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GENERALIZED SET-VALUED VARIATIONAL INEQUALITIES

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In this paper, we introduce and study a new class of variational inequalities, which is called generalized set-valued variational inequality. The projection technique is used to establish the equivalence among generalized set-valued variational inequalities, fixed point problems and generalized set-valued Wiener-Hopf equations. This equivalence is used to study the existence of a solution of set- valued variational inequalities and to suggest a number of iterative algorithms for solving variational inequalities. We also consider the auxiliary principle technique to study the existence of a solution of the generalized set-valued variational inequalities and to suggest a general and novel iterative algorithm. In addition, we have shown that the auxiliary principle technique can be used to find the equivalent differentiable optimization problem for the generalized set-valued variational inequalities. The results proved in this paper represent a significant refinement and improvement of the previous results.

1. Introduction.

One of the most important developments in applicable mathematics over the last few decades has been the emergence of the theory of variational inequalities, which constitutes a significant and important extension of the

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calculus of variations, the origin of which can be traced back to Fermat, Newton, Leibniz, Bernoulli, Euler and Lagrange. This theory provides us with a simple, natural, unified and general frame to study a wide class of unrelated linear and nonlinear problems arising in fluid flow through porous media, elasticity, transportation, economics, operations research, optimization, regional, physical and applied sciences, see [1] – [46] and the references therein. The ideas and techniques of variational inequalities are being applied in a variety of diverse fields and proved to be productive and innovative.

Inspired and motivated by the research going on in these fields, Noor [22] introduced a new class of variational inequalities, known as the general strongly nonlinear variational inequality, which enabled him to study the odd-order and nonsymmetric obstacle, contact, unilateral, free and moving problems, see [24], [25], [26], [29]. On the other hand, Fang and Peterson [9] considered the multivalued version of the variational inequality problem. In this paper, we introduce and study a new class of variational inequalities which unifies both these problems. This new class has many important and significant applications in various branches of pure and applied sciences. Equally important is the area of mathematical sciences known as the Wiener-Hopf equations or normal maps, which was introduced by Shi [41], [42] and Robinson [38] independently in different settings. Shi [41] and Robinson [38] also established the equivalence between the variational inequalities and the Wiener-Hopf equations using essentially the projection technique. The Wiener-Hopf equations (normal maps) techniques are being used to develop powerful and efficient numerical techniques for solving variational inequalities and the complementarity problems, see [20], [25], [28], [31], [32], [34], [35], [37] – [42], [44] and the references therein. Noor [28] has modified and generalized the Wiener-Hopf equations technique to suggest and analyze a number of new iterative algorithms for various classes of variational and quasi variational inequalities. Recently Robinson [29] and Noor [31], [32] used this technique to study the sensitivity analysis of variational inequalities via different methods. The Wiener-Hopf equations technique provides a simple and convenient device for formulating a wide variety of important problems from applications in a single and unified manner. For related work on the Wiener-Hopf equations, see [44].

In recent years, considerable interest has been shown in developing various extensions and generalizations of variational inequalities and the Wiener-Hopf equations, both for their own sake and for their applications. There are significant developments of these problems related to multivalued operators, nonconvex optimization, iterative methods and structural analysis. Inspired and motivated by the recent research work going on in these fields, we introduce and study a new class of variational inequalities, which is called the generalized set-valued variational inequality. This class is the most general and includes the previously studied classes of variational inequalities as special cases. This class has important applications in structural analysis and optimization theory. In particular, we show that if the nonsmooth and nonconvex superpotential of the structure is quasidifferentiable, then these problems can be studied via the generalized multivalued variational inequalities. In this formulation, the ascending and descending branches of non- monotone multivalued and boundary conditions are considered separately. The solution of the multivalued variational inequalities gives the position of the state equilibrium of the structure. For the formulation and applications of the generalized set-valued variational inequalities, see the references.

Using essentially the projection technique and its variant forms, we establish the equivalence between generalized set-valued variational inequalities, and fixed points; and Wiener-Hopf equations. These alternate formulations are used to prove the existence of a solution of generalized set-valued variational inequalities as well as to analyze a number of iterative algorithms. We remark that the scope of the iterative projection algorithms is limited. In some cases, one cannot even find the projection of the solution. In these cases, we used the auxiliary principle technique of Noor [21], [23], [24], [26], [27], [29] and Glowinski, Lions and Tremolieres [11] to study the existence of a solution of the generalized set-valued variational inequalities. This technique deals with the auxiliary variational inequality and proves that the solution of the auxiliary problem is the solution of the original generalized set-valued variational inequality. This technique also enables to suggest a novel and general iterative algorithm for computing the approximate solution of the variational inequality and related complementarity problems. Furthermore, we also show that the auxiliary principle technique can be used to find the equivalent differentiable optimization problems for the generalized set-valued variational inequalities. These equivalent differentiable optimization problems can be used to suggest general descent and Newton methods with line search to solve the generalized set-valued variational inequalities and complementarity problems.

In Section 2, we formulate the variational inequality and review some basic results. In Section 3, we establish the equivalence between the set-valued variational inequalities, the fixed points and the Wiener-Hopf equations. This equivalence is used to study the existence of a solution of the set-valued variational inequalities. The auxiliary principle technique is also discussed. In Section 4, we suggest a number of iterative algorithms for solving the set-valued variational inequalities.

