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DECOMPOSITIONS AND THE FIXED POINT PROPERTY FOR MULTIFUNCTIONS

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Relations between the fixed point properties for some classes of multifunctions of a compact Hausdorff space X, of a decomposition space $X/\mathcal{D}$, where $\mathcal{D}$ is an upper semi-continuous decomposition of X, and of the members of $\mathcal{D}$ are studied. Results are applied to some special decompositions of metric continua.

1. Introduction.

In the present paper we study interrelations between the fixed point properties for some classes of multifunctions of a continuum X, of a decomposition space $X/\mathcal{D}$, where $\mathcal{D}$ is an upper semi-continuous decomposition of X, and of the members of $\mathcal{D}$. Namely we consider the following three conditions:

(1.1) the continuum X has the fixed point property,

(1.2) the decomposition space has the fixed point property,

(1.3) every member of the decomposition $\mathcal{D}$ has the fixed point property,

and we find conditions under which (1.2) and (1.3) imply (1.1) for some classes of multifunctions. The reader is referred to the author's earlier paper ([4],

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Remark 4, p. 132) to see that no two of the three requirements (1.1), (1.2) and (1.3) imply the other one, even for single-valued mappings.

The result of the paper are based on the following simple idea which is formulated in a form of an observation below.

Observation 1.4. *Let $\mathcal{D}$ be an upper semi-continuous decomposition of a continuum X, and let $\mathcal{M}$ be a class of multifunctions $F : X \to X$. If for each $F \in \mathcal{M}$ there exists an element $D_0 \in \mathcal{D}$ such that*

(1.5) *D_0 has a fixed point under F;*

(1.6) *$F(D_0) \subset D_0$,*

then the continuum X has the fixed point property for the class $\mathcal{M}$ of multifunctions.

In general, condition (1.5), or even a stronger one, viz.

(1.7) every element D of $\mathcal{D}$ has the fixed point property for $F \in \mathcal{M}$,

depends on the structure of elements of the decomposition $\mathcal{D}$, while condition (1.6) depends on some special requirements concerning the decomposition space $X/\mathcal{D}$, namely it is implied by the fixed point property for single-valued mappings of $X/\mathcal{D}$ (see Proposition 4.1 below).

The paper consists of five sections. After Introduction and Preliminaries (where necessary definitions are collected), we show some general results on multifunctions (called decomposition concordant, d.c.) related to upper semi-continuous decompositions of compact Hausdorff spaces. These results are used in the next section to prove a theorem on the fixed point property for some classes of d.c. multifunctions of continua, which is turn applied in Section 5 to get the fixed point theorems for some multifunctions related to certain special decompositions of metric continua.

2. Preliminaries.

All spaces considered in this paper are Hausdorff compact. A family of nonempty closed and mutually disjoint subsets of a space X whose union is the whole space is called a *decomposition* of X. The topology in a decomposition $\mathcal{D}$ is defined by means of the following assumption: a subset $\mathcal{A}$ of $\mathcal{D}$ is open in $\mathcal{D}$ if and only if the union of all subsets of X which belong to $\mathcal{A}$ is open in X. The decomposition $\mathcal{D}$ equipped with the above described topology is called the *decomposition space*, and it is then denoted by $X/\mathcal{D}$. A mapping $q : X \to X/\mathcal{D}$ defined by $q(x) = D \in \mathcal{D}$ if and only if $x \in D$ is called the *natural projection*. It is easily seen that q is continuous (see [7], § 43, III, p. 64). A decomposition $\mathcal{D}$ of a space X is called *upper semi-continuous* if for each member M of $\mathcal{D}$ and for each open subset G of X containing M there is an open subset U of X such that $M \subset U \subset G$ and U is the union of some members of $\mathcal{D}$ (see [6], § 19, I, p. 183 and II, p. 185). This definition is equivalent to closedness of the natural projection $q : X \to X/\mathcal{D}$ (see [7], § 43, III, p. 64). A decomposition $\mathcal{D}$ of a space X is said to be *monotone* provided all members of $\mathcal{D}$ are connected subsets of $\mathcal{D}$.

By a *multifunction* F on set X we mean a relation on X into Y whose domain is X, i.e., F is a subset of $X \times Y$ such that for each $x \in X$ there is $y \in Y$ with $(x, y) \in F$. In other words, F assigns to each point $x \in X$ a nonempty subset $F(x)$ of Y, and we will write $F : X \to Y$. We shall denote multifunctions by upper case letters $F, G, \ldots$, while lower case ones $f, g, \ldots$ will denote single-valued functions. The term of a *mapping* is used to denote a single-valued continuous function. For each $A \subset X$ we put $F(A) = \cup\{F(x) : x \in A\}$, and for each $B \subset Y$ we put $F^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. A space X is said to have the *fixed point property for a certain class $\mathcal{M}$ of multifunctions* of X into itself if for every multifunction F belonging to $\mathcal{M}$ there exists a point p in X with $p \in F(p)$. In particular, if single-valued functions $f : X \to X$ are under consideration, then the above mentioned condition can be written as $f(p) = p$.

