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SOBOLEV'S ORIGINAL DEFINITION OF HIS SPACES REVISITED AND A COMPARISON WITH NOWADAYS DEFINITION

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Filippo Chiarenza in memoriam.
Un matematico grande e un uomo particolare.

1. Introduction.

The definition of Sobolev spaces used nowadays in literature reads as follows. If $\Omega \subseteq \mathbb{R}^n (n \geq 1)$ is an open set, if $m \in \mathbb{N}$ and if $1 \leq p < \infty$ then

$$W^{m,p}(\Omega) := \Big\{ u \in L^p(\Omega) : \exists\, D^\alpha u \in L^p(\Omega) \quad \text{for} \quad |\alpha| \leq m \Big\}$$

is called a Sobolev space (all functions we consider are assumed to be real valued). Here $D^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdot \ldots \cdot \partial x_n^{\alpha_n}}$ denotes the weak (distributional) derivative of u corresponding to the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| := \alpha_1 + \ldots + \alpha_n$ denotes its order. By

$$\|u\|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \quad \text{if} \quad 1 \leq p < +\infty,$$

for $u \in W^{m,p}(\Omega)$ a norm is defined on $W^{m,p}(\Omega)$. By means of the definition of weak derivative and of the completeness of L^p-spaces it is readily seen that $(W^{m,p}(\Omega), ||.||_{W^{m,p}(\Omega)})$ is a Banach space and in case of $p = 2$ a Hilbert space with suitable inner product. This definition fits perfectly with the weak formulation of many boundary value problems for partial differential equations in *bounded* domains. But as soon as *unbounded* domains are considered it turns out that the spaces $W^{m,p}(\Omega)$ are too "narrow". As an example consider for $n \geq 2$ the exterior domain

$$\Omega := \Big\{ x \in \mathbb{R}^n : |x| > 1 \Big\}, \tag{1.1}$$

and the functions

$$h(x) := \begin{cases} 1 - |x|^{2-n} & \text{if} \quad n \geq 3, x \in \Omega, \\ \ln|x| & \text{if} \quad n = 2, x \in \Omega. \end{cases} \tag{1.2}$$

Then $h \in C^\infty(\bar{\Omega})$,

$$\begin{cases} h \in L^q_{\text{loc}}(\Omega) & \text{for} \quad 1 \leq q < \infty \\ h \notin L^s(\Omega) & \text{for all} \quad 1 \leq s < \infty, \quad \text{but} \\ \nabla h \in L^p(\Omega)^n & \text{for all} \quad \frac{n}{n-1} < p < \infty \\ \partial_i \partial_j h \in L^p(\Omega) & \text{for all} \quad 1 \leq p < \infty, i, j = 1, \dots, n. \end{cases} \tag{1.3}$$

Therefore $h \notin W^{1,p}(\Omega)$ and $h \notin W^{2,p}(\Omega)$ for all $1 \leq p < \infty$. On the other hand, $\triangle h = 0$ in Ω and $h|_{\partial\Omega} = 0$.

A functional analytical setting of the Dirichlet problem for the Laplacian in the sense of weak or strong L^p-solutions in exterior domains has clearly to cover such an example. But this is obviously *not* possible within the framework of $W^{m,p}(\Omega)$-spaces (compare [14]).

Another example arises from the Hilbert space setting of the weak Neumann problem. Let $\Omega \subset \mathbb{R}^n$ be a domain and let $\underline{f} := (f_1, \dots, f_n) \in L^2(\Omega)^n$ be given. We would call any $u \in W^{1,2}(\Omega)$ satisfying

$$< \nabla u, \nabla \Phi >_\Omega = < \underline{f}, \nabla \Phi >_\Omega \quad \text{for all} \quad \Phi \in W^{1,2}(\Omega) \tag{1.4}$$

$$\left(\text{Here } < \nabla u, \nabla \Phi >_\Omega := \int_\Omega \sum_{i=1}^n \partial_i u \partial_i \Phi \, dx\right)$$

a weak L^2-solution of the Neumann problem

$$\triangle u = \text{div}\underline{f} \text{ in } \Omega, \qquad \frac{\partial u}{\partial \underline{N}} \,|_{\partial\Omega} = \underline{f_N} \mid \partial\Omega$$

(where $\underline{N}$ should denote the exterior normal of Ω (if it exists), $\frac{\partial u(x)}{\partial \underline{N}} := \sum_{i=1}^{n} \partial_i u(x) N_i(x) \mid_{x\in\partial\Omega}$ and $\underline{f}_N(x) := \sum_{i=1}^{n} f_i(x) N_i(x) \mid_{x\in\partial\Omega}$). To solve the functional equation (1.4) it would suffice to consider a suitable *Hilbert space* so that $< \nabla., \nabla. >_\Omega$ becomes an inner product on it. In case that e.g. $|\Omega| < \infty$ to rule out the constants it would suffice to consider the subspace

$$W_\Omega^{1,2}(\Omega) := \left\{ u \in W^{1,2}(\Omega) : \int_\Omega u\,dy = 0 \right\}. \tag{1.5}$$

Then $< \nabla., \nabla. >_\Omega$ is clearly an inner product on the space defined by (1.5) (see Theorem A below). But the question arises whether $W_\Omega^{1,2}(\Omega)$ equipped with this inner product is *complete*. This question will be studied systematically in Section 4.

