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THEORY OF MULTIVARIABLE BESSEL FUNCTIONS AND ELLIPTIC MODULAR FUNCTIONS

G. DATTOLI - A.TORRE - S. LORENZUTTA

The theory of multivariable Bessel functions is exploited to establish further links with the elliptic functions. The starting point of the present investigations is the Fourier expansion of the theta functions, which is used to derive an analogous expansion for the Jacobi functions (sn,dn,cn...) in terms of multivariable Bessel functions, which play the role of Fourier coefficients. An important by product of the analysis is an unexpected link with the elliptic modular functions.

1. Introduction.

The theory of generalized Bessel function (GBF) has been reviewed in Ref. [6]. The importance of these functions stems from their wide use in application [10], [7] and from their implications in different fields of pure and applied mathematics, ranging from the theory of generalized Hermite polynomials [2] to the theory of elliptic functions [1], [5].

As to this last point, it has been shown that [5] exponents of the Jacobi functions exhibit expansions of the Jacobi-Anger type in which the ordinary B.F. is replaced by an infinite variable GBF.

This paper is addressed to a further investigation on the link exixting between multivariable GBF and elliptic functions. In particular we will prove that the Jacobi functions can be written in terms of a trigonometric series

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whose expansion coefficients are infinite variable GBF. We will also prove that these functions provide a natural basis for the expansion of the elliptic modular functions.

Before entering into the specific details of the problem, we will review the main points of the multivariable GBF theory. The few elements we will discuss in the following are both aimed at making the paper self-contained and fixing the notation we will exploit in the following.

A two variable one index GBF is specified by the following generating function

$$(1)\quad \exp\left\{\frac{1}{2}\left[x_1\left(t-\frac{1}{t}\right)+x_2\left(t^2-\frac{1}{t^2}\right)\right]\right\}=\sum_{n=-\infty}^{+\infty} t^n J_n(x_1,x_2),\ 0<|t|<\infty$$

or equivalently by the following series

$$(2)\qquad J_n(x_1,x_2)=\sum_{\ell=-\infty}^{+\infty} J_{n-2\ell}(x_1)J_\ell(x_2).$$

The extension to more variables is accomplished rather straightforwardly and in fact

$$(3)\qquad \begin{aligned} &\exp\left[\sum_{s=1}^{3}\frac{x_s}{2}\left(t^s-\frac{1}{t^s}\right)\right]=\sum_{n=-\infty}^{+\infty} t^n J_n(x_1,x_2,x_3),\\ &J_n(x_1,x_2,x_3)=\sum_{\ell=-\infty}^{+\infty} J_{n-3\ell}(x_1,x_2)J_\ell(x_3). \end{aligned}$$

In a similar way we can construct GBF with $4, 5, \ldots, m$ variables. The recurrence relations of a m-variable GBF can be written as

$$(4)\qquad \begin{aligned} &\frac{\partial}{\partial x_s} J_n\left(\{x_s\}_1^m\right)=\frac{1}{2}\left[J_{n-s}\left(\{x_s\}_1^m\right)-J_{n+s}\left(\{x_s\}_1^m\right)\right],\\ &2nJ_n\left(\{x_s\}_1^m\right)=\sum_{s=1}^{m} s x_s\left[J_{n-s}\left(\{x_s\}_1^m\right)+J_{n+s}\left(\{x_s\}_1^m\right)\right]. \end{aligned}$$

The extension to infinite variables has been shown possible, under the assumption that the series $\sum_{s=1}^{\infty} s|x_s|$ be convergent ([9]). The link of many-variable GBFs with trigonometric series is almost natural, since the following generalized form of the Jacobi Anger expansion holds [9]

$$\exp\left[i\sum_{s=1}^{\infty}x_s\sin(s\theta)\right]=\sum_{n=-\infty}^{+\infty}e^{in\theta}J_n\Big(\{x_s\}_1^{\infty}\Big). \tag{5}$$

