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OPTIMAL INTEGRABILITY IN B_p^q CLASSES

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Equations for the best integrability exponent, for monotonic functions in one-dimensional Gehring and Muckenhoupt classes, are unified in more general Reverse Holder Inequality classes.
Furthermore, the result is extended by removing the monotonicity assumptions.

1. Introduction.

Let E be a measurable set of $\mathbb{R}^n$ with positive Lebesgue measure, and $p, q \in \mathbb{R} - \{0\}$ such that $p < q$. For $K > 1$ we will denote with $B_p^q(K)$ the class of nonnegative measurable functions $f \in L^q(E)$ satisfying the *Reverse Holder Inequality*

$$\left(\fint_Q f^q(x)\, dx \right)^{1/q} \leq K \left(\fint_Q f^p(x)\, dx \right)^{1/p} \tag{1.1}$$

for all cubes $Q \subset E$.

Well-known particular cases of B_p^q classes are the *Gehring class* $G_q(K)$ of functions f such that

$$\left(\fint_Q f^q(x)\, dx \right)^{1/q} \leq K \fint_Q f(x)\, dx \quad \forall Q \subset E \tag{1.2}$$

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and the *Muckenhoupt class* $A_p(K)$ of functions f such that

$$\fint_Q f(x)\,dx \left(\fint_Q f^{1/(1-p)}(x)\,dx\right)^{p-1} \le K \quad \forall Q \subset E \tag{1.3}$$

where clearly is

$$G_q(K) = B_1^q(K)$$

and

$$A_p(K) = B^1_{\frac{1}{1-p}}(K).$$

In these two classes, respectively, the *forward* and the *backward propagation* property hold; namely, for $f \in G_q(K)$, there exists $q_0 > q$ such that $f \in L^s(E)$ for all $s \in [q, q_0)$ and, for $f \in A_p(K)$ there exists $p_0 < p$ such that $f \in L^s(E)$ for all $s \in (p_0, p]$.

In [1] and [2] Bojarski proved an asymptotic dependence of $\varepsilon = (q_0 - q)$, as $K \to 1$.

In one-dimensional case, where E is an interval of $\mathbb{R}$, the problem of finding the exact value of q_0 and p_0 has been completely solved, for monotonic functions, by following two parallel theorems.

Theorem 1.1 (D'Apuzzo - Sbordone)**.** *Let $f \in G_q(K)$ be a nonnegative and nonincreasing function on $E \subset R$. Then $f \in L^s(E)$ for $q \le s < q_0$, where q_0 is the unique solution of equation*

$$1 - K^q \frac{x-q}{x}\left(\frac{x}{x-1}\right)^q = 0. \tag{$E1$}$$

Theorem 1.2 (Korenovskii)**.** *Let $f \in A_p(K)$ be a nonnegative and nondecreasing function on $E \subset R$. Then $f \in L^r(E)$ for $p_0 < r \le p$, where p_0 is the unique solution of equation*

$$\frac{p-x}{p-1}(Kx)^{1/(p-1)} = 1. \tag{$E2$}$$

Besides, in [9] is proved that Theorem 1.1 and equation (E1) still hold in weighted Gehring classes.

Aim of this paper is to unify the previous theorems in the class $B_p^q(K)$. Indeed we prove the following

Theorem 1.3. *Let $f \in B_p^q(K)$ be a nonnegative function on $E \subset \mathbb{R}$. Then there exists x_0 such that, if $p > 0$ $[p < 0]$, $f \in L^s(E)$ for all s such that $q \le s < x_0$ $[x_0 < s \le p]$, where x_0 is given by the unique solution of equation*

$$(E3) \qquad \left(\frac{x}{x-q}\right)^{1/q} = K\left(\frac{x}{x-p}\right)^{1/p}.$$

By applying simple transformations, it's easy to see that equation (E1) and (E2) are particular cases of equation (E3).

Moreover, we remark that Theorem 1.3 is an improvement of Theorem 1.1 and Theorem 1.2 since the monotonicity assumption is removed.

