
&+ \sum_{k=1}^{\infty} \left\{ \frac{(2k)^{-m}}{1-\beta} |b_{2k}| B_{2k} + \frac{(2k-1)^{-m-1}(2k-1+\beta)}{1-\beta} |b_{2k-1}| B_{2k-1} \right\} \\
&\le \sum_{k=2}^{\infty} \left\{ \frac{(2k-2)^{-m}}{1-\beta} |a_{2k-2}| + \frac{(2k-1)^{-m-1}(2k-1-\beta)}{1-\beta} |a_{2k-1}| \right\} \\
&+ \sum_{k=1}^{\infty} \left\{ \frac{(2k)^{-m}}{1-\beta} |b_{2k}| + \frac{(2k-1)^{-m-1}(2k-1+\beta)}{1-\beta} |b_{2k-1}| \right\} \\
&\le \sum_{k=2}^{\infty} \left\{ \frac{(2k-2)^{-m}}{1-\alpha} |a_{2k-2}| + \frac{(2k-1)^{-m-1}(2k-1-\alpha)}{1-\alpha} |a_{2k-1}| \right\} \\
&+ \sum_{k=1}^{\infty} \left\{ \frac{(2k)^{-m}}{1-\alpha} |b_{2k}| + \frac{(2k-1)^{-m-1}(2k-1+\alpha)}{1-\alpha} |b_{2k-1}| \right\} \le 1.
\end{aligned}$$

since $0 \le \beta \le \alpha < 1$ and $f_m \in \overline{SH_s}(m;\alpha)$. Therefore $f_m * F_m \in \overline{SH_s}(m;\alpha) \subset \overline{SH_s}(m;\beta)$. □

Now, we show that the class $\overline{SH_s}(m;\alpha)$ is closed under convex combinations of its members.

Theorem 4.2. *The class $\overline{SH_s}(m;\alpha)$ is closed under convex combination.*

Proof. For $i = 1,2,...,$ let $f_{m_i} \in \overline{SH_s}(m;\alpha)$, where f_{m_i} is given by

$$f_{m_i}(z) = z - \sum_{k=2}^{\infty} a_{k_i} z^k + (-1)^m \sum_{k=1}^{\infty} b_{k_i} \overline{z}^k \, (|b_{1_i}| < 1;\ z \in U).$$

Then by using Theorem 2.3, we have

$$\sum_{k=2}^{\infty} \left\{ \frac{(2k-2)^{-m}}{1-\alpha} |a_{i,2k-2}| + \frac{(2k-1)^{-m-1}(2k-1-\alpha)}{1-\alpha} |a_{i,2k-1}| \right\} + \sum_{k=1}^{\infty} \left\{ \frac{(2k)^{-m}}{1-\alpha} |b_{i,2k}| + \frac{(2k-1)^{-m-1}(2k-1+\alpha)}{1-\alpha} |b_{i,2k-1}| \right\} \leq 1. \quad (16)$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{m_i}(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{k_i}| \right) z^k + (-1)^m \sum_{k=1}^{\infty} (\sum_{i=1}^{\infty} t_i |b_{k_i}|) \overline{z}^k. \quad (17)$$

Then, by using (16), we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \left\{ \frac{(2k-2)^{-m}}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i |a_{i,2k-2}| \right) \right. \\
&\left. + \frac{(2k-1)^{-m-1}(2k-1-\alpha)}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i |a_{i,2k-1}| \right) \right\} \\
&+ \sum_{k=1}^{\infty} \left\{ \frac{(2k)^{-m}}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i |b_{i,2k}| \right) \right. \\
&\left. + \frac{(2k-1)^{-m-1}(2k-1+\alpha)}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i |b_{i,2k-1}| \right) \right\} \\
&= \sum_{i=1}^{\infty} t_i \left[\sum_{k=2}^{\infty} \left\{ \frac{(2k-2)^{-m}}{1-\alpha} |a_{i,2k-2}| + \frac{(2k-1)^{-m-1}(2k-1-\alpha)}{1-\alpha} |a_{i,2k-1}| \right\} \right. \\
&\left. + \sum_{k=1}^{\infty} \left\{ \frac{(2k)^{-m}}{1-\alpha} |b_{i,2k}| + \frac{(2k-1)^{-m-1}(2k-1+\alpha)}{1-\alpha} |b_{i,2k-1}| \right\} \right] \\
&\leq \sum_{i=1}^{\infty} t_i = 1,
\end{aligned}$$

this is the condition required by (9) and so $\sum_{i=1}^{\infty} t_i f_{m_i}(z) \in \overline{SH_s}(m;\alpha)$. This completes the proof of Theorem 6. □

5. Properties of an integral operator

Finally, we study properties of an integral operator.

Theorem 5.1. *Let the functions $f_m(z)$ defined by (5) be in the class $\overline{SH_s}(m;\alpha)$ and let c be a real number such that $c > -1$. Then the function $F_m(z)$ defined by*

$$F_m(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f_m(t)dt \tag{18}$$

belongs to the class $\overline{SH_s}(m;\alpha)$.