The difficulties arising in both examples above we can avoid if we remember Sobolev's *original* definition given in his pioneering works [15], [16] from 1936-1938 and in his monography [17] from 1950. For $\Omega \subset \mathbb{R}^n$ a *domain*, $m \in \mathbb{N}$ and $1 \le p < \infty$ Sobolev defines

$$L^{m,p}(\Omega) := \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \exists\, D^\alpha u \in L^p(\Omega) \text{ for all } \alpha \quad \text{with} \quad |\alpha| = m \right\}. \tag{1.6}$$

He assumes for Ω in addition that

i) Ω ist bounded;

ii) Ω is a finite union of domains each of which is starshaped with respect to a ball (see also [2], [8], [17]).

In 1964 it was proved by Gröger [5] that assumption i) can be dropped, but assumption ii) seems to be essential (see [2], [7] too). Last assumption is needed because of the use of Sobolev's ingenious, but rather difficult method of spherical projection operators. The definition (1.6) is slightly more general then Sobolev's original definition [15], [16], [17], where he used functions $u \in L^1(\Omega)$ in place of $L^1_{\mathrm{loc}}(\Omega)$. We should mention that the letters W and L for the notation of the above spaces in [17] are changed in contemporary literature. Our definition coincides with that given in [7], [8]. If $\Omega \subset \mathbb{R}^n$ is defined by (1.1) and h by (1.2) then we see by (1.3), (1.6): $h \in L^{1,p}(\Omega)$ for $\frac{n}{n-1} < p < \infty$ and $h \in L^{2,p}(\Omega)$ for $1 \le p < \infty$.

To define a norm on $L^{m,p}(\Omega)$, we choose an arbitrary but fixed $G \subset\subset \Omega$ (here and in the sequel we always assume for those sets $G \neq \emptyset$) and we define

$$\|u\|_{m,p;\Omega,G} := \|u\|_{L^1(G)} + |u|_{m,p;\Omega}, \tag{1.7}$$

where

$$|u|_{m,p;\Omega} := \left(\sum_{|\alpha|=m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

(note that $\|u\|_{m,p;\Omega,G} = 0$ implies in particular $|u|_{m,p;\Omega} = 0$, hence $u = P$ (= polynomial of degree $\leq m-1$; cf. Theorem B below) a.e. on Ω, and $0 = \|u\|_{L^1(G)} = \|P\|_{L^1(G)}$ gives $P \equiv 0$).
The following problems occur while studying the spaces $L^{m,p}(\Omega)$:

1. *existence of intermediate derivatives* $D^\beta u \in L^p_{\text{loc}}(\Omega)$ *(*$|\beta| \leq m-1$*) for any* $u \in L^{m,p}(\Omega)$*;*
2. *completeness of* $L^{m,p}(\Omega)$ *with respect to the norm* $\|\cdot\|_{m,p;\Omega,G}$*;*
3. *equivalence of the norms* $\|\cdot\|_{m,p;\Omega,G_k}$ *for arbitrary* $G_k \subset\subset \Omega$ *(*$k = 1, 2$*);*
4. *possible other choice of equivalent norms more adopted to a "natural" decomposition of* $L^{m,p}(\Omega)$ *(see* (3.4) *below).*

The first aim of this paper is to give report on recent joint work with Naumann [10], where we presented an entirely different and quite elementary method to solve problems 1–4 avoiding at the same moment the above mentioned restrictions i) and ii) concerning Ω. This method is essentially based on Poincaré's inequality for balls or cubes (compare Appendix 2), which can be proved by elementary calculus arguments. The second aim of our paper is to study very weak conditions on Ω so that $L^{1,p}(\Omega) = W^{1,p}(\Omega)$.

