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ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS IN UNBOUNDED DOMAINS OF $\mathbb{R}^2$

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Dedicated to Professor Francesco Guglielmino on his seventieth birthday

In this paper we are concerned with second order elliptic equations in unbounded domains Ω of $\mathbb{R}^2$. We establish existence and uniqueness theorems under the assumptions that the leading coefficients are bounded and measurable in Ω and satisfy a suitable condition at infinity.

Introduction.

Let Ω a sufficiently regular open subset of $\mathbb{R}^2$.

In Ω we consider the second order linear differential operator

$$Lu := -\sum_{i,j=1}^{2} a_{ij} u_{x_i x_j} + \sum_{i=1}^{2} a_i u_{x_i} + au, \tag{1}$$

which is uniformly elliptic with symmetric, bounded and measurable leading coefficients, i.e.

$$a_{ji} = a_{ij} \in L^\infty(\Omega), \quad \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \quad \text{a.e. in } \Omega \quad \forall \xi \in \mathbb{R}^2, \tag{2}$$

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where ν is a positive constant.

As well known, the Dirichlet problem

$$u \in W^2(\Omega) \cap W_0^1(\Omega), \quad Lu = f, \quad f \in L^2(\Omega), \tag{3}$$

has been exhaustively studied (see [9]) under the only assumption (2) in the case of a bounded domain Ω.

Indeed, assuming that a_{ij} satisfy (2), whilst a_i and a are bounded and measurable, G. Talenti [9] has established for a solution u of (3) the estimate

$$|u_{xx}|_{2,\Omega} \leq c(|f|_{2,\Omega} + |u|_{2,\Omega}) \tag{4}$$

with c independent of u; by using (4) and an uniqueness result of C. Pucci (see [8]), he has also shown that problem (3) is uniquely solvable when

$$\operatorname*{essinf}_{\Omega} a \geq 0.$$

In this paper we study the same problem (3) when Ω is an unbounded domain.

After recalling (see Sec. 1) definitions and properties of the spaces of Morrey type $M^p(\Omega)$, $VM^p(\Omega)$, $\widetilde{M}^p(\Omega)$, $M_0^p(\Omega)$, introduced and studied in [11], [14], we prove (see Sec. 2) that the a-priori bound (4) still holds true, assuming (2) and

$$a_i \in \widetilde{M}^s(\Omega) \quad \text{for some } s > 2, \quad a \in \widetilde{M}^2(\Omega). \tag{5}$$

Plainly, in the case of unbounded domains the above estimate (4) is not sufficient to get an existence and uniqueness result.

In order to do this our method proceeds through a L^∞-bound of Pucci type

$$\sup_{\Omega} |u| \leq c|f|_{2,\Omega} \tag{6}$$

and a W^2-estimate of type

$$\|u_{xx}\|_{W^2(\Omega)} \leq c(|f|_{2,\Omega} + |u|_{2,\Omega_0}), \tag{7}$$

where Ω_0 is a bounded open subset of Ω, to be satisfied by a solution u of problem (3) in an unbounded domain Ω, with c and Ω_0 independent of u and f.

By virtue of the already given assumptions on the coefficients of the operator L, the a-priori bound (6) is contained in a recent paper (see [15]).

For the estimate (7) we need further conditions (at infinity) about the boundary of Ω and the behavior of the coefficients of L. Precisely, in order to get (7), we suppose $\partial\Omega$ has non-negative curvature outside some closed ball $\overline{B}_{r_0}$ of sufficiently large radius r_0 and centered at the origin, a.e. with respect to the one-dimensional Hausdorff measure on $\partial\Omega$, and the coefficients of L satisfy (2) together with the following conditions:

$$a_i \in M_0^s(\Omega) \text{ for some } s > 2, \quad a = a' + b \in \widetilde{M}^s(\Omega), \quad a' \in M_0^2(\Omega), \tag{8}$$

and

$$\mu^{-2} \operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_1}} \sum_{i=1}^{2} (e_{ij} - g a_{ij})^2 + \mu_1^{-2} \operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_1}} (e - gb)^2 < 1 \tag{9}$$

for a sufficiently large r_1 with $\mu, \mu_1 \in \mathbb{R}_+$ and $e_{ij}, e \in L^\infty(\Omega)$ such that

$$e_{ji} = e_{ij}, \quad \sum_{i,j=1}^{2} e_{ij}\xi_i\xi_j \geq \mu|\xi|^2 \quad \text{a.e. in } \Omega \quad \forall \xi \in \mathbb{R}^2,$$

$$(e_{ij})_{x_k}, e_{x_k} \in M_0^s(\Omega) \text{ for some } s > 2, \quad \operatorname*{essinf}_{\Omega} e \geq \mu_1,$$

$$g \in L^\infty(\Omega), \quad \operatorname*{essinf}_{\Omega} g > 0.$$

We notice that (9) implies

$$\operatorname*{essinf}_{\Omega\setminus\overline{B}_{r_1}} b > 0. \tag{10}$$

On the other side, we remark that (9) holds true for any b satisfying (10) if the coefficients a_{ij} converge at infinity (see Remark 3.5) and that for any matrix-function with coefficients a_{ij} satisfying (2) there exists a b verifying (9) (see (2.5)).

