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LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE GENERALIZED GAMMA FUNCTION

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By a simple approach, two classes of functions involving generalization Euler's gamma function and originating from certain problems are proved to be logarithmically completely monotonic and a class of functions involving the psi function is showed to be completely monotonic.

1. Introduction and Preliminaries

Euler's classical gamma function, defined for positive x by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt,$$

The logarithmic derivative of the gamma function is called the digamma function. It is know as the psi function and is denoted by $\psi(x)$.

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

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The following integral and series representations are valid (see [1]):

$$\psi(x) = -\gamma + \int_0^{\infty} \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n \geq 1} \frac{x}{n(n+x)} \quad (1)$$

It's derivatives are given by

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^{\infty} \frac{t^n e^{-xt}}{1 - e^{-t}} dt \quad (2)$$

for $x > 0$ and $n \in \mathbb{N}$, where $\gamma = 0.57721566490153286\dots$ is the Euler-Mascheroni constant.

Euler, gave another equivalent definition for the $\Gamma(x)$ (see [7],[8],[13]),

$$\Gamma_p(x) = \frac{p! p^x}{x(x+1)\cdots(x+p)} = \frac{p^x}{x(1+\frac{x}{1})\cdots(1+\frac{x}{p})}, \quad x > 0, \quad (3)$$

where

$$\Gamma(x) = \lim_{p \rightarrow \infty} \Gamma_p(x). \quad (4)$$

The p -analogue of the psi function is defined as the logarithmic derivative of the Γ_p function (see [7]), that is

$$\psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \frac{\Gamma'_p(x)}{\Gamma_p(x)}. \quad (5)$$

The function ψ_p defined in (5) satisfies the following properties (see [7]). It has the following series representation

$$\psi_p(x) = \ln p - \sum_{k=0}^p \frac{1}{x+k}. \quad (6)$$

It is increasing on $(0, \infty)$ and it is strictly completely monotonic on $(0, \infty)$. It's derivatives are given by

$$\psi_p^{(n)}(x) = \sum_{k=0}^p \frac{(-1)^{n-1} \cdot n!}{(x+k)^{n+1}}. \quad (7)$$

Definition 1.1. A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \quad (8)$$

for $x \in I$ and $n \geq 0$.

Let C denote the set of completely monotonic functions.

Definition 1.2. A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n [\ln f(x)]^{(n)} \geq 0 \tag{9}$$

for $x \in I$, and $n \geq 0$.

Let L on $(0, \infty)$ stand for the set of logarithmically completely monotonic functions. The notion "logarithmically completely monotonic function" was posed explicitly in [10] and published formally in [9] and a much useful and meaningful relation $L \subset C$ between the completely monotonic functions and the logarithmically completely monotonic functions was proved in [9, 10].

Kershaw (see [5]) prove that for positive x and $0 \leq s \leq 1$,

$$\exp\left((1-s)\psi\left(x+s\frac{1}{2}\right)\right) \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq \exp\left((1-s)\psi\left(x+\frac{1+s}{2}\right)\right).$$

2. Main Results

Chen [3] proved that the function

$$x \mapsto \frac{1}{[\Gamma(x+1)]^{\frac{1}{x}}}$$

is logarithmically completely monotonic in $(0, \infty)$. The following theorem is a generalized result of Chen [3].

Theorem 2.1. *The function*

$$f(x) = \frac{1}{[\Gamma_p(x+1)]^{\frac{1}{x}}} \tag{10}$$

is logarithmically completely monotonic in $(0, \infty)$.

Proof. Using Leibnitz rule

$$[u(x)v(x)]^{(n)} = \sum_{k=0}^n \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x),$$

we obtain

$$\begin{aligned}
 [\ln f(x)]^{(n)} &= \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{x}\right)^{(k)} \left(-\ln \Gamma_p(x+1)\right)^{(n-k)} \\
 &= -\frac{1}{x^{n+1}} \sum_{k=0}^n \binom{n}{k} (-1)^k k! x^{n-k} \psi_p^{(n-k-1)}(x+1) \\
 &\triangleq -\frac{1}{x^{n+1}} g(x)
 \end{aligned}$$