2. Preliminaries.

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot\,, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. Let 2^H be the family of all nonempty compact subsets of H.

For given multivalued operators $T, V : H \to 2^H$, single-valued operators $g : H \to H$,and $N : H \times H \to H$, we consider the problem of finding $u \in H$ such that $\omega \in T(u)$, $y \in V(u)$, $g(u) \in K$ and

$$\langle N(w, y), g(v) - g(u)\rangle \geq 0, \qquad \text{for all } g(v) \in K, \tag{2.1}$$

where $v \in H$. The inequality of type (2.1) is called the generalized set-valued variational inequality and has many important and potential applications in mechanics, elasticity, fluid flow through porous media, oceanography, pure and applied sciences. Furthermore, there are problems arising in structural analysis, which can be studied only by the variational inequality (2.1).

Example 2.1. For simplicity and to convey an idea of the applications of the multivalued variational inequality (2.1), we consider an elastoplasticity problem, which is mainly due to Panagiotopoulos and Stavroulakis [36]. For simplicity, it is assumed that a general hyperelastic material law holds for the elastic behaviour of the elastoplastic material under consideration. For the basic definitions and concepts, see [36]. Let us assume the decomposition

$$E = E^e + E^p, \tag{2.2}$$

where E^e denotes the elastic and E^p, the plastic deformation of the three-dimensional elasto-plastic body. We write the complementary virtual work expression for the body in the form

$$\langle E^e, \tau - \sigma\rangle + \langle E^p, \tau - \sigma\rangle = \langle f, \tau - \sigma\rangle, \qquad \text{for all } \tau \in Z. \tag{2.3}$$

Here we have assumed that the body on a part Γ_U of its boundary has given displacements, that is, $\mu_i = U_i$ on Γ_U and that on the rest of its boundary $\Gamma_F = \Gamma - \Gamma_U$, the boundary tractions are given, that is, $S_i = F_i$ on Γ_F, where

$$\langle E, \sigma\rangle = \int_\Omega \varepsilon_{ij}\sigma_{ij}\, d\Omega \tag{2.4}$$

$$\langle f, \sigma\rangle = \int_{\Gamma_U} U_i S_i\, d\Gamma \tag{2.5}$$

$$Z = \Big\{\tau : \tau_{i_j, j} + f_i = 0 \text{ on } \Omega, i, j = 1, 2, 3, \qquad T_i = F_i \text{ on } \Gamma_F, i = 1, 2, 3\Big\}, \tag{2.6}$$

is the set of statically admissible stresses and Ω is the structure of the body. Let us assume that the material of the structure Ω is hyperelastic such that

$$\langle E^e, \tau - \sigma\rangle \leq \langle W_m'(\sigma), \tau - \sigma\rangle, \quad \text{for all } \tau \in \mathbb{R}^6, \tag{2.7}$$

where W_m is the superpotential which produces the constitutive law of the hyperelastic material and is assumed to be quasidifferentiable [36], that is, there exist convex and compact subsets $\underline{\partial}' W_m$ and $\overline{\partial}' W_m$ such that

$$\langle W_m'(\sigma), \tau - \sigma\rangle = \max_{W_1^e \in \underline{\partial}' W_m} \langle W_1^e, \tau - \sigma\rangle + \\ + \min_{W_2^e \in \overline{\partial}' W_m} \langle W_2^e, \tau - \sigma\rangle. \tag{2.8}$$

We also introduce the generally nonconvex yield function $P \subset Z$, which is defined by means of the general quasidifferentiable function $F(\sigma)$, that is,

$$P = \{\sigma \in Z;\ F(\sigma) \leq 0\}. \tag{2.9}$$

Here W_m is a generally nonconvex and nonsmooth, but quasidifferentiable function for the case of plasticity with convex yield surface and hyperelasticity. Combining (2.2) – (2.9), Panagiotopoulos and Stavroulakis [36] have obtained the following multivalued variational inequality problem:
Find $\sigma \in P$ such that $W_1^e \in \underline{\partial}' W_m(\sigma)$, $W_2^e \in \overline{\partial}' W_m(\sigma)$ and

$$\langle W_1^e + W_2^e, \tau - \sigma\rangle \geq \langle f, \tau - \sigma\rangle, \quad \text{for all } \tau \in P,$$

which is exactly the problem (2.1), with $N(w, y) = W_1^e + W_2^e$, $g = I$,

$$T(u) = \underline{\partial}' W_m(\sigma),\ V(u) = \overline{\partial}' W_m(\sigma), \quad \text{and} \quad K = P.$$

For other applications of the set-valued variational inequalities in mechanics, structural engineering and economics, see [7].

Special Cases.

I. If $V \equiv I$, the identity operator, $T, A, g : H \to H$ are single-valued operators and $N(w, y) = Tu + A(u)$, then problem (2.1) reduces to finding $u \in H$ such that $g(u) \in K$ and

$$\langle Tu + A(u), g(v) - g(u)\rangle \geq 0, \qquad \text{for all } g(v) \in K, \tag{2.10}$$

which is called the general strongly nonlinear variational inequality problem considered and studied by Noor [22]. For recent applications, formulation, iterative methods and sensitivity analysis, see for example [24] – [29]. It has been shown in [24], [25] that the odd-order obstacle, unilateral, contact and free boundary problems arising in oceanography, elasticity, fluid flow through porous media and mechanics can be studied via the general strongly nonlinear variational inequality technique.