Alternatively, we prove (7), for a sufficiently regular domain Ω, when conditions (2), (8), (10) are verified and the operator

$$L_0 u := -\sum_{i,j=1}^{2} a_{ij} u_{x_i x_i} \tag{11}$$

can be approximated (at infinity) by means of an uniformly elliptic operator $-\sum_{i,j=1}^{2} \alpha_{ij} u_{x_i x_i}$ having coefficients α_{ij} such that

$$\alpha_{ji} = \alpha_{ij} \in L^\infty(\Omega), \quad (\alpha_{ij})_{x_k} \in M_0^s(\Omega) \text{ for some } s > 2.$$

Finally, by using (7) together with the results of [15], we show that (3) is a zero index problem, uniquely solvable when $a' = 0$.

We also remark that such conclusions can fail when (10) is not satisfied. For instance (see [2]) we have uniqueness, but not always existence, when we consider the Dirichlet problem

$$u \in W^2(\mathbb{R}^2), \quad -\Delta u = f, \quad f \in L^2(\mathbb{R}^2).$$

1. The spaces of Morrey type $M^p(\Omega)$, $VM^p(\Omega)$, $\widetilde{M}^p(\Omega)$, $M_0^p(\Omega)$.

In this section we introduce the notations which will be used throughout the paper.

For $x \in \mathbb{R}^2$ and $r \in \mathbb{R}_+$ we set

$$B(x,r) := \{y \in \mathbb{R}^2 : |y - x| < r\},$$

in particular $B_r := B(0,r)$.

We denote by ζ_1 a function of class $C_0^\infty(\mathbb{R}^2)$ such that

$$0 \le \zeta_1 \le 1, \quad \zeta_1 = 1 \text{ on } \overline{B}_1, \quad \zeta_1 = 0 \text{ on } \mathbb{R}^2 \setminus B_2,$$

and put

$$\zeta_r(x) := \zeta_1(x/r), \quad x \in \mathbb{R}^2.$$

For an open subset Ω of $\mathbb{R}^2$ we let

$$\Omega(x,r) := \Omega \cap B(x,r), \quad \Omega(x) := \Omega(x,1), \quad \Omega_r := \Omega(0,r)$$

and denote by $\Sigma(\Omega)$ the σ-algebra of the Lebesgue-measurable subsets of Ω.

For $p \in [1, +\infty]$, if $A \in \Sigma(\Omega)$ and $g \in L^p(A)$, we put

$$|A| := \text{ Lebesgue-measure of } A,$$

$$\chi_A := \text{ characteristic function of } A,$$

$$|g|_{p,A} := \|g\|_{L^p(A)}.$$

Introducing $\mathcal{D}(\overline{\Omega})$, the class of the restrictions to Ω of the functions in $C_0^\infty(\mathbb{R}^2)$, and $L^p_{\mathrm{loc}}(\overline{\Omega})$, the class of the functions $g : \Omega \to \mathbb{R}$ such that $\zeta g \in L^p(\Omega)$ for every $\zeta \in \mathcal{D}(\overline{\Omega})$, we define $M^p(\Omega)$ as the space of the functions $g \in L^p_{\mathrm{loc}}(\overline{\Omega})$ such that

$$\|g\|_{M^p(\Omega)} := \sup_{x\in\Omega} |g|_{p,\Omega(x)} < +\infty, \tag{1.1}$$

endowed with the norm given in (1.1).

We also need the following subspaces of $M^p(\Omega)$:

$VM^p(\Omega)$, the subspace of the functions $g \in M^p(\Omega)$ such that

$$\eta_p[g,\Omega](\tau) := \sup_{x\in\Omega} |g|_{p,\Omega(x,\tau)} \to 0 \quad \text{as } \tau \to 0;$$

$\widetilde{M}^p(\Omega)$, the subspace of the functions $g \in M^p(\Omega)$ such that

$$\sigma_p[g,\Omega](\tau) := \sup_{\substack{A\in\Sigma(\Omega)\\ |A(x)|\le\tau \ \forall x\in\Omega}} \|\chi_A g\|_{M^p(\Omega)} \to 0 \text{ as } \tau \to 0;$$

$M^p_0(\Omega)$, the subspaces of the functions $g \in M^p(\Omega)$ such that

$$\theta_p[g,\Omega](r) := \|(1-\zeta_r)u\|_{M^p(\Omega)} \to 0 \quad \text{as } r \to \infty.$$

