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COMBINED EFFECTS AND DEGENERATE PHENOMENA IN NONLINEAR STATIONARY PROBLEMS

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In this survey paper we are concerned with several nonlinear stationary problems involving nonhomogeneous differential operators. We report on some recent qualitative results related with various nonlinear problems in Orlicz-Sobolev spaces. Our analysis combines spectral analysis techniques with variational methods.

1. Basic properties of Orlicz-Sobolev spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set with smooth boundary. In Orlicz [31], the standard Lebesgue spaces $L^p(\Omega)$ were replaced by more general function spaces denoted $L_\Phi(\Omega)$ and which are now called *Orlicz spaces*. The spaces $L_\Phi(\Omega)$ were thoroughly studied in the monograph by Kranosel'skii & Rutickii [18] and also in the doctoral thesis of Luxemburg [23]. If the role played by $L^p(\Omega)$ in the definition of the Sobolev spaces $W^{m,p}(\Omega)$ is assigned instead to an Orlicz space $L_\Phi(\Omega)$, the resulting space is denoted by $W^m L_\Phi(\Omega)$ and called an *Orlicz-Sobolev space*. Many properties of Sobolev spaces have been extended to Orlicz-Sobolev spaces, mainly by Donaldson & Trudinger [12] and O'Neill [30]. Orlicz-Sobolev spaces have been used in the last decades to model various

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phenomena, such as image restoration and electrorheological fluids [1, 9, 25, 38].

We recall in what follows the definition and the main properties of Orlicz-Sobolev spaces. Consider the mapping $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(t) := \log(1 + |t|^q) \cdot |t|^{p-2}t$. Set $\Phi(t) := \int_0^t \phi(s)ds$ A straightforward computation yields

$$\Phi(t) = \frac{1}{p}\log(1+|t|^q)\cdot|t|^p - \frac{q}{p}\int_0^{|t|}\frac{s^{p+q-1}}{1+s^q}\,ds,$$

for all $t \in \mathbb{R}$. We observe that ϕ is an odd, increasing homeomorphism of $\mathbb{R}$ into $\mathbb{R}$, while Φ is convex and even on $\mathbb{R}$ and increasing from $\mathbb{R}_+$ to $\mathbb{R}_+$.

Set

$$\Phi^\star(t) := \int_0^t \phi^{-1}(s)\,ds, \qquad \text{for all } t \in \mathbb{R}.$$

The functions Φ and $\Phi^\star$ are complementary N-functions (see Kranosel'skii & Rutickii [18]).

Define the Orlicz class

$$K_\Phi(\Omega) := \{u : \Omega \to \mathbb{R},\ \text{measurable};\ \int_\Omega \Phi(|u(x)|)\,dx < \infty\}$$

and the Orlicz space

$$L_\Phi(\Omega) := \text{ the linear hull of } K_\Phi(\Omega).$$

The space $L_\Phi(\Omega)$ is a Banach space endowed with the Luxemburg norm

$$\|u\|_\Phi := \inf\left\{k > 0;\ \int_\Omega \Phi\left(\frac{u(x)}{k}\right)\,dx \le 1\right\}$$

or the equivalent norm (the Orlicz norm)

$$\|u\|_{(\Phi)} := \sup\left\{\left|\int_\Omega uv dx\right|;\ v \in K_{\overline{\Phi}}(\Omega),\ \int_\Omega \overline{\Phi}(|v|)dx \le 1\right\},$$

where $\overline{\Phi}$ denotes the conjugate Young function of Φ, that is,

$$\overline{\Phi}(t) = \sup\{ts - \Phi(s);\ s \in \mathbb{R}\}.$$

By Lemma 2.4 and Example 2 in Clément, de Pagter, Sweers & de Thélin [11, p. 243] we have

$$1 < \liminf_{t\to\infty}\frac{t\phi(t)}{\Phi(t)} \le \sup_{t>0}\frac{t\phi(t)}{\Phi(t)} < \infty. \tag{1}$$

These inequalities imply that Φ satisfies the Δ_2-condition. By Lemma C.4 in [11] it follows that $\Phi^\star$ also satisfies the Δ_2-condition. Then, according to Adams [2, p. 234], it follows that $L_\Phi(\Omega) = K_\Phi(\Omega)$. Moreover, by Theorem 8.19 in Adams [2], $L_\Phi(\Omega)$ is reflexive.

We denote by $W^1L_\Phi(\Omega)$ the Orlicz-Sobolev space defined by

$$W^1L_\Phi(\Omega) := \left\{ u \in L_\Phi(\Omega);\ \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega),\ i = 1, \dots, N \right\}.$$

Then $W^1L_\Phi(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \||\nabla u|\|_\Phi.$$

We also define the Orlicz-Sobolev space $W_0^1L_\Phi(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^1L_\Phi(\Omega)$. By Lemma 5.7 in [16] we obtain that on $W_0^1L_\Phi(\Omega)$ we may consider an equivalent norm $\|u\| := \||\nabla u|\|_\Phi$. The space $W_0^1L_\Phi(\Omega)$ is also a reflexive Banach space.

We refer to Adams [2], Luxemburg [23], and Kranosel'skii & Rutickii [18] for more details.

2. Crucial role of nonlinearities sign

Let 2^* denote the critical Sobolev exponent, that is, $2^* := 2N/(N-2)$ if $N \geq 3$ and $2^* := +\infty$ if $N \in \{1, 2\}$. If $2 < r < 2^*$, consider the Dirichlet problems

$$\begin{cases} -\Delta u = -\lambda u + u^{r-1}, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \\ u > 0, & \text{in } \Omega \end{cases} \tag{2}$$

and

$$\begin{cases} -\Delta u = \lambda u - u^{r-1}, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \\ u > 0, & \text{in } \Omega. \end{cases} \tag{3}$$

A direct application of the mountain pass theorem implies that problem (2) has at least one solution for any $\lambda > 0$. By multiplication with the first eigenfunction $\varphi_1 > 0$ of the Laplace operator in (3) we obtain

$$\lambda_1 \int_\Omega u\varphi_1 dx = \lambda \int_\Omega u\varphi_1 dx - \int_\Omega u^{r-1}\varphi_1 dx.$$

Thus, a necessary condition that problem (3) has a solution is that λ is sufficiently large.

In this section, we describe the corresponding setting in the framework of nonhomogeneous differential operators (see Mihăilescu & Rădulescu [26]).

We first consider the boundary value problem

$$\begin{cases} -\mathrm{div}(\log(1+|\nabla u|^q)|\nabla u|^{p-2}\nabla u) = -\lambda|u|^{p-2}u+|u|^{r-2}u, & \text{in} \quad \Omega \\ u=0, & \text{on} \quad \partial\Omega. \end{cases} \tag{4}$$

We say that $u \in W_0^1 L_\Phi(\Omega)$ is a *weak solution* of problem (4) if

$$\int_\Omega \log(1+|\nabla u(x)|^q)|\nabla u(x)|^{p-2}\nabla u \nabla v\, dx + \lambda \int_\Omega |u(x)|^{p-2}u(x)v(x)\, dx \\ - \int_\Omega |u(x)|^{r-2}u(x)v(x)\, dx = 0$$

for all $v \in W_0^1 L_\Phi(\Omega)$.

The property corresponding to problem (2) is the following multiplicity result.