A multifunction $F : X \to Y$ is said to be:

- *point closed* if $F(x)$ is closed for each $x \in X$;
- *point connected* if $F(x)$ is connected for each $x \in X$;
- *lower semi-continuous* (l.s.c.) if $F^{-1}(B)$ is open for each open subset $B \subset Y$;
- *upper semi-continuous* (u.s.c.) if $F^{-1}(B)$ is closed for each closed subset $B \subset Y$;
- *continuous* if F is both l.s.c. and u.s.c.

The reader is referred to ([8], Proposition 1.1, p. 228) for connections between these (and related) concepts of multifunctions.

3. Decomposition concordant multifunctions.

Let $\mathcal{D}$ and $\mathcal{E}$ be decompositions of spaces X and Y, respectively. A multifunction $F : X \to Y$ is said to be *decomposition concordant* (d.c.) provided that it maps every element of $\mathcal{D}$ into an element of $\mathcal{E}$, i.e., if for each element D of $\mathcal{D}$ there exist and element E of $\mathcal{E}$ such that $F(D) \subset E$. In other words, if $q : X \to X/\mathcal{D}$ and $r : Y \to Y/\mathcal{E}$ are the natural projections, then $F : X \to Y$ is d.c. if for every $s \in q(X)$ there is a $t \in r(Y)$ such that

$F(q^{-1}(s)) \subset r^{-1}(t)$. The following properties of d.c. multifunctions can easily be observed.

Proposition 3.1. *The composition of d.c. multifunctions is a d.c. multifunction, i.e., if multifunctions $F : X \to Y$ and $G : Y \to Z$ are d.c., then the composition $G \circ F : X \to Z$ is d.c.*

Proposition 3.2. *If a multifunction $F : X \to Y$ is d.c. and if $r : Y \to r(Y) = Y/\mathcal{E}$ is the natural projection, then*

(3.3) *the composition $r \circ F : x \to r(Y)$ is a single-valued d.c. function;*

(3.4) *if F is either l.s.c. or u.s.c., then $r \circ F$ is continuous.*

The next theorem generalizes earlier author's results ([2], Property 2, p. 338 and [3], Proposition 14, p. 125).

Theorem 3.5. *Let $\mathcal{D}$ and $\mathcal{E}$ be upper semi-continuous decompositions of spaces X and Y respectively, and let $q : X \to X/\mathcal{D}$ and $r : Y \to Y/\mathcal{E}$ be the natural projections. Then for each d.c. multifunction $F : X \to Y$ there exists one and only one (single-valued) function g between the decomposition spaces $X/\mathcal{D}$ and $Y/\mathcal{E}$, i.e., $g : q(X) \to r(Y)$ (called the function induced by F), such that*

(3.6) $$g(q(x)) = r(F(x)) \quad \text{for each} \quad x \in X,$$

that is, the following diagram commutes

(3.7) $$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\ q(X) & \xrightarrow[g]{} & r(Y) \end{array}$$

Furthermore,

(3.8) *if the d.c. multifunction $F : X \to Y$ is either l.s.c. or u.s.c., then the induced function g continuous.*

Conversely,

(3.9) *if a multifunction $F : X \to Y$ is either l.s.c. or u.s.c. and if there exists a continuous function $g : q(X) \to (Y)$ such that diagram* (3.7) *commutes, then F is d.c., and g is a function induced by F.*

Proof. The needed mapping $g : q(X) \to r(Y)$ is defined by

$$g(s) = r(F(q^{-1}(s))) \quad \text{for each} \quad s \in q(X), \tag{3.10}$$

whence equality (3.6), and thus commutativity of diagram (3.7), follows.

Observe that the decomposition $\mathcal{D}$ of the space X (or, equivalently, the natural projection $q : X \to X/\mathcal{D}$) defines an equivalence relation on X: two points of X are in the relation if they belong to the same element of $\mathcal{D}$, or - in other words - if they are mapped into the same point of $X/\mathcal{D}$ under q. Now uniqueness of g is a consequence of ([5], Theorem 7.7, p. 17). To see (3.8) note that the composition $r \circ F$ is continuous by (3.4) of Proposition 3.2, and since $g \circ q = r \circ F$, it follows that $g \circ q$ is continuous. Consequently g is continuous, the quotient mapping q being a closed one (compare also ([5], Theorem 4.3, p. 126). Finally (3.9) is a consequence of the above quoted Theorem 7.7 of ([5], p. 17).