II. If $g, V \equiv I$, the identity operators, $T, A : H \to H$ are nonlinear operators and $N(w, y) = Tu + A(u)$, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu + A(u), v - u \rangle \geq 0, \qquad \text{for all } v \in K. \tag{2.11}$$

The inequality of the type (2.11) is known as the strongly (mildly) nonlinear variational inequality, which is mainly due to Noor [18], [19]. For the physical formulation, applications, generalizations, numerical methods and sensitivity analysis, see, for example [4], [8], [13], [20] – [30].

III. If $K^* = \{u \in H : \langle u, v \rangle \geq 0 \text{ for all } v \in K\}$ is a polar cone of the convex cone in H, then problem (2.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in V(u)$, $g(u) \in K$ such that

$$N(w, y) \in K^*, \qquad \langle N(w, y), g(u) \rangle = 0, \tag{2.12}$$

which is known as the set-valued complementarity problem and appears to be a new one.

IV. If $V \equiv 0$, the identity operator and $T : H \to H$ is a single-valued operator, then problem (2.1) collapses to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \qquad \text{for all } v \in K, \tag{2.13}$$

which is known as the classical variational inequality problem, and is originally due to Stampacchia [45]. For recent applications, generalizations and numerical techniques, see [1] – [46] and the references therein.

For a suitable choice of operators T, A, g, V and convex set K, one can obtain various classes of variational inequalities and complementarity problems as special cases of the problem (2.1). In addition, problem (2.1) also enables to study a wide number of problems arising in regional, physical, mathematical, and engineering sciences in a unified framework, see [7].

Lemma 2.1 ([2], [14])**.** *Let K be a closed convex set in H. Then, for a given $z \in H$, $u = P_K z$, if and only if $u \in K$ satisfies*

$$\langle u - z, v - u\rangle \geq 0, \qquad \textit{for all } v \in K,$$

where P_K is the projection of H into K.

Definition 2.1. *For all $u_1, u_2 \in H$, the operator $N(\cdot\,,\cdot)$ is said to be strongly monotone and Lipschitz continuous with respect to the first argument, if there exist constants $\alpha > 0$, $\beta > 0$ such that*

$$\langle N(w_1, \cdot) - N(w_2, \cdot), u_1 - u_2\rangle \geq \alpha\|u_1 - u_2\|^2,$$

for all $w_1 \in T(u_1)$, $w_2 \in T(u_2)$

$$\|N(u_1, \cdot) - N(u_2, \cdot)\| \leq \beta\|u_1 - u_2\|.$$

In a similar way, we can define the strongly monotonicity and Lipschitz continuity of the operator $N(\cdot\,,\cdot)$ with respect to the second argument.

Definition 2.2. *The set-valued operator $V : H \to C(H)$ is said to be M-Lipschitz continuous, if there exists a constant $\xi > 0$ such that*

$$M(V(u), V(v)) \leq \xi\|u - v\|, \quad \textit{for all } u, v \in H,$$

where $C(H)$ is the family of all nonempty compact subsets of H and $M(\cdot\,,\cdot)$ is the Hausdorff metric on $C(H)$.

3. Equivalence and existence theory.

In this section, we use the projection technique to establish the equivalence between generalized set-valued variational inequality (2.1) and generalized fixed point problem. This equivalence is used to prove the existence of a solution of the problem (2.1). For this purpose, we need the following result, which can be proved by invoking Lemma 2.1.

Lemma 3.1. *Let K be a closed nonempty convex set in H. Then (u, w, y) is a solution of* (2.1) *if and only if (u, w, y) satisfies the relation*

$$g(u) = P_K[g(u) - \rho N(w, y)], \tag{3.1}$$

where $\rho > 0$ is a constant and P_K is the projection of H into K.

Lemma 3.1 implies that the generalized set-valued variational inequality (2.1) is equivalent to the fixed point problem. This equivalent formulation of the generalized set-valued variational inequality (2.1) is very important from both theoretical and numerical analysis point of views. This formulation enables us not only to study the existence of a solution of the set-valued variational inequality (2.1), but also to develop iterative algorithms for computing the approximate solutions of various classes of variational inequalities and complementarity problems.

Theorem 3.1. *Let K be a closed convex set in H. Let the operator $N(.,.)$ be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$ with respect to the first argument. Let the single-valued operator $g : H \to H$ be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$. Assume that the operator $N(.,.)$ is Lipschitz continuous with constant $\eta > 0$ with respect to the second argument and V is M-Lipschitz continuous with constant $\xi > 0$. Let $T : H \to 2^H$ be a M-Lipschitz continuous with constant $\mu > 0$. If $\gamma < \alpha$, where γ is defined by* (3.11) *and*

$$\left|\rho - \frac{\alpha - (1-k)\eta\xi}{\beta^2\mu^2 - \eta^2\xi^2}\right| < \\ < \frac{\sqrt{[\alpha - (1-k)\eta\xi]^2 - k(\beta^2\mu^2 - \eta^2\xi^2)(2-k)}}{\beta^2\mu^2 - \eta^2\xi^2} \tag{3.2}$$

$$\alpha > (1-k)\eta\xi + \sqrt{k(\beta^2\mu^2 - \eta^2\xi^2)(2-k)} \tag{3.3}$$

$$\rho\eta\xi < 1 - k \tag{3.4}$$

$$k = 2\sqrt{1 - 2\sigma + \delta^2}, \tag{3.5}$$

then the generalized set-valued variational inequality (2.1) *has a unique solution $u \in H$ such that $w \in T(u)$, $y \in V(u)$ and $g(u) \in K$.*