2. Preliminary results.

Theorem 2.1. *Let g be a nonnegative function on interval (a, b), and*

$$G(x) = \frac{1}{x-a}\int_a^x g(t)\,dt\,.$$

Then, for α and β such that $\alpha\beta < 0$ or $|\alpha| < |\beta|$, we have

$$(2.1) \qquad \int_a^b (x-a)^{\alpha-1}G^\beta(x)\,dx \le \left(\frac{\beta}{\beta-\alpha}\right)^\beta \int_a^b (x-a)^{\alpha-1}g^\beta(x)\,dx\,.$$

Proof. Integrating by parts we have

$$\alpha\int_a^b (x-a)^{\alpha-1}G^\beta = c - \beta\int_a^b (x-a)^\alpha G^{\beta-1}G'$$

where $c = (b-a)^\alpha G^\beta(b) > 0$. Now, since $(x-a)G' = g - G$,

$$\alpha\int_a^b (x-a)^{\alpha-1}G^\beta = c - \beta\left[\int_a^b (x-a)^{\alpha-1}G^{\beta-1}g - \int_a^b (x-a)^{\alpha-1}G^\beta\right]$$

and then

$$(\beta-\alpha)\int_a^b (x-a)^{\alpha-1}G^\beta = \beta\int_a^b (x-a)^{\alpha-1}G^{\beta-1}g - c.$$

Let us first suppose $\beta > 0$; by our assumptions $(\beta - \alpha) > 0$ so, since $c > 0$

$$\int_a^b (x-a)^{\alpha-1} G^\beta \le \frac{\beta}{\beta-\alpha} \int_a^b (x-a)^{\alpha-1} G^{\beta-1} g \tag{2.2}$$

from Holder's inequality we have

$$\int_a^b (x-a)^{\alpha-1} G^{\beta-1} g \le \left(\int_a^b (x-a)^{\alpha-1} G^\beta \right)^{\frac{\beta-1}{\beta}} \left(\int_a^b (x-a)^{\alpha-1} g^\beta \right)^{\frac{1}{\beta}} \tag{2.3}$$

and finally, by (2.2) and (2.3), we get

$$\left(\int_a^b (x-a)^{\alpha-1} G^\beta \right)^{1/\beta} \le \frac{\beta}{\beta-\alpha} \left(\int_a^b (x-a)^{\alpha-1} g^\beta \right)^{1/\beta}$$

that proves the theorem for $\beta > 0$.
If $\beta < 0$, by our assumptions $(\beta - \alpha) < 0$ so

$$\int_a^b (x-a)^{\alpha-1} G^\beta \ge \frac{\beta}{\beta-\alpha} \int_a^b (x-a)^{\alpha-1} G^{\beta-1} g\,. \tag{2.4}$$

By the other hand, Holder's inequality for $\beta(\frac{\beta}{\beta-1}) < 0$ gives

$$\int_a^b (x-a)^{\alpha-1} G^{\beta-1} g \ge \left(\int_a^b (x-a)^{\alpha-1} G^\beta \right)^{\frac{\beta-1}{\beta}} \left(\int_a^b (x-a)^{\alpha-1} g^\beta \right)^{1/\beta}. \tag{2.5}$$

Then from (2.4) and (2.5) we find

$$\left(\int_a^b (x-a)^{\alpha-1} G^\beta \right)^{1/\beta} \ge \frac{\beta}{\beta-\alpha} \left(\int_a^b (x-a)^{\alpha-1} g^\beta \right)^{1/\beta}.$$

By raising both members to the negative exponent β we get the result. □

Remark 2.1. For p and q such that $1 < p < q, \alpha = q/p$ and $\beta = q$, Theorem 2.1 gives the classical Hardy's inequality

$$\int_a^b (x-a)^{(q/p)-1} G^q(x)\,dx \le \left(\frac{p}{p-1} \right)^q \int_a^b (x-a)^{(q/p)-1} g^q(x)\,dx$$

and, for $\alpha = q/p$ and $\beta = -q$, the inequality

$$\int_a^b (x-a)^{(q/p)-1} G^{-q}(x)\,dx \le \left(\frac{p+1}{p} \right)^q \int_a^b (x-a)^{(q/p)-1} g^{-q}(x)\,dx$$

proved in [6].