Clearly, it turns out that $\widetilde{M}^p(\Omega) \subset VM^p(\Omega)$ and for every $g \in \widetilde{M}^p(\Omega)$

$$\eta_p[g,\Omega](\tau) \le \sigma_p[g,\Omega](\tau);$$

moreover (see Lemma 2.1 of [11])

$$M^p_0(\Omega) \subset \widetilde{M}^p(\Omega).$$

Furthermore we call:

modulus of continuity of $g \in VM^p(\Omega)$ any function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\eta(\tau) \to 0 \quad \text{as } \tau \to 0, \ \eta_p[g,\Omega](\tau) \le \eta(\tau) \,\forall \tau \in \mathbb{R}_+;$$

modulus of continuity of $g \in \widetilde{M}^p(\Omega)$ any function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sigma(\tau) \to 0 \quad \text{as } \tau \to 0, \ \sigma_p[g,\Omega](\tau) \le \sigma(\tau) \quad \forall \tau \in \mathbb{R}_+;$$

modulus of continuity of $g \in M^p_0(\Omega)$ any function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\theta(r) \to 0 \quad \text{as } r \to +\infty, \ \sigma_p[g,\Omega](1/r) + \theta_p[g,\Omega](r) \le \theta(r) \quad \forall r \in \mathbb{R}_+ .$$

The above-mentioned spaces have been introduced in [10] and represent the particular case $\lambda = 0$ of the spaces $M^{p,\lambda}(\Omega)$, which have been defined in [14].

From [10] and [14] we also infer the following two lemmas.

Lemma 1.1. *$\widetilde{M}^p(\Omega)$ is the closure of $L^\infty(\Omega)$ in $M^p(\Omega)$; $M_0^p(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $M^p(\Omega)$.*

Lemma 1.2. *Let $k \in \mathbb{N}$, $p \in [2, +\infty[$, with $p > 2$ if $k = 1$, and suppose Ω endowed with the cone property. Then for every $g \in M^p(\Omega)$ and $u \in W^k(\Omega)$ we have $gu \in L^2(\Omega)$ and*

$$|gu|_{2,\Omega} \le c\|g\|_{M^p(\Omega)}\|u\|_{W^2(\Omega)},$$

where c is a positive constant depending only on p, k and the characteristic cone of Ω.

From the previous lemmas we easily deduce the following further results.

Lemma 1.3. *If the assumptions of Lemma 1.2 are verified and $g \in \widetilde{M}^p(\Omega)$, then for any $\varepsilon \in \mathbb{R}_+$ the bound*

$$|gu|_{2,\Omega} \le \varepsilon\|u\|_{W^2(\Omega)} + c(\varepsilon)|u|_{2,\Omega}, \quad u \in W^k(\Omega),$$

holds true with a positive constant $c(\varepsilon)$ depending only on ε, p, k, the modulus of continuity of $g \in \widetilde{M}^p(\Omega)$ and the characteristic cone of Ω.

Lemma 1.4. *If the assumptions of Lemma 1.2 are verified and $g \in M_0^p(\Omega)$, then there exist $c(\varepsilon) \in \mathbb{R}_+$ and an open subset $\Omega(\varepsilon) \subset\subset \Omega$ such that for any $\varepsilon \in \mathbb{R}_+$*

$$|gu|_{2,\Omega} \le \varepsilon\|u\|_{W^2(\Omega)} + c(\varepsilon)|u|_{2,\Omega(\varepsilon)}, \quad \forall u \in W^k(\Omega),$$

with $c(\varepsilon)$ and $\Omega(\varepsilon)$ depending only on ε, p, k, the modulus of continuity of $g \in M_0^p(\Omega)$ and the characteristic cone of Ω.

Lemma 1.5. *If the assumptions of Lemma 1.2 are verified, then for every $g \in M_0^p(\Omega)$ the operator*

$$u \in W^k(\Omega) \to gu \in L^2(\Omega)$$

is compact.

For a function u defined on Ω having derivatives in the sense of the distributions, we will make use of the following notations:

$$u_x = \left(u_{x_1}^2 + u_{x_2}^2\right)^{\frac{1}{2}}, \quad u_{xx} = \left(u_{x_1x_1}^2 + 2u_{x_1x_2}^2 + u_{x_2x_2}^2\right)^{\frac{1}{2}}.$$

2. Preliminary lemmas.

In the sequel we suppose the open subset Ω of $\mathbb{R}^2$ has the uniform C^2-regularity property according to R.A. Adams [1] (see 4.6):

i_1) there exist $d \in \mathbb{R}_+$, $k \in \mathbb{N}$, an open covering $\{U_i\}_{i\in\mathbb{N}}$ of $\partial\Omega$ and diffeomorphisms $\Phi_i : U_i \to B_1$, $i \in \mathbb{N}$, of class C^2 such that

1) $\{x \in \Omega / \text{dist}(x, \partial\Omega) < d\} \subset \bigcup_{i\in\mathbb{N}} \Phi_i^{-1}(B(0, 1/2))$;
2) every collection of $k + 1$ of the sets U_i has empty intersection;
3) $\Phi_i(U_i \cap \Omega) = \{x \in B_1 / x_2 > 0\}$, $i \in \mathbb{N}$;
4) the components of Φ_i and Φ_i^{-1}, together with first and second derivatives, are all bounded by a constant independent of $i \in \mathbb{N}$.

Let us consider the differential operator L defined in (1) with principal term L_0 given by (11).

