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SPACE-TIME FINITE ELEMENTS NUMERICAL SOLUTION OF BURGERS PROBLEMS

M. MORANDI CECCHI - R. NOCIFORO - P. PATUZZO GREGO

A finite-element numerical method to solve a weak formulation of quasi-linear parabolic problems on space-time domain governed by Burgers equation is given. Stability and errors estimates theorems for the numerical solution are proved for smooth initial conditions and numerical examples are presented.

1. Introduction.

In this paper a class of quasi-linear parabolic problems controlled by the Burgers equation, as presented in [2], is solved applying the space-time finite element discretization as in [3]. Numerically this problem has a considerable interest ([1],[4],[5],[6],[13]) because the Burgers equations have the some convective and dissipative form as the incompressible Navier-Stokes equations, although the pressure gradient terms are not retained. A variational formulation is given and by direct integration, having used a linear approximation in time, one obtains the non linear system to get the approximate solution. Stability and error estimate theorems are proved for this approximate solution. An iterative algorithm is implemented and applied to solve some numerical examples.

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2. The space-time finite element method.

Let $\Omega = (a, b)$ be a bounded one dimensional set of $\mathbb{R}$. In $Q = \Omega \times]0, T[$, $T < \infty$, we consider the Burgers problem $\mathscr{P}$ with homogeneous boundary conditions:

$$
(2.1) \quad \mathscr{P} : \begin{cases} u_t(x,t) - \dfrac{1}{Re} u_{xx}(x,t) + u(x,t)\, u_x(x,t) = f(x,t), \; (x,t) \in Q \\ u(x,0) = u_0(x) \qquad x \in \Omega \\ u(a,t) = u(b,t) = 0 \qquad t \in [0,T] \end{cases}
$$

where $f(x,t) : \overline{Q} \to \mathbb{R}$, $u_0(x) : \Omega \to \mathbb{R}$ are given functions, with $u_0(a) = u_0(b) = 0$. Let be $V = H_0^1(\Omega)$, performing an inner product on $H = L^2(\Omega)$ of both sides of the equations of $\mathscr{P}$ by a test function $v \in V$, it results

$$
(2.2) \qquad (u_t, v) + \frac{1}{Re}(u_x, v_x) + (uu_x, v) = (f, v), \; \forall v \in V, \; a.e. \; t \in [0, T]
$$

where $(\cdot, \cdot)$ is the inner product in H. This is a "weak problem" of the Burgers equations and a solution $u \in L^2(Q)$, if it satisfies the conditions of $\mathscr{P}$, is called a "weak solution" of the Burgers problem $\mathscr{P}$.

In [10] we prove that, if in the problem $\mathscr{P}$ we have $u_0(x) \in L^\infty(\Omega)$ and $f(x,t) \in L^\infty(Q)$, there exist a unique solution $u(x,t) \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$ of the correspondent weak problem. This solution is proved to be limit of the Cauchy sequence in $L^\infty(0, T; L^2(\Omega))$ of smooth functions $u_n(x,t) \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$. Each smooth function $u_n(x,t)$ is the weak solution of the problem $\mathscr{P}$, corresponding to initial data $u_{0n}(x) \in H_0^1(\Omega)$, $|u_{0n}(x)| \le M$, with

$$
M = \max \left\{ M_1 = |u_0(x)|_{L^\infty(\Omega)}, M_2 = |f|_{L^\infty(Q)} \right\} .
$$

We prove also that the sequence $\{u_{n,x}(x,t)\}$ is convergent to $u_x(x,t)$ in $L^2(Q)$.

For simplicity we consider the homogeneous Burgers equation.

A weak equivalent form of the problem:

$$
(2.3) \quad \begin{cases} (u_t, v) + \dfrac{1}{Re}(u_x, v_x) + (uu_x, v) = 0, \; \forall v \in H_0^1(\Omega), a.e. \; t \in [0, T]. \\ u(x,0) = u_0(x) \qquad x \in \Omega \\ u(a,t) = u(b,t) = 0 \qquad t \in [0,T] \end{cases}
$$

can be obtained [7], multiplying the equation of (2.3) by a test function $\phi \in L^2(0, T; H_0^1(\Omega))$ such that $\phi_t \in L^2(0, T; L^2(\Omega))$ and $\phi(T) = 0$ having integrated over Q.

The variational formulation of problem (2.3) is then:

to find $u \in L^2(0, T; H_0^1(\Omega))$ such that

$$\int_0^T \left\{(-u, \phi_t) + \frac{1}{Re}(u_x, v_x) + (uu_x, \phi)\right\} dt = (u_0(x), \phi(0)) \quad \forall \phi \in L^2(0, T; H_0^1(\Omega)), \phi_t \in L^2(0, T; L^2(\Omega)), \phi(T) = 0. \tag{2.4}$$

The research of an approximation of u is based on a feasible discretization of the variational form (2.4) given by the space-time finite element method that now is described and that was used in ([3], [9], [11]) for linear parabolic equations and in [12] for the Stefan problem.