Modified forms of many variable GBF can be introduced as well using the following Jacobi-Anger expansion ([1])

$$\exp\left[\sum_{s=1}^{\infty}x_s\cos(s\varphi)\right]=\sum_{n=-\infty}^{+\infty}e^{in\varphi}I_n\Big(\{x_s\}_1^{\infty}\Big), \tag{6}$$

which is valid under the same restriction on the $\{x_s\}$ as in the J-case. Function $I_n\Big(\{x_s\}_1^{\infty}\Big)$ satisfies the recurrences

$$\begin{aligned}&\frac{\partial}{\partial x_s}I_n\Big(\{x_s\}_1^{\infty}\Big)=\frac{1}{2}\Big[I_{n-s}\Big(\{x_s\}_1^{\infty}\Big)+I_{n+s}\Big(\{x_s\}_1^{\infty}\Big)\Big],\\&2nI_n\Big(\{x_s\}_1^{\infty}\Big)=\sum_{s=1}^{\infty}sx_s\Big[I_{n-s}\Big(\{x_s\}_1^{\infty}\Big)-I_{n+s}\Big(\{x_s\}_1^{\infty}\Big)\Big].\end{aligned} \tag{7}$$

A particular type of many-variable GBF which will be largely exploited in the following sections is defined below

$$\exp\left[\sum_{s=1}^{m}x_{2s-1}\Big(t^{2s-1}-\frac{1}{t^{2s-1}}\Big)\right]=\sum_{n=-\infty}^{+\infty}t^{n\,(0)}J_n\Big(\{x_{2s-1}\}_1^{m}\Big), \tag{8}$$

where the superscript (0) stands for odd. In the case of $m=2$ the function ${}^{(0)}J_n(x_1,x_3)$ is specified by the series

$${}^{(0)}J_n(x_1,x_3)=\sum_{\ell=-\infty}^{+\infty}J_{n-3\ell}(x_1)J_\ell(x_3) \tag{9a}$$

and the extension to a larger number of variables is obvious.

([1]) The $I_n\Big(\{x_s\}_1^{m}\Big)$ functions are modified GBF of first kind and play the same role as $I_n(x)$ in the one variable case.

Analogously, one can define the modified version, ${}^{(0)}I_n(\{x_{2s-1}\}_{s=1}^m)$ of ${}^{(0)}J_n(\{x_{2s-1}\}_{s=1}^m)$ and then consider the relevant extensions to the infinite-variable case denoted by ${}^{(0)}J_n(\{x_{2s-1}\}_{s\geq 1})$ and ${}^{(0)}I_n(\{x_{2s-1}\}_{s\geq 1})$, respectively, for which the following Jacobi-Anger expansions hold true

$$
\begin{aligned}
e^{i\sum\limits_{s=1}^{\infty} x_{2s-1}\sin[(2s-1)\varphi]} &= \sum_{n=-\infty}^{+\infty} e^{in\varphi\,(0)}J_n(\{x_{2s-1}\}_{s\geq 1}),\\
e^{\sum\limits_{s=1}^{\infty} x_{2s-1}\cos[(2s-1)\varphi]} &= \sum_{n=-\infty}^{+\infty} e^{in\varphi\,(0)}I_n(\{x_{2s-1}\}_{s\geq 1}),
\end{aligned}
\tag{9b}
$$

under the assumption that the series $\sum\limits_{s=1}^{\infty}(2s-1)|x_{2s-1}|$ be convergent.

An interesting result, to be immediately quoted, is the possibility of exploiting functions of the above type, to establish "non-linear" Jacobi-Anger expansions, i.e. expansions relevant to the generating functions

$$
\begin{aligned}
F(x,\theta;n) &= e^{ix(\sin\theta)^n},\\
G(x,\theta;n) &= e^{x(\cos\theta)^n}.
\end{aligned}
\tag{10}
$$

In the specific cases of $n = 3$ we obtain

$$
\begin{aligned}
e^{x(\cos\theta)^3} &= e^{\frac{x}{4}[3\cos\theta+\cos(3\theta)]} = \sum_{n=-\infty}^{+\infty} e^{in\theta\,(0)}I_n\Big(\frac{3}{4}x,\frac{x}{4}\Big),\\
e^{ix(\sin\theta)^3} &= e^{i\frac{x}{4}[3\sin\theta-\sin(3\theta)]} = \sum_{n=-\infty}^{+\infty} e^{in\theta\,(0)}J_n\Big(\frac{3}{4}x,\frac{x}{4}\Big),
\end{aligned}
\tag{11}
$$

After these few remarks we are able to introduce the specific topic of the paper namely the link between GBFs and elliptic functions of the Jacobi-type.