2. Notations. Ingredients.

For $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we put

$$\mathcal{P}(m) := \left\{ P = P(x) : P(x) = \sum_{|\alpha| \leq m} a_\alpha x^\alpha, : x \in \mathbb{R}^n, \quad a_\alpha \in \mathbb{R} \right\}$$

$$= \textit{vector space of real polynomials of degree } \leq m \textit{ in } \mathbb{R}^n.$$

For $G \subseteq \mathbb{R}^n$ and $x \in G$ let

$$d_x := \begin{cases} \frac{1}{4}\operatorname{dist}(x, \partial G) & \text{if } G \neq \mathbb{R}^n, \\ 1 & \text{if } G = \mathbb{R}^n. \end{cases}$$

We put for $x_0 \in \mathbb{R}^n$ and $\mathbb{R} > 0$

$$B_R(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < R\}.$$

In particular, we let denote $B_{d_x} = B_{d_x}(x)$ for $x \in G \subseteq \mathbb{R}^n$.
First we start with two standard arguments, whose proof can e.g. be found in [10] or [14].

Theorem A. *Let $G \subset \mathbb{R}^n$ be a bounded open set. Then for any $u \in W^{m,p}(G)$ there exists a uniquely determined polynomial $P_u \in \mathcal{P}(m-1)$ such that*

$$\int_G D^\alpha (u - P_u)\, dx = 0 \qquad \forall |\alpha| \leq m-1, \tag{2.1}$$

$$\| P_u \|_{W^{m-1,p}(G)} \leq C \| u \|_{W^{m-1,p}(G)}, \tag{2.2}$$

where the constant $C > 0$ depends only on $m,\ n,\ p$ and $|G|$.

Theorem B. *Let $G \subseteq \mathbb{R}^n$ be a domain. Let $u \in L^{m,p}(G)$ satisfy $D^\alpha u = 0$ a.e. in G for all $|\alpha| = m$. Then there exists exactly one $P \in \mathcal{P}(m-1)$ such that*

$$u = P \quad \text{a.e. in } G.$$

In addition we need

Theorem C. (Poincaré's inequality). *Let $B_R = B_R(x_0)$ be any fixed ball. Then there exists a constant $C(R) > 0$ (depending on $m,\ n,\ p$ too) such that*

$$\begin{cases} ||u||_{W^{m-1,p}(B_R)} \leq C(R) |u|_{m,p;B_R} \\ \forall u \in W^{m,p}(B_R) \text{ with } \displaystyle\int_{B_R} D^\beta u\, dx = 0 \quad \forall |\beta| \leq m-1. \end{cases} \tag{2.3}$$

An elementary proof of this theorem for $m = 1$ which is based on potential estimates, may be found in [4]. The proof for $m \geq 2$ follows by induction. For $x \in \mathbb{R}^n$ we may replace the Euclidean norm $|x| = |x|_2 = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}$ by the equivalent norm $|x|_\infty := \max \{|x_i|,\ i = 1, \ldots, n\}$. Then a "ball" $B_R(x_0)$ with respect to $|\cdot|_\infty$-norm is the cube $W_R(x_0) := \{x \in \mathbb{R}^n : |x_i - x_{0i}| < R, i = 1, \ldots, n\}$. With this change all our arguments remain valid. But for $W_R(x_0)$ Poincaré's inequality admits a very simple proof by induction on n (see Appendix 2).

3. The spaces $L^{m,p}(\Omega)$ and their properties.

The proof of the following statements rests only Theorems A–C and is given in detail in [10].

Theorem 3.1. *Let $u \in L^{m,p}(\Omega)$. Then there exist the weak derivatives*

$$D^{\beta} u \in L^{p}_{\mathrm{loc}}(\Omega) \quad \forall |\beta| \le m-1.$$

Let now $G \subset\subset \Omega$. Because of Theorem 3.1 we may define for $u \in L^{m,p}(\Omega)$

$$|u|_{m-1;G} := \sum_{|\beta|\le m-1} \left| \int_G D^{\beta} u \, dx \right| \tag{3.1}$$

and

$$|u|_{m,p;\Omega,G} := |u|_{m-1;G} + |u|_{m,p;\Omega}. \tag{3.2}$$

Both expressions are semi-norms on $L^{m,p}(\Omega)$. If $P \in \mathcal{P}(m-1)$ and $|P|_{m-1;G} = 0$, then it is readily seen $P \equiv O$. Therefore by (3.1) a norm is defined on $\mathcal{P}(m-1)$. Suppose now that $u \in L^{m,p}(\Omega)$ and $|u|_{m,p;\Omega,G} = 0$. Then $|u|_{m,p;\Omega} = 0$ and by Theorem B we see $u = P \in \mathcal{P}(m-1)$. Since $0 = |u|_{m-1;G} = |P|_{m-1;G}$ we conclude $u = P = 0$. Therefore by (3.2) a norm is defined on $L^{m,p}(\Omega)$ (all other properties of a norm are obvious). Let us now define

$$L^{m,p}_G(\Omega) := \left\{ u \in L^{m,p}(\Omega) \;\middle|\; \int_G D^{\beta} u \, dx = 0 \quad \forall |\beta| \le m-1 \right\}. \tag{3.3}$$