(a) *Uniqueness*. Let $u_1, u_2 \in H$, $u_1 \neq u_2$ be two solutions of the variational inequality (2.1), then

$$\langle N(w_1, y_1), g(v) - g(u_1)\rangle \geq 0 \tag{3.6}$$

$$\langle N(w_2, y_2), g(v) - g(u_2)\rangle \geq 0. \tag{3.7}$$

Now taking $v = u_2$ in (3.6) and $v = u_1$ in (3.7) and adding the resultant inequalities, we have

$$\langle N(w_1, y_1) - N(w_2, y_2), g(u_1) - g(u_2)\rangle \leq 0,$$

which can be written as

$$\begin{aligned}&\langle N(w_1, y_1) - N(w_2, y_1), u_1 - u_2\rangle \leq \\ &\leq \langle N(w_1, y_1) - N(w_2, y_1), u_1 - u_2 - (g(u_1) - g(u_2))\rangle \\ &\qquad + \langle N(w_2, y_1) - N(w_2, y_2), g(u_1) - g(u_2)\rangle.\end{aligned}$$

Using the strongly monotonicity and Lipschitz continuity of the operator $N(.,.)$ with respect to the first argument, we have

$$\begin{aligned}&\alpha\|u_1 - u_2\|^2 \leq \\ &\leq \|N(w_1, y_1) - N(w_2, y_1)\|\|u_1 - u_2 - (g(u_1) - g(u_2))\| \\ &\quad +\|N(w_2, y_1) - N(w_2, y_2)\|\|g(u_1) - g(u_2)\| \\ &\quad \leq \beta\|w_1 - w_2\|\|u_2 - u_1 - (g(u_1) - g(u_2)\| \\ &\quad +\|N(w_2, y_1) - N(w_2, y_2)\|\|g(u_1) - g(u_2)\| \\ &\quad \leq \beta M(T(u_1), T(u_2))\|u_1 - u_2 - (g(u_1) - g(u_2))\| \\ &\quad +\|N(w_2, y_1) - N(w_2, y_2)\|\|g(u_1) - g(u_2)\| \\ &\quad \leq \mu\beta\|u_1 - u_2\|\|u_1 - u_2 - (g(u_1) - g(u_2)\| \\ &\quad +\|N(w_2, y_1) - N(w_2, y_2)\|\|g(u_1) - g(u_2)\|.\end{aligned} \tag{3.8}$$

Since $g : H \to H$ is strongly monotone Lipschitz continuous, so

$$\begin{aligned}&\|u_1 - u_2 - (g(u_1) - g(u_2))\|^2 = \|u_1 - u_2\|^2 \\ &- 2\langle g(u_1) - g(u_2), u_1 - u_2\rangle + \|g(u_1) - g(u_2)\|^2 \leq (1 - 2\sigma + \delta^2)\|u_1 - u_2\|^2.\end{aligned} \tag{3.9}$$

Using the Lipschitz continuity of the operator $N(.,.)$ with respect to the second argument and the M-Lipschitz continuity of V, we have

$$\begin{aligned}&\|N(w_2, y_1) - N(w_2, y_2)\| \leq \eta\|y_1 - y_2\| \leq \\ &\leq \eta M(V(u_1), V(u_2)) \leq \eta\xi\|u_1 - u_2\|.\end{aligned} \tag{3.10}$$

Combining (3.8), (3.9), (3.10) and using the Lipschitz continuity of the operator $g : H \to H$, we obtain

$$\alpha\|u_1 - u_2\|^2 \leq \{\beta\mu\sqrt{1 - 2\sigma + \delta^2} + \eta\xi\delta\}\|u_1 - u_2\|^2 = \gamma\|u_1 - u_2\|^2,$$

where

$$\gamma = \eta\xi\delta + \beta\mu\sqrt{1 - 2\sigma + \delta^2}. \tag{3.11}$$

Hence

$$(a - \gamma)\|u_1 - u_2\|^2 \leq 0,$$

which shows that $u_1 = u_2$, the uniqueness of the solution, since $\gamma < \alpha$.

(b) *Existence.* From Lemma 3.1, it follows that the generalized set-valued variational inequality (2.1) is equivalent to the fixed point problem

$$u = F(u) \equiv u - g(u) + P_K[g(u) - \rho N(w, y)]. \tag{3.12}$$

In order to prove the existence of a solution of (2.1), it is sufficient to show that the problem (3.12) has a fixed point. Thus, for all $u_1, u_2 \in H$, $u_1 \neq u_2$, we have

$$\begin{aligned}
\|F(u_1) - F(u_2)\| &= \|u_1 - u_2 - (g(u_1) - g(u_2)) + \\
&\quad + P_K[g(u_1) - \rho(N(w_1, y_1))] - P_K[g(u_2) - \rho(N(w_2, y_2))]\| \\
&\leq \|u_1 - u_2 - (g(u_1) - g(u_2))\| \\
&\quad + \|P_K[g(u_1) - \rho(N(w_1, y_1))] - P_K[g(u_2) - \rho(N(w_2, y_2))]\| \\
&\leq \|u_1 - u_2 - (g(u_1) - g(u_2))\| + \|g(u_1) - g(u_2) - \rho(N(w_1, y_1) - N(w_2, y_1))\| \\
&\quad + \rho\|N(w_2, y_1) - N(w_2, y_2)\| \\
&\leq 2\|u_1 - u_2 - (g(u_1) - g(u_2))\| + \|u_1 - u_2 - \rho(N(w_1, y_1) - N(w_2, y_1))\| \\
&\quad + \rho\|N(w_2, y_1) - N(w_2, y_2)\| \leq \left\{2\sqrt{1 - 2\sigma + \delta^2} + \rho\eta\xi\right\}\|u_1 - u_2\| \\
&\quad + \|u_1 - u_2 - \rho(N(w_1, y_1) - N(w_2, y_1))\|,
\end{aligned} \tag{3.13}$$

where we have used (3.9) and (3.10).