Lemma 2.1. *Let $h(x)$ be a nonnegative function in $L^\infty(a,b)$. Then for $\lambda \neq 0$*

$$\int_a^b (x-a)^\lambda \left(\fint_a^x h(t)\,dt \right) dx = \frac{1}{\lambda}\left[(b-a)^\lambda \int_a^b h(x)\,dx - \int_a^b (x-a)^\lambda h(x)\,dx \right].$$

Proof. From Fubini's theorem we have:

$$\begin{aligned}\int_a^b (x-a)^\lambda \left(\fint_a^x h(t)\,dt \right) dx &= \int_a^b (x-a)^{\lambda-1} \left(\int_a^x h(t)\,dt \right) dx = \\ &= \int_a^b h(t) \left(\int_t^b (x-a)^{\lambda-1}\,dx \right) dt\end{aligned}$$

that easily leads to the result. □

Lemma 2.2. *For $C > 1$ and $a, b \in \mathbb{R} - \{0\}$ with $b > a > 0$ or $b < 0 < a$, let γ_C be defined for $x \in [0,1]$ as*

$$\gamma_C(a,b,x) = 1 - C^b(1-x)\left(\frac{b}{b-ax}\right)^{b/a}. \tag{2.6}$$

Then, there exists an unique solution x_b of equation

$$\gamma_c(a,b,x) = 0. \tag{2.7}$$

Moreover

$$\gamma_C(a,b,x) > 0 \Leftrightarrow x \in (x_b, 1].$$

Proof. Let us consider the auxiliary function

$$w(x) = (1-x)\left(\frac{b}{b-ax}\right)^{b/a}.$$

This function has range $[0,1]$ and, since

$$w'(x) = -\left(\frac{b}{b-ax}\right)^{b/a} \frac{(b-a)x}{b-ax},$$

w is decreasing in $[0,1]$ so, for $C^{-b} \in [0,1]$ there exists an unique solution of equation $w(x) = C^{-b}$ given by $x_b = w^{-1}(C^{-b})$ that is (2.7). Since w decreases

$$\gamma_C(a,b,x) > 0 \Leftrightarrow w(x) < C^{-b} \Leftrightarrow x > x_b$$

that completes the proof. $\square$

Lemma 2.3. *Let $f \in B_p^q(K)$ be a nonnegative function in $L^\infty(a,b)$. Then, there exist $\alpha_q, \alpha_p \in (0,1)$, such that for $p > 0$*

$$\int_a^b (x-a)^{\alpha-1} f^q(x)\,dx \le \frac{(b-a)^{\alpha-1}}{\gamma_K(p,q,\alpha)} \int_a^b f^q(x)\,dx \quad \forall \alpha \in (\alpha_q, 1] \tag{2.8}$$

and for $p < 0$

$$\int_a^b (x-a)^{\alpha-1} f^p(x)\,dx \le \frac{(b-a)^{\alpha-1}}{\gamma_{1/K}(q,p,\alpha)} \int_a^b f^p(x)\,dx \quad \forall \alpha \in (\alpha_p, 1] \tag{2.9}$$

with γ_C defined as in Lemma 2.2.

Proof. Let $f \in B_p^q(K)$ with $p > 0$. Then

$$\int_a^b (x-a)^{\alpha-1}\Big(⨍_a^x f^q(t)\,dt\Big)dx \le K^q \int_a^b (x-a)^{\alpha-1}\Big(⨍_a^x f^p(t)\,dt\Big)^{q/p} dx\,.$$