Theorem 2.1. *Assume that $p,\ q > 1$, $p+q < N$, $p+q < r$ and $r < (Np-N+p)/(N-p)$. Then, for every $\lambda > 0$ problem (4), has infinitely many weak solutions.*

We remark that in the particular case $q = 1$, $\lambda = 0$, $1 < p < N-1$, and $p < r \leq [N(p-1)+p]/(N-p)$, problem (4) has a nontrivial weak solution, by means of Theorem 1.2 in Clément, García-Huidobro, Manásevich & Schmitt [10]. On the other hand, Theorem 1.2 in [10] also applies for solving equations involving more general differential operators $\mathrm{div}(a(|\nabla u(x)|)\nabla u(x))$.

Next, we consider the problem

$$\begin{cases} -\mathrm{div}(\log(1+|\nabla u|^q)|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u-|u|^{r-2}u, & \text{in} \quad \Omega \\ u=0, & \text{on} \quad \partial\Omega. \end{cases} \tag{5}$$

We say that $u \in W_0^1 L_\Phi(\Omega)$ is a *weak solution* of problem (5) if

$$\int_\Omega \log(1+|\nabla u(x)|^q)|\nabla u(x)|^{p-2}\nabla u \nabla v\, dx - \lambda \int_\Omega |u(x)|^{p-2}u(x)v(x)\, dx \\ + \int_\Omega |u(x)|^{r-2}u(x)v(x)\, dx = 0$$

for all $v \in W_0^1 L_\Phi(\Omega)$.

The following result shows that problem (5) has a solution provided that λ is large enough.

Theorem 2.2. *Assume that the hypotheses of Theorem 2.1 are fulfilled. Then there exists $\lambda_\star > 0$ such that for any $\lambda \geq \lambda_\star$, problem (5) has a nontrivial weak solution.*

We sketch in what follows the proof of Theorem 2.1. The key argument is the following $\mathbb{Z}_2$-symmetric version (for even functionals) of the Mountain Pass Lemma (see Theorem 9.12 in Rabinowitz [35]).

Mountain Pass Lemma. *Let X be an infinite dimensional real Banach space and let $I \in C^1(X, \mathbb{R})$ be even, satisfying the Palais-Smale condition (that is, any sequence $\{x_n\} \subset X$ such that $\{I(x_n)\}$ is bounded and $I'(x_n) \to 0$ in $X^\star$ has a convergent subsequence) and $I(0) = 0$. Suppose that*

(I1) there exist two constants ρ, $b > 0$ such that $I(x) \geq b$ if $\|x\| = \rho$;

(I2) for each finite dimensional subspace $X_1 \subset X$, the set $\{x \in X_1;\, I(x) \geq 0\}$ is bounded.

Then I has an unbounded sequence of critical values.

Let E denote the Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$. Let $\lambda > 0$ be arbitrary but fixed.

The energy functional associated to problem (4) is $J_\lambda : E \to \mathbb{R}$ defined by

$$J_\lambda(u) := \int_\Omega \Phi(|\nabla u(x)|)\, dx + \frac{\lambda}{p}\int_\Omega |u(x)|^p\, dx - \frac{1}{r}\int_\Omega |u(x)|^r\, dx.$$

We split the proof of Theorem 2.1 into several steps.

Step 1. There exist $\eta > 0$ and $\alpha > 0$ such that $J_\lambda(u) \geq \alpha > 0$ for any $u \in E$ with $\|u\| = \eta$.

Step 2. Assume that E_1 is a finite dimensional subspace of E. Then the set $S = \{u \in E_1;\, J_\lambda(u) \geq 0\}$ is bounded.

Step 3. Assume that $\{u_n\} \subset E$ is a sequence which satisfies the properties

$$|J_\lambda(u_n)| < M \tag{6}$$

$$J_\lambda'(u_n) \to 0 \quad \text{as } n \to \infty, \tag{7}$$

where M is a positive constant. Then $\{u_n\}$ possesses a convergent subsequence.

Proof of Theorem 2.1 completed. The energy functional J_λ is even and verifies $J_\lambda(0) = 0$. Step 3 implies that J_λ satisfies the Palais-Smale condition. On the other hand, Steps 1 and 2 show that conditions (I1) and (I2) are satisfied. Thus, the mountain pass lemma can be applied to the functional J_λ. We conclude that equation (4) has infinitely many weak solutions in E. The proof of Theorem 2.1 is complete. □

We point out that the Orlicz-Sobolev space E cannot be replaced by a classical Sobolev space. Indeed, in such a case, condition (I1) in the mountain pass lemma cannot be satisfied (see the proof of Remark 4 in Clément, García-Huidobro, Manásevich & Schmitt [10, p. 56-57]).

Fix $\lambda > 0$ and consider the energy functional associated to problem (5), that is,

$$I_\lambda(u) := \int_\Omega \Phi(|\nabla u(x)|)\, dx - \frac{\lambda}{p}\int_\Omega |u(x)|^p\, dx + \frac{1}{r}\int_\Omega |u(x)|^r\, dx \qquad \text{for all } u \in E.$$

Standard arguments show that I_λ is coercive and lower semi-continuous. Thus, there exists a global minimizer $u_\lambda \in E$ of I_λ, hence a weak solution of problem (5). We show that u_λ is not trivial for λ large enough. Indeed, letting $t_0 > 1$ be a fixed real and Ω_1 be an open subset of Ω with $|\Omega_1| > 0$ we deduce that there exists $u_1 \in C_0^\infty(\Omega) \subset E$ such that $u_1(x) = t_0$ for any $x \in \overline{\Omega}_1$ and $0 \le u_1(x) \le t_0$ in $\Omega \setminus \Omega_1$. We have

$$\begin{aligned} I_\lambda(u_1) &= \textstyle\int_\Omega \Phi(|\nabla u_1(x)|)\, dx - \frac{\lambda}{p}\int_\Omega |u_1(x)|^p\, dx + \frac{1}{r}\int_\Omega |u_1(x)|^r\, dx \\ &\le L - \textstyle\frac{\lambda}{p}\int_{\Omega_1} |u_1(x)|^p\, dx \\ &\le L - \textstyle\frac{\lambda}{p} \cdot t_0^p \cdot |\Omega_1| \end{aligned}$$

where L is a positive constant. Thus, there exists $\lambda_\star > 0$ such that $I_\lambda(u_1) < 0$ for any $\lambda \in [\lambda_\star, \infty)$. It follows that $I_\lambda(u_\lambda) < 0$ for any $\lambda \ge \lambda_\star$ and thus u_λ is a nontrivial weak solution of problem (5) for λ large enough. The proof of Theorem 2.2 is complete. □

A careful analysis of the proofs shows that Theorems 2.1 and 2.2 still remain valid for more general classes of differential operators. Indeed, we can replace $\operatorname{div}(\log(1+|\nabla u(x)|^q)|\nabla u(x)|^{p-2}\nabla u(x))$ by $\operatorname{div}(a(|\nabla u(x)|)\nabla u(x))$, where $a(t)$ is so that the assumption (1) is fulfilled. Some potentials $a(t)$ satisfying this hypothesis are $a(t) = |t|^{\alpha-1}$ ($\alpha > 0$) and $a(t) = |t|^\alpha / \log(1+|t|^\beta)$ ($0 < \beta < \alpha$).