For $u \in L^{m,p}(\Omega)$, by Theorem A there exists a uniquely determined $P_u \in \mathcal{P}(m-1)$ such that

$$\int_G D^{\beta}(u - P_u)\, dx = 0 \quad \forall |\beta| \le m-1.$$

Then $u_0 := (u - P_u) \in L^{m,p}_G(\Omega)$ and $u = u_0 + P_u$.
If $v \in L^{m,p}_G(\Omega) \cap \mathcal{P}(m-1)|_{\Omega}$, i.e. $v \in \mathcal{P}(m-1)$ and $\int_G D^{\beta} v \, dx = 0$ for all $|\beta| \le m-1$, it follows that $v \equiv 0$. Therefore we see the direct decomposition

$$\begin{aligned} L^{m,p}(\Omega) &= L^{m,p}_G(\Omega) \oplus \mathcal{P}(m-1)|_{\Omega} \\ u &= u_0 + P_u. \end{aligned} \tag{3.4}$$

With that decomposition we have

$$|u|_{m,p;\Omega,G} = |u_0|_{m,p;\Omega} + |P_u|_{m-1;G}. \tag{3.5}$$

Furthermore $|\cdot|_{m,p;\Omega}$ is a norm on $L^{m,p}_G(\Omega)$. The most important tool of this section is the following Theorem 3.2 whose proof (see [10], Theorem 4.2) rests solely on Theorems A–C. This result is a straight forward generalization of an argument, we used a couple of years ago in our proof of the Helmholtz-decompositon (see [13], Lemma 2.2).

Theorem 3.2. *Let (u_k) be a sequence of functions in $L^{m,p}(\Omega)$ such that*

$$|u_k - u_l|_{m,p;\Omega} \to 0 \quad \text{as } k, l \to \infty.$$

Let $x_0 \in \Omega$ be arbitrary, but fixed, and let $P_{u_k} = P^{(x_0)}_{u_k} \in \mathcal{P}(m-1)$ the polynomial according to Theorem A:

$$\int\limits_{B_{dx_0}} D^\beta(u_k - P_{u_k})\,dx = 0 \quad \forall |\beta| \le m-1 \quad (k = 1, 2, \ldots).$$

Then there exists a $u \in L^{m,p}(\Omega)$ such that

$$\left\| u - (u_k - P_{u_k}) \right\|_{W^{m-1,p}(\Omega')} \to 0 \quad \text{as } k \to \infty, \ \forall \Omega' \subset\subset \Omega, \tag{3.6}$$

$$|u - u_k|_{m,p;\Omega} \to 0 \quad \text{as } k \to \infty. \tag{3.7}$$

If we put $G := B_{d_{x_0}}$ then with our notation $(u_k - P_{u_k}) \in L^{m,p}_{B_{d_{x_0}}}(\Omega)$. Clearly

$$|(u_k - P_{u_k}) - (u_l - P_{u_l})|_{m,p;\Omega} = |u_k - u_l|_{m,p;\Omega} \to 0 \qquad \text{as } k, l \to \infty$$

If we choose for (3.6) $\Omega' := B_{d_{x_0}}$ then $||u - (u_k - P_{u_k})||_{W^{m-1,p}(B_{d_{x_0}})} \to 0$ as $k \to \infty$ and therefore

$$\int_{B_{d_{x_0}}} D^\beta u\,dx = \lim_{k\to\infty} \int_{B_{d_{x_0}}} D^\beta(u_k - P_{u_k})\,dx = 0 \quad \text{for } |\beta| \le m-1.$$

Therefore $u \in L^{m,p}_{B_{d_{x_0}}}(\Omega)$ and we derived as a first consequence

Corollary 3.3. *Let $(u_k) \subset L^{m,p}_{B_{d_{x_0}}}(\Omega)$ be Cauchy with respect to the norm $|\cdot|_{m,p;\,\Omega}$ (coinciding with $|\!|\cdot|\!|_{m,p;\Omega,B_{d_{x_0}}}$ on $L^{m,p}_{B_{d_{x_0}}}(\Omega)$). Then there exists $u \in L^{m,p}_{B_{d_{x_0}}}(\Omega)$ such that*

$$\|u - u_k\|_{W^{m-1,p}(\Omega')} \to 0 \quad \text{as } k \to \infty, \quad \forall\, \Omega' \subset\subset \Omega,$$
$$|u - u_k|_{m,p;\Omega} \to 0 \quad \text{as } k \to \infty.$$