For $\lambda = \alpha - 1$ and $h = f^q$ from Lemma 2.1 we have

$$\int_a^b (x-a)^{\alpha-1}\Big(⨍_a^x f^q(t)\,dt\Big)dx = \\ = \frac{1}{\alpha-1}\Big[(b-a)^{\alpha-1}\int_a^b f^q(x)\,dx - \int_a^b (x-a)^{\alpha-1} f^q(x)\,dx\Big]$$

while, from Theorem 2.1 for $\beta = q/p$ and $g = f^p$

$$\int_a^b (x-a)^{\alpha-1}\Big(⨍_a^x f^p(t)\,dt\Big)^{q/p} dx \le \Big(\frac{q}{q-p\alpha}\Big)^{q/p} \int_a^b (x-a)^{\alpha-1} f^q(x)\,dx\,.$$

Then

$$\frac{1}{\alpha-1}\Big[(b-a)^{\alpha-1}\int_a^b f^q - \int_a^b (x-a)^{\alpha-1} f^q\Big] \le \\ \le K^q\Big(\frac{q}{q-p\alpha}\Big)^{q/p} \int_a^b (x-a)^{\alpha-1} f^q$$

from which follows that

$$(b-a)^{\alpha-1}\int_a^b f^q \ge \Big[1 - K^q(1-\alpha)\Big(\frac{q}{q-p\alpha}\Big)^{q/p}\Big]\int_a^b (x-a)^{\alpha-1} f^q$$

that is

$$\gamma_K(p,q,\alpha)\int_a^b (x-a)^{\alpha-1} f^q \le (b-a)^{\alpha-1}\int_a^b f^q\,.$$

From Lemma 2.2 there exists $\alpha_p \in (0,1)$ such that $\gamma_K(p,q,\alpha) > 0$ if $\alpha \in (\alpha_q, 1]$ so relation (2.8) is proved.
If now is $b < 0$ we have

$$\int_a^b (x-a)^{\alpha-1}\Big(\fint_a^x f^p\Big) \le K^{-p}\int_a^b (x-a)^{\alpha-1}\Big(\fint_a^x f^q\Big)^{p/q}.$$

As before, applying Lemma 2.1, for $\lambda = \alpha - 1$ and $h = f^p$, and Theorem 2.1, for $\beta = p/q$ and $g = f^q$ we get

$$\frac{1}{\alpha-1}\Big[(b-a)^{\alpha-1}\int_a^b f^p - \int_a^b (x-a)^{\alpha-1} f^p\Big] \le$$
$$\le K^{-p}\Big(\frac{p}{p-q\alpha}\Big)^{p/q}\int_a^b (x-a)^{\alpha-1} f^p$$

and then

$$\gamma_{1/K}(q,p,\alpha)\int_a^b (x-a)^{\alpha-1} f^p \le (b-a)^{\alpha-1}\int_a^b f^p\,.$$

Finally, for Lemma 2.2, there exists $\alpha_p \in (0,1)$ such that $\gamma_{1/K}(q,p,\alpha) > 0$ for $\alpha \in (\alpha_p, 1]$; so inequality (2.9) holds. □

Lemma 2.4 (Hardy-Littlewood-Polya). *Let $f \in L^s(E)$ be a nonnegative and nonincreasing [nondecreasing] function. Then, for* $0 < r < s$ $[s < r < 0]$

$$\Big(\int_a^b f^s(x)\,dx\Big)^{r/s} \le \frac{r}{s}\int_a^b (x-a)^{(r/s)-1} f^r(x)\,dx\,.$$

3. Main results.

We first prove a monotonic version of Theorem 1.3.

Theorem 3.1. *Let $f \in B_p^q(K)$, with $pq > 0$ $[pq < 0]$, be a nonnegative and nonincreasing [nondecreasing] function on $E \subset \mathbb{R}$. Then $f \in L^s(E)$ for $q \le s < q_0$ $[p_0 < s \le p]$, where q_0 $[p_0]$ is the unique solution of equation*

$$(E3)\qquad \Big(\frac{x}{x-q}\Big)^{1/q} = K\Big(\frac{x}{x-p}\Big)^{1/p}.$$

Proof. Let us first suppose $p > 0$ and $q > 0$, and let f be a nonincreasing function in $B_p^q(K)$. By using truncated functions (see [12]) we can construct a sequence of nonincreasing functions $f_h \in L^\infty(E)$ converging to f in L^q and verifying (1.1) for each h with the same constant K.
Hence functions f_h verify conditions of Lemma 2.3 so, for each h, inequality