2. Generalized Bessel functions and theta elliptic functions.

In a previous paper ([5]) it has been proved that the Jacobi functions $sn\ u$, $dn\ u$ and $cn\ u$ are linked to the infinite variable GBF by Jacobi-Anger like expansion. In other words, using the Fourier expansions of the elliptic functions, for real z it can be shown that

$$
\begin{aligned}
e^{ixsn\,u} &= \sum_{n=-\infty}^{+\infty} e^{inz(0)} J_n\left[\left\{\frac{yq^{m-\frac{1}{2}}}{1-q^{2m-1}}\right\}_{m\geq 1}\right],\\
e^{xcn\,u} &= \sum_{n=-\infty}^{+\infty} e^{inz(0)} I_n\left[\left\{\frac{yq^{m-1/2}}{1+q^{2m-1}}\right\}_{m\geq 1}\right],\\
e^{xdn\,u} &= e^{\frac{\pi x}{2K}} \sum_{n=-\infty}^{+\infty} e^{2inz} I_n\left[\left\{ky\frac{q^m}{1+q^{2m}}\right\}_{m\geq 1}\right],
\end{aligned}
\tag{12}
$$

where

$$
z = \frac{\pi u}{2K}, \quad q = \exp\left\{-\frac{\pi K'}{K}\right\}, \quad y = \frac{2\pi}{kK}x \tag{13}
$$

and K and iK' are the quarter periods of the elliptic functions, specified by the integrals

$$
\begin{aligned}
K &= \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},\\
K' &= \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k'^2\sin^2\varphi}}, \quad k^2+k'^2=1.
\end{aligned}
\tag{14}
$$

Expansion analogous to (12) can be derived for the theta-functions too.

It is to be noted that Eqs. (12) can be viewed as trigonometric series associated to the exponents of the Jacobi functions. In this paper we will derive trigonometric series of $cn\ u$, $dn\ u$ etc. whose coefficients are provided by infinite variable GBFs.

The starting point of our analysis is the following representation of theta-functions in terms of infinite products, namely

$$
\begin{aligned}
\theta_1(z) &= 2q^{1/4}\sin z G(q) \underset{n=1}{\overset{\infty}{\pi}} (1-2q^{2n}\cos(2z)+q^{4n}),\\
\theta_2(z) &= 2q^{1/4}\cos z G(q) \underset{n=1}{\overset{\infty}{\pi}} (1+2q^{2n}\cos(2z)+q^{4n}),\\
\theta_3(z) &= G(q) \underset{n=1}{\overset{\infty}{\pi}} (1+2q^{2n-1}\cos(2z)+q^{4n-2}),\\
\theta_4(z) &= G(q) \underset{n=1}{\overset{\infty}{\pi}} (1-2q^{2n-1}\cos(2z)+q^{4n-2}),\\
&\left(G(q) = \underset{n=1}{\overset{\infty}{\pi}} \left(1-q^{2n}\right)\right).
\end{aligned}
\tag{15}
$$

Taking the logarithm of the first of Eqs. (15) we obtain

$$\ln\left[\frac{\theta_1(z)}{2q^{1/4}G(q)\sin z}\right] = \sum_{n=1}^{\infty} \ln\left(1 - 2q^{2n}\cos(2z) + q^{4n}\right). \tag{16}$$

The r.h.s. of the above equation can be cast in a more convenient form by means of the expansion [11]

$$\ln\left(\left[1 - 2x\cos y + x^2\right]^{-1/2}\right) = \sum_{m=1}^{\infty} \frac{x^m}{m}\cos(my) \tag{17}$$

which holds for

$$|x| < \exp(-\mathcal{I}my) \tag{18a}$$

or supposing $|x| < 1$

$$-\mathcal{R}e\left(\ln\frac{1}{x}\right) < \mathcal{I}my < \mathcal{R}e\left(\ln\frac{1}{x}\right). \tag{18b}$$