Let $\{t_n\}_0^S$ be a finite sequence of real numbers with $t_0 = 0$, $t_S = T$ and $t_n < t_{n+1}$ for $n \geq 0$. Set $I_n = (t_n, t_{n+1}]$ and denote by $S_h^p(\Omega) \subset H_0^1(\Omega)$ and $S_k^q([0, T])$ two finite element spaces of continuous piecewise polynomials respectively of degree less or equal to p in Ω, that are zero on the boundary of Ω and of degree q in time, h e k are the mesh parameters and the uniform time step respectively. Let V_{hk} denote the tensor product space

$$V_{hk} = S_h^p(\Omega) \otimes S_k^q([0, T]) \tag{2.5}$$

The discretization of (2.4) is performed in each slab $\Omega \times I_n$, assuming the continuity of the approximate solution in time when moving from one slab to the successive one. Finally the method is to find $u^{hk} \in V_{hk}$ such that

$$\begin{aligned}\int_{I_n} \Big\{ - (u^{hk}, \phi_t^{hk}) + \frac{1}{Re}(u_x^{hk}, \phi_x^{hk}) + (u^{hk}u_x^{hk}, \phi^{hk})\Big\}dt = \\ = (u^{hk}|_{t=t_n}, \phi^{hk}|_{t=t_n}) \quad \forall \phi^{hk} \in S_h^p \otimes P^q(I_n),\ \phi^{hk}|_{t=t_{n+1}} = 0\,,\end{aligned} \tag{2.6}$$

where $P^q(I_n)$ is the set of polynomials of degree q in time and $u^{hk} \mid t = 0$ is an approximation of $u_0(x)$ in $S_h^p(\Omega)$. In this way an iterative process is built, which allows to compute the solution at time t_{n+1} from the knowledge of the solution at time t_n.

3. Discretization with respect to the space variable.

Let us consider the semidiscrete problem $\mathscr{P}_h$, that is:
to find $u_h(t) \in \mathscr{S}_h(\Omega) = \mathscr{S}_h^p(\Omega)$ such that

$$(3.1) \qquad \mathscr{P}_h : \begin{cases} (u_{ht}, v_h) + \dfrac{1}{Re}(u_{hx}, v_{hx}) + (u_h u_{hx}, v_h) = 0\,, \\ \qquad\qquad\qquad\qquad \forall\, v_h \in \mathscr{S}_h(\Omega),\, t \geq 0 \\ u_h(a,t) = u_h(b,t) = 0\,, \qquad t \in [0,T] \\ u_h(x,0) = u_{0h}\,,\,, \qquad x \in \Omega \end{cases}$$

where u_{0h} is an approximation of u_0 in $\mathscr{S}_h(\Omega)$, with

$$\left|u_{0h}\right|_{L^\infty(\Omega)} \leq M = \left|u_0\right|_{L^\infty(\Omega)}\,,$$

that, for example, could be the projection of u_0 on $\mathscr{S}_h(\Omega)$ for the norm of H.

The family $\{\mathscr{S}_h(\Omega)\}$ of subspaces of $H_0^1(\Omega)$ of finite dimension N_h, namely the family of the subspaces of the continuous functions that are piecewise polynomials of degree $\leq p$ over any mesh, that are zero on the extremum point of the interval (a, b), has the following property of approximation for h small enough:

$$(3.2) \qquad \inf_{\chi \in \mathscr{S}_h} \{\|v - \chi\| + h\|(v - \chi)_x\|\} \leq c\, h^s \|v\|_s$$

for $1 \leq s \leq p + 1 = r$, $v \in H^s(\Omega) \cap H_0^1(\Omega)$ where $\|\cdot\|$ is the norm on H and $\|\cdot\|_s$ is the norm on $H^s(\Omega)$. The problem $\mathscr{P}_h$ has at least a solution $u_h = u_h(t) \in \mathscr{S}_h(\Omega)$ (Lemma 4.3, p. 52 of [8]). To show the uniqueness of the solution u_h, it is enough to follow the same proof of the continuous case [10].

The spatial discretization is stable, since taking $v_h = u_h$ in (3.1), one has:

$$\frac{1}{2}\,\frac{d}{dt}\,\|u_h\|^2 + \frac{1}{Re}\|u_{hx}\|^2 = 0.$$

By the fact that the second term is non-negative, the first term has to be non-positive, therefore $\|u_h\|^2$ is non-increasing and

$$\|u_h(t)\| \leq \|u_{0h}\|.$$

It is possible to prove, as done in [10], that the semidiscrete solution u_h too comply with the maximum principle, i.e. $\left|u_h\right|_{L^\infty(Q)} \leq M$.

We shall prove the following estimate for the error between the solutions of the semidiscrete and for the continuous problem.