4. Fixed points of d.c. multifunctions on continua.

We start with the following proposition which generalizes Lemma 14 of [4], p. 135.

Proposition 4.1. *Let a decomposition $\mathcal{D}$ of a continuum X be upper semi-continuous and such that*

(4.2) *the decomposition space $X/\mathcal{D}$ has the fixed point property for single-valued mappings.*

Then for every d.c. either l.s.c. or u.s.c. multifunction $F : X \to X$ of X to itself there exists an element of $\mathcal{D}$ which contains its image under F.

Proof. We apply Proposition 3.2 and Theorem 3.5 with $Y = F(X) \subset X$ and $r = q|F(X)$. Then by (3.3) and (3.4) the composition $(q|F(X)) \circ F$ is a single-valued d.c. continuous function, and by (3.8) the induced function $g : q(X) \to q(F(X))$ is also continuous. Since $q(X)$ is assumed to have the fixed point property for single-valued mappings, there exists a point $s_0 \in q(X)$ such that $g(s_0) = s_0$. To show that $q^{-1}(s_0)$ is just the needed element D_0 of $\mathcal{D}$ which contains its image, note that since F is d.c., $F(D_0)$ is contained in an element D of $\mathcal{D}$. Put $s = q(D) \in X$. Since diagram (3.7) commutes, we have $g(q(D_0)) = (q|F(X))(F(D_0)) \subset q(D) = s$. On the other hand, $g(q(D_0)) = g(s_0) = s_0$, so $s = s_0$, and thereby $D = D_0$. The proof is complete.

Proposition 4.1 will be used to prove the following result.

Theorem 4.3. *Let $\mathcal{D}$ be an upper semi-continuous decomposition of a continuum X. If*

(4.2) *the decomposition space $X/\mathcal{D}$ has the fixed point property for single-valued mappings,*

and

(4.4) *every element of the decomposition $\mathcal{D}$ has the fixed point property for a class $\mathcal{C}$ of either l.s.c. or u.s.c. multifunctions,*

then

(4.5) *the continuum X has the fixed point property for d.c. multifunctions belonging to the class $\mathcal{C}$.*

Proof. Let $q : X \to q(X) = X/\mathcal{D}$ be the natural projection, and let $F : X \to F(X) \subset X$ be a given d.c. either l.s.c. or u.s.c. multifunction belonging to $\mathcal{C}$. By (4.2) and Proposition 4.1 we infer that there exists a member D_0 of $\mathcal{D}$ such that $F(D_0) \subset D_0$, i.e., that condition (1.6) is satisfied. Condition (1.5) holds by (4.4). Then (4.5) is a consequence of (1.6) of Observation 1.4. The proof is then complete.

Recall that a λ-*dendroid* means a hereditarily decomposable and hereditarily unicoherent metric continuum. The following result is known (see [8], Corollary 2.5, p. 239).

Theorem 4.6. *A hereditarily decomposable continuum is a λ-dendroid if and only if it has the fixed point property for point-connected, point-closed u.s.c. multifunctions. In particular, every λ-dendroid has the fixed point property for single-valued mappings.*

Combining Theorems 4.3 and 4.6 we get the following two corollaries.

Corollary 4.7. *Let $\mathcal{D}$ be an upper semi-continuous decomposition of a continuum X such that every element of $\mathcal{D}$ is a λ-dendroid. If condition* (4.2) *holds, then the continuum X has the fixed point property for d.c. point-connected, point-closed u.s.c. multifunctions.*

Corollary 4.8. *Let $\mathcal{D}$ be an upper semi-continuous decomposition of a continuum X such that the decomposition space $X/\mathcal{D}$ is a λ-dendroid. The condition* (4.4) *implies* (4.5).

If variants of the fixed point property for multifunctions are studied, one can consider multifunctions F from a space X to itself having, for a given upper semi-continuous decomposition $\mathcal{D}$ of X, the following property (which has been proposed to the author by W.J. Charatonik).

(4.9) For each point $x \in X$ there exists an element D of $\mathcal{D}$ such that $F(x) \subset D$.

Call multifunctions $F : X \to X$ satisfying (4.9) *point-decomposition concordant*, shortly p.d.c. Then it would be interesting to know if analogs of Proposition 4.1 and Theorem 4.3 are true, in the following form.

Question 4.10. Let a continuum X, an upper semi-continuous decomposition $\mathcal{D}$ of X, and an u.s.c. p.d.c. multifunction $F : X \to X$ be given. Assume that

(4.11) X has the fixed point property for single-valued mappings.

Is then true the following assertion (4.12) ?

(4.12) There exists an element D_0 of $\mathcal{D}$ which contains a point $x_0 \in D_0$ such that $F(x_0) \subset D_0$.