$$\begin{aligned}
 g'(x) &= \sum_{k=0}^n \binom{n}{k} (-1)^k k! (n-k) x^{n-k-1} \psi_p^{(n-k-1)}(x+1) + \\
 &+ \sum_{k=0}^n \binom{n}{k} (-1)^k k! x^{n-k} \psi_p^{(n-k)}(x+1) \\
 &= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k k! (n-k) x^{n-k-1} \psi_p^{(n-k-1)}(x+1) + \\
 &+ x^n \psi_p^{(n)}(x+1) + \sum_{k=0}^n \binom{n}{k} (-1)^k k! x^{n-k} \psi_p^{(n-k)}(x+1) \\
 &= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k k! (n-k) x^{n-k-1} \psi_p^{(n-k-1)}(x+1) + \\
 &+ x^n \psi_p^{(n)}(x+1) + \sum_{k=0}^{n-1} \binom{n}{k+1} (-1)^{k+1} (k+1)! x^{n-k-1} \psi_p^{(n-k-1)}(x+1) \\
 &= \sum_{k=0}^{n-1} \left[\binom{n}{k} (n-k) - \binom{n}{k+1} (k+1) \right] (-1)^k k! x^{n-k-1} \psi_p^{(n-k-1)}(x+1) \\
 &+ x^n \psi_p^{(n)}(x+1) = x^n \psi_p^{(n)}(x+1) \\
 &= x^n (-1)^n \sum_{k=1}^{p+1} \frac{n!}{(x+k)^{n+1}}
 \end{aligned}$$

If n is odd, then for $x > 0$,

$$\begin{aligned}
 g'(x) > 0 &\Rightarrow g(x) < g(0) = 0 \Rightarrow (\ln f(x))^{(n)} < 0 \Rightarrow \\
 &\Rightarrow (-1)^n (\ln f(x))^{(n)} > 0.
 \end{aligned}$$

If n is even, then for $x > 0$,

$$\begin{aligned}
 g'(x) < 0 &\Rightarrow g(x) < g(0) = 0 \Rightarrow (\ln f(x))^{(n)} > 0 \Rightarrow \\
 &\Rightarrow (-1)^n (\ln f(x))^{(n)} > 0.
 \end{aligned}$$

Hence,

$$(-1)^n (\ln f(x))^{(n)} > 0$$

for all real $x \in (0, \infty)$ and all integers $n \geq 1$. The proof is completed. □

For the next result we will make use of the following Lemma.

Lemma 2.2. *Let f'' be completely monotonic on $(0, \infty)$, then for $0 \leq s \leq 1$, the functions*

$$x \mapsto \exp \left(-f(x+1) - f(x+s) - (1-s)f' \left(x + \frac{1+s}{2} \right) \right)$$

$$x \mapsto \exp \left(f(x+1) - f(x+s) - \frac{1-s}{2} f'(x+1) + f'(x+s) \right)$$

are logarithmically completely monotonic on $(0, \infty)$.

Proof. See [2]. □

The following Theorems are Γ_p analogues of the results from [2].

Theorem 2.3. *For $0 \leq s \leq 1$, the functions*

$$x \mapsto \frac{\Gamma_p(x+s)}{\Gamma_p(x+1)} \exp \left((1-s)\psi_p \left(x + \frac{1+s}{2} \right) \right)$$

and

$$x \mapsto \frac{\Gamma_p(x+1)}{\Gamma_p(x+s)} \exp \left(-\frac{1-s}{2} \left(\psi_p(x+1) + \psi_p(x+s) \right) \right)$$

are logarithmically completely monotonic.

Proof. Applying Lemma 2.2 to $f(x) = \log \Gamma_p(x)$, and using the fact that $f''(x) = \psi_p'(x)$ is completely monotonic on $(0, \infty)$ (see [7]), one obtains the proof. □

Theorem 2.4. *For positive x and $0 \leq s \leq 1$, then*

$$\exp \left(\frac{1-s}{2} \left(\psi_p(x+1) + \psi_p(x+s) \right) \right) \leq \frac{\Gamma_p(x+1)}{\Gamma_p(x+s)} \leq \exp \left((1-s)\psi_p \left(x + \frac{1+s}{2} \right) \right).$$