Based on this result, using Theorem A and the fact, that any two norms on the finite dimensional vector space $\mathcal{P}(m-1)$ are equivalent, we readily derive

Theorem 3.4. *Let $G \subset\subset \Omega$. Let $(u_k) \subset L^{m,p}_G(\Omega)$ be Cauchy with respect to the norm $|\cdot|_{m,p;\Omega}$. Then there exists $u \in L^{m,p}_G(\Omega)$ such that*

$$\|u - u_k\|_{W^{m-1,p}(\Omega')} \to 0 \quad \text{as } k \to \infty, \quad \forall\, \Omega' \subset\subset \Omega, \tag{3.8}$$
$$|u - u_k|_{m,p;\Omega} \to 0 \quad \text{as } k \to \infty.$$

Due to the direct decomposition (3.4) and using the fact that dim $\mathcal{P}(m-1) < \infty$ it follows from Theorem 3.4

Theorem 3.5. *Let $G \subset\subset \Omega$. Then $L^{m,p}(\Omega)$ is a Banach space with respect to the norm $|\!|\cdot|\!|_{m,p;\Omega,G}$.*

A further trivial consequence of (3.8) is the following Poincaré - type inequality which is of its own interest. We observe that in next theorem it ist *not* assumed that $G \subset \Omega'$ or $G \cap \Omega' \neq \emptyset$.

Theorem 3.6. *Let $G \subset\subset \Omega$. Then for every $\Omega' \subset\subset \Omega$ there exists a constant $C_{\Omega'} > 0$, such that*

$$\|u\|_{W^{m-1,p}(\Omega')} \le C_{\Omega'} |u|_{m,p;\,\Omega} \quad \forall\, u \in L^{m,p}_G(\Omega). \tag{3.9}$$

Based on (3.9) and the decomposition (3.4), we readily prove that our norms (3.2), depending on the choice of the sets G, are equivalent one to the other. Moreover they are equivalent with Sobolev's norm (1.7).

Theorem 3.7. *Let $G_i \subset\subset \Omega$ $(i = 1, 2)$. Then there exists a constant $K = K_{G_1,G_2} > 0$ such that*

$$|\!|u|\!|_{m,p;\Omega,G_1} \le K |\!|u|\!|_{m,p;\Omega,G_2} \qquad \forall\, u \in L^{m,p}(\Omega).$$

Theorem 3.8. *Let $G \subset\subset \Omega$. Then there exist constants $K_i > 0$ $(i = 1, 2)$ such that*

$$K_1 \|u\|_{m,p;\Omega,G} \leq |u|_{m,p;\Omega,G} \leq K_2 \|u\|_{m,p;\Omega,G} \quad \forall u \in L^{m,p}(\Omega).$$

Besides the compatibility (3.5) of our norm (3.2) with the direct decomposition (3.4) another advantage may be seen if we consider the quotient space

$$L^{m,p}(\Omega)/\mathcal{P}(m-1) := \{[u] : u \in L^{m,p}(\Omega)\}.$$

where as usual $[u] := \{v \in L^{m,p}(\Omega) : u - v \in \mathcal{P}(m-1)\}$, $[u] + [v] := [u + v]$ and $\lambda[u] := [\lambda u]$ for $\lambda \in \mathbb{R}$. The norm is given by (where $G \subset\subset \Omega$ is fixed)

$$\|[u]\|_{m,p;\Omega,G} := \inf\{\|v\|_{m,p;\Omega,G} : v \in [u]\}.$$

For $u \in L^{m,p}(\Omega)$ let $P_u \in \mathcal{P}(m-1)$ be the (by Theorem A even unique) polynomial so that $u_0 := (u - P_u) \in L_G^{m,p}(\Omega)$. Then $[u] = [u_0]$.
If $v_0 \in L_G^{m,p}(\Omega)$ satisfies $v_0 \in [u_0]$, then $v_0 = u_0 + q$ with $q \in \mathcal{P}(m-1)$. Then $|q|_{m-1;G} = |v_0 - u_0|_{m-1;G} = 0$, therefore $q = 0$ and $v_0 = u_0$. Therefore

$$[u_0] = \{u_0 + P : P \in \mathcal{P}(m-1)\}.$$

Then by (3.5)

$$||u_0 + P||_{m,p;\Omega,G} = |u_0|_{m,p;\Omega} + |P|_{m-1;G} \geq |u_0|_{m,p;\Omega} \qquad \forall P \in \mathcal{P}(m-1).$$