3. Eigenvalue problems in Orlicz-Sobolev spaces

In this section we are concerned with a related nonlinear eigenvalue problem in a new framework, corresponding to Orlicz-Sobolev spaces. The main result establishes a curious phenomenon, which does not hold in the standard setting corresponding to the Laplace operator. More precisely, we prove that there exist two constants $0 < \lambda_0 \le \lambda_1$ such that any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue, while any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of our problem.

Consider the nonlinear eigenvalue problem

$$\begin{cases} -\operatorname{div}((a_1(|\nabla u|) + a_2(|\nabla u|))\nabla u) = \lambda |u|^{q(x)-2}u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{8}$$

We assume that for any $i = 1,2$, the functions $a_i : (0,\infty) \to \mathbb{R}$ are such that the mappings $\phi_i : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi_i(t) = \begin{cases} a_i(|t|)t, & \text{for} \quad t \neq 0 \\ 0, & \text{for} \quad t = 0, \end{cases}$$

are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. We also suppose throughout this section that $\lambda > 0$ and $q : \overline{\Omega} \to (0,\infty)$ is a continuous function.

We work with functions Φ_i and $(\Phi_i)^\star$, $i = 1,2$, satisfying the Δ_2-condition (at infinity), namely

$$1 < \liminf_{t\to\infty} \frac{t\phi_i(t)}{\Phi_i(t)} \le \limsup_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)} < \infty.$$

Then $L_{\Phi_i}(\Omega)$ and $W_0^1 L_{\Phi_i}(\Omega)$, $i = 1,2$, are reflexive Banach spaces.

Now we introduce the Orlicz-Sobolev conjugate $(\Phi_i)_\star$ of Φ_i, $i = 1,2$, defined as

$$(\Phi_i)_\star^{-1}(t) = \int_0^t \frac{(\Phi_i)^{-1}(s)}{s^{(N+1)/N}}\, ds.$$

We assume that

$$\lim_{t\to 0} \int_t^1 \frac{(\Phi_i)^{-1}(s)}{s^{(N+1)/N}}\, ds < \infty, \quad \text{and} \quad \lim_{t\to\infty} \int_1^t \frac{(\Phi_i)^{-1}(s)}{s^{(N+1)/N}}\, ds = \infty,\ i = 1,2. \tag{9}$$

Finally, we define

$$(p_i)_0 := \inf_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)} \ \text{ and } \ (p_i)^0 := \sup_{t>0} \frac{t\phi_i(t)}{\Phi_i(t)},\ i = 1,2.$$

We study problem (8) under the following basic assumptions:

$$1 < (p_2)_0 \le (p_2)^0 < q(x) < (p_1)_0 \le (p_1)^0, \quad \forall\, x \in \overline{\Omega} \tag{10}$$

and

$$\lim_{t\to\infty} \frac{|t|^{q^+}}{(\Phi_2)_\star(kt)} = 0, \quad \text{for all } k > 0.. \tag{11}$$

We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of problem (8) if there exists $u \in W_0^1 L_{\Phi_1}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega (a_1(|\nabla u|) + a_2(|\nabla u|))\nabla u \nabla v\, dx - \lambda \int_\Omega |u|^{q(x)-2} uv\, dx = 0,$$

for all $v \in W_0^1 L_{\Phi_1}(\Omega)$. We point out that if λ is an eigenvalue of problem (4) then the corresponding $u \in W_0^1 L_{\Phi_1}(\Omega) \setminus \{0\}$ is a *weak solution* of (8).

Define

$$\lambda_1 := \inf_{u\in W_0^1 L_{\Phi_1}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla u|)\,dx}{\displaystyle\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx}.$$

The main result in this section is the following (see Mihăilescu & Rădulescu [27]).

Theorem 3.1. *Assume that conditions (9), (10) and (11) are fulfilled. Then* $\lambda_1 > 0$*. Moreover, any* $\lambda \in [\lambda_1, \infty)$ *is an eigenvalue of problem (8). Furthermore, there exists a positive constant* λ_0 *such that* $\lambda_0 \le \lambda_1$ *and any* $\lambda \in (0, \lambda_0)$ *is not an eigenvalue of problem (8).*

Proof. Let E denote the generalized Sobolev space $W_0^1 L_{\Phi_1}(\Omega)$. Denote by $\|\cdot\|_1$ the norm on $W_0^1 L_{\Phi_1}(\Omega)$ and by $\|\cdot\|_2$ the norm on $W_0^1 L_{\Phi_2}(\Omega)$.

Define the energy functionals J, I, J_1, $I_1 : E \to \mathbb{R}$ by

$$J(u) = \int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla u|)\,dx,$$

$$I(u) = \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx,$$

$$J_1(u) = \int_\Omega a_1(|\nabla u|)|\nabla u|^2\,dx + \int_\Omega a_2(|\nabla u|)|\nabla u|^2\,dx,$$

$$I_1(u) = \int_\Omega |u|^{q(x)}\,dx.$$

Then J, $I \in C^1(E, \mathbb{R})$ and for all u, $v \in E$,

$$\langle J'(u), v\rangle = \int_\Omega (a_1(|\nabla u|) + a_2(|\nabla u|))\nabla u \nabla v\,dx,$$

$$\langle I'(u), v\rangle = \int_\Omega |u|^{q(x)-2}uv\,dx.$$

We split the proof of Theorem 3.1 into four steps.
Step 1. We have $\lambda_1 > 0$.

A straightforward computation combined with relation (10) implies

$$\begin{aligned} 2\cdot c\cdot(\Phi_1(|\nabla u(x)|) + \Phi_2(|\nabla u(x)|)) &\ge 2\cdot(|\nabla u(x)|^{(p_1)_0} + |\nabla u(x)|^{(p_2)^0}) \\ &\ge |\nabla u(x)|^{q^+} + |\nabla u(x)|^{q^-} \end{aligned}$$

and

$$|u(x)|^{q^+} + |u(x)|^{q^-} \ge |u(x)|^{q(x)}.$$

Integrating these inequalities we find

$$2c\cdot\int_\Omega(\Phi_1(|\nabla u(x)|)+\Phi_2(|\nabla u(x)|))\,dx\geq\int_\Omega(|\nabla u|^{q^+}+|\nabla u|^{q^-})\,dx,\quad\forall\,u\in E \tag{12}$$

and

$$\int_\Omega(|u|^{q^+}+|u|^{q^-})\,dx\geq\int_\Omega|u|^{q(x)}\,dx\quad\forall\,u\in E. \tag{13}$$

On the other hand, there exist two positive constants λ_{q^+} and λ_{q^-} such that

$$\int_\Omega|\nabla u|^{q^+}\,dx\geq\lambda_{q^+}\int_\Omega|u|^{q^+}\,dx,\quad\forall\,u\in W_0^{1,q^+}(\Omega) \tag{14}$$

and

$$\int_\Omega|\nabla u|^{q^-}\,dx\geq\lambda_{q^-}\int_\Omega|u|^{q^-}\,dx,\quad\forall\,u\in W_0^{1,q^-}(\Omega). \tag{15}$$

Using again the fact that $q^-\leq q^+<(p_1)_0$, we deduce that E is continuously embedded both in $W_0^{1,q^+}(\Omega)$ and in $W_0^{1,q^-}(\Omega)$. Thus, inequalities (14) and (15) hold true for any $u\in E$.