Proof. Let $f_p(x) = \frac{\Gamma_p(x+s)}{\Gamma_p(x+1)} \exp\left((1-s)\psi_p\left(x + \frac{1+s}{2}\right)\right)$ and

$$g_p(x) = \frac{\Gamma_p(x+1)}{\Gamma_p(x+s)} \exp\left(-\frac{1-s}{2}\left(\psi_p(x+1) + \psi_p(x+s)\right)\right).$$

Since,

$$\lim_{x \rightarrow \infty} f_p(x) = \lim_{x \rightarrow \infty} g_p(x) = 1$$

and $f_p(x), g_p(x)$ are decreasing from theorem 2.3 we have

$$\begin{aligned} \exp\left(\frac{1-s}{2}\left(\psi_p(x+1) + \psi_p(x+s)\right)\right) &\leq \frac{\Gamma_p(x+1)}{\Gamma_p(x+s)} \leq \\ &\leq \exp\left(\left(1-s\right)\psi_p\left(x + \frac{1+s}{2}\right)\right) \end{aligned}$$

for $0 \leq s \leq 1$. The proof is completed. \square

Let s and t be two real numbers with $s \neq t$, $\alpha = \min\{s, t\}$ and $\beta \geq -\alpha$, for $x \in (-\alpha, \alpha)$, define

$$h_{\beta,p}(x) = \begin{cases} \left[\frac{\Gamma_p(\beta+t)}{\Gamma_p(\beta+s)} \cdot \frac{\Gamma_p(x+s)}{\Gamma_p(x+t)} \right]^{\frac{1}{x-\beta}} & x \neq \beta \\ \exp[\psi_p(\beta+s) - \psi_p(\beta+t)] & x = \beta \end{cases}$$

for $p > 0$.

The following theorem is a generalization of a result of [12].

Theorem 2.5. *The function $h_{\beta,p}(x)$ is logarithmically completely monotonic on $(-\alpha, +\infty)$ if $s > t$.*

Proof. For $x \neq \beta$, taking logarithm of the function $h_{\beta,p}(x)$ we have

$$\begin{aligned}
 \ln h_{\beta,p}(x) &= \frac{1}{x-\beta} \left[\ln \frac{\Gamma_p(\beta+t)}{\Gamma_p(\beta+s)} - \ln \frac{\Gamma_p(x+s)}{\Gamma_p(x+t)} \right] \\
 &= \frac{\ln \Gamma_p(x+s) - \ln \Gamma_p(\beta+s)}{x-\beta} - \frac{\ln \Gamma_p(x+t) - \ln \Gamma_p(\beta+t)}{x-\beta} \\
 &= \frac{1}{x-\beta} \int_{\beta}^x \psi_p(u+s) du - \frac{1}{x-\beta} \int_{\beta}^x \psi_p(u+t) du \\
 &= \frac{1}{x-\beta} \int_{\beta}^x [\psi_p(u+s) - \psi_p(u+t)] du \\
 &= \frac{1}{x-\beta} \int_{\beta}^x \int_t^s \psi'_p(u+v) dv du \\
 &= \frac{1}{x-\beta} \int_{\beta}^x \varphi_{p,s,t}(u) du \\
 &= \int_0^1 \varphi_{p,s,t}((x-\beta)u + \beta) du,
 \end{aligned}$$

and by differentiating $\ln h_{\beta,p}(x)$ with respect to x ,

$$[\ln h_{\beta,p}(x)]^{(k)} = \int_0^1 u^k \varphi_{p,s,t}^{(k)}((x-\beta)u + \beta) du \tag{11}$$

If $x = \beta$ formula (11) is valid.

Since functions ψ'_p and $\varphi_{p,s,t}$ are completely monotonic in $(0, \infty)$ and $(-t, \infty)$ respectively, then $(-1)^i [\varphi_{p,s,t}(x)]^{(i)} \geq 0$ holds for $n \in (-t, \infty)$ for any nonnegative integer i .

Thus

$$(-1)^k [\ln h_{\beta,p}(x)]^{(k)} = \int_0^1 u^k (-1)^k \varphi_{p,s,t}^{(k)}((x-\beta)u + \beta) du \geq 0$$

in $(-t, \infty)$ for $k \in \mathbb{N}$. The proof is completed. □

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