Theorem 3.1. *Let u_h and u be the solutions of* (3.1) *and* (2.3)*, respectively. Then*

$$\|u_h(t) - u(t)\| \leq C \left(\|u_{0h} - u_0\| + h^{r-1}\right)$$

with $C = C(u)$.

Proof. We require, of course, that the solution of the continuous problem has the necessary regularity. For the purpose of the proof, we introduce the Ritz projection P_1 into $\mathscr{S}_h$ as the orthogonal projection with respect to the inner product (v_x, u_x) so that:

$$\big((P_1 u)_x, \chi_x\big) = (u_x, \chi_x) \qquad \forall\, \chi \in \mathscr{S}_h. \tag{3.3}$$

For the projection P_1 the following lemma holds (see [14]):

Lemma 3.1. *With P_1 defined by* (3.3) *we have*

$$\begin{aligned}
&\|(P_1 v - v)_x\| \leq \overline{C} h^{s-1} \|v\|_s \\
&\|P_1 v - v\| \leq \overline{C} h^{s} \|v\|_s
\end{aligned}$$

for $1 \leq s \leq r$ and $v \in H^s(\Omega) \cap H_0^1(\Omega)$.

We write

$$u_h - u = (u_h - P_1 u) + (P_1 u - u) = \vartheta + \rho. \tag{3.4}$$

The second term is easily bounded by Lemma 3.1 and obvious estimates:

$$\begin{aligned}
\|\rho(t)\| &\leq C_1 h^r \|u(t)\|_r = C_1 h^r \left\| u_0 + \int_0^t u_t\, ds \right\|_r \\
&\leq C_1 h^r \left\{ \|u_0\|_r + \int_0^t \|u_t\|_r\, ds \right\} \leq C_1(u) h^r \\
\|\rho_x\| &\leq C_1(u) h^{r-1}.
\end{aligned}$$

In order to estimate ϑ, we note that

$$\begin{aligned}
&(\vartheta_t, \chi) + \frac{1}{Re}(\vartheta_x, \chi_x) \\
&\quad = (u_{h,t}, \chi) + \frac{1}{Re}(u_{h,x}, \chi_x) - (P_1 u_t, \chi) - \frac{1}{Re}\big((P_1 u)_x, \chi_x\big) \\
&\quad = -(u_h u_{h,x}, \chi) + (u_t - P_1 u_t, \chi) + (u u_x, \chi) \\
&\quad = -(\rho_t, \chi) + (u u_x - u_h u_{h,x}, \chi)
\end{aligned} \tag{3.5}$$

for $\chi \in \mathscr{S}_h$. In this derivations we have used the definition of P_1 and the easily established fact that this operator commutes with time differentiations. Since ϑ belongs to $\mathscr{S}_h$, we may choose $\chi = \vartheta$ in (3.5) and conclude

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\vartheta\|^2 + \frac{1}{Re}\|\vartheta_x\|^2 \\ &\qquad = -(\rho_t, \vartheta) + (uu_x - u_h u_{h,x}, \vartheta) \\ &\qquad = -(\rho_t, \vartheta) + (u(u-u_h)_x, \vartheta) + (u_{hx}(u-u_h), \vartheta) \\ &\qquad \le \|\rho_t\|\|\vartheta\| + C_2\left\{\|\rho\| + \|\rho_x\| + \|\vartheta\| + \|\vartheta_x\|\right\}\|\vartheta\| \\ &\qquad \le \frac{1}{Re}\|\vartheta_x\|^2 + C_3\left(\|\rho\|^2 + \|\rho_x\|^2 + \|\rho_t\|^2 + \|\vartheta\|^2\right). \end{aligned} \tag{3.6}$$

Hence, using Gronwall's lemma,

$$\|\vartheta(t)\|^2 \le C_3\|\vartheta(0)\|^2 + C_3\int_0^t \left(\|\rho\|^2 + \|\rho_x\|^2 + \|\rho_t\|^2\right) dt$$
$$\|\rho_t\| = \|P_1 u_t - u_t\| \le C_4 h^r \|u_t\|_r \le C_4(u) h^r$$

and further

$$\begin{aligned} \|\vartheta(0)\| &= \|u_{0h} - P_1 u_0\| \le \|u_{0h} - u_0\| + \|P_1 u_0 - u_0\| \\ &\le \|u_{0h} - u_0\| + \overline{C} h^r \|u_0\|_r . \end{aligned}$$

In view of Lemma 3.1 and together with these estimates, we show

$$\|\vartheta(t)\| \le C\left(\|u_{0h} - u_0\| + h^{r-1}\right)$$

and this completes the proof.