Since $N(\cdot, \cdot)$ is a strongly monotone Lipschitz continuous operator with respect to the first argument, so

$$\begin{aligned}
\|u_1 - u_2 - \rho(N(w_1, y_1) - N(w_2, y_1))\|^2 &= \\
= \|u_1 - u_2\|^2 - 2\rho\langle N(w_1, y_1) - N(w_2, y_1), u_1 - u_2\rangle \\
+ \rho^2\|N(w_1, y_1) - N(w_2, y_1)\|^2 \leq (1 - 2\rho\alpha + \rho^2\beta^2\mu^2)\|u_1 - u_2\|^2.
\end{aligned} \tag{3.14}$$

From (3.13) and (3.14), we have

$$\begin{aligned}
\|F(u_1) - F(u_2)\| &\leq \left\{2\sqrt{1 - 2\sigma + \delta^2} + \rho\eta\xi + \right. \\
&\quad \left. + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}\right\}\|u_1 - u_2\| \\
&= \{k + \rho\eta\xi + t(\rho)\}\|u_1 - u_2\| = \theta\|u_1 - u_2\|,
\end{aligned}$$

where

$$\theta = k + \rho\eta\xi + t(\rho) \tag{3.15}$$

$$k = 2\sqrt{1 - 2\sigma + \delta^2}$$

$$t(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}. \tag{3.16}$$

From (3.2) – (3.5), it follows that $\theta < 1$, so the map $F(u)$ defined by (3.12) has a fixed point $u \in H$ such that $w \in T(u)$, $y \in V(u)$ satisfying the generalized set-valued variational inequality (2.1). This completes the proof. $\square$

Related to generalized set-valued variational inequality (2.1), we consider the problem of solving generalized set-valued Wiener-Hopf equations. Let P_K be the projection of H into the convex set K and $Q_K \equiv I - P_K$, where I is the identity operator.

Given multivalued operators $T, V : H \to 2^H$, single-valued operators $g : H \to H$, and $N : H \times H \to H$, consider the problem of finding $z, u \in H$ such that $w \in T(u)$, $y \in V(u)$ and

$$N(w, y) + \rho^{-1} Q_K z = 0. \tag{3.17}$$

The equations of type (3.17) are called the generalized set-valued Wiener-Hopf equations, which were introduced by Noor [28] related to the generalized multivalued variational inequalities. For the general treatment, formulation and applications of the Wiener-Hopf equations, see [25], [28], [29], [37], [39] and the references therein.

Using Lemma 2.1, 3.1 and the techniques of Shi [41], [42] and Noor [20], [25], [28], we prove the following result.

Theorem 3.2. *The generalized set-valued variational inequality* (2.1) *has a solution* $u \in H$ *such that* $w \in T(u)$, $y \in V(u)$, $g(u) \in K$, *if and only if, the generalized set-valued Wiener-Hopf equations* (3.17) *have a solution* $z, u \in H$ *such that* $w \in T(u)$, $y \in V(u)$, *where*

$$g(u) = P_K z, \tag{3.18}$$

and

$$z = g(u) - \rho N(w, y). \tag{3.19}$$

Proof. Let $u \in H$ such that $w \in T(u)$, $y \in V(u)$, and $g(u) \in K$ be a solution of (2.1). Then by Lemmas 2.1 and 3.1, we have

$$g(u) = P_K[g(u) - \rho N(w, y)]. \tag{3.20}$$

Using the fact $Q_K \equiv I - P_K$ and equations (3.20), we obtain

$$\begin{aligned} Q_K[g(u) - \rho N(w, y)] = g(u) - \rho N(w, y) - \\ - P_K[g(u) - \rho N(w, y)] = -\rho N(w, y), \end{aligned}$$

which implies that

$$N(w, y) + \rho^{-1} Q_K z = 0, \quad \text{with } z = g(u) - \rho N(w, y),$$

the required (4.1).

Conversely, let $z \in H$ such that $w \in T(u)$, and $y \in V(u)$ be a solution of (3.17), then

$$\rho N(w, y) = -Q_K z = P_K z - z. \tag{3.21}$$

Now invoking Lemma 2.1 and (3.21), for all $v \in K$, we have

$$0 \le \langle P_K z - z, v - P_K z\rangle = \rho\langle N(w, y), v - P_K z\rangle.$$

Thus (u, w, y), where $u = g^{-1} P_K z$, is a solution of (2.1), the required result. □

We now consider another technique to study the existence of a solution of the variational inequality (2.1), which does not depend on the projection method. This technique is known as the auxiliary principle technique, which has been developed by Noor [21], [23], [24], [26] in recent years. This technique is being used to develop a variational formulation of the variational inequalities, which in turn enables us to develop a general frame of descent type algorithms, see, for example, [10], [23], [26] and the references therein.