Accordingly, the n^{th} term of the summation in Eq. (16) writes

$$\ln\left(1 - 2q^{2n}\cos(2z) + q^{4n}\right) = -2\sum_{m=1}^{\infty} \frac{q^{2nm}}{m}\cos(2mz) \tag{19}$$

whose validity is limited to the region

$$-\pi\,\mathcal{I}m\tau < \mathcal{I}mz < \pi\,\mathcal{I}m\tau \qquad (q = e^{i\pi\tau}). \tag{20}$$

Inserting therefore Eq. (19) into Eq. (16) we end up with

$$\ln\left[\frac{\theta_1(z)}{2q^{1/4}G(q)\sin z}\right] = -2\sum_{n=1}^{\infty} \frac{q^{2n}}{n(1-q^{2n})}\cos(2nz) \tag{21}$$

which, according to Eq. (20), holds for

$$-\pi\,\mathcal{I}m\tau < \mathcal{I}mz < \pi\,\mathcal{I}m\tau. \tag{22}$$

Using the same reasoning we are able to write

$$\ln\left[\frac{\theta_4(z)}{G(q)}\right] = -2\sum_{n=1}^{\infty} \frac{q^n}{n(1-q^{2n})}\cos(2nz) \tag{23}$$

whose validity is limited to the region

$$-\frac{\pi}{2}\mathcal{I}m\tau < \mathcal{I}mz < \frac{\pi}{2}m\tau. \tag{24}$$

Finally exploiting the identities

$$\begin{aligned} \theta_2(z) &= \theta_1(z+\pi/2), \\ \theta_3(z) &= \theta_4(z+\pi/2), \end{aligned} \tag{25}$$

we find

$$\begin{aligned} \ln\left[\frac{\theta_2(z)}{2q^{1/4}G(q)\cos z}\right] &= -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{q^{2n}}{1-q^{2n}}\cos(2nz), \\ \ln\left[\frac{\theta_3(z)}{G(q)}\right] &= -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{q^n}{1-q^{2n}}\cos(2nz). \end{aligned} \tag{26}$$

According to the previous results, for z real we can express the theta functions $(\vartheta_i(z) = \vartheta_i(z)\tau)$ in terms of infinite variable GBF as reported below

$$\begin{aligned} \theta_1(z) &= 2q^{1/4}G(q)\sin z \sum_{n=-\infty}^{+\infty} e^{2inz} I_n\left[\left\{-2\frac{q^{2m}}{m(1-q^{2m})}\right\}_{m\geq 1}\right], \\ \theta_2(z) &= 2q^{1/4}G(q)\cos z \sum_{n=-\infty}^{+\infty} e^{2inz} I_n\left[\left\{\frac{2(-1)^{m+1}q^{2m}}{m(1-q^{2m})}\right\}_{m\geq 1}\right], \\ \theta_3(z) &= G(q) \sum_{n=-\infty}^{+\infty} e^{2inz} I_n\left[\left\{-\frac{2(-q)^m}{m(1-q^{2m})}\right\}_{m\geq 1}\right], \\ \theta_4(z) &= G(q) \sum_{n=-\infty}^{+\infty} e^{2inz} I_n\left[\left\{-\frac{2q^m}{m(1-q^{2m})}\right\}_{m\geq 1}\right]. \end{aligned} \tag{27}$$

The above formulae provide the direct link between GBF and theta elliptic functions. In the next section we will show how similar expression can be obtained for $sn\ u$, $cn\ u$, $dn\ u$, ..., Jacobi functions.

3. Generalized Bessel function and Jacobi elliptic functions.

The principal Jacobi function sn, cn and dn, are quotient of θ theta functions, indeed [11]

$$sn\ u = \frac{1}{\sqrt{k}}\frac{\theta_1(z)}{\theta_4(z)}, \quad cn\ u = \sqrt{\frac{k'}{k}}\frac{\theta_2(z)}{\theta_4(z)}, \quad dn\ u = \sqrt{k'}\frac{\theta_3(z)}{\theta_4(z)}, \tag{28}$$

where $z = \frac{\pi u}{2K}$.