4. Numerical method via finite elements.

Let us take $q = 1$ (linear polynomials in time are the ones more often used in calculations) and let $\left\{\Phi_j(x)\right\}_1^{N_h}$ be the basis functions of $\mathscr{S}_h^p(\Omega)$, every function $u^{hk} \in V_{hk}$ may be written inside the slab $\Omega \times I_n$ as

$$u^{hk}(x,t) = u_n^{hk}(x)\frac{(t_{n+1} - t)}{k} + u_{n+1}^{hk}(x)\frac{(t - t_n)}{k},$$

where

$$u_n^{hk}(x) = \sum_{j=1}^{N_h} u^{hk}(x_j, t_n)\Phi_j = \sum_{j=1}^{N_h} u_{n,j}^{hk}\Phi_j$$

and similarly $u_{n+1}^{hk}(x)$ are elements of $\mathscr{S}_h^p(\Omega)$. The test functions are expressed as

$$\phi^{hk} = \phi_n^{hk}(x)\frac{(t_{n+1} - t)}{k}$$

so that the condition $\phi(x, t_{n+1}) = 0$ holds. Substituting these expressions inside (2.6), separating the inner product with respect to the space variable from the integrals in the time variable and noting that $\phi_n^{hk}(x)$, being a linear combination of Φ_j, may be thought as an arbitrary element of $\mathscr{S}_h^p(\Omega)$, we get

$$U^n = u_n^{hk}(x), \quad U^{n+1} = u_{n+1}^{hk}(x), \quad \chi = \phi_n^{hk}(x)$$

and the following nonlinear system that gives the nodal values of the approximate solution at the time t_{n+1}, when the solution at the preceding time is known:

$$\begin{aligned}(4.1) \qquad & \left(\frac{U^{n+1} - U^n}{k}, \chi\right) + \frac{1}{3Re}\left(U_x^{n+1} + 2U_x^n, \chi_x\right) + \\ & + \frac{1}{6}\left((U^{n+1} + U^n)(U_x^{n+1} + U_x^n), \chi\right) + \frac{1}{3}(U^n U_x^n, \chi) = 0\end{aligned}$$

$\forall \chi \in \mathscr{S}_h^p(\Omega)$, $n \geq 0$ and $U^0 = u_{0h}$. The numerical process (4.1) is stable, in fact we have the following theorem:

Theorem 4.1 (Stability). *Under the appropriate regularity assumptions, exists a constant $C = C(h)$ independent of k, the uniform time step, and dependent of initial data, such that, if*

$$(4.2) \qquad k \leq h^2 C(h)$$

then the numerical process (4.1) *is stable.*

Proof. From (4.1) we get

$$\begin{aligned}& \left(U^{n+1} - U^n, \chi\right) - \frac{2k}{3Re}\left(U_x^{n+1} - U_x^n, \chi_x\right) + \frac{k}{Re}\left(U_x^{n+1}, \chi_x\right) + \\ & + \frac{k}{2}(U^n U_x^n, \chi) + \frac{k}{6}(U^n U_x^{n+1}, \chi) + \frac{k}{6}(U^{n+1} U_x^n, \chi) + \frac{k}{6}(U^{n+1} U_x^{n+1}, \chi) = 0.\end{aligned}$$

Taking $\chi = U^{n+1}$ we obtain

$$\begin{aligned}& \left(U^{n+1} - U^n, U^{n+1}\right) + \frac{k}{Re}\left\|U_x^{n+1}\right\|^2 - \frac{2k}{3Re}\left(U_x^{n+1} - U_x^n, U_x^{n+1}\right) = \\ & = \frac{k}{6}\left((U^{n+1} - U^n)(U_x^{n+1} - U_x^n), U^{n+1}\right) + \frac{k}{3}\left(U^n U_x^{n+1}, U^n\right) = \\ & = \frac{k}{6}\left((U^{n+1} - U^n)(U_x^{n+1} - U_x^n), U^{n+1}\right) + \frac{k}{3}\left(U^n U_x^{n+1}, U^n - U^{n+1}\right) + \\ & \qquad + \frac{k}{3}\left((U^n - U^{n+1})U_x^{n+1}, U^{n+1}\right).\end{aligned}$$

By the boundedness of U^{n+1} and U^n we obtain:

$$\begin{aligned}&\left(U^{n+1}-U^n,U^{n+1}\right)+\frac{k}{Re}\left\|U_x^{n+1}\right\|^2\leq\\ &\leq\frac{2k}{3Re}\|U_x^{n+1}\|\left\|U_x^{n+1}-U_x^n\right\|+\frac{C_1k}{6}\left\|U^{n+1}-U^n\right\|\left\|U_x^{n+1}-U_x^n\right\|+\\ &\qquad+\frac{2C_1k}{3}\left\|U_x^{n+1}\right\|\left\|U^{n+1}-U^n\right\|\end{aligned}$$

and therefore:

$$\left(U^{n+1}-U^n,U^{n+1}\right)\leq\frac{k}{2Re}\left\|U_x^{n+1}-U_x^n\right\|^2+\frac{C_1^2\mathrm{Re}k}{4}\left\|U^{n+1}-U^n\right\|^2.$$