The main and basic idea in the auxiliary principle technique is to consider an auxiliary variational inequality problem related to the given variational inequality. This formulation defines a mapping connecting the solutions of both these problems and one has to show that this mapping is a contraction mapping and consequently it has a fixed point, which is the solution of the original variational inequality problem. It is known that for a given problem, one can consider a number of auxiliary variational problems. The main advantage of this technique is that it enables us to prove the existence of a solution of various classes of variational inequalities as well as provides us a variational formulation of the variational inequalities. This alternate formulation allows us to develop a large number of numerical techniques for solving variational inequalities and related optimization problems. To be more precise, for a given $u \in H$, we consider the problem of finding a unique $z \in H$, $w \in T(u)$, $y \in V(u)$, $g(z) \in K$ satisfying the auxiliary variational inequality

$$\langle z, v - z\rangle \geq \langle u, v - z\rangle - \rho\langle N(w, y), g(v) - g(z)\rangle, \tag{3.22}$$

for all $g(v) \in K$, $v \in H$, where $\rho > 0$ is a constant.

From now onward, we assume that g is a linear operator. We note that the solution of the auxiliary variational inequality (3.22) is equivalent to finding the minimum of the functional $F[z]$ on K in H, where

$$F[z] = \frac{1}{2}\langle z - u, z - u\rangle + \rho\langle N(w, y), g(z) - g(u)\rangle, \tag{3.23}$$

which is called the auxiliary differentiable functional associated with the problem (3.22). Using the technique of Fukushima [10], one can prove that the generalized set-valued variational inequality (2.1) is equivalent to finding the minimum of the functional $N[u]$ on K in H, where

$$N[u] = \frac{1}{2}\langle u - z(u), z(u) - u\rangle + \rho\langle N(w, y), g(u) - g(z(u))\rangle, \tag{3.24}$$

where $z = z(u) \in H$ such that $w \in T(z(u))$, $y \in V(z(u))$, and $g(z(u)) \in K$ is the solution of the auxiliary variational inequality (3.22). The functional $N[u]$ defined by (3.23) is known as the gap (merit) function associated with the generalized set-valued variational inequality (2.1). These gap functions can be used to develop general framework for descent and Newton methods with line search for solving the generalized set-valued variational inequality (2.1) using the technique of Fukushima [10], Larsson and Patriksson [16] and Zhu and Marcotte [46].

Using the ideas and techniques of Noor [23], [24] and Cohen [5], we can propose and analyze a general algorithm. For a given $u \in H$, we introduce the following general auxiliary problem of finding the minimum of the functional $M[z]$ on K in H, where

$$\begin{aligned} M[z] = E(z) - E(u) - \langle E'(u), z - u\rangle + \\ + \rho\langle N(w, y), g(z) - g(u)\rangle. \end{aligned} \tag{3.25}$$

Here $E(z)$ is a differentiable convex function. Thus we can associate to (2.1), the equivalent optimization problem

$$\max\{M(z), z \in K\},$$

which is known as the variational principle in the terminology of Blum and Oettli [3]. One can easily show that the minimum of $M[z]$, defined by (3.24), on K can be characterized by a variational inequality of the type

$$\langle E(z), v - z\rangle \geq \langle E'(u), v - z\rangle - \rho\langle N(w, y), g(v) - g(z)\rangle, \tag{3.26}$$

for all $v \in H$ such that $g(v) \in K$.

It is clear that the auxiliary variational inequality (3.22) is a special case of (3.24). We also remark that if $z = u$, then z is a solution of the variational inequality (2.1). In many applications, the auxiliary variational inequalities of the type (3.22) and (3.24) occur, which do not arise as result of minimization problems. The main motivation of this section is to suggest a general auxiliary generalized set-valued variational inequality problem, which includes (3.22), (3.23) – (3.25) as special cases.

Auxiliary Principle 3.1. For a given $u \in H$, we consider the problem of finding $z \in H$ such that $w \in T(u)$, $y \in V(u)$, $g(z) \in K$ and

$$\langle B(z), v - z\rangle \geq \langle B(u), v - z\rangle - \rho\langle N(w, y), g(v) - g(z)\rangle, \tag{3.27}$$

for all $v \in H$ such that $g(v) \in K$ and $B : H \to H$ is a single-valued nonlinear operator.

We remark that if $z = u$, then z is a solution of the generalized set-valued variational inequality (2.1).

4. Iterative Algorithms.

In this section, we invoke Lemma 3.1 and Theorem 3.2 to suggest a number of iterative algorithms for solving generalized set-valued variational inequality (2.1) and its various special cases. From Lemma 3.1, it is clear that the variational inequality (2.1) is equivalent to the fixed point problem of the type

$$u = u - g(u) + P_K[g(u) - \rho N(w, y)],$$

which implies that

$$u = (1 - \lambda)u + \lambda\{u - g(u) + P_K[g(u) - \rho N(w, y)]\}, \tag{4.1}$$

where $0 < \lambda < 1$ is a parameter and $\rho > 0$ is a constant.

We use this fixed point formulation to suggest the following unified algorithm for the variational inequalities (2.1).