Using inequalities (14), (15) and (13) we obtain a positive constant μ such that

$$\int_\Omega(|\nabla u|^{q^+}+|\nabla u|^{q^-})\,dx\geq\mu\int_\Omega|u|^{q(x)}\,dx\quad\forall\,u\in E. \tag{16}$$

Next, inequalities (16) and (12) yield

$$\int_\Omega(\Phi_1(|\nabla u(x)|)+\Phi_2(|\nabla u(x)|))\,dx\geq\frac{\mu}{2c}\int_\Omega|u|^{q(x)}\,dx\quad\forall\,u\in E. \tag{17}$$

The above inequality implies

$$J(u)\geq\frac{\mu\cdot q^-}{2c}I(u)\quad\forall\,u\in E. \tag{18}$$

The last inequality assures that $\lambda_1>0$ and thus, step 1 is verified.

We point out that by the definitions of $(p_i)_0$, $i=1,2$, we have

$$a_i(t)\cdot t^2=\phi_i(t)\cdot t\geq(p_i)_0\Phi_i(t),\quad\forall\,t>0.$$

The above inequality and relation (17) imply

$$\lambda_0=\inf_{v\in E\setminus\{0\}}\frac{J_1(v)}{I_1(v)}>0. \tag{19}$$

Step 2. We show that λ_1 is an eigenvalue of problem (8).

We start with some auxiliary results.

Lemma 3.2. *The following relations hold true:*

$$\lim_{\|u\|\to\infty}\frac{J(u)}{I(u)}=\infty \tag{20}$$

and

$$\lim_{\|u\|\to 0}\frac{J(u)}{I(u)}=\infty. \tag{21}$$

Proof of lemma. Since E is continuously embedded in $L^{q^\pm}(\Omega)$ it follows that there exist two positive constants c_1 and c_2 such that

$$\|u\|_1\geq c_1\cdot|u|_{q^+},\quad \forall\, u\in E \tag{22}$$

and

$$\|u\|_1\geq c_2\cdot|u|_{q^-},\quad \forall\, u\in E. \tag{23}$$

For any $u\in E$ with $\|u\|_1>1$, relations (13), (22), (23) imply that

$$\frac{J(u)}{I(u)}\geq\frac{\|u\|_1^{(p_1)_0}}{\dfrac{|u|_{q^+}^{q^+}+|u|_{q^-}^{q^-}}{q^-}}\geq\frac{\dfrac{\|u\|_1^{p_1^-}}{p_1^+}}{\dfrac{c_1^{-q^+}\|u\|_1^{q^+}+c_2^{-q^-}\|u\|_1^{q^-}}{q^-}}.$$

Since $(p_1)_0>q^+\geq q^-$, passing to the limit as $\|u\|_1\to\infty$ in the above inequality we deduce that relation (20) holds true.

Next, the space $W_0^1L_{\Phi_1}(\Omega)$ is continuously embedded in $W_0^1L_{\Phi_2}(\Omega)$. Thus, $\|u\|_1<1$ is small enough, then $\|u\|_2<1$. On the other hand, since (11) holds true we deduce that $W_0^1L_{\Phi_2}(\Omega)$ is continuously embedded in $L^{q^\pm}(\Omega)$. It follows that there exist two positive constants d_1 and d_2 such that

$$\|u\|_2\geq d_1\cdot|u|_{q^+},\quad \forall\, u\in W_0^1L_{\Phi_2}(\Omega) \tag{24}$$

and

$$\|u\|_2\geq d_2\cdot|u|_{q^-},\quad \forall\, u\in W_0^1L_{\Phi_2}(\Omega). \tag{25}$$

Thus, for any $u\in E$ with $\|u\|_1<1$ small enough, relations (13), (24), (25) imply

$$\frac{J(u)}{I(u)}\geq\frac{\int_\Omega\Phi_2(|\nabla u|)\,dx}{\dfrac{|u|_{q^+}^{q^+}+|u|_{q^-}^{q^-}}{q^-}}\geq\frac{\|u\|_2^{(p_2)^0}}{\dfrac{d_1^{-q^+}\|u\|_2^{q^+}+d_2^{-q^-}\|u\|_2^{q^-}}{q^-}}.$$

Since $(p_2)^0<q^-\leq q^+$, passing to the limit as $\|u\|_1\to 0$ (and thus, $\|u\|_2\to 0$) in the above inequality we deduce that relation (21) holds true. The proof of Lemma 3.2 is complete. □

Lemma 3.3. *There exists $u \in E \setminus \{0\}$ such that $\frac{J(u)}{I(u)} = \lambda_1$.*

Proof of lemma. Let $\{u_n\} \subset E \setminus \{0\}$ be a minimizing sequence for λ_1, that is,

$$\lim_{n\to\infty} \frac{J(u_n)}{I(u_n)} = \lambda_1 > 0. \tag{26}$$

By relation (20) we deduce that $\{u_n\}$ is bounded in E. Since E is reflexive it follows that there exists $u \in E$ such that u_n converges weakly to u in E. On the other hand, the functional J is weakly lower semi-continuous. Therefore

$$\liminf_{n\to\infty} J(u_n) \geq J(u). \tag{27}$$

By Remark 1 it follows that E is compactly embedded in $L^{q(x)}(\Omega)$. Thus, u_n converges strongly in $L^{q(x)}(\Omega)$, hence

$$\lim_{n\to\infty} I(u_n) = I(u). \tag{28}$$

Relations (27) and (28) imply that if $u \not\equiv 0$ then

$$\frac{J(u)}{I(u)} = \lambda_1.$$

Thus, in order to conclude that the lemma holds true it is enough to show that u can not be trivial. Assume by contradiction the contrary. Then u_n converges weakly to 0 in E and strongly in $L^{q(x)}(\Omega)$. In other words, we have

$$\lim_{n\to\infty} I(u_n) = 0. \tag{29}$$

Letting $\varepsilon \in (0, \lambda_1)$ be fixed by relation (26) we deduce that for n large enough we have

$$|J(u_n) - \lambda_1 I(u_n)| < \varepsilon I(u_n),$$

or

$$(\lambda_1 - \varepsilon) I(u_n) < J(u_n) < (\lambda_1 + \varepsilon) I(u_n).$$

Passing to the limit in the above inequalities and taking into account that relation (29) holds true we find $\lim_{n\to\infty} J(u_n) = 0$. That implies that actually u_n converges strongly to 0 in E, that is, $\lim_{n\to\infty} \|u_n\|_1 = 0$. So, by (21),

$$\lim_{n\to\infty} \frac{J(u_n)}{I(u_n)} = \infty,$$

and this is a contradiction. Thus, $u \not\equiv 0$. The proof of Lemma 3.3 is complete. $\square$

By Lemma 3.3 we conclude that there exists $u \in E \setminus \{0\}$ such that

$$\frac{J(u)}{I(u)} = \lambda_1 = \inf_{w \in E \setminus \{0\}} \frac{J(w)}{I(w)}. \tag{30}$$

Then, for any $v \in E$ we have

$$\frac{d}{d\varepsilon} \frac{J(u+\varepsilon v)}{I(u+\varepsilon v)} |_{\varepsilon=0} = 0 .$$

A simple computation yields

$$\int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u \nabla v \, dx \cdot I(u) - J(u) \cdot \int_\Omega |u|^{q(x)-2} uv \, dx = 0, \quad \forall\, v \in E. \tag{31}$$

Relation (31) combined with the fact that $J(u) = \lambda_1 I(u)$ and $I(u) \neq 0$ implies the fact that λ_1 is an eigenvalue of problem (8). Thus, step 2 is verified.