Exploiting the previous relations for the logarithm of the theta-functions, we get from Eqs. (28)

$$\begin{aligned} \ln(sn\ u) &= \ln\left[\frac{2q^{1/4}}{\sqrt{k}}\sin z\right] + 2\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^n}\cos(2nz), \\ \ln(cn\ u) &= \ln\left[2q^{1/4}\sqrt{\frac{k'}{k}}\cos z\right] + 2\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+(-q)^n}\cos(2nz), \\ \ln(dn\ u) &= \frac{1}{2}\ln(k') + 4\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{4n-2}}\frac{\cos(4n-2)z}{(2n-1)} \end{aligned} \tag{29}$$

which for real z yield

$$\begin{aligned} sn\ u &= \frac{2q^{1/4}}{\sqrt{k}}\sum_{n=-\infty}^{+\infty}\sin[(2n+1)z]I_n\left[\left\{\frac{2}{m}\frac{q^m}{1+q^m}\right\}_{m\geq 1}\right], \\ cn\ u &= 2q^{1/4}\sqrt{\frac{k'}{k}}\sum_{n=-\infty}^{+\infty}\cos[(2n+1)z]I_n\left[\left\{\frac{2}{m}\frac{q^m}{1+(-q)^m}\right\}_{m\geq 1}\right], \\ dn\ u &= \sqrt{k'}\sum_{n=-\infty}^{+\infty}e^{2inz}\ {}^{(0)}I_n\left[\left\{\frac{4}{2m-1}\frac{q^{2m-1}}{1-q^{4m-2}}\right\}_{m\geq 1}\right]. \end{aligned} \tag{30}$$

Analogous expression relevant to the product of elliptic function can be found keeping e.g. the derivatives of both sides of Eqs. (30) with respect to u, thus getting

$$cn\ u\ dn\ u = \frac{\pi q^{1/4}}{K\sqrt{k}} \sum_{n=-\infty}^{+\infty} (2n+1)\cos[(2n+1)z] \cdot \\ \cdot I_n\Big[\Big\{\frac{2}{m}\frac{q^m}{1+q^m}\Big\}_{m\geq 1}\Big],$$

$$sn\ u\ dn\ u = \frac{\pi q^{1/4}}{K} \sum_{n=-\infty}^{+\infty} (2n+1)\sin[(2n+1)z] \cdot \\ \cdot I_n\Big[\Big\{\frac{2}{m}\frac{q^m}{1+(-q)^m}\Big\}_{m\geq 1}\Big], \tag{31}$$

$$-k^2 sn\ u\ cn\ u = \frac{i\pi\sqrt{k'}}{K} \sum_{n=-\infty}^{+\infty} n e^{2inz} {}^{(0)}I_n\Big[\Big\{\frac{4}{2m-1}\frac{q^{2m-1}}{1-q^{4m-2}}\Big\}_{m\geq 1}\Big].$$

We can now specialize the above relations for particular values of u to get further interesting identities. By setting $u = 0$ in Eq. (30) we find

$$\frac{1}{2}\Big(\frac{k}{k'}\Big)^{1/2} q^{-1/4} = \sum_{n=-\infty}^{+\infty} I_n\Big[\Big\{\frac{2}{m}\frac{q^m}{1+(-q)^m}\Big\}_{m\geq 1}\Big], \\ \frac{1}{\sqrt{k'}} = \sum_{n=-\infty}^{+\infty} {}^{(0)}I_n\Big[\Big\{\frac{4}{2m-1}\frac{q^{2m-1}}{1-q^{4m-2}}\Big\}_{m\geq 1}\Big]. \tag{32a}$$

In a similar way, $u = K$ yields

$$\frac{1}{2}k^{1/2} q^{-1/4} = \sum_{n=-\infty}^{+\infty} (-1)^n I_n\Big[\Big\{\frac{2}{m}\frac{q^m}{1+q^m}\Big\}_{m\geq 1}\Big], \\ \sqrt{k'} = \sum_{n=-\infty}^{+\infty} (-1)^n\, {}^{(0)}I_n\Big[\Big\{\frac{4}{2m-1}\frac{q^{2m-1}}{1-q^{4m-2}}\Big\}_{m\geq 1}\Big]. \tag{32b}$$

These last relations provide the basis for the link between multivariable GBF and elliptic modular functions.

4. Concluding remarks.

In the previous sections we have analyzed the link existing between GBF and Jacobi elliptic functions. Further interesting relations will be discussed in these concluding remarks. We believe that an important by product of the already developed analysis is the possibility of expressing the so called elliptic modular functions (EMF) in series of GBF with infinite variables.