If (2) is verified, $a_i \in M^s(\Omega)$ for some $s > 2$, $a \in M^2(\Omega)$, then we put

$$\beta := \max\{\max_{i,j} |a_{ij}|_{\infty,\Omega}, \max_i \|a_i\|_{M^s(\Omega)}, \|a\|_{M^2(\Omega)}\}.$$

Lemma 2.1. *Assuming i_1), (2), $a_i \in VM^s(\Omega)$ for some $s > 2$, $a \in M^2(\Omega)$ and*

$$a_0 := \operatorname*{ess\,inf}_{\Omega} a > 0, \tag{2.1}$$

we have the bound

$$\sup_{\Omega} |u| \le c|Lu|_{2,\Omega}, \quad \forall u \in W^2(\Omega) \cap W_0^1(\Omega), \tag{2.2}$$

where c is a constant depending only on ν, β, a_0 and the moduli of continuity of $a_i \in VM^s(\Omega)$.

Proof. As a consequence of well known results about Sobolev spaces. A function $u \in W^2(\Omega) \cap W_0^1(\Omega)$ has the following properties:

$$u \in C^0(\overline{\Omega}), \quad u = 0 \quad \text{on } \partial\Omega, \quad \lim_{|x|\to+\infty} u(x) = 0.$$

So we deduce the assertion from the results of [15]. □

Let us suppose

i_2) the coefficient of L verify (2) and (5).

It is known (e.g., see [4], [9]) that the uniform ellipticity of L in an open subset Ω of $\mathbb{R}^2$ is equivalent to Cordes' hypothesis:

$$\operatorname*{essinf}_{\Omega} \frac{\left(\sum_{i=1}^{2} a_{ii}\right)^2}{\sum_{i,j=1}^{2} a_{ij}^2} > 1. \tag{2.3}$$

If we put

$$\varepsilon_0 := \operatorname*{essinf}_{\Omega} \frac{\left(\sum_{i=1}^{2} a_{ii}\right)^2}{\sum_{i,j=1}^{2} a_{ij}^2} - 1, \quad \gamma := \operatorname*{essinf}_{\Omega} \frac{\sum_{i=1}^{2} a_{ii}}{\sum_{i,j=1}^{2} a_{ij}^2}, \tag{2.4}$$

we have

$$\operatorname*{esssup}_{\Omega} \sum_{i,j=1}^{2} (\delta_{ij} - \gamma a_{ij})^2 = 1 - \varepsilon_0$$

and so (2.3) is equivalent to the condition

$$\operatorname*{esssup}_{\Omega} \sum_{i,j=1}^{2} (\delta_{ij} - \gamma a_{ij})^2 < 1. \tag{2.5}$$

Lemma 2.2. *Assuming i_1) and i_2), we have bound*

$$|u_{xx}|_{2,\Omega} \leq c(|Lu + \lambda u|_{2,\Omega} + |u|_{2,\Omega}), \quad \forall u \in W^2(\Omega) \cap W_0^1(\Omega) \text{ and } \forall \lambda \in [0, +\infty[, \tag{2.6}$$

where c is a constant depending only on Ω, ν, β and the moduli of continuity of $a_i \in \widetilde{M}^s(\Omega)$, $i = 1, 2$, and of $a \in \widetilde{M}^2(\Omega)$.

Proof. From Theorem 3 of [12] we have (2.6) with L_0 instead of L, and so we obtain the result by applying Lemma 1.3. □

3. Conditions at infinity on the coefficients a_{ij}.

Let $\mu \in \mathbb{R}_+$ and $k \in \mathbb{N}$.

We denote by $E_k(\mu, \Omega)$ the class of the $k \times k$ matrix-functions $((e_{ij}))$ such that

$$e_{ji} = e_{ij} \in L^\infty(\Omega), \quad \sum_{i,j=1}^{k} e_{ij}\xi_i\xi_j \geq \mu|\xi|^2 \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^k,$$

$$(e_{ij})_{x_k} \in M_0^s(\Omega) \quad \text{for some } s > 2.$$

Moreover we put

$$\mathcal{G}(\Omega) := \{g \in L^\infty(\Omega) : \operatorname*{essinf}_{\Omega} g > 0\}.$$

We will use the pair (a_{ij}, b) to indicate the operator

$$L_0 u + bu, \quad u \in W^2(\Omega),$$

with L_0 given by (11) and $b \in \widetilde{M}^2(\Omega)$ such that $\operatorname*{essinf}_{\Omega\setminus\overline{B}_r} b > 0$ for some $r \in \mathbb{R}_+$.

Hypothesis 3.1. There exist $\mu, \mu_1, r_1 \in \mathbb{R}_+$, $e_{ij} \in E_2(\mu, \Omega)$, $e \in E_1(\mu_1, \Omega)$, $g \in \mathcal{G}(\Omega)$ such that

$$\mu^{-2} \operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_1}} \sum_{i,j=1}^{2} (e_{ij} - g a_{ij})^2 + \mu_1^{-2} \operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_1}} (e - gb)^2 < 1. \tag{3.1}$$

To be more explicit, we will also say that (a_{ij}, b) verifies Hypothesis 3.1 (with respect to (e_{ij}, e, g)).