Then $||[u_0]||_{m,p;\Omega,G} = |u_0|_{m,p;\Omega}$.
If $u \in L^{m,p}(\Omega)$, $u_0 \in L_G^{m,p}(\Omega)$, $P_u \in \mathcal{P}(m-1)$ and $u = u_0 + P_u$, then, as we have seen above, an isometric isomorphic map is defined by

$$J : L^{m,p}(\Omega)/\mathcal{P}(m-1) \to L_G^{m,p}(\Omega)$$
$$[u] \to J[u] := u_0.$$

In case $p = 2$ we can define an inner product on $L^{m,2}(\Omega)$. Let again $G \subset\subset \Omega$. We set

$$(3.10) \qquad < u, v >_{m;\Omega} := \sum_{|\alpha|=m} < D^\alpha u, D^\alpha v >_\Omega \text{ for } u, v \in L^{m,2}(\Omega),$$

where $< f, g >_\Omega := \int\limits_\Omega f(x)g(x)\,dx$ for $f, g \in L^2(\Omega)$, and

$$\textbf{<} u, v \textbf{>}_{m;G} := \sum_{|\beta|\le m-1} \Big(\int\limits_G D^\beta u dx\Big)\cdot\Big(\int\limits_G D^\beta v dx\Big). \tag{3.11}$$

Then by

$$<< u, v >>_{m;\Omega,G} := < u, v >_{m;\Omega} + \textbf{<} u, v \textbf{>}_{m;G} \tag{3.12}$$

an inner product is defined on $L^{m,2}(\Omega)$. If $u = u_0 + P_u$, $v = v_0 + P_v$ with $u_0, v_0 \in L^{m,2}(\Omega)$ and $P_u, P_v \in \mathcal{P}(m-1)$ then

$$<< u, v >>_{m;\Omega,G} = < u_0, v_0 >_{m;\Omega} + \textbf{<} P_u, P_v \textbf{>}_{m;G}\ . \tag{3.13}$$

Further for $u_0, v_0 \in L^{m,2}_G(\Omega)$ we see

$$<< u_0, v_0 >>_{m;\Omega,G} = < u_0, v_0 >_{m;\Omega}$$

Clearly (3.4) holds in the sense of an *orthogonal decomposition*. By

$$|||u|||_{m,2;\Omega,G} := (<< u, u >>_{m;\Omega,G})^{\frac{1}{2}} \quad \text{for } u \in L^{m,2}(\Omega)$$

a norm is defined. Let us denote by $c(n, m)$ the number of multi-indices $\beta = (\beta_1, \dots \beta_n)$ with $|\beta| \le m-1$. Since

$$|||u|||_{m,2;\Omega,G} \le (1 + c(n, m))|u|_{m,2;\Omega,G} \tag{3.14}$$

and by Schwarz's inequality

$$|u|_{m,2;\Omega,G} \le (1 + c(n, m)^2)^{\frac{1}{2}} |||u|||_{m,2;\Omega,G} \tag{3.15}$$

we have equivalence of norms and hence $(L^{m,2}(\Omega), << .,. >>_{m;\Omega,G})$ is a Hilbert space. Finally, by Theorem 3.7, for any two $G_i \subset\subset \Omega$ $i = 1, 2)$ the corresponding inner products (3.12) are equivalent. Clearly, if $G \subset\subset \Omega$ and $\Omega \subseteq \mathbb{R}^n$ is *any domain*, then $L^{m,2}_G(\Omega) \subset L^{m,2}(\Omega)$ is a closed subspace. We regard now the case $m = 1$ and the functional equation (1.4) considered in the introduction. Since for $u_0, \Phi_0 \in L^{1,2}_G(\Omega)$ by (3.13)

$$<< u_0; \Phi_0 >>_{1;\Omega,G} = < u_0, \Phi_0 >_{1,\Omega} \equiv < \nabla u_0, \nabla\Phi_0 >_\Omega$$

we see by the Riesz-representation-theorem applied to the Hilbert space

$$(L^{1,2}_G(\Omega), < \nabla., \nabla. >_\Omega)$$

that for any $\underline{f} \in L^2(\Omega)^n$ there exists a unique $u_0 \in L^{1,2}_G(\Omega)$ such that

$$< \nabla u_o, \nabla \Phi_o >_\Omega =< \underline{f}, \nabla \Phi_o >_\Omega \qquad \forall \, \Phi_o \in L^{1,2}_G(\Omega). \tag{3.16}$$