The EMF are usually denoted by ([11], [8])

$$f(\tau) = \frac{\theta_2^4(0|\tau)}{\theta_3^4(0|\tau)} = k^2,$$

$$g(\tau) = \frac{\theta_4^4(0|\tau)}{\theta_3^4(0|\tau)} = k'^2, \tag{33}$$

$$h(\tau) = -\frac{f(\tau)}{g(\tau)} = -\frac{k^2}{k'^2} = -\frac{k^2}{1-k^2},$$

and satisfy the properties

$$\begin{aligned} &f(\tau+2) = f(\tau), \quad g(\tau+2) = g(\tau),\\ &f(\tau)+g(\tau) = 1, \quad f(\tau+1) = h(\tau) \end{aligned} \tag{34}$$

which follow from the fact that

$$k^2 + k'^2 = 1, \quad q = e^{i\pi\tau} = e^{i\pi(\tau+2)} = -e^{i\pi(\tau+1)}. \tag{35}$$

The link between EMF and GBF is readily obtained. The first of Eqs. (32a) yields indeed

$$\left(\frac{k}{k'}\right)^{1/2} = \sqrt[4]{-h(\tau)} = 2e^{i\frac{\pi}{4}\tau} \sum_{n=-\infty}^{\infty} I_n\Big[\Big\{\frac{2}{m}\frac{e^{im\pi\tau}}{1+(-1)^m e^{im\pi\tau}}\Big\}_{m\geq 1}\Big], \tag{36}$$

while Eqs. (32b) provide the identities

$$\begin{aligned} k^{1/2} &= \sqrt[4]{f(\tau)} = 2e^{i\frac{\pi}{4}\tau} \sum_{n=-\infty}^{\infty} (-1)^n I_n\Big[\Big\{\frac{2}{m}\frac{e^{im\pi\tau}}{1+e^{im\pi\tau}}\Big\}_{m\geq 1}\Big],\\ k'^{1/2} &= \sqrt[4]{g(\tau)} = \sum_{n=-\infty}^{\infty} (-1)^{n(0)} I_n\Big[\Big\{\frac{4}{2m-1}\frac{e^{(2m-1)i\pi\tau}}{1-e^{2(2m-1)i\pi\tau}}\Big\}_{m\geq 1}\Big]. \end{aligned} \tag{37}$$

By keeping the fourth power of both sides of Eqs. (36) and (37) and by using the Jacobi-Anger expansion for the GBF we get

$$
\begin{aligned}
h(\tau) &= -16e^{i\pi\tau}\sum_{n=-\infty}^{\infty} I_n\Big[\Big\{\frac{8}{m}\frac{e^{im\pi\tau}}{1+(-1)^m e^{im\pi\tau}}\Big\}_{m\geq 1}\Big],\\
f(\tau) &= 16e^{i\pi\tau}\sum_{n=-\infty}^{\infty}(-1)^n I_n\Big[\Big\{\frac{8}{m}\frac{e^{im\pi\tau}}{1+e^{im\pi\tau}}\Big\}_{m\geq 1}\Big],\\
g(\tau) &= \sum_{n=-\infty}^{\infty}(-1)^{n\,(0)}I_n\Big[\Big\{\frac{16}{2m-1}\frac{e^{(2m-1)i\pi\tau}}{1-e^{2(2m-1)i\pi\tau}}\Big\}_{m\geq 1}\Big].
\end{aligned} \tag{38}
$$

It is worth noting that the properties of the EMF can be directly inferred from the above series definition, in fact

$$
\begin{aligned}
f(\tau+1) &= 16e^{i\pi(\tau+1)}\sum_{n=-\infty}^{\infty}(-1)^n I_n\Big[\Big\{\frac{8}{m}\frac{e^{im\pi(\tau+1)}}{1+e^{im\pi(\tau+1)}}\Big\}_{m\geq 1}\Big]\\
&= -16e^{i\pi\tau}\sum_{n=-\infty}^{\infty}(-1)^n I_n\Big[\Big\{\frac{8}{m}\frac{(-1)^m e^{im\pi\tau}}{1+(-1)^m e^{im\pi\tau}}\Big\}_{m\geq 1}\Big]
\end{aligned} \tag{39a}
$$

and since

$$
I_n\Big(\Big\{(-1)^s x_s\Big\}_{s\geq 1}\Big) = (-1)^n I_n\big(\{x_s\}_{s\geq 1}\big) \tag{39b}
$$

we end up with

$$
f(\tau+1) = -16e^{i\pi\tau}\sum_{n=-\infty}^{\infty} I_n\Big[\Big\{\frac{8}{m}\frac{e^{im\pi\tau}}{1+(-1)^m e^{im\pi\tau}}\Big\}_{m\geq 1}\Big] \tag{39c}
$$

namely, the last of Eqs. (34).

