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COMMON BLOCKS FOR $ASQS(12)$

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An ASQS(v) is a particular Steiner system featuring a set of v vertices and two separate families of blocks, $\mathcal{B}$ and $\mathcal{G}$, whose elements have a respective cardinality of 4 and 6. It has the property that any three vertices of X belong either to a B-block or to a G-block. The parameter cb is the number of common blocks in two separate ASQSs, both defined on the same set of vertices X. In this paper it is shown that $cb \leq 29$ for any pair of ASQSs(12).

1. Introduction.

An *Atypical Steiner Quadruple System* (ASQS) [2] is a Steiner system defined by the triple $(X, \mathcal{G}, \mathcal{B})$ where X is a finite set of v points called *vertices*, $\mathcal{G}$ is a family of subsets of X called *G-blocks* with a cardinality of 6, which partitions X, and $\mathcal{B}$ is a family of subsets of X whose elements all have a cardinality of 4 and are called *B-blocks*. In this system each triple of vertices of X is contained either in a G-block or in a B-block. In general, if there is no need to specify whether an element belongs to $\mathcal{G}$ or $\mathcal{B}$ it will be called a *block*.

The number of G-blocks in the system is $|\mathcal{G}| = \frac{v}{6}$, while the number of B-blocks is $|\mathcal{B}| = \frac{1}{4}\left[\binom{v}{3} - 20 \cdot \frac{v}{6}\right]$.

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For classical SQS(v)s the authors of [3] posed the problem of determining the number of possible blocks in common between two separate SQSs defined on the same set of vertices. In this paper a similar problem is dealt with for ASQS(12) and for each pair of such systems it is determined that the number of blocks in common is $cb \leq 29$.

2. Preliminary Results.

2.1. 1-factorizations.

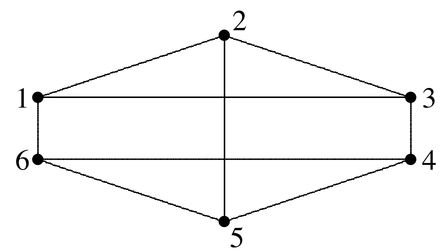
It has been proved that all the 1-factorizations of K_6 are isomorphous [5]. From a 1-factorization, $\mathcal{E}$, of K_6 it is possible to obtain another 1-factorization, $\mathcal{F}$, by permutation of the 1-factors of $\mathcal{E}$, in which case we say that $\mathcal{E}$ and $\mathcal{F}$ are *different by permutation*. If two 1-factorizations are different, but not by permutation, they are said to be *strict sense different*.

Lemma 1. *In a 1-factorization of K_6 any two 1-factors define a circuit with a length of 6 in K_6.*

Proof. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_5\}$ be a 1-factorization of K_6 and F_i and F_j two 1-factors of $\mathcal{F}$; if we remove from K_6 all the edges not belonging to F_i and F_j we obtain a regular graph of order two and, by Petersen's theorem [1] it can be made up either of two K_3's or of a circuit with a length of 6. A graph made up of two K_3's cannot contain two 1-factors and the lemma is proved. □

Lemma 2. *If two different 1-factorizations $\mathcal{E}$ and $\mathcal{F}$ of K_6 have two equal 1-factors, then if they are different they are so by permutation.*

Proof. Let F_i and F_j be the two 1-factors present in $\mathcal{E}$ and $\mathcal{F}$; by the previous lemma the graph obtained from K_6 by removing the edges present in F_i and F_j is as follows:



In it there are three 1-factors and it can easily be seen that the edges (1,2), (2,3), (4,5) and (5,6) belong to a single 1-factor. □

2.2. Doubling Construction.

An ASQS = $(X, \mathcal{G}, \mathcal{B})$ where $X = 2v$ can be obtained from the two systems $ASQS_1 = (X_1, \mathcal{G}_1, \mathcal{B}_1)$ and $ASQS_2 = (X_2, \mathcal{G}_2, \mathcal{B}_2)$, where $|X_1| = |X_2| = v$ with $v = 2h$, $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = \emptyset$ by means of the following double construction.

Let $\mathcal{F} = \{F_1, F_2, \ldots, F_{v-1}\}$ and $\mathcal{E} = \{E_1, E_2, \ldots, E_{v-1}\}$ be two 1-factorizations of K_v on the sets of vertices X_1 and X_2 and α a permutation of the set $\{1, 2, \ldots, v-1\}$.

Let $\mathcal{B}_0$ be the family of B-blocks thus defined: two vertices $(x_1, x_2) \in \mathcal{F}_i$ belong to the B-block of $\mathcal{B}_0$ $\{x_1, x_2, y_l, y_m\}$ if and only if the pair $(y_l, y_m) \in E_j$ and $i\alpha = j$.