$$\int_a^b (x-a)^{\alpha-1} f_h^q \le \frac{(b-a)^{\alpha-1}}{\gamma_K(p,q,\alpha)} \int_a^b f_h^q$$

for all $(a,b) \in E$ holds true and, passing to limit as $h \to +\infty$

$$\int_a^b (x-a)^{\alpha-1} f^q \le \frac{(b-a)^{\alpha-1}}{\gamma_K(p,q,\alpha)} \int_a^b f^q$$

with $\alpha_q < \alpha \le 1$.
If $\alpha_q = q/q_0$ and $\alpha = q/s$, so that $\alpha_q < \alpha \le 1$, we have $q \le s < q_0$. Then we can apply Lemma 2.4 for $r = q$ and obtain

$$\left(\int_a^b f^s\right)^{q/s} \le \frac{q}{s} \frac{(b-a)^{(q/s)-1}}{\gamma_K(p,q,q/s)} \int_a^b f^q \quad q \le s < q_0$$

and finally

$$\left(⨍_a^b f^s\right)^{q/s} \le \frac{q}{s} \frac{1}{\gamma_K(p,q,q/s)} ⨍_a^b f^q \quad q \le s < q_0$$

with q_0 unique solution of equation

$$\gamma_K(p,q,q/x) = 0$$

that easily leads to (E3).
Let us suppose $p < 0$ and $q > 0$, and f a nondecreasing function in $B_p^q(K)$. By using nondecreasing functions and inequality (2.9) of Lemma 2.3 ($p < 0$), we get

$$\int_a^b (x-a)^{\alpha-1} f^p \le \frac{(b-a)^{\alpha-1}}{\gamma_{1/K}(q,p,\alpha)} \int_a^b f^p$$

with $\alpha_p < \alpha \le 1$. Now, if $\alpha_p = p/p_0$ and $\alpha = p/s$, so that $\alpha_p < \alpha \le 1$, we have $p_0 < s \le p$. By applying again Lemma 2.4 for $r = p$ we have

$$\left(⨍_a^b f^s\right)^{p/s} \le \frac{p}{s} \frac{1}{\gamma_{1/K}(q,p,p/s)} ⨍_a^b f^p$$

with p_0 unique solution of equation

$$\gamma_{1/K}(q, p, p/x) = 0$$

that is again equation (E3). □

It is immediate to see that, for $p = 1$, Theorem 3.1 reduces to the Theorem 1.1 and equation (E3) to the equation (E1) for $f \in B_1^q(K) = G^q(K)$. Moreover, also for $q = 1$ and $f \in B^1_{\frac{1}{1-p}}(K) = A_p(K)$ the equation (E3) becomes

$$\left(\frac{x}{x-1}\right) = K\left(\frac{x}{x-(1-p)^{-1}}\right)^{1-p}$$

that, applying the transform $t \to (1-t)^{-1}$, returns the equation (E2).

To remove the monotonicity assumption in the previous theorem we need an important result, due to Korenovskii, on relationships between functions in Reverse Jensen Inequality classes and their rearrangements.

Namely, let Φ be the class of nonnegative convex functions φ on $(0, +\infty)$ and for $\varphi \in \Phi$ let $L_\varphi(E)$ be the related Orlicz class of functions f such that $\varphi(f) \in L^1(E)$. Then we will say that a function $f \in L_\varphi(E)$ belongs to the class $B_\varphi(S)$ if it satisfies the *Reverse Jensen Inequality*

$$⨍_Q \varphi(f) \le S\varphi\left(⨍_Q f\right) \quad \forall Q \in E$$

with

$$S = S(\varphi, f, E) = \sup_{Q \subset E} \frac{⨍_Q \varphi(f)}{\varphi\left(⨍_Q f\right)} < \infty$$

where the supremum is taken over all cubes $Q \subset E$.
It's easy to show that for $\varphi_G(t) = t^q (q > 1)$ we have

$$B_{\varphi_G}(S) = G_q(S^{1/q}) = B_1^q(S^{1/q})$$

and for $\varphi_M(t) = t^{p/(1-p)} (p > 1)$ we have

$$B_{\varphi_M}(S) = A_p(S^{p-1}) = B^1_{\frac{1}{p-1}}(S^{p-1}).$$

Let us again restrict ourself to functions of one real variable, and let E be an interval. In [6] Korenovskii proved the following