Remark 3.2. As a consequence of (2.5), in order that (a_{ij}, b) verifies Hypothesis 3.1 (with respect to (e_{ij}, e, γ), where γ has been defined in (2.4)) it is sufficient that there exist $\mu_1, r_0 \in \mathbb{R}_+$, $e \in E_1(\mu_1, \Omega)$, such that

$$\operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_0}} |e - \gamma b| < \mu_1\sqrt{\varepsilon_0}. \tag{3.2}$$

Remark 3.3. Let $\mu, r \in \mathbb{R}_+$, $e_{ij} \in E_2(\mu, \Omega)$, $g \in \mathcal{G}(\Omega)$, such that

$$\alpha = 1 - \mu^{-2} \operatorname*{esssup}_{\Omega\setminus\overline{B}_r} \sum_{i,j=1}^{2} (e_{ij} - g a_{ij})^2 > 0. \tag{3.3}$$

As a consequence of Remark 4.1 of [3], Hypothesis 3.1 is satisfied (by (a_{ij}, b)) if there exist $r_0 \in \mathbb{R}_+$ such that

$$\operatorname*{essinf}_{\Omega\setminus\overline{B}_{r_0}}(gb) > (1-\sqrt{\alpha})\operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_0}}(gb). \tag{3.4}$$

Remark 3.4. From (2.5) and Remark 3.3 we deduce that Hypothesis 3.1 is satisfied (by (a_{ij}, b)) if there exists $r \in \mathbb{R}_+$ such that

$$\frac{\operatorname*{essinf}_{\Omega\setminus\overline{B}_r}(\gamma b)}{\operatorname*{esssup}_{\Omega\setminus\overline{B}_r}(\gamma b)} > 1 - \sqrt{\operatorname*{essinf}_{\Omega\setminus\overline{B}_r}\frac{(a_{11}+a_{22})^2}{a_{11}^2+2a_{12}^2+a_{22}^2} - 1}\ .$$

Remark 3.5. As a consequence of Remark 3.3, Hypothesis 3.1 is satisfied (by (a_{ij}, b)), whatever b is, in the case of

$$a_{ij} = a'_{ij} + a''_{ij}\,,\ (a'_{ij})_{x_k} \in M_0^s(\Omega) \text{ for some } s > 2,\ \lim_{|x|\to+\infty} a''_{ij} = a^0_{ij} \in \mathbb{R},$$

because (3.3) and (3.4) can be satisfied by taking $\mu = \nu/2, r_0 \in \mathbb{R}_+,\ e_{ij} = a'_{ij} + a^0_{ij},\ g = 1$, such that

$$\operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_0}} |a''_{ij} - a^0_{ij}| < \frac{\nu}{2}\left[1-\left(1-\frac{\operatorname*{essinf}_{\Omega\setminus\overline{B}_{r_0}} b}{\operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_0}}}\right)^2\right]^{\frac{1}{2}}\ .$$

We also observe (see note (1) of M. Giaquinta [5] and Proposition 1 of M. Chicco [2]) that, if we set

$$g_0 := \frac{\mu^{-2}\sum\limits_{i,j=1}^{2} e_{ij}a_{ij} + \mu_1^{-2}eb}{\mu^{-2}\sum\limits_{i,j=1}^{2} a_{ij}^2 + \mu_1^{-2}b^2}, \tag{3.5}$$

then for any function $f : \Omega \to \mathbb{R}$ we have

$$\mu^{-2}\sum_{i,j=1}^{2}(e_{ij}-g_0a_{ij})^2+\mu_1^{-2}(e-g_0b)^2 \le \mu^{-2}\sum_{i,j=1}^{2}(e_{ij}-fa_{ij})^2+\mu_1^{-2}(e-fb)^2.$$

Therefore a pair (a_{ij}, b) verifies Hypothesis 3.1 with respect to (e_{ij}, e, g) if and only if (a_{ij}, b) does it with respect to (e_{ij}, e, g_0).

Moreover

$$\mu^{-2}\sum_{i,j=1}^{2}(e_{ij}-g_0a_{ij})^2+\mu_1^{-2}(e-g_0b)^2=$$

$$=\mu^{-2}\sum_{i,j=1}^{2}e_{ij}^2+\mu_1^{-2}e^2-\frac{\left(\mu^{-2}\sum_{i,j=1}^{2}e_{ij}a_{ij}+\mu_1^{-2}eb\right)^2}{\mu^{-2}\sum_{i,j=1}^{2}a_{ij}^2+\mu_1^{-2}b^2},$$

and so (a_{ij}, b) verifies Hypothesis 3.1 with respect to (e_{ij}, e, g) if and only if

$$\operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_0}}\left[\mu^{-2}\sum_{i,j=1}^{2}e_{ij}^2+\mu_1^{-2}e^2-\frac{\left(\mu^{-2}\sum_{i,j=1}^{2}e_{ij}a_{ij}+\mu_1^{-2}eb\right)^2}{\mu^{-2}\sum_{i,j=1}^{2}a_{ij}^2+\mu_1^{-2}b^2}\right]<1.$$

4. A-priori bounds.

We state in advance some lemmas.