Proof. From the representation of $F_m(z)$, it follows that

$$\begin{aligned}
F_m(z) &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left\{h(t)+\overline{g_m(t)}\right\}dt \\
&= \frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t-\sum_{k=2}^{\infty}a_k t^k\right)dt + (-1)^m \overline{\int_0^z t^{c-1}\left(\sum_{k=1}^{\infty}b_k t^k\right)dt}\right) \\
&= \frac{c+1}{z^c}\left(\int_0^z t^c dt - \sum_{k=2}^{\infty}a_k\int_0^z t^{c+k-1}dt + (-1)^m \overline{\sum_{k=1}^{\infty}b_k\int_0^z t^{c+k-1}dt}\right) \\
&= z-\sum_{k=2}^{\infty}A_k z^k + (-1)^m\sum_{k=1}^{\infty}B_k\overline{z}^k,
\end{aligned}$$

where $A_k = \dfrac{c+1}{c+k}a_k,\ B_k = \dfrac{c+1}{c+k}b_k$. Therefore

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left\{(2k-2)^{-m}A_{2k-2}+(2k-1)^{-m-1}(2k-1-\alpha)A_{2k-1}\right\} \\
&\qquad\qquad + \sum_{k=1}^{\infty}\left\{(2k)^{-m}B_{2k}+(2k-1)^{-m-1}(2k-1+\alpha)B_{2k-1}\right\}
\end{aligned}$$

$$= \sum_{k=2}^{\infty} \left\{ (2k-2)^{-m} \left(\tfrac{c+1}{c+2k-2} \right) a_{2k-2} \right.$$

$$\left. + (2k-1)^{-m-1} (2k-1-\alpha) \left(\tfrac{c+1}{c+2k-1} \right) a_{2k-1} \right\}$$

$$+ \sum_{k=1}^{\infty} \left\{ (2k)^{-m} \left(\tfrac{c+1}{c+2k} \right) b_{2k} + (2k-1)^{-m-1} (2k-1+\alpha) \left(\tfrac{c+1}{c+2k-1} \right) b_{2k-1} \right\}$$

$$\leq \sum_{k=2}^{\infty} \left\{ (2k-2)^{-m} a_{2k-2} + (2k-1)^{-m-1} (2k-1-\alpha) a_{2k-1} \right\}$$

$$+ \sum_{k=1}^{\infty} \left\{ (2k)^{-m} b_{2k} + (2k-1)^{-m-1} (2k-1+\alpha) b_{2k-1} \right\} \leq 1-\alpha.$$

Since $f_m(z) \in \overline{SH_s}(m;\alpha)$, therefore by Theorem 2.3 $F_m(z) \in \overline{SH_s}(m;\alpha)$. □

Remark 5.2. Putting $m = -n-1$ $(n \in \mathbb{N}_0)$ in Theorems 3.1, 3.2, 4.1, 4.2 and 5.1, we obtain the results obtained by AL-Khal and Al-Kharsani [1] in Theorems 2.7, 2.3, 2.5, 2.6, and 2.8, respectively.

REFERENCES

[1] R. A. Al-Khal - H. A. Al-Kharsani, *Salagean-type harmonic univalent functions with respect to symmetric points*, The Australian J. Math. Analysis Appl. 4 (1) (2007), 1–13.

[2] M. K. Aouf - F. M. Al-Oboudi - M. M. Hadain, *An application of certain integral operator*, Math. (Cluj), 47 (70) (2) (2005), 121–124.

[3] M. K. Aouf, *Some applications of Salagean integral operator*, Studia Univ. Babes-Bolyai Math. 55 (1) (2010), 21–30.

[4] M. K. Aouf, *The Salagean integral operator and strongly starlike functions*, Studia Univ. Babes-Bolyai Math., 56 (1) (2011), 109–115.

[5] Y. Avci - E. Zlotkiewicz, *On harmonic univalent mappings*, Ann. Univ. Mariae-Curie Sklodowska Sect. A, 44 (1990), 1–7.

[6] J. Clunie - T. Shell-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. 9 (1984), 3–25.

[7] L. I. Cotirla, *Harmonic univalent functions defined by an integral operator*, Acta Universitatis Apulensis, 17 (2009), 95–105.

[8] R. N. Das - P. Singh, *On subclasses of Schlicht mapping*, Indian J. Pure Appl. Math. 8 (1977), 864–872.

[9] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan, 11 (1959), 72–75.

[10] G. S. Salagean, *Subclasses of univalent functions*, Lecture Notes in Math. 1013, Springer-Verlag 1983, 362-372.

MOHAMED K. AOUF
Department of Mathematics, Faculty of Science
Mansoura University
Mansoura 35516, Egypt
e-mail: mkaouf127@yahoo.com

RABHA M. EL-ASHWAH
Department of Mathematics, Faculty of Science at Damietta
Mansoura University
New Damietta 34517, Egypt
e-mail: r_elashwah@yahoo.com

ALI SHAMANDY
Department of Mathematics, Faculty of Science
Mansoura University
Mansoura 35516, Egypt
e-mail: shamandy16@hotmail.com

SHEZA M. EL-DEEB
Department of Mathematics, Faculty of Science at Damietta
Mansoura University
New Damietta 34517, Egypt
e-mail: shezaeldeeb@yahoo.com