Theorem 3.2. *For $\varphi \in \Phi$ and $f \in B_\varphi(S)$ we have*

$$S(\varphi, f_*, [0, |E|]) = S(\varphi, f^*, [0, |E|]) \le S(\varphi, f, E)$$

where f_* and f^* are, respectively, the nondecreasing and nonincreasing rearrangements of f.

We are now able to prove our main result.

Proof of Theorem 1.3. Let $f \in B_p^q(K)$ so, for all $J = (a, b) \subset E$

$$\left(\fint_J f^q\right)^{1/q} \le K\left(\fint_J f^p\right)^{1/p} \tag{3.1}$$

and let us first suppose $p > 0$. If we set $g = f^p$, (3.1) can be written as

$$\left(\fint_J g^{q/p}\right) \le K^q\left(\fint_J g\right)^{q/p}. \tag{3.2}$$

Let us introduce the function $\varphi(t) = t^{q/p}$; for our assumptions $\varphi \in \Phi$ so (3.2) means that $g \in B_\varphi(K^q)$. Then, applying Theorem 3.2 to the nonincreasing rearrangement g^* we have

$$\fint_0^{|J|} (g^*)^{q/p} \le K^q\left(\fint_0^{|J|} g^*\right)^{q/p} \tag{3.3}$$

that implies $g^* \in B_1^{\frac{q}{p}}(K^p)$.
Hence we can invoke Theorem 3.1 and say that there exists $q_0 > q/p$ such that $g^* \in L^s$ for all $s \in [q/p, q_0)$ where q_0 is the solution of equation

$$\left(\frac{y}{y - q/p}\right)^{p/q} = K^p\left(\frac{y}{y-1}\right). \tag{3.4}$$

Then if we put $y = x/p$, equation (3.4) becomes

$$\left(\frac{x}{x-q}\right)^{1/q} = K\left(\frac{x}{x-p}\right)^{1/p}. \tag{$E3$}$$

Therefore, if $x_0 = q_0 p$ is the root of (E3), we proved that $g^* \in L^{s/p}$ for all $s/p \in [q/p, x_0/p)$, namely for all $s \in [q, x_0)$. Finally since

$$g^* \in L^{s/p} \Rightarrow g \in L^{s/p} \Rightarrow f^p \in L^{s/p} \Rightarrow f \in L^s$$

theorem is proved for $p > 0$.
Now let $p < 0$; from (3.1) we have

$$\left(\fint_J f^p\right) \le K^{-p}\left(\fint_J f^q\right)^{p/q}$$

and, for $g = f^q$,

$$\left(\fint_J g^{p/q}\right) \le K^{-p}\left(\fint_J g\right)^{p/q}. \tag{3.5}$$

If we define $\psi(t) = t^{p/q}$, (3.5) means that $g \in B_\psi(K^{-p})$. Applying again Theorem 3.2 to the nondecreasing rearrangement g_* of g we deduce that $g_* \in B^1_{\frac{p}{q}}(K^q)$. As before, from Theorem 3.1, there exists $p_0 < p/q$ such that $g_* \in L^s$ for all $s \in (p_0, p/q]$ with p_0 root of equation

$$\left(\frac{y}{y-1}\right) = K^q\left(\frac{y}{y-p/q}\right)^{q/p}. \tag{3.6}$$

For $y = x/q$, we get again equation (E3).
Therefore, $g_* \in L^{s/q}$ and, by the same arguments used in the previous case, we can conclude that $f \in L^s$ for any $s \in (x_0, p]$ where $x_0 = p_0 q$. □

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