Lemma 4.1. *If Ω has the uniform C^2-regularity property, then each $u \in W^2(\Omega)\cap W_0^1(\Omega)$ is the limit in $W^2(\Omega)$ of a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that*

$$u_n \in W^2(\Omega)\cap C^2(\overline{\Omega}),\quad u_n = 0 \text{ on } \partial\Omega.$$

Proof. Let us take $v_n \in \mathcal{D}(\overline{\Omega})$, $n \in \mathbb{N}$, such that

$$v_n \to u \quad \text{in } W^2(\Omega). \tag{4.1}$$

By virtue of Theorem 5.4 of [11] for each $n \in \mathbb{N}$ there exists a solution $u_n \in W^2(\Omega)\cap W_0^1(\Omega)$ of the equation

$$-\Delta u_n + u_n = -\Delta v_n + v_n; \tag{4.2}$$

from Theorem 5.1 of [2] we deduce that $u_n \in W^{2,p}(\Omega)$ for every $p \in [2, +\infty[$; so in particular $u_n \in C^0(\overline{\Omega})$, whence, by known results (see [6]), $u_n \in C^2(\overline{\Omega})$.

On the other side, as a consequence of Theorem 4.2 of [11], the solution

$$u_n - u \in W^2(\Omega) \cap W_0^1(\Omega)$$

of the equation

$$-\Delta(u_n - u) + (u_n - u) = -\Delta(v_n - u) + (v_n - u)$$

satisfies a bound of the type

$$\|u_n - u\|_{W^2(\Omega)} \le c| - \Delta(v_n - u) + (v_n - u)|,$$

with $c \in \mathbb{R}_+$ independent of n, whence the result. □

Lemma 4.2. *Let Ω have the uniform C^2-regularity property and $r_0 \in \mathbb{R}_+$ be such that the curvature is non-negative on $\partial\Omega \setminus \overline{B}_{r_0}$ a.e. with respect to the one-dimensional Hausdorff measure on $\partial\Omega$.*

Let $u \in W^2(\Omega) \cap W_0^1(\Omega)$ and $r > r_0$.

If $e_{ij} \in E_2(\mu, \Omega \setminus \overline{B}_{r_0})$, then the function

$$u_r := (1 - \zeta_r)u$$

satisfies the inequality

$$\mu^2 \int_\Omega (u_r)_{xx}^2 \, dx \le \int_\Omega \Big| - \sum_{i,j=1}^{2} e_{ij} (u_r)_{x_i x_j} \Big|^2 dx + \tag{4.3}$$

$$+ \sum_{i,j,h,k=1}^{2} \int_\Omega [(e_{ij} e_{hk})_{x_j} (u_r)_{x_i} (u_r)_{x_h x_k} - (e_{ij} e_{hk})_{x_h} (u_r)_{x_i} (u_r)_{x_k x_j}] \, dx.$$

Proof. By virtue of Lemma 4.1 we can suppose

$$u \in W^2(\Omega) \cap C^2(\overline{\Omega}), \quad u = 0 \text{ on } \partial\Omega.$$

Setting

$$w_\rho := \zeta_\rho u_r, \quad \rho \in \mathbb{R}_+,$$

from classical results we deduce that

$$
\begin{aligned}
(4.4)\qquad & \mu^2 \int_\Omega (w_\rho)_{xx}^2 \, dx + \\
& + \int_{\partial\Omega} \sum_{i,j,h,k=1}^{2} e_{ij} e_{hk} [(w_\rho)_{x_h x_k} (w_\rho)_{x_i} n_j - (w_\rho)_{x_j x_k} (w_\rho)_{x_i} n_h] \, d\ell \le \\
& \le \int_\Omega | - \sum_{i,j=1}^{2} e_{ij} (w_\rho)_{x_i x_j} |^2 \, dx + \\
& + \sum_{i,j,h,k=1}^{2} \int_\Omega [(e_{ij} e_{hk})_{x_j} (w_\rho)_{x_i} (w_\rho)_{x_h x_k} - (e_{ij} e_{hk})_{x_h} (w_\rho)_{x_i} (w_\rho)_{x_k x_j}] \, dx,
\end{aligned}
$$

with $n = (n_1, n_2)$ the unit outward normal to $\partial\Omega$.

By proceeding as in [7] and using the assumption on the curvature, the line integral along $\partial\Omega$ turns out to be non-negative, and so (4.4) yields (4.3) for w_ρ (in the place of u_r).

From this we get the result, letting $\rho \to +\infty$, by the dominated convergence theorem of Lebesgue. □

We will consider the following two conditions alternatively:

i_3) Hypothesis 3.1 is satisfied and there exists $r_0 \in \mathbb{R}_+$ such that the curvature is non-negative on $\partial\Omega \setminus B_{r_0}$ a.e. with respect to the one-dimensional measure of Hausdorff on $\partial\Omega$;

i_3') there exist $\mu, \mu_1 \in \mathbb{R}_+$, $((\alpha_{ij})) \in E_2(\mu, \Omega)$ and, for any $\varepsilon \in \mathbb{R}_+$, $r_\varepsilon \in \mathbb{R}_+$ such that

$$
\operatorname*{esssup}_{\Omega \setminus \overline{B}_{r_\varepsilon}} |\alpha_{ij} - a_{ij}| \le \varepsilon, \quad \operatorname*{esssup}_{\Omega \setminus \overline{B}_{r_\varepsilon}} b \ge \mu_1 .
$$

Remark 4.3. Condition i_3') implies Hypothesis 3.1. In fact, if i_3') holds true, then (3.1) is satisfied choosing $\mu, \mu_1 \in \mathbb{R}_+$, as given by i_3'), $e_{ij} = \alpha_{ij}$, $e = \mu$, $g = 1$, for a sufficiently large r_1.