Algorithm 4.1. Let K be a nonempty closed convex set in H. Assume that $g : H \to H$, $V, T : H \to 2^H$ operators. For a given $u_0 \in H$, let us consider $w_0 \in T(u_0)$, $y_0 \in V(u_0)$,$g(u_0) \in K$ and

$$u_1 = (1-\lambda)u_0 + \lambda\big\{u_0 - g(u_0) + P_K[g(u_0) - \rho N(w_0, y_0)]\big\}.$$

Since $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, so there exist $w_1 \in T(u_1)$, $y_1 \in V(u_1)$ such that

$$\|w_0 - w_1\| \le M(T(u_0), T(u_1))$$

$$\|y_0 - y_1\| \le M(V(u_0), V(u_1)).$$

Let

$$u_2 = (1-\lambda)u_1 + \lambda\big\{u_1 - g(u_1) + P_K[g(u_1) - \rho N(w_1, y_1)]\big\}.$$

Continuing this way, we can obtain the sequences $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ such that

$$w_n \in T(u_n) : \|w_n - w_{n+1}\| \le M(T(u_n), T(u_{n+1})) \tag{4.2}$$

$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \le M(V(u_{n+1}), V(u_n)) \tag{4.3}$$

$$u_{n+1} = (1-\lambda)u_n + \lambda\{u_n - g(u_n) + P_K[g(u_n) - \rho N(w_n, y_n)]\}, \tag{4.4}$$

for $n = 0, 1, 2, \ldots$.

If $T, V : H \to 2^H$, and $g \equiv I$, the identity operator, then Algorithm 4.1 reduces to:

Algorithm 4.2. For a given $u_0 \in K$ such that $w_0 \in T(u_0)$, $y_0 \in V(u_0)$, compute $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ from the iterative schemes

$$w_n \in T(u_n) : \|w_n - w_{n+1}\| \le M(T(u_n), T(u_{n+1}))$$
$$y_n \in V(u_n) : \|y_{n+1} - y_n\| \le M(V(u_{n+1}), V(u_n))$$
$$u_{n+1} = (1-\lambda)u_n + \lambda P_K[u_n - \rho N(w_n, y_n)],$$

$n = 0, 1, 2, \ldots$.

Theorem 3.2 implies that generalized set-valued variational inequalities (2.1) and Wiener-Hopf equations (3.17) are equivalent. This equivalence is quite general and flexible. We use this equivalence to suggest a number of iterative algorithms for solving the generalized set-valued variational inequalities and the complementarity problems.

I. The equations (3.17) can be written as

$$Q_K z = -\rho N(w, y),$$

from which it implies that

$$\begin{aligned} z &= P_K z - \rho N(w, y) \\ &= g(u) - \rho N(w, y), \qquad \text{using (3.18).} \end{aligned}$$

This fixed point formulation enables us to suggest the following iterative method.

Algorithm 4.3. For given $z_0, u_0 \in H$ such that $w_0 \in T(u_0)$, and $y_0 \in V(u_0)$, compute the sequences $\{z_n\}$, $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ by the iterative schemes

$$\begin{aligned} u_n &= u_n - g(u_n) + P_K z_n \\ w_n &\in T(u_n) : \|w_n - w_{n+1}\| \leq M(T(u_n), T(u_{n+1})) \\ y_n &\in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n)) \\ z_{n+1} &= g(u_n) - \rho N(w_n, y_n), \end{aligned}$$

$n = 0, 1, 2, \ldots$.

II. The equations (3.17) may be written as

$$Q_K z = -N(w, y) + (1 - \rho^{-1}) Q_K z,$$

which implies that

$$\begin{aligned} z &= P_K z - N(w, y) + (1 - \rho^{-1}) Q_K z \\ &= g(u) - N(w, y) + (1 - \rho^{-1}) Q_K z, \qquad \text{using (3.18).} \end{aligned}$$

Using this fixed point formulation, we can suggest the following iterative scheme.

Algorithm 4.4. For given $z_0, u_0 \in H$ such that $w_0 \in T(u_0)$, and $y_0 \in V(u_0)$, compute the approximate solutions $\{z_n\}$, $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ by the iterative schemes

$$\begin{aligned} u_n &= u_n - g(u_n) + P_K z_n \\ w_n &\in T(u_n) : \|w_n - w_{n+1}\| \leq M(T(u_n), T(u_{n+1})) \\ y_n &\in V(u_n) : \|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n)) \\ z_{n+1} &= g(u_n) - N(w_n, y_n) + (1 - \rho^{-1}) Q_K z_n, \end{aligned}$$

$n = 0, 1, 2, \ldots$.

For a suitable choice of operators T, A, g, V and convex set K, one can obtain a wide class of iterative algorithms for solving various classes of variational inequalities and complementarity problems.

For the sake of completeness and to convey an idea, we now study the convergence of Algorithm 4.1. In a similar way, one can study the convergence analysis of other algorithms.