It can easily be shown that the triple $(X, \mathcal{G}, \mathcal{B})$, where $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ and $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$ is an ASQS$(2v)$.

3. Common Blocks.

Let us consider two generic systems, $ASQS_1 = (X, \mathcal{G}_1, \mathcal{B}_1)$ and $ASQS_2 = (X, \mathcal{G}_2, \mathcal{B}_2)$.

Definition 1. *The two systems $ASQS_1$ and $ASQS_2$ are said to be different if there exist in them blocks that are not in common.*

Definition 2. *Two different systems $ASQS_1$ and $ASQS_2$ are said to be of the first species if $\mathcal{G}_1 = \mathcal{G}_2$, of the second species if the opposite is true.*

Definition 3. *Let $ASQS_1$ and $ASQS_2$ be two different systems of the second species, both defined on a set X of 12 vertices, and let g_i and g_j be any two G-blocks belonging respectively to $\mathcal{G}_1$ and $\mathcal{G}_2$. We will then say that the two systems are of the first type if g_i and g_j have one or five vertices in common, of the second type if they have two or four vertices in common, and of the third type if they have three vertices in common.*

For two separate ASQSs - $ASQS_1$ and $ASQS_2$ - let us indicate the number of blocks they have in common as cb and try to establish the maximum value cb can take when pairs of ASQS(12)s are considered.

For two systems $ASQS_1$ and $ASQS_2$ with 12 vertices we can make the following observations:

1. If $\mathcal{G}_1 \neq \mathcal{G}_2$ the two systems cannot have G-blocks in common;
2. if g_{11} and g_{12} belong to the $\mathcal{G}_1$ of $ASQS_1$, in each B-block of B_1 there will be two vertices of g_{11} and two of g_{12};

3. if $g_{11} = \{x, y, k, z, t, w\}$ then each pair of g_{11} is in three B-blocks and in these blocks the vertices of g_{12} define a 1-factor of K_6 on the vertices of g_{12}; in addition, the pairs (x, y), (x, k), (x, z), (x, t) and (x, w) define a 1-factorization of K_6 on the vertices of g_{12} (due to the symmetry of the system it is possible to repeat what was said above for the vertices of the G-block g_{12});
4. the three types of different systems of the second species identified by Definition 3 are the only three possible.

Proposition 1. *For two different systems $ASQS_1$ and $ASQS_2$, of the first type of the second species, $cb \leq 25$ and this inequality is the best possible.*

Proof. $ASQS_1$ and $ASQS_2$ are two systems of the first type of the second species with $\mathscr{G}_1 = \{g_{11}, g_{12}\}$ and $\mathscr{G}_2 = \{g_{21}, g_{22}\}$; it is therefore possible to determine an $x \in g_{11}$ and a $y \in g_{12}$ such that $g_{21} = (g_{11} - \{x\}) \cup \{y\}$ and $g_{22} = (g_{12} - \{y\}) \cup \{x\}$. Then if $g_{21} = \{y, p, t, k, z, w\}$, each pair of vertices of g_{21} which does not contain y can be in two B-blocks common to $ASQS_1$ and $ASQS_2$, while each pair of vertices of g_{21} containing y can at most be in one common B-block. Therefore, as there are 15 different pairs of vertices in g_{21} and only 5 contain y, $cb \leq 25$. This limit is the best possible; if, in fact, we consider a system ASQS = $(X, \mathscr{G}, \mathscr{B})$ with $\mathscr{G} = \{g_1, g_2\}$ and we choose an $x \in g_1$ and a $y \in g_2$, then the system $ASQS'$ obtained from ASQS by exchanging x with y and y with x in the blocks where these vertices are present, has 25 blocks in common with ASQS. $\square$

With analogous demonstration it is possible to prove that:

Proposition 2. *If two different systems $ASQS_1$ and $ASQS_2$ of the second species are of the second or third type, $cb \leq 27$.*

Unlike Proposition 1 this upper limit for cb is not the best possible.

Let us now consider two different systems $ASQS_1$ and $ASQS_2$ of the first species. They have two G-blocks in common. If $\mathcal{C}B$ is the set of B-blocks in common, the two systems $(X, \mathscr{B}_1 - \mathcal{C}B)$ and $(X, \mathscr{B}_2 - \mathcal{C}B)$ are two *disjoint mutually balanced* (DMB) PQSs, i.e. two partial systems of quadruples in which if a triple of vertices is in one block of one of the two systems it will also be in one block of the other system and, in addition, the two systems do not have blocks in common. From the results given in [3], *cb* can take the following values: $\{0, 1, 2, \ldots, 32, 33, 35, 39\}$.