If $c \in \mathbb{R} = \mathcal{P}(0) \subset L^{1,2}(\Omega)$, then $\nabla c = 0$ and (3.16) holds even for all $\Phi \in L^{1,2}(\Omega)$. In case $1 < p < \infty$ instead of (3.1), (3.2) for $u \in L^{m,p}(\Omega)$ we could introduce

$$|||u|||_{m,p;\Omega,G} := \left(|u|^p_{m,p;\Omega} + \sum_{|\beta| \le m-1} | \int_G D^\beta u \, dx \, |^p \right)^{\frac{1}{p}} \tag{3.17}$$

defining again a norm on $L^{m,p}(\Omega)$. We see similarly to (3.14), (3.15) that the norms defined by (3.2) and (3.17) are equivalent. For our purposes the choice of (3.2) seemed to be simpler. But if we observe that in case $1 < p < \infty$, $p' := \frac{p}{p-1}$, the right hand side of (3.10) is well defined for $u \in L^{m,p}(\Omega)$ and $v \in L^{m,p'}(\Omega)$, then $<< u, v >>_{m;\Omega;G}$ is defined. Hence by Hölder's inequality

$$| << u, v >>_{m;\Omega,G} | \le |||u|||_{m,p;\Omega,G} |||v|||_{m,p';\Omega,G}$$

for all $u \in L^{m,p}(G)$, $v \in L^{m,p'}(G)$.
A further problem is the density of a suitable subspace of smooth functions. In case of the spaces $W^{m,p}(\Omega)$ a positive answer was given by the famous "H = W"-paper by Meyers and Serrin [9]. If we carry over carefully their proof to the underlying situation, we see

Theorem 3.9. *For a domain $\Omega \subseteq \mathbb{R}^n$, $m \in \mathbb{N}$, $1 \le p < \infty$, we put*

$$C^{m,p}(\Omega) := \{\varphi \in C^\infty(\Omega) : D^\alpha \varphi \in L^p(\Omega) \, for \, |\alpha| = m\}.$$

Then $C^{m,p}(\Omega) \subset L^{m,p}(\Omega)$. In addition for any $G \subset\subset \Omega$,

$$\overline{C^{m,p}(\Omega)}^{\,|.|_{m,p;\Omega,G}} = L^{m,p}(\Omega),$$

Moreover, given $u \in L^{m,p}(\Omega)$ and $\varepsilon > 0$. Then there exists $\varphi \in C^{m,p}(\Omega)$ such that

$$||u - \varphi||_{W^{m,p}(\Omega)} \le \varepsilon. \tag{3.18}$$

Estimate (3.18) is a surprise, since neither u nor φ need to belong to $W^{m,p}(\Omega)$. But the method of proof developed in [9] is so powerful that the approximation φ of u even satisfies $(u - \varphi) \in W^{m,p}(\Omega)$ and the estimate. This result is stronger than that of [7], Chapter 1, Section 1.1.5.

4. Necessary and sufficient conditions for $L^{1,p}(\Omega) = W^{1,p}(\Omega)$.

Definition 4.1. Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ and $\partial\Omega \neq \emptyset$. We say that $\partial\Omega \in C^0$ if for each $x_0 \in \partial\Omega$ there exists an orthogonal matrix $\tilde{S}$ so that with the map $S : \mathbb{R}^n \to \mathbb{R}^n$, $Sx := \tilde{S}(x - x_o)$ the following conditions are satisfied:

(For $\alpha > 0$ let $Q'_\alpha := \{y' \in \mathbb{R}^{n-1} : |y_i| < \alpha, i = 1, ., n-1\}$). There exist $\alpha, \beta > 0$ and a continuous map $a : Q'_\alpha \to \mathbb{R}$ so that with

$$M_{\alpha\beta} : = \left\{(y', a(y') + t) : y' \in Q'_\alpha \quad |t| < \beta\right\}$$
$$S^{-1}(M_{\alpha\beta}) \cap \partial\Omega = S^{-1}\left(\left\{(y', a(y')) : y' \in Q'_\alpha\right\}\right) \subset \partial\Omega$$
$$S^{-1}(M_{\alpha\beta}) \cap \Omega = S^{-1}\left(\left\{(y', a(y') + t) : y' \in Q'_\alpha, 0 < t < \tau\right\}\right) \subset \Omega$$
$$S^{-1}(M_{\alpha\beta}) \cap (\mathbb{R}^n \backslash \bar{\Omega}) = S^{-1}\left(\left\{(y', a(y') - t) : y' \in Q'_\alpha, -\tau < t < 0\right\}\right) \subset \subset \mathbb{R}^n \backslash \bar{\Omega}.$$

Roughly speaking this condition means that after shifting the origin to $x_0 \in \partial\Omega$ and performing a suitable rotation of coordinates ($\tilde{S}$), the intersection of a neighborhood of x_0 with $\partial\Omega$ can be represented as the graph of a continuous function. Using a suitable representation in local coordinates (compare e.g. [11], Chap. 2, Théorème 7.6) and a standard covering argument one proves easily