We will set

$$
\beta' \ge \max\{\beta, |e_{ij}|_{\infty,\Omega}, |e|_{\infty,\Omega}, |g|_{\infty,\Omega}\},
$$

with β defined in Section 2, and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, $\gamma(\tau) \to 0$ as $\tau \to 0$, such that

$$
\gamma(\tau) \ge \theta_s[(e_{ij})_x, \Omega] + \theta_s[e_x, \Omega],
$$

if i_3) is verified, whilst

$$\beta' \geq \max\{\beta, |\alpha_{ij}|_{\infty,\Omega}, \operatorname*{esssup}_{\Omega\setminus\overline{B}_{r_\varepsilon}} b\},$$

and

$$\gamma(\tau) \geq \theta_s[(\alpha_{ij})_x, \Omega],$$

if i_3') is verified.

We will also make use of the following condition:

i_4) $a_i \in M_0^s(\Omega)$ for some $s > 2$, $a = a' + b$, with $a' \in M_0^2(\Omega)$.

Lemma 4.4. *If conditions i_1), i_2), i_3), i_4) are verified, then there exists $r^* \in \mathbb{R}_+$ such that*

$$\|(1-\zeta_r)u\|_{W^2(\Omega)} \leq c|L[(1-\zeta_r)u]|_{2,\Omega} \tag{4.5}$$

for every $u \in W^2(\Omega) \cap W_0^1(\Omega)$ and $r > r^*$, where c is a positive constant depending only on Ω, μ, μ_1, β', $\gamma(\tau)$, $\operatorname*{essinf}_{\Omega} g$, and the moduli of continuity of $a_i \in M_0^s(\Omega)$, $a' \in M_0^2(\Omega)$, $b \in \widetilde{M}^2(\Omega)$.

Proof. Starting from inequality (4.3) and proceeding as in the proof of Lemma 6 of [13], we can find a bounded open subset Ω_0 of Ω such that

$$\|(1-\zeta_r)u\|_{W^2(\Omega)} \leq c(|L[(1-\zeta_r)u]|_{2,\Omega} + |(1-\zeta_r)u|_{2,\Omega_0}) \tag{4.6}$$

for $r > \max\{r_0, r_1\}$, whence the result follows at once. □

Theorem 4.5. *If conditions i_1), i_2), i_3) or i_3') (alternatively, i_4) are verified, then there exist $c \in \mathbb{R}_+$ and a bounded open subset Ω_0 of Ω such that*

$$\|u\|_{W^2(\Omega)} \leq c(|Lu|_{2,\Omega} + |u|_{2,\Omega_0}), \quad \forall u \in W^2(\Omega) \cap W_0^1(\Omega), \tag{4.7}$$

with c and Ω_0 depending only on Ω, μ, μ_1, β', $\gamma(\tau)$, $\operatorname*{essinf}_{\Omega} g$, and the moduli of continuity of $a_i \in M_0^s(\Omega)$, $a' \in M_0^2(\Omega)$, $b \in \widetilde{M}^2(\Omega)$.

Proof. Firstly, we consider the case when i_1), i_2), i_3), i_4) are verified.

Let $r^* \in \mathbb{R}_+$ as in Lemma 4.4. By applying Lemma 2.2 to $\zeta_r u$ and using (4.5), for $r > r^*$ we have:

$$\|u\|_{W^2(\Omega)} \leq c_1(|Lu|_{2,\Omega} + |L(\zeta_r u)|_{2,\Omega} + |\zeta_r u|_{2,\Omega}). \tag{4.8}$$

From i_2), by virtue of Lemma 1.3, we deduce that

$$|L(\zeta_r u)|_{2,\Omega} \leq |Lu|_{2,\Omega} + \varepsilon \|u\|_{W^2(\Omega)} + c(\varepsilon)|u|_{2,\Omega_r} \tag{4.9}$$

with Ω_r a bounded open subset of Ω, whence (4.7) in the present case.

Now, let us suppose that i_1), i_2), i'_3), i_4) are verified.

In this case (see, e.g., Theorem 4.4 of [11]) there exist c_2 and a bounded open subset Ω' of Ω such that

$$\|(1-\zeta_r)u\|_{W^2(\Omega)} \leq c_2\Big(|L[(1-\zeta_r)u] + \\ + \sum_{i,j=1}^{2} (a_{ij} - \alpha_{ij})[(1-\zeta_r)u]_{x_i x_i}|_{2,\Omega} + |(1-\zeta_r)u|_{2,\Omega'}\Big). \tag{4.10}$$

whence, by virtue of i'_3), choosing a sufficiently large $r_\varepsilon \in \mathbb{R}_+$ we get

$$\|(1-\zeta_r)u\|_{W^2(\Omega)} \leq c_2(|L[(1-\zeta_r)u]|_{2,\Omega} + |(1-\zeta_r)u|_{2,\Omega'}) + \varepsilon\|(1-\zeta_r)u\|_{W^2(\Omega)},$$

for $r \geq r_\varepsilon$, which yields an inequality of type (4.6) and so (4.5).