Proposition 3. *For two different systems $ASQS_1$ and $ASQS_2$ of the first species, $cb \leq 29$ and this inequality is the best possible.*

Proof. Let $\mathcal{G}_1 = \{g_{11}, g_{12}\}$ and $\mathcal{G}_2 = \{g_{21}, g_{22}\}$ be the two G-blocks of the systems $ASQS_1$ and $ASQS_2$. First we make the following consideration: if a pair (x, y) with $x, y \in g_{11}$ (or $x, y \in g_{12}$) is in one block not common to the two systems, then it will be in at least two non-common blocks. If, in fact, we assume this to be untrue, it will be in three common blocks, which would be absurd.

Let us consider the two PQS systems $\mathcal{M}_1 = (X, \mathcal{B}_1 - \mathcal{C}B)$ and $\mathcal{M}_2 = (X, \mathcal{B}_2 - \mathcal{C}B)$. They are DMB and by Theorem 2.7 in [3] the number of blocks present in these systems, if $|\mathcal{C}B| \geq 29$ is 8, 12, 14, 15, 16, or 17.

Let us assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ have 8 or 14 or 16 or 17 blocks. As these integers are not multiples of three, in $g_{11} = g_{21}$ (or $g_{12} = g_{22}$) there exists a pair (x, y) which is only present in two blocks of $\mathcal{M}_1$ and $\mathcal{M}_2$. If $g_{11} = g_{21} = \{x, y, k, z, t, w\}$ (or $g_{12} = g_{22} = \{x, y, k, z, t, w\}$), then (x, y) identifies in g_{12} (g_{11}) a 1-factor in $ASQS_1$ and a 1-factor in $ASQS_2$ which only have one edge in common. This implies that the pairs (x, y), (x, k), (x, z), (x, t), (x, w), by Lemma 2, identify in $ASQS_1$ and $ASQS_2$ two strict sense different 1-factorizations which can have at most one 1-factor in common and therefore in g_{12} (or g_{11}) we can count at least 8 different pairs present in at least two blocks of $\mathcal{M}_1$ and $\mathcal{M}_2$, and so these systems cannot have 8 or 14 blocks.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ have 12 or 15 blocks, then each pair of g_{11} and g_{12} present in one block of $\mathcal{M}_1$ and $\mathcal{M}_2$ is present in three blocks. The only DMB PQSs with 12 and 15 blocks were determined in [4] and it can easily be seen that these systems do not have the same features.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ have 16 blocks, as 16 cannot be divided by 3 a pair (x, y) of g_{11} (or g_{12}) is only present in two blocks. So, according to what was said previously, there exist exactly 8 different pairs in g_{11} and g_{12} each only present in two blocks. If $g_{11} = \{x, y, k, t, w, z\}$ and the pair (x, y) is in two non-common blocks there are in g_{11} (or g_{12}) at least 4 different pairs in which x is present, three different pairs in which y is present and two in which k or t or w or z is present. These nine pairs are all different from each other and have the property of being present in two blocks. This is absurd as there are 16 blocks in $\mathcal{M}_1$ and $\mathcal{M}_2$.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ have 17 blocks, there must be at least one pair of g_{11} (or g_{12}) present in two blocks and one and only one pair (x, y) present in three non-common blocks. This implies that the 1-factorizations of g_{12} (or g_{11}) identified by the pairs (x, y), (x, t), (x, k), (x, z), (x, w) in $ASQS_1$ and $ASQS_2$ are strict sense different and in g_{12} (or g_{11}) there are at least nine different pairs present in non-common blocks, which would be absurd.

Therefore $cb \leq 29$ and this inequality is the best possible. It is sufficient, in fact, to consider a system, $ASQS'$, obtained from a double construction which uses the 1-factorization $\mathcal{E}$. If two 1-factors of $\mathcal{E}$ are exchanged, a new ASQS - $ASQS''$ - is obtained which has exactly 29 blocks in common with $ASQS'$. $\square$

From the last three propositions we get:

Theorem 1. *For two different systems $ASQS_1$ and $ASQS_2$, $cb \leq 29$.*

REFERENCES

[1] C. Berge, *Hypergraph: combinatories of finite sets,* North Holland, 1989.

[2] M. C. Di Domenico, *Su atipici sistemi di Steiner: ASTS, ASQS,* Thesis, (1995).

[3] M. Gionfriddo - C. C. Lindner, *Construction of Steiner quadruple systems having a prescribed number of blocks in common,* Discrete Mathematics, 34 (1981), pp. 31–42.

[4] M. Gionfriddo, *Construction of all disjoint and mutually balanced partial quadruple systems with 12, 14 or 15 blocks,* Rend. Sem. Brescia, 7 (1984), pp. 343–353.

[5] E. Mendelsohn - A. Rosa, *One-factorization of the complete graph – a survey,* Journal of Graph Theory, 9 (1985), pp. 43–65.

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