Question 4.13. Let a continuum X, and an upper semi-continuous decomposition $\mathcal{D}$ of X be given. Assume that

(4.4) every element of the decomposition $\mathcal{D}$ has the fixed point property for a class $\mathcal{C}$ of either l.s.c. or u.s.c. multifunctions,

and that

(4.11) X has the fixed point property for single-valued mappings.

Is then true the following assertion (4.14) ?

(4.14) The continuum X has the fixed point property for p.d.c. multifunctions belonging to the class $\mathcal{C}$.

5. Applications to the canonical decompositions.

We apply Proposition 4.1 and Theorem 4.3 to get fixed point theorems for d.c. multifunctions related to some special decompositions, as in [4] for single-valued mappings. To formulate them we recall the needed concepts.

A continuum I is said to be *irreducible between points a and b* (or shortly *irreducible*) provided it contains a and b, but no proper subcontinuum of I contains both of them. For every metric irreducible continuum I there exists a monotone (i.e., having connected point-inverses) mapping $g : I \to [0, 1]$ of I into the closed unit interval $[0, 1]$ of reals (see [7], p. 199) whose point inverses $g^{-1}(t)$ for $t \in [0, 1]$, called *layers* of I, have the property that the decomposition of I into layers is the finest possible of all linear upper semi-continuous monotone decompositions of I (see [7], § 48, IV, Theorem 3, p. 200; compare also some related results in [9], Theorem 2.2, p. 185 and in [10], Theorem 3, p. 8.

Let X be a metric continuum. A decomposition $\mathcal{D}$ of X is said to be *admissible* (see [3], § 4, p. 115–117) provided that

(5.1) $\mathcal{D}$ is upper semi-continuous,

(5.2) $\mathcal{D}$ is monotone,

(5.3) for every irreducible continuum $I \subset X$ every layer L_t of X (where $t \in [0, 1]$) is contained in some element of $\mathcal{D}$.

It is known that for every metric continuum X there exists one and only one admissible decomposition $\mathcal{D}^*$, called the *canonical decomposition* of X (see [3], § 4, Theorems 2, 3 and 4, pp. 117–121) such that

(5.4) $\mathcal{D}^*$ is the minimal decomposition among all decompositions of X which satisfy conditions (5.1), (5.2) and (5.3).

Elements of the canonical decompositions $\mathcal{D}^*$ of a (metric) continuum X are called *strata* of X, and the natural projection $q^* : X \to X/\mathcal{D}^* = q^*(X)$ of X onto the decomposition space $X/\mathcal{D}^*$ is called the *canonical mapping*. Extending the definition of a strata concordant (single-valued) mapping given in [4], p. 133, we say that a multifunction $F : X \to Y$ between metric continua X and Y is *strata concordant* if it is d.c. with respect to the canonical decompositions of X and Y correspondingly. Recall that a *dendroid* means an arcwise connected and hereditarily unicoherent metric continuum. In other words, a dendroid is an arcwise connected λ-dendroid. As a corollary to Proposition 4.1 we get the following result.

Proposition 5.5. *Let $\mathcal{D}$ be an admissible decomposition of a metric continuum X, and let the decomposition space $X/\mathcal{D}$ contain no simple closed curve. Then for every d.c. either l.s.c. or u.s.c. multifunction $F : X \to X$ there exists an element of $\mathcal{D}$ which contains its image under F.*

Proof. By [3], Theorem 1, p. 116, the decomposition space $X/\mathcal{D}$ is hereditarily arcwise connected. Since it contains no simple closed curve by the assumption, it is a dendroid (see [1], Theorem T26, p. 197), and therefore it has the fixed point property for single-valued mappings according to Theorem 4.6. Thus Proposition 4.1 can be applied, and the conclusion follows. The proof is complete.

The next result is a consequence of Observation 1.4 and Proposition 5.5. It is a generalization of Theorem 17 of [4], p. 136, where it had been proved for a particular case of the canonical decomposition of a continuum and for single-valued mappings.

Theorem 5.6. *Let $\mathcal{D}$ be an admissible decomposition of a metric continuum X, and let the decomposition space $X/\mathcal{D}$ contain no simple closed curve. If*

(4.4) *every element of the decomposition $\mathfrak{D}$ has the fixed point property for a class $\mathcal{C}$ of either l.s.c. or u.s.c. multifunctions,*

then

(4.5) *the continuum X has the fixed point property for d.c. multifunctions belonging to the class $\mathcal{C}$.*

The following is a version of Corollary 23 of [4], p. 137, for multifunctions.

Corollary 5.7. *Let the canonical decomposition of an irreducible continuum X be given. Then condition* (4.4) *implies* (4.5).

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