Step 3. Any $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem (8).

Fix $\lambda \in (\lambda_1, \infty)$. Define $T_\lambda : E \to \mathbb{R}$ by

$$T_\lambda(u) = J(u) - \lambda I(u).$$

Thus, λ is an eigenvalue of problem (8) if and only if there exists $u_\lambda \in E \setminus \{0\}$ a critical point of T_λ.

With similar arguments as in the proof of relation (20) we deduce that T_λ is coercive, that is, $\lim_{\|u\| \to \infty} T_\lambda(u) = \infty$. On the other hand, T_λ is weakly lower semi-continuous. Thus, there exists $u_\lambda \in E$ a global minimum point of T_λ and hence, a critical point of T_λ. It remains to show that u_λ is not trivial. Indeed, since $\lambda_1 = \inf_{u \in E \setminus \{0\}} \frac{J(u)}{I(u)}$ and $\lambda > \lambda_1$ it follows that there exists $v_\lambda \in E$ such that $J(v_\lambda) < \lambda I(v_\lambda)$, or, equivalently, $T_\lambda(v_\lambda) < 0$. Thus, $\inf_E T_\lambda < 0$ and we conclude that u_λ is a nontrivial critical point of T_λ, that is, λ is an eigenvalue of problem (8). Thus, step 3 is verified.

Step 4. Any $\lambda \in (0, \lambda_0)$, where λ_0 is given by relation (19), is not an eigenvalue of problem (8).

Indeed, assuming by contradiction that there exists $\lambda \in (0, \lambda_0)$ an eigenvalue of problem (8) it follows that there exists $u_\lambda \in E \setminus \{0\}$ such that

$$\langle J'(u_\lambda), v \rangle = \lambda \langle I'(u_\lambda), v \rangle, \quad \forall\, v \in E.$$

Thus, for $v = u_\lambda$ we find

$$\langle J'(u_\lambda), u_\lambda \rangle = \lambda \langle I'(u_\lambda), u_\lambda \rangle,$$

or

$$J_1(u_\lambda) = \lambda I_1(u_\lambda).$$

The fact that $u_\lambda \in E \setminus \{0\}$ assures that $I_1(u_\lambda) > 0$. Since $\lambda < \lambda_0$, the above information implies

$$J_1(u_\lambda) \geq \lambda_0 I_1(u_\lambda) > \lambda I_1(u_\lambda) = J_1(u_\lambda).$$

Clearly, the above inequalities lead to a contradiction. Thus, step 4 is verified.

By steps 2, 3 and 4 we deduce that $\lambda_0 \leq \lambda_1$. The proof of Theorem 3.1 is now complete. □

4. Neumann problems in Orlicz-Sobolev spaces

In this section we study the nonhomogeneous Neumann problem

$$\begin{cases} -\mathrm{div}(a(x,|\nabla u(x)|)\nabla u(x)) + a(x,|u(x)|)u(x) = \lambda\, g(x,u(x)), & \text{for } x \in \Omega \\ \dfrac{\partial u}{\partial \nu}(x) = 0, & \text{for } x \in \partial\Omega, \end{cases} \tag{32}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and ν is the outward unit normal to $\partial\Omega$. We assume that the function $a(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is such that $\varphi(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$,

$$\varphi(x,t) = \begin{cases} a(x,|t|)t, & \text{for } \quad t \neq 0 \\ 0, & \text{for } \quad t = 0, \end{cases}$$

and satisfies

(φ) for all $x \in \Omega$, $\varphi(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$;

and $\Phi(x,t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$,

$$\Phi(x,t) = \int_0^t \varphi(x,s)\, ds, \quad \forall\, x \in \overline{\Omega},\ t \geq 0,$$

belongs to *class* Φ, that is, Φ satisfies the following conditions

(Φ_1) for all $x \in \Omega$, $\Phi(x,\cdot) : [0,\infty) \to \mathbb{R}$ is a nondecreasing continuous function, with $\Phi(x,0) = 0$ and $\Phi(x,t) > 0$ whenever $t > 0$; $\lim_{t\to\infty} \Phi(x,t) = \infty$;

(Φ_2) for every $t \geq 0$, $\Phi(\cdot,t) : \Omega \to \mathbb{R}$ is a measurable function.

We also assume that there exist two positive constants φ_0 and φ^0 such that

$$1 < \varphi_0 \leq \frac{t\varphi(x,t)}{\Phi(x,t)} \leq \varphi^0 < \infty, \quad \forall\, x \in \overline{\Omega},\ t \geq 0. \tag{33}$$

Furthermore, we assume that Φ satisfies the following condition:

$$\text{for each } x \in \overline{\Omega}, \text{ the function } [0,\infty) \ni t \to \Phi(x,\sqrt{t}) \text{ is convex}. \tag{34}$$

Relation (16) assures that $L^{\Phi}(\Omega)$ is an uniformly convex space and thus, a reflexive space.

We study problem (32) in the particular case when Φ satisfies

$$M \cdot |t|^{p(x)} \le \Phi(x,t), \quad \forall\, x \in \overline{\Omega},\ t \ge 0, \tag{35}$$

where $p(x) \in C(\overline{\Omega})$ with $p(x) > 1$ for all $x \in \overline{\Omega}$ and $M > 0$ is a constant.

On the other hand, we assume that the function g from problem (32) satisfies the hypotheses

$$|g(x,t)| \le C_0 \cdot |t|^{q(x)-1}, \quad \forall\, x \in \Omega,\ t \in \mathbb{R} \tag{36}$$

and

$$C_1 \cdot |t|^{q(x)} \le G(x,t) := \int_0^t g(x,s)\, ds \le C_2 \cdot |t|^{q(x)}, \quad \forall\, x \in \Omega,\ t \in \mathbb{R}, \tag{37}$$

where C_0, C_1 and C_2 are positive constants and $q(x) \in C(\overline{\Omega})$ satisfies $1 < q(x) < \frac{Np^-}{N-p^-}$ for all $x \in \overline{\Omega}$.

We say that $u \in W^{1,\Phi}(\Omega)$ is a *weak solution* of problem (32) if

$$\int_\Omega a(x,|\nabla u|)\nabla u \nabla v\, dx + \int_\Omega a(x,|u|)uv\, dx - \lambda \int_\Omega g(x,u)v\, dx = 0,$$

for all $v \in W^{1,\Phi}(\Omega)$.

The main results of this section are the following (see Mihăilescu & Rădulescu [28]).

Theorem 4.1. *Assume φ and Φ verify conditions (φ), (Φ_1), (Φ_2), (33), (34) and (35) and the functions g and G satisfy conditions (36) and (37). Furthermore, we assume that $q^- < \varphi_0$. Then there exists $\lambda_\star > 0$ such that for any $\lambda \in (0,\lambda_\star)$ problem (32) has a nontrivial weak solution.*

Theorem 4.2. *Assume φ and Φ verify conditions (φ), (Φ_1), (Φ_2), (33), (34) and (35) and the functions g and G satisfy conditions (36) and (37). Furthermore, we assume that $q^+ < \varphi_0$. Then there exists $\lambda_\star > 0$ and $\lambda^\star > 0$ such that for any $\lambda \in (0,\lambda_\star) \cup (\lambda^\star,\infty)$ problem (32) has a nontrivial weak solution.*

Let E denote the generalized Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$.