Lemma 4.2. Let $\Omega \subset \mathbb{R}^n$ be a domain with $\partial\Omega \in C^0$. For $R > 0$ let $\Omega_R := \Omega \cap B_R$ (where $B_R := B_R(0)$). Suppose that $\Omega_R \neq \emptyset$ and $R' > R$. If $u \in L^p_{\text{loc}}(\Omega)(1 \leq p < \infty)$ and $\nabla u \in L^p(\Omega_{R'})^n$ then $u \in L^p(\Omega_R)$ and there exist $\Omega' = \Omega'(R, R', \partial\Omega) \subset\subset \Omega_{R'}$, $C_{i,R'} = C_i(R, R', \partial, \Omega) > 0 \quad (i = 1, 2)$, independently of u, so that

$$||u||^p_{p;\Omega_R} \leq C_{1,R}||\nabla u||^p_{p;\Omega_{R'}} + C_{2,R}||u||^p_{p;\Omega'} \tag{4.1}$$

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $\partial\Omega \in C^0$, let $1 \leq p < \infty$ and $m \in \mathbb{N}$. Let $R > 0$ with $\Omega_R := \Omega \cap B_R \neq \emptyset$. Then $u \mid_{\Omega_R} \in W^{m,p}(\Omega_R)$ for $u \in L^{m,p}(\Omega)$. If $G \subset\subset \Omega$, then there is a constant $C_R = C(R, \Omega, G, p) > 0$ so that*

$$||u||_{W^{m-1,p}(\Omega \cap B_R)} \leq C_R|u|_{m,p;\Omega,G} \quad \forall u \in L^{m,p}(\Omega). \tag{4.2}$$

Proof. i) We choose $R' > R$ and for $k \in \mathbb{N}_0$, $0 \le k \le m-1$, let

$$R_k := R + k \cdot \frac{(R' - R)}{m}$$

and let $\Omega_k := \Omega \cap B_{R_k}$. Let $v \in L^{m,p}(\Omega)$ and assume in addition that

$$D^\alpha v \in L^p(\Omega_{R_{k+1}}) \qquad \forall |\alpha| = k+1.$$

Let $|\beta| = k$. Then by Lemma 4.2 we see $D^\beta v \in L^p(\Omega_k)$ and with a suitable $\Omega_k' \subset\subset \Omega_{k+1}$, and with $C_{i,k} > 0$ we get

$$||D^\beta v||^p_{p;\Omega_k} \le C_{1,k} ||\nabla D^\beta v||^p_{p;\Omega_{k+1}} + C_{2,k} ||D^\beta v||^p_{p;\Omega_{k'}}.$$

With a constant $C_k := C(n, k) > 0$ we have

$$\sum_{|\beta|=k} ||\nabla D^\beta v||^p_{p;\Omega_{k+1}} \le C_k |v|^p_{k+1,p;\Omega_{k+1}}.$$

Summation over $|\beta| = k$ yields with $D_{1,k} := C_{1,k} \cdot C_k$

$$|v|^p_{k,p;\Omega_k} \le D_{1,k} |v|^p_{k+1,p;\Omega_{k+1}} + C_{2,k} |v|^p_{k,p;\Omega_k'}. \tag{4.3}$$

By (3.4) we write $v = v_0 + P_v$ with $v_0 \in L^{m,p}_G(\Omega)$ and $P_v \in \mathcal{P}(m-1)$. Because of equivalence of norms on $\mathcal{P}(m-1)$ there is $K_k = K(\Omega_k', k) > 0$ so that

$$|P|_{k,p;\Omega_k'} \le K_k |P|_{m-1;G} \qquad \forall P \in \mathcal{P}(m-1).$$

By Theorem 3.6 there is $M_k = M_k(\Omega_k', m, p) > 0$ so that

$$|v_0|_{k,p;\Omega_k'} \le M_k |v_0|_{m,p;\Omega}.$$

With $D_{2,k} = \max(K_k, M_k)^p$ because of

$$|v|_{k,p;\Omega_k'} \le |v_0|_{k,p;\Omega_k'} + |P_v|_{k,p;\Omega_k'} \le$$
$$\le M_k |v_0|_{m,p;\Omega} + K_k |P_v|_{m-1;G} \le D_{2,k}^{\frac{1}{p}} |v|_{m,p;\Omega,G}$$

we get from (4.3)

$$|v|^p_{k,p;\Omega_k} \le D_{1,k} |v|^p_{k+1,p;\Omega_{k+1}} + D_{2,k} |v|_{m,p;\Omega;G} \tag{4.4}$$

ii) We put $k := m - j$ and we prove by induction on $j = 1, \dots, m$ that $D^\beta u \in L^p(\Omega_{m-j})$ for $m - j \le |\beta| \le m$ and

$$\sum_{i