Recalling the identity

$$\left(U^{n+1}-U^n,U^{n+1}\right)=\frac{1}{2}\left\|U^{n+1}\right\|^2-\frac{1}{2}\|U^n\|^2+\frac{1}{2}\left\|U^{n+1}-U^n\right\|^2$$

and using the inequality

$$\|U_{h,x}\|^2\leq\frac{C_2}{h^2}\|U_h\|^2\quad\forall U_h\in\mathscr{S}_h$$

we have:

$$\left\|U^{n+1}\right\|^2-\|U^n\|^2+\left\|U^{n+1}-U^n\right\|^2\leq\left(\frac{kC_2}{Re\,h^2}+\frac{C_1^2Re\,k}{2}\right)\left\|U^{n+1}-U^n\right\|^2$$

hence:

$$\left\|U^{n+1}\right\|^2-\|U^n\|^2+\left(1-\frac{kC_2}{Re\,h^2}-\frac{C_1^2Re\,k}{2}\right)\left\|U^{n+1}-U^n\right\|^2\leq 0.$$

Under the condition

$$1-\frac{kC_2}{Re\,h^2}-\frac{C_1^2Re\,k}{2}\geq 0$$

namely,

$$\frac{k}{h^2}\leq\frac{2Re}{2C_2+C_1^2Re^2h^2}$$

we have

$$\left\|U^{n+1}\right\|^2\leq\|U^n\|^2\leq\left\|U^0\right\|^2\leq\|U_{0h}\|^2,$$

which completes the proof.

We shall prove the following error estimate.

Theorem 4.2 (Error estimate). *Denoting by U^{n+1} and $u_{n+1} = u((n+1)k)$ the solutions of* (4.1) *and* (2.3)*, respectively, we have for $n \geq 0$ and for small k*

$$\left\| U^{n+1} - u_{n+1} \right\| \leq C \left(\|u_{0h} - u_0\| + h^{r-1} + k \right)$$

with $C = C(u)$.

Proof. We denote, us before, by U^{n+1} and U_n the approximations at time $(n+1)k$ and nk, respectively, and by $u_{n+1}, u_n, u_{n+\frac{1}{2}}$ the solutions of (2.3) at time $(n+1)k$, nk, $(n+\frac{1}{2})k$. We introduce the notation:

$$\overline{U}^{\,n+1} = \frac{U^{n+1} + U^n}{2}, \overline{u}_{n+1} = \frac{u_{n+1} + u_n}{2}, \overline{\partial}_t U^{n+1} = \frac{U^{n+1} - U^n}{k}.$$

As before we have:

$$\begin{aligned} U^{n+1} - u_{n+1} &= \left(U^{n+1} - P_1 u_{n+1} \right) + \left(P_1 u_{n+1} - u_{n+1} \right) \\ &= \vartheta^{n+1} + \rho^{n+1} \end{aligned}$$

where $P_1 u_{n+1}$ is the elliptic projection of u_{n+1} defined by (3.3). Then

$$\overline{\vartheta}^{\,n+1} = \overline{U}^{\,n+1} - P_1 \overline{u}_{n+1} = \frac{1}{2} \left(\vartheta^{n+1} + \vartheta^n \right).$$

We first recall that Lemma 3.1 holds, then

$$\|\rho^{n+1}\| \leq C(u) h^r$$

Hence it remains to estimate ϑ^{n+1}. With the above notation we have for $\chi \in \mathscr{S}_h$

$$\begin{aligned} &(4.3) \qquad \left(\overline{\partial}_t \vartheta^{n+1}, \chi \right) + \frac{2}{3Re} \left(\overline{\vartheta}_x^{\,n+1}, \chi_x \right) = \\ &= \left(\overline{\partial}_t (U^{n+1} - P_1 u_{n+1}), \chi \right) + \frac{2}{3Re} \left((\overline{U}^{\,n+1} - P_1 \overline{u}_{n+1})_x, \chi_x \right) = \\ &= -\frac{1}{3Re} (U_x^n, \chi_x) - \frac{2}{3} \left(\overline{U}^{\,n+1} \overline{U}_x^{\,n+1}, \chi \right) - \frac{1}{3} \left(\overline{U}^{\,n} \overline{U}_x^{\,n}, \chi \right) \\ &- \left(\overline{\partial}_t P_1 u_{n+1} - u_t(t_{n+\frac{1}{2}}), \chi \right) - \left(u_t(t_{n+\frac{1}{2}}), \chi \right) - \frac{2}{3Re} ((\overline{u}_{n+1,x}, \chi_x) \\ &= - \left(\overline{\partial}_t P_1 u_{n+1} - u_t(t_{n+\frac{1}{2}}), \chi \right) - \frac{2}{3} \left(\overline{U}^{\,n+1} \overline{U}_x^{\,n+1} - u_{n+\frac{1}{2}} (u_{n+\frac{1}{2}})_x, \chi \right) \\ &- \frac{1}{3} \left(U^n U_x^n - u_{n+\frac{1}{2}} (u_{n+\frac{1}{2}})_x, \chi \right) - \frac{1}{3Re} \left(U_x^n - (u_{n+\frac{1}{2}})_x, \chi_x \right) \\ &- \frac{2}{3Re} \left((\overline{u}_{n+1})_x - (u_{n+\frac{1}{2}})_x, \chi_x \right) \end{aligned}$$