For each $\lambda > 0$ we define the energy functional $J_\lambda : E \to \mathbb{R}$ by

$$J_\lambda(u) = \int_\Omega [\Phi(x,|\nabla u|) + \Phi(x,|u|)]\, dx - \lambda \int_\Omega G(x,u)\, dx.$$

Then J_λ is well-defined on E, $J_\lambda \in C^1(E,\mathbb{R})$, and

$$\langle J_\lambda^{'}(u), v\rangle = \int_\Omega a(x,|\nabla u|)\nabla u \cdot \nabla v\, dx + \int_\Omega a(x,|u|)uv\, dx - \lambda \int_\Omega g(x,u)v\, dx\,,$$

for all u, $v \in E$. Standard arguments show that J_λ is weakly lower semi-continuous.

We also define the functional $\Lambda : E \to \mathbb{R}$ by

$$\Lambda(u) = \int_\Omega [\Phi(x,|\nabla u|) + \Phi(x,|u|)]\, dx.$$

Then Λ is well defined on E, $\Lambda \in C^1(E,\mathbb{R})$ is weakly lower semi-continuous, and for all u, $v \in E$,

$$\langle \Lambda^{'}(u), v\rangle = \int_\Omega a(x,|\nabla u|)\nabla u \cdot \nabla v\, dx + \int_\Omega a(x,|u|)uv\, dx\,.$$

Proof of Theorem 4.1. We split the proof into several steps.

Step 1. There exists $\lambda_\star > 0$ such that for all $\lambda \in (0,\lambda_\star)$, there are ρ, $\alpha > 0$ such that $J_\lambda(u) \geq \alpha > 0$, for any $u \in E$ with $\|u\| = \rho$. The value of $\lambda_\star$ is given by

$$\lambda_\star = \frac{\rho^{\varphi^0 - q^-}}{2 \cdot C_2 \cdot c_1^{q^-}}\,. \tag{38}$$

Step 2. There exists $\theta \in E$ such that $\theta \geq 0$, $\theta \neq 0$ and $J_\lambda(t\theta) < 0$, for $t > 0$ small enough.

Step 3. Conclusion.

Fix $\lambda \in (0,\lambda_\star)$. Then, by Step 1, it follows that on the boundary of the ball centered in the origin and of radius ρ in E, denoted by $B_\rho(0)$, we have $\inf\limits_{\partial B_\rho(0)} J_\lambda > 0$. On the other hand, by Step 2, there exists $\theta \in E$ such that $J_\lambda(t \cdot \theta) < 0$ for all $t > 0$ small enough. Moreover, our hypotheses imply that for any $u \in B_\rho(0)$ we have

$$J_\lambda(u) \geq \|u\|^{\varphi^0} - \lambda \cdot C_2 \cdot c_1^{q^-} \|u\|^{q^-}\,.$$

It follows that

$$-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} J_\lambda < 0\,.$$

We let now $0 < \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$. Applying Ekeland's variational principle we find $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$J_\lambda(u_\varepsilon) < \inf_{\overline{B_\rho(0)}} J_\lambda + \varepsilon$$

$$J_\lambda(u_\varepsilon) < J_\lambda(u) + \varepsilon \cdot \|u - u_\varepsilon\|, \quad u \neq u_\varepsilon.$$

Since

$$J_\lambda(u_\varepsilon) \leq \inf_{\overline{B_\rho(0)}} J_\lambda + \varepsilon \leq \inf_{B_\rho(0)} J_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda \,,$$

we deduce that $u_\varepsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \varepsilon \cdot \|u - u_\varepsilon\|$. Then u_ε is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\varepsilon + t \cdot v) - I_\lambda(u_\varepsilon)}{t} \geq 0$$

for small $t > 0$ and any $v \in B_1(0)$. Therefore

$$\frac{J_\lambda(u_\varepsilon + t \cdot v) - J_\lambda(u_\varepsilon)}{t} + \varepsilon \cdot \|v\| \geq 0.$$

Letting $t \to 0$ it follows that $\langle J_\lambda^{'}(u_\varepsilon), v\rangle + \varepsilon \cdot \|v\| > 0$ and we infer that $\|J_\lambda^{'}(u_\varepsilon)\| \leq \varepsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$J_\lambda(w_n) \to \underline{c} \quad \text{and} \quad J_\lambda^{'}(w_n) \to 0. \tag{39}$$

It is clear that $\{w_n\}$ is bounded in E. Thus, there exists $w \in E$ such that, up to a subsequence, $\{w_n\}$ converges weakly to w in E. Since E is compactly embedded in $L^{q(x)}(\Omega)$, it follows that $\{w_n\}$ converges strongly to w in $L^{q(x)}(\Omega)$. Thus, by (36) and Hölder's inequality,

$$\begin{aligned} \left| \int_\Omega g(x, w_n) \cdot (w_n - w)\, dx \right| &\leq C_0 \cdot \int_\Omega |w_n|^{q(x)-1} |w_n - w|\, dx \\ &\leq C_0 \cdot \big|\, |w_n|^{q(x)-1} \,\big|_{\frac{q(x)}{q(x)-1}} \cdot |w_n - w|_{q(x)} \to 0, \\ &\text{as } n \to \infty. \end{aligned} \tag{40}$$

On the other hand, by (39) we have

$$\lim_{n\to\infty} \langle J_\lambda^{'}(w_n), w_n - w\rangle = 0\,. \tag{41}$$

Relations (40) and (41) imply $\lim_{n\to\infty}\langle \Lambda^{'}(w_n), w_n - w\rangle = 0$. Thus, $\{w_n\}$ converges strongly to w in E. So, by (39), $J_\lambda(w) = \underline{c} < 0$ and $J_\lambda^{'}(w) = 0$. We conclude that w is a nontrivial weak solution for problem (32) for any $\lambda \in (0, \lambda_\star)$. The proof of Theorem 4.1 is complete. □

Proof of Theorem 4.2. Since $q^+ < \varphi_0$ it follows that $q^- < \varphi_0$. Thus, by Theorem 4.1, there exists $\lambda_\star > 0$ such that for any $\lambda \in (0, \lambda_\star)$ problem (32) has a nontrivial weak solution.

Next, we observe that J_λ is coercive and weakly lower semi-continuous in E, for all $\lambda > 0$. Thus, there exists $u_\lambda \in E$ a global minimizer of I_λ, hence a weak solution of problem (32).

We show that u_λ is not trivial for λ large enough. Indeed, letting $t_0 > 1$ be a fixed real and $u_0(x) = t_0$, for all $x \in \Omega$ we have $u_0 \in E$ and

$$\begin{aligned} J_\lambda(u_0) = \Lambda(u_0) - \lambda \int_\Omega G(x,u_0)\,dx \quad &\le \int_\Omega \Phi(x,t_0)\,dx - \lambda \cdot C_1 \cdot \int_\Omega |t_0|^{q(x)}\,dx \\ &\le L - \lambda \cdot C_1 \cdot t_0^{q^+} \cdot |\Omega_1|\,, \end{aligned}$$

where L is a positive constant. Thus, there exists $\lambda^\star > 0$ such that $J_\lambda(u_0) < 0$ for any $\lambda \in [\lambda^\star, \infty)$. It follows that $J_\lambda(u_\lambda) < 0$ for any $\lambda \ge \lambda^\star$ and thus u_λ is a nontrivial weak solution of problem (32) for λ large enough. The proof of Theorem 4.2 is complete. □

We conclude this section with several examples of functions φ and Φ for which the results in this section do apply.