Theorem 4.1. *Let the operator $N(.,.)$ be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$ with respect to the first argument. Let the single-valued operator $g : H \to H$ be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$. Let the operator $N(.,.)$ be Lipschitz continuous with constant $\eta > 0$ with respect to the second argument and $V : H \to 2^H$ be M-Lipschitz continuous with constant $\xi > 0$. Let the multivalued operator T be M-Lipschitz continuous with constant $\mu > 0$. If the relations* (3.2) – (3.5) *hold, then there exists $u \in H$ such that $w \in T(u)$, $y \in V(u)$, and $g(u) \in K$ satisfying the generalized set-valued variational inequality* (2.1) *and the sequences $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ generated by Algorithm* 4.1 *converge to u, w and y strongly in H respectively.*

Proof. From Algorithm 4.1, we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &\le (1-\lambda)\|u_n - u_{n-1}\| + \lambda\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) \\
&\quad + P_K[g(u_n) - \rho N(w_n, y_n)] - P_K[g(u_{n-1}) - \rho N(w_{n-1}, y_{n-1})]\| \\
&\le (1-\lambda)\|u_n - u_{n-1}\| + \lambda\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| \\
&\quad + \lambda\|P_K[g(u_n) - \rho N(w_n, y_n)] - P_K[g(u_{n-1}) - \rho N(w_{n-1}, y_{n-1})]\| \\
&\le (1-\lambda)\|u_n - u_{n-1}\| + 2\lambda\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| \\
&\quad + \lambda\|u_n - u_{n-1} - \rho\{N(w_n, y_n) - N(w_{n-1}, y_n)\}\| \\
&\quad + \lambda\rho\|N(w_{n-1}, y_n) - N(w_{n-1}, y_{n-1})\| \\
&\le (1-\lambda)\|u_n - u_{n-1}\| + \Big\{2\lambda\sqrt{1 - 2\sigma + \delta^2} + \lambda\sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2} \\
&\quad + \lambda\rho\eta\xi\Big\}\|u_n - u_{n-1}\|, \quad \text{using (3.9), (3.10) and (3.14)} \\
&= \{(1-\lambda) + \lambda(k + \rho\eta\xi + t(\rho))\}\|u_n - u_{n-1}\|, \quad \text{using (3.15) - (3.16)} \\
&= \{(1-\lambda) + \lambda\theta\}\|u_n - u_{n-1}\| = \{1 - \lambda(1-\theta)\}\|u_n - u_{n-1}\|.
\end{aligned}$$

Thus

$$\|u_{n+1} - u_n\| \le h\|u_n - u_{n-1}\|, \tag{4.5}$$

where

$$h = 1 - \lambda(1 - \theta).$$

Now from (3.2) - (3.5), we have $0 < \theta < 1$. It follows that $h < 1$. Consequently, from (4.5), we know that the sequence $\{u_n\}$ is a Cauchy sequence in H, that is, there exists $u \in H$ with $u_{n+1} \to u$. Also from (4.3), we have

$$\|y_{n+1} - y_n\| \leq M(V(u_{n+1}), V(u_n)) \leq \xi \|u_{n+1} - u_n\|, \tag{4.6}$$

which implies that the sequence $\{y_n\}$ is a Cauchy sequence in H, so that there exists $y \in H$ such that $y_n \to y$. Now by using the continuity of the operators T, A, g, V, P_K and Algorithm 4.1, we have

$$u = (1 - \lambda)u + \lambda\{u - g(u) + P_K[g(u) - \rho N(w, y)]\},$$

that is

$$g(u) = P_K[g(u) - \rho N(w, y)] \in K.$$

Now we show that $y \in V(u)$. In fact,

$$\begin{aligned} d(y, V(u)) &\leq \|y - y_n\| + d(y_n, V(u)) \leq \|y - y_n\| + M(V(u_n), V(u)) \\ &\leq \|y - y_n\| + \xi\|u_n - u\| \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

where $d(y, V(u)) = \inf\{\|y - z\| : z \in V(u)\}$. Since the sequences $\{u_n\}$ and $\{y_n\}$ are the Cauchy sequences, it follows from the above inequality that $d(y, V(u)) = 0$. This implies that $y \in V(u)$. In a similar way, we can show that $w \in T(u)$. By Lemma 3.1, it follows that $u \in H$ such that $w \in T(u)$, $y \in V(u)$, $g(u) \in K$, which satisfies the inequality (2.1) and $u_n \to u$, $w_n \to w$, $y_n \to y$ strongly in H, the required result. □

Remark 4.1. It is worth mentioning that various methods including projection, linear approximation, relaxation and decomposition methods for solving variational inequalities can be derived from the auxiliary principle technique by a suitable and appropriate rearrangement of the operators T, g, V and the convex set K, see, for example, [23] and the references therein. In recent years, the auxiliary principle technique has been used to find the equivalent differentiable optimization problems for the variational inequalities. These equivalent differentiable optimization problems are being used to develop the general descent and Newtons methods with line search to solve the variational inequalities and complementarity problem. For recent development in this direction, see Larsson and Patriksson [16], Zhu and Marcotte [46] and Noor [23], [26]. In brief, we conclude that by a suitable choice of the auxiliary principle problem, one can find not only a number of equivalent formulations for various types of variational inequalities, but can also study the existence of the solution. These facts show that the auxiliary principle technique is quite general and flexible.

Remark 4.2. In many applications of the variational inequalities, the convex set K also depends explicitly or implicitly on the solution u. In this case, problem (2.1) is called generalized set-valued quasi variational inequality. We remark that with a suitable modification, the techniques discussed in Section 3 and Section 4 can be extended to obtain similar results for quasi variational inequalities. An extension of the auxiliary principle technique for generalized set-valued quasi variational inequalities is still an open problem, which is another direction for future research in this interesting area. The development and implementation of efficient iterative algorithms for generalized set-valued variational inequalities deserve further research efforts.

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