Now

$$
\begin{aligned}
&\left|\left(\overline{U}^{\,n+1}\overline{U}_x^{\,n+1} - u_{n+\frac{1}{2}}(u_{n+\frac{1}{2}})_x,\, \chi\right)\right| \leq \\
&\leq \left(\left|\overline{U}_x^{\,n+1}\right|_{L^\infty(Q)} \left\|\overline{U}^{\,n+1} - u_{n+\frac{1}{2}}\right\| + \left|u_{n+\frac{1}{2}}\right|_{L^\infty(Q)} \left\|\overline{U}_x^{\,n+1} - (u_{n+\frac{1}{2}})_x\right\|\right) \|\chi\| \\
&\leq C(u)\left(\left\|\overline{U}^{\,n+1} - u_{n+\frac{1}{2}}\right\| + \left\|\overline{U}_x^{\,n+1} - (u_{n+\frac{1}{2}})_x\right\|\right) \|\chi\|
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\left|\left(U^n U_x^n - u_{n+\frac{1}{2}}(u_{n+\frac{1}{2}})_x,\, \chi\right)\right| \leq \\
&\qquad \leq C(u)\left(\left\|U^n - u_{n+\frac{1}{2}}\right\| + \left\|U_x^n - (u_{n+\frac{1}{2}})_x\right\|\right) \|\chi\|.
\end{aligned}
$$

Setting $\chi = \overline{\vartheta}^{\,n+1}$ and noting

$$
\left(\overline{\partial}_t \vartheta^{n+1}, \overline{\vartheta}^{\,n+1}\right) = \frac{1}{2}\overline{\partial}_t \left\|\vartheta^{n+1}\right\|^2,
$$

we find

$$
\begin{aligned}
&\frac{1}{2}\,\overline{\partial}_t \left\|\vartheta^{n+1}\right\|^2 + \frac{2}{3Re}\left\|\overline{\vartheta}_x^{\,n+1}\right\|^2 \\
&\leq C(u)\left\|\bar{\vartheta}^{n+1}\right\| \Big(\left\|\overline{\partial}_t P_1 u_{n+1} - u_t(t_{n+\frac{1}{2}})\right\| + \left\|\overline{U}^{\,n+1} - u_{n+\frac{1}{2}}\right\| \\
&+ \left\|\left(\overline{U}^{\,n+1} - u_{n+\frac{1}{2}}\right)_x\right\| + \left\|U^n - u_{n+\frac{1}{2}}\right\| \\
&+ \left\|\left(U^n - u_{n+\frac{1}{2}}\right)_x\right\| + \left\|\left(\overline{u}_{n+1} - u_{n+\frac{1}{2}}\right)_x\right\| \Big).
\end{aligned}
$$

As in [14], we have the estimates:

$$
\begin{aligned}
&\left\|\overline{\partial}_t u_{n+1} - u_t(t_{n+\frac{1}{2}})\right\| \\
&\qquad = k^{-1}\left\|\int_{t_n}^{t_{n+\frac{1}{2}}} (s - t_n) u_{tt}(s)ds + \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} (s - t_{n+1}) u_{tt}(s)ds\right\| \\
&\qquad \leq C\int_{t_n}^{t_{n+1}} \|u_{tt}(s)\| ds \leq C(u)\,k
\end{aligned}
$$

$$
\begin{aligned}
\left\|\overline{\partial}_t P_1 u_{n+1} - u_t(t_{n+\frac{1}{2}})\right\| &\leq \left\|\overline{\partial}_t \rho^{n+1}\right\| + \left\|\overline{\partial}_t u_{n+1} - u_t(t_{n+\frac{1}{2}})\right\| \\
&\leq C(u)(h^r + k)
\end{aligned}
$$

$$\left\|\overline{u}_{n+1} - u_{n+\frac{1}{2}}\right\| \leq C \int_{t_n}^{t_{n+1}} \|u_t(s)\| ds \leq C(u)\, k$$

$$\begin{aligned}\left\|\overline{U}^{\,n+1} - u_{n+\frac{1}{2}}\right\| &\leq \left\|\overline{\vartheta}^{\,n+1}\right\| + \left\|\overline{\rho}^{\,n+1}\right\| + \left\|\overline{u}^{\,n+1} - u_{n+\frac{1}{2}}\right\| \\ &\leq \left\|\overline{\vartheta}^{\,n+1}\right\| + C(u)(h^r + k)\end{aligned}$$