Example 4.3. Define

$$\varphi(x,t) = p(x)|t|^{p(x)-2}t \quad \text{and} \quad \Phi(x,t) = |t|^{p(x)}\,,$$

with $p(x) \in C(\overline{\Omega})$ satisfying $2 \le p(x) < N$, for all $x \in \overline{\Omega}$.

Example 4.4. Define

$$\varphi(x,t) = p(x)\frac{|t|^{p(x)-2}t}{\log(1+|t|)}$$

and

$$\Phi(x,t) = \frac{|t|^{p(x)}}{\log(1+|t|)} + \int_0^{|t|} \frac{s^{p(x)}}{(1+s)(\log(1+s))^2}\,ds\,,$$

with $p(x) \in C(\overline{\Omega})$ satisfying $3 \le p(x) < N$, for all $x \in \overline{\Omega}$.

Example 4.5. Define

$$\varphi(x,t) = p(x) \cdot \log(1+\alpha+|t|) \cdot |t|^{p(x)-1}t\,,$$

and

$$\Phi(x,t) = \log(1+\alpha+|t|) \cdot |t|^{p(x)} - \int_0^{|t|} \frac{s^{p(x)}}{1+\alpha+s}\,dx\,,$$

where $\alpha > 0$ is a constant and $p(x) \in C(\overline{\Omega})$ satisfying $2 \le p(x) < N$, for all $x \in \overline{\Omega}$.

5. Variational analysis versus nonlinear eigenvalue problems

Consider the eigenvalue problem

$$\begin{cases} -\mathrm{div}(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u = \lambda f(x,u) & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \qquad (N^f_{\alpha,\lambda})$$

We assume that $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\alpha : (0,\infty) \to \mathbb{R}$ is such that the mapping $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(t) = \begin{cases} \alpha(|t|)t, & \text{for} \quad t \neq 0 \\ 0, & \text{for} \quad t = 0, \end{cases}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

The main result in this section (see Bonanno, Molica Bisci & Rădulescu [7]) establishes that if $p > N+1$ and $\lambda > 0$ is arbitrary, then there exists a sequence of pairwise distinct solutions of problem $(N^f_{\alpha,\lambda})$ that converges to zero in $W^1L_\Phi(\Omega)$. We also refer to Bonanno & Molica Bisci [6] for a related property for the *p*–Laplace operator.

Throughout this section we assume that Φ satisfies the following hypotheses:

(Φ_0)
$$1 < \liminf_{t\to\infty} \frac{t\phi(t)}{\Phi(t)} \leq p^0 := \sup_{t>0} \frac{t\phi(t)}{\Phi(t)} < \infty;$$

(Φ_1)
$$N < p_0 := \inf_{t>0} \frac{t\phi(t)}{\Phi(t)} < \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}.$$

Let

$$A := \liminf_{\xi\to 0^+} \frac{\displaystyle\int_\Omega \max_{|t|\leq\xi} F(x,t)\,dx}{\xi^{p^0}}, \quad B := \limsup_{\xi\to 0^+} \frac{\displaystyle\int_\Omega F(x,\xi)\,dx}{\xi^{p_0}}.$$

The following multiplicity result has been established in [7].

Theorem 5.1. *Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous function, Φ be a Young function satisfying the structural hypotheses (Φ_0)–(Φ_1) and let ρ be a positive constant such that*

(Φ_ρ)
$$\lim_{t\to 0^+} \frac{\Phi(t)}{t^{p_0}} < \rho.$$

Further, assume that

$$(\mathrm{h}_0) \qquad \liminf_{\xi\to 0+} \frac{\displaystyle\int_\Omega \max_{|t|\le\xi} F(x,t)\,dx}{\xi^{p^0}} < \frac{1}{(2c)^{p^0}\rho\,|\Omega|} \limsup_{\xi\to 0^+} \frac{\displaystyle\int_\Omega F(x,\xi)\,dx}{\xi^{p_0}}.$$

Then, for every λ belonging to

$$\left]\frac{\rho\,|\Omega|}{B}, \frac{1}{(2c)^{p^0}A}\right[,$$

the problem $(N^f_{\alpha,\lambda})$ admits a sequence of pairwise distinct weak solutions which strongly converges to zero in $W^1L_\Phi(\Omega)$.

The key ingredient in the proof of Theorem 5.1 is the following result of Bonanno & Molica Bisci [5, Theorem 2.1], which is a refinement of Ricceri's variational principle [37]. Ricceri's result goes back to an elementary property established by Pucci and Serrin [33, 34], which asserts that if a functional of class C^1 defined on a real Banach space has two local minima, then it has a third critical point. At our best knowledge, the first *three critical point* property was found by Krasnoselskii [17]. He showed that if f is a coercive C^1 functional defined on a finite dimensional space having a nondegenerate critical point x_0 (that is, the *topological index* $\operatorname{ind} f'(x_0)(0)$ is different from zero) which is not a global minimum, then f admits a third critical point. This result was extended to infinite dimensional Banach spaces by Amann [3]. We refer to Bonanno & Marano [4], Livrea & Marano [22], and Marano & Motreanu [24] for related results and applications of Ricceri's variational principle. The recent book by Kristály, Rădulescu & Varga [20] contains several applications of Ricceri's variational principle.

Theorem 5.2. (Bonanno & Molica Bisci [5, Theorem 2.1]). *Let X be a reflexive real Banach space, let $J, I : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that J is strongly continuous, sequentially weakly lower semicontinuous and coercive and I is sequentially weakly upper semicontinuous. For every $r > \inf_X J$, put*

$$\varphi(r) := \inf_{u\in J^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in J^{-1}(]-\infty,r[)} I(v)\right) - J(u)}{r - J(u)},$$

and $\delta := \liminf_{r\to(\inf_X J)^+} \varphi(r)$.

Then, if $\delta < +\infty$, for each $\lambda \in \left]0, \frac{1}{\delta}\right[$, the following alternative holds:

either
(c_1) *there is a global minimum of J which is a local minimum of $g_\lambda := J - \lambda I$,*
or
(c_2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of g_λ which weakly converges to a global minimum of J, with* $\lim_{n\to+\infty} J(u_n) = \inf_X J$.

Define

$$\phi(t) = \frac{|t|^{p-2}}{\log(1+|t|)}t \quad \text{for } t \neq 0, \text{ and } \phi(0) = 0.$$

A straightforward computation shows that the assumptions (Φ_0), (Φ_1), and (Φ_ρ) are fulfilled. A direct application of Theorem 5.1 implies the following multiplicity property.

Corollary 5.3. *Let $p > N+1$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous non-negative function with potential $G(\xi) := \int_0^\xi g(t)\,dt$. Assume that*

$$\liminf_{\xi\to 0^+} \frac{G(\xi)}{\xi^p} = 0, \quad \text{and} \quad \limsup_{\xi\to 0^+} \frac{G(\xi)}{\xi^{p-1}} = +\infty.$$

Let $h : \overline{\Omega} \to \mathbb{R}$ be a continuous and positive function.