By arguing as in the first part of this proof, then we obtain (4.7). □

Theorem 4.6. *Let us suppose that the conditions of Theorem 4.5 are verified and assume*

$$a_0 := \operatorname*{ess\,inf}_{\Omega} a > 0. \tag{4.11}$$

Then we have the estimate

$$\|u\|_{W^2(\Omega)} \leq c|Lu|_{2,\Omega}\,, \quad \forall u \in W^2(\Omega) \cap W^1_0(\Omega), \tag{4.12}$$

with c depending only on a_0 the parameters occurring in the constant of the bound (4.7).

Proof. The result is an obvious consequence of Theorem 4.5 and Lemma 4.2, since a modulus of continuity in $M^s_0(\Omega)$ is a modulus of continuity in $VM^s(\Omega)$, too. □

5. Existence theorems.

In this section we consider the problem

$$u \in W^2(\Omega) \cap W_0^1(\Omega), \quad Lu = f, \quad f \in L^2(\Omega). \tag{5.1}$$

Theorem 5.1. *If the conditions of Theorem 4.5 are verified, then* (5.1) *is a zero index problem.*

If in addition (4.11) *is verified, then problem* (5.1) *is uniquely solvable.*

Proof. Firstly, we consider the case when (4.11) is verified.

Let us set

$$L_\tau u := \tau Au + (1-\tau)Lu, \quad \tau \in [0,1], \tag{5.2}$$

where

$$Au := -\sum_{i,j=1}^{2} e_{ij} u_{x_i x_i} + eu, \tag{5.3}$$

if we consider i_3),

$$Au := -\sum_{i,j=1}^{2} a_{ij} u_{x_i x_i} + bu, \tag{5.4}$$

if we consider i_3').

In the case of assumption i_3), we observe that for every $\tau \in [0,1]$

$$\nu^{-2} \sum_{i,j=1}^{2} [e_{ij} - g_\tau(\tau e_{ij} + (1-\tau)a_{ij})]^2 + \tag{5.5}$$

$$+ \mu^{-2}[e - g_\tau(\tau e + (1-\tau)b)]^2 \le \nu^{-2} \sum_{i,j=1}^{2} (e_{ij} - g_0 a_{ij})^2 + \mu^{-2}(e - g_0 b)^2,$$

where

$$g_\tau := \frac{\mu^{-2} \sum\limits_{i,j=1}^{2} e_{ij}[\tau e_{ij} + (1-\tau)a_{ij}] + \mu_1^{-2} e[\tau e + (1-\tau)b]}{\mu^{-2} \sum\limits_{i,j=1}^{2} [\tau e_{ij} + (1-\tau)a_{ij}]^2 + \mu_1^{-2}[\tau e + (1-\tau)b]^2},$$

which is reduced to (3.5) for $\tau = 0$.

Since (a_{ij}, b) verifies Hypothesis 3.1 with respect to (e_{ij}, e, g_0), then for every $\tau \in [0, 1]$ the pair $([\tau e_{ij} + (1 - \tau)a_{ij}], [\tau e + (1 - \tau)b])$ verifies Hypothesis 3.1 with respect to (e_{ij}, e, g_τ).

Furthermore, since $\tau \to g_\tau$ is a continuous function, from Theorem 4.6 we deduce that there exists $c \in \mathbb{R}_+$ such that

$$(5.6) \qquad \|u\|_{W^2(\Omega)} \le c|L_\tau u|_{2,\Omega} \quad \forall u \in W^2(\Omega) \cap W_0^1(\Omega) \text{ and } \forall \tau \in [0, 1].$$

In the case of assumption $i_3')$, the coefficients of L_τ satisfy condition $i_3')$ uniformly with respect to $\tau \in [0, 1]$ and so again Theorem 4.2 yields (5.6).

Now, we recall that, as a consequence of known results, the problem

$$(5.7) \qquad u \in W^2(\Omega) \cap W_0^1(\Omega), \ Au = f, \ f \in L^2(\Omega),$$

is uniquely solvable. For instance, we can get this result observing that the proof of Theorem 5.4 of [11] remains unchanged if we suppose the coefficient of u belongs to $\widetilde{M}^2(\Omega)$ rather than to $M^{t_0}(\Omega)$ for some $t_0 > 2$.

From the uniqueness and existence result for problem (5.7), together with (5.6), we can apply the classical method of continuity along a parameter in order to establish that problem (5.1) is uniquely solvable if (4.11) is verified.

If (4.11) is not verified, by applying the above conclusions to the operator $Lu - a'u$ and observing that, as a consequence of Lemma 1.5, the operator $u \in W^k(\Omega) \to a'u \in L^2(\Omega)$ is compact, we deduce that (5.1) is a zero index problem from well known results of functional analysis. □

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