$$\begin{aligned}\left\|\left(\overline{u}_{n+1} - u_{n+\frac{1}{2}}\right)_x\right\| &= \left\|\left(\frac{1}{2}u(t_{n+1}) + \frac{1}{2}u(t_n) - u(t_{n+\frac{1}{2}})\right)_x\right\| \\ &\leq C \int_{t_n}^{t_{n+1}} \left\|\frac{\partial}{\partial x} u_t\right\| ds \leq C(u)\, k\end{aligned}$$

$$\begin{aligned}\left\|\left(\overline{U}^{\,n+1} - u_{n+\frac{1}{2}}\right)_x\right\| &\leq \left\|\overline{\vartheta}_x^{\,n+1}\right\| + \left\|\overline{\rho}_x^{\,n+1}\right\| + \left\|\left(\overline{u}^{n+1} - u_{n+\frac{1}{2}}\right)_x\right\| \\ &\leq \left\|\overline{\vartheta}_x^{\,n+1}\right\| + C(u)(h^r + k)\end{aligned}$$

$$\begin{aligned}\left\|U^n - u_{n+\frac{1}{2}}\right\| &\leq \|\vartheta^n\| + \|\rho^n\| + \left\|u^n - u_{n+\frac{1}{2}}\right\| \\ &\leq \|\vartheta^n\| + C(u)(h^r + k)\end{aligned}$$

$$\begin{aligned}\left\|\left(U^n - u_{n+\frac{1}{2}}\right)_x\right\| &\leq \|\vartheta_x^n\| + \|\rho_x^n\| + \left\|u^n - u_{n+\frac{1}{2}}\right\| \\ &\leq \|\vartheta_x^n\| + C(u)\left(h^{r-1} + k\right)\end{aligned}$$

Recalling that $\overline{\vartheta}^{\,n+1}$ and ϑ^n belongs to $\mathscr{S}_h(\Omega)$ and that in $\mathscr{S}_h(\Omega)$ all the norms are equivalent, we conclude that

$$\overline{\partial}_t \left\|\vartheta^{n+1}\right\|^2 \leq C \left\|\overline{\vartheta}^{\,n+1}\right\|^2 + C\left(h^{r-1} + k\right)^2.$$

Therefore

$$(1 - Ck)\left\|\vartheta^{n+1}\right\|^2 \leq (1 + Ck)\|\vartheta^n\|^2 + Ck\left(h^{r-1} + k\right)^2$$

and, for small k,

$$\left\|\vartheta^{n+1}\right\|^2 \leq C\left\|\vartheta^0\right\|^2 + C(n+1)k\left(h^{r-1} + k\right)^2$$

which complete the proof.

5. Numerical examples.

To solve numerically the quasi-linear Burgers problem with homogeneous boundary Dirichlet conditions, we have implemented an iterative algorithm, that computes the unknown solution on the first step starting from known initial data, and it proceeds iteratively.
The domain $Q = \Omega \times [0, T]$ of the problem is partitioned into subdomains $\theta_e = \Omega_j \times I_n$ with $1 \leq e \leq JS$, where the domain Ω is partitioned into element Ω_j, with $1 \leq j \leq J$ where J is the number of the space discretization and the time domain $[0, T]$ is partitioned as $I_n = [t_n, t_{n+1}]$, with $0 \leq n \leq S - 1$, where S is the number of time-slabs. To obtain an iterative scheme, we express the unknown functions u^{hk} and ϕ^{hk} of the discrete formulations (2.6) on θ_e by $u^{hk} = Nu_e$ and $\phi^{hk} = N\phi_e$, where N is the matrix of the piecewice bilinear lagrangian shape functions and u_e and ϕ_e are the vectors of the nodal values of u and ϕ on θ_e. Therefore the approximation spaces V_{hk}, that have been used for the following numerical examples, are both linear in space and time ($p = 1$ and $q = 1$). Apply now at (2.6) on θ_e the explicit time integration and performing the classic finite element assembling process, we obtain the numerical solution solving iteratively a non-linear system by the modified Jacobi method. The iterative scheme is implemented on VAX 8650 in Fortran 77.
The program is applied to solve one-dimensional test problems with homogeneous boundary condition, those show to decay as an arbitrary periodic initial disturbance as a sine wave (Ex. 1) and the shock waves approaching a steady state (Ex. 2).