By noting that, the last of eqs. (32a) yields

$$
\frac{1}{\sqrt{k'}} = \sum_{n=-\infty}^{+\infty}{}^{(0)}I_n\Big[\Big\{\frac{4}{2m-1}\frac{e^{(2m-1)i\pi\tau}}{1-e^{(4m-2)i\pi\tau}}\Big\}_{m\geq 1}\Big] \tag{40}
$$

it appears convenient to introduce the further EMF

$$
\mathrm{l}(\tau) = \sum_{n=-\infty}^{+\infty}{}^{(0)}I_n\Big[\Big\{\frac{16}{2m-1}\frac{e^{(2m-1)i\pi\tau}}{1-e^{(4m-2)i\pi\tau}}\Big\}_{m\geq 1}\Big] \tag{41}
$$

and note that

$$l(\tau + 1) = g(\tau). \tag{42}$$

Furthermore, from the Jacobi-Anger expansion of multivariable GBF it also follows that

$$h(\tau) = -16e^{i\pi\tau} \exp\Big\{4 \sum_{m=1}^{\infty} \frac{2}{m} \frac{e^{im\pi\tau}}{1 + (-1)^m e^{im\pi\tau}}\Big\} \tag{43}$$

and similarly for the other functions.

A more general treatment regarding EMF and GBF will be presented elsewhere. Before closing this paper it is worth giving further identities, which albeit an almost direct consequence of the considerations developed in the previous sections, provide a deeper insight into the link between GBF and elliptic functions.

By integrating both sides of Eqs. (30) we obtain

$$\begin{aligned}
&{}^{(0)}I_n\Big[\Big\{\frac{4}{2m-1}\frac{q^{2m-1}}{1-q^{4m-2}}\Big\}_{m\geq 1}\Big] = \frac{1}{2K\sqrt{k'}} \int_0^{2K} dn(u,k) e^{-i\frac{\pi u n}{K}}\, du, \\
&I_n\Big[\Big\{\frac{2}{m}\frac{q^m}{1+(-1)^m q^m}\Big\}_{m\geq 1}\Big] = \frac{1}{2} q^{-1/4} \frac{1}{2K} \big(\frac{k}{k'}\big)^{1/2} \cdot \\
&\qquad \cdot \int_0^{4K} cn(u,k) \cos[(2n+1)\frac{\pi u}{2K}]\, du.
\end{aligned} \tag{44}$$

Analogous relations involving the theta functions can also be obtained, thus getting e.g.

$$\begin{aligned}
&I_n\big[\{(-1)^m \tilde{q}_m\}_{m\geq 1}\big] = \frac{1}{2\pi G(q)} \int_{-\pi}^{+\pi} \theta_3(z) e^{-2inz}\, dz, \\
&I_n\big[\{\tilde{q}_m\}_{m\geq 1}\big) = \frac{1}{\pi G(q)} \int_0^{+\pi} \theta_4(z) \cos(2nz)\, dz, \\
&\Big(\tilde{q}_m = -\frac{2q^m}{m(1-q^{2m})}\Big).
\end{aligned} \tag{45}$$

This paper is a further contribution to the theory of GBF and to their link with elliptic functions. All the possible implications are far from being completely understood, a forthcoming note will be devoted to a deeper insight.

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G. Dattoli - A. Torre,
ENEA, Dipartimento Innovazione,
Settore Fisica Applicata,
Centro Ricerche Frascati,
P.O. Box 65,
00044 Frascati, Roma (ITALY)

S. Lorenzutta,
ENEA, Dipartimento Innovazione,
Settore Fisica Applicata,
Bologna (ITALY)