Then, for each $\lambda > 0$, the Neumann problem

$$\begin{cases} -\text{div}\Big(\dfrac{|\nabla u|^{p-2}}{\log(1+|\nabla u|)}\nabla u\Big) + \dfrac{|u|^{p-2}}{\log(1+|u|)}u = \lambda h(x)g(u) & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

admits a sequence of pairwise distinct weak solutions which strongly converges to zero in $W^1L_\Phi(\Omega)$.

The reader interested in nonlinear PDE's in Orlicz-Sobolev spaces may consult the following very related references: Byun, Yao & Zhou [8], Fukagai, Ito & Narukawa [13], Le [21], Kristály, Mihăilescu & Rădulescu [19], Mihăilescu, Rădulescu & Repovš [29], Pucci & Rădulescu [32], and Xing & Ding [39]. For many examples and related properties we also refer to the books by Ghergu & Rădulescu [14, 15].

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REFERENCES

[1] E. Acerbi - G. Mingione, *Gradient estimates for the $p(x)$-Laplacean system*, J. Reine Angew. Math. 584 (2005), 117–148.

[2] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[3] H. Amann, *A note on degree theory for gradient mappings*, Proc. Amer. Math. Soc. 85 (1982), 591–597.

[4] G. Bonanno - S. A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition*, Appl. Anal. 89 (2010), 1–10.

[5] G. Bonanno - G. Molica Bisci, *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities*, Bound. Value Probl. 2009 (2009), 1–20.

[6] G. Bonanno - G. Molica Bisci, *Infinitely many solutions for a Dirichlet problem involving the p-Laplacian*, Proc. Roy. Soc. Edinburgh 140 A (2010), 737–752.

[7] G. Bonanno - G. Molica Bisci - V. Rădulescu, *Arbitrarily small weak solutions for a nonlinear eigenvalue problem in Orlicz-Sobolev spaces*, Monatshefte für Mathematik, in press.

[8] S. S. Byun - F. Yao - S. Zhou, *Gradient estimates in Orlicz space for nonlinear elliptic equations*, J. Funct. Anal. 255 (2008), 1851–1873.

[9] Y. Chen - S. Levine - M. Rao, *Variable exponent, linear growth functionals in image processing*, SIAM J. Appl. Math. 66 (2006), 1383–1406.

[10] Ph. Clément - M. García Huidobro - R. Manásevich - K. Schmitt, *Mountain pass type solutions for quasilinear elliptic equations*, Calc. Var. 11 (2000), 33–62.

[11] Ph. Clément - B. de Pagter - G. Sweers - F. de Thélin, *Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces*, Mediterr. J. Math. 1 (2004), 241–267.

[12] T. K. Donaldson - N. S. Trudinger, *Orlicz-Sobolev spaces and imbedding theorems*, J. Functional Analysis 8 (1971), 52–75.

[13] N. Fukagai - M. Ito - K. Narukawa, *Quasilinear elliptic equations with slowly growing principal part and critical Orlicz-Sobolev nonlinear term*, Proc. Roy. Soc. Edinburgh Sect. A 139 (2009), 73–106.

[14] M. Ghergu - V. Rădulescu, *Singular Elliptic Problems. Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and Its Applications, vol. 37, Oxford University Press, New York, 2008.

[15] M. Ghergu - V. Rădulescu, *Nonlinear Analysis and Beyond. Partial Differential Equations Applied to Biosciences*, Springer, Heidelberg, 2011.

[16] J. P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Trans. Amer. Math. Soc. 190 (1974), 163–205.

[17] M. A. Krasnoselskii, *The Operator of Translation along the Trajectories of Differential Equations*, Nauka, Moscow, 1963.

[18] M. A. Kranosel'skii - Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, Noordhoff, Gröningen, 1961.

[19] A. Kristály - M. Mihăilescu - V. Rădulescu, *Two nontrivial solutions for a nonhomogeneous Neumann problem: an Orlicz-Sobolev setting*, Proceedings of the Royal Society of Edinburgh 139 A (2009), 367–379.

[20] A. Kristály - V. Rădulescu - Cs. Varga,*Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and its Applications, No. 136, Cambridge University Press, Cambridge, 2010.

[21] V. K. Le, *A global bifurcation result for quasilinear elliptic equations in Orlicz-Sobolev spaces*, Topol. Methods Nonlinear Anal. 15 (2000), 301–327.

[22] R. Livrea - S. A. Marano, *On a min-max principle for non-smooth functions and applications*, Commun. Appl. Anal. 13 (2009), 411–430.

[23] W. Luxemburg, *Banach Function Spaces*, Ph. D. Thesis, Technische Hogeschool te Delft, The Netherlands, 1955.

[24] S. A. Marano - D. Motreanu, *On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems*, Nonlinear Anal. 48 (2002), 37–52.

[25] M. Mihăilescu - V. Rădulescu, *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Proc. R. Soc. Lond. 462 A (2006), 2625–2641.

[26] M. Mihăilescu - V. Rădulescu, *Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: an Orlicz-Sobolev space setting*, J. Math. Anal. Appl. 330 (2007), 416–432.

[27] M. Mihăilescu - V. Rădulescu, *Eigenvalue problems associated to nonhomogeneous differential operators in Orlicz-Sobolev spaces*, Analysis and Applications 6 (2008), 83–98.

[28] M. Mihăilescu - V. Rădulescu, *Neumann problems associated to nonhomogeneous differential operators in Orlicz-Sobolev spaces*, Ann. Inst. Fourier 58 (2008), 2087–2111.

[29] M. Mihăilescu - V. Rădulescu - D. Repovš, *On a non-homogeneous eigenvalue problem involving a potential: an Orlicz-Sobolev space setting*, J. Math. Pures Appliquées 93 (2010), 132–148.

[30] R. O'Neill, *Fractional integration in Orlicz spaces*, Trans. Amer. Math. Soc. 115 (1965), 300–328.

[31] W. Orlicz, *Ueber eine gewisse Klasse von Räumen vom Typus* $\mathscr{B}$, Bull. Intern. Acad. Pol. 8/9 A (1932), 207–220.

[32] P. Pucci - V. Rădulescu, *Remarks on eigenvalue problems for nonlinear polyharmonic equations*, C. R. Acad. Sci. Paris, Ser. I 348 (2010), 161–164.

[33] P. Pucci - J. Serrin, *Extensions of the mountain pass theorem*, J. Funct. Anal. 59 (1984), 185–210.

[34] P. Pucci - J. Serrin, *A mountain pass theorem*, J. Differential Equations 60 (1985),

142–149.

[35] P. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, Expository Lectures from the CBMS Regional Conference held at the University of Miami, American Mathematical Society, Providence, RI, 1984.

[36] V. Rădulescu, *Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations*, Contemporary Mathematics and Its Applications 6, Hindawi Publ. Corp., 2008.

[37] B. Ricceri, *A general variational principle and some of its applications*, J. Comput. Appl. Math. 113 (2000), 401–410.

[38] M. Ružička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2000.

[39] Y. Xing - S. Ding, *Caccioppoli inequalities with Orlicz norms for solutions of harmonic equations and applications*, Nonlinearity 23 (2010), 1109–1119.

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