Example 1. In this example we solve in $Q = [0, 2] \times [0, T]$ the quasi-linear Burgers problem with homogeneous boundary conditions and periodic initial condition [4] Re $= 100$, $\Delta x = 0.05$ and $\Delta t = 0.01$.

$$\begin{aligned} & u_t - \frac{1}{Re}\, u_{xx} + uu_x = 0, && u(x,t) \in Q \\ & u(x,0) = u_0 = u_{ex}(x,0), && x \in [0,2] \\ & u(0,t) = u(2,t) = 0, && t \in (0,T). \end{aligned}$$

The exact solution is

$$u_{ex}(x,t) = \frac{2\pi}{Re} \left\{ \frac{\frac{1}{4}\sin \pi x e^{-(\pi/Re)^2 t} + \sin 2\pi x e^{-4(\pi/\mathrm{Re})^2 t}}{1 + \frac{1}{4}\cos \pi x e^{-(\pi/Re)^2 t} + \frac{1}{2}\cos 2\pi x e^{-4(\pi/Re)^2 t}} \right\}.$$

In fig. 1a the compared results of iterative scheme and the exact solution at the time $t = 0.01$ are plotted and in fig. 1b the numerical solutions at several timesteps between $t = 0.01$ and $t = 1$ are plotted.

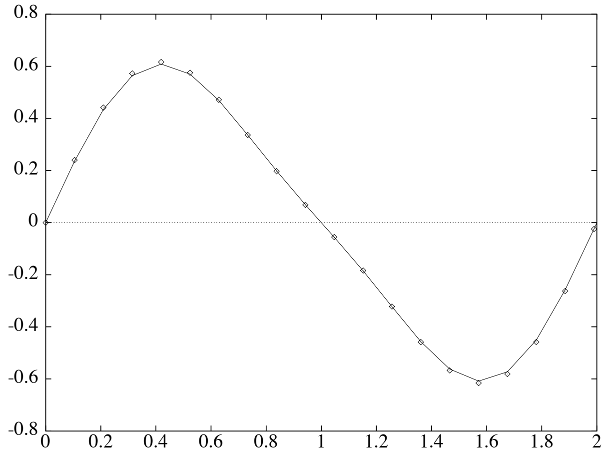

Fig. 1a

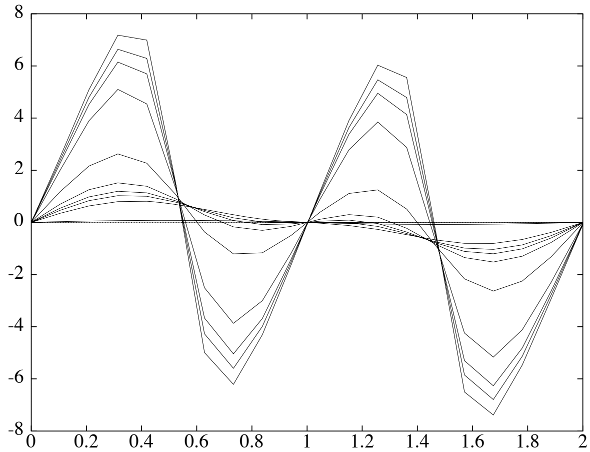

Fig. 1b

Example 2. In this example the numerical scheme is applied to solve in $Q = [0, 1] \times [0, T]$ the quasi-linear Burgers problem with homogeneous boundary conditions and discontinuous initial data [13]

$$\begin{aligned} & u_t - \tfrac{1}{Re} u_{xx} + u u_x = 0, && u(x,t) \in Q \\ & u(0,t) = u(1,t) = 0, && t \in (0,T) \\ & u(x,0) = \begin{cases} 0 & 0 < x < 1 \\ 1 & x = 0 \text{ and } x = 1 \end{cases} \end{aligned}$$

In fig. 2 the results of the numerical scheme at nine time-steps between $t = 0.0001$ and $t = 1$ for the $Re = 100$, for $\Delta t = 0.01$ and $\Delta x = 0.1$ are plotted.

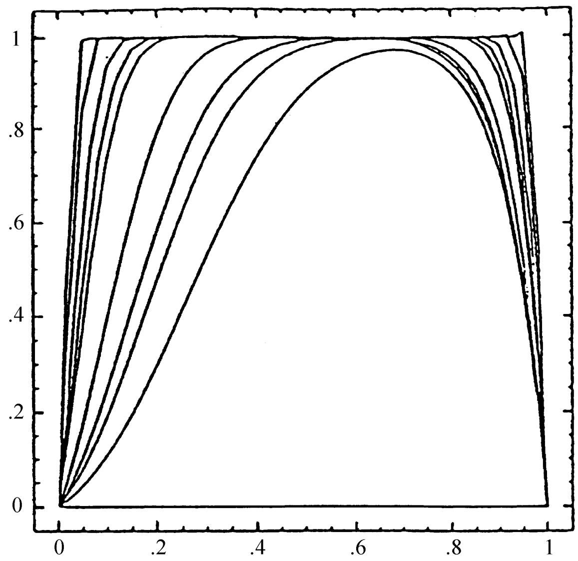

Fig. 2

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M. Morandi Cecchi and P. Patuzzo Grego
Dipartimento di Matematica Pura ed Applicata,
Università di Padova,
Via Belzoni 7,
35131 Padova (ITALY)

R. Nociforo
Dipartimento di Matematica,
Università di Catania,
Viale A. Doria 6,
95125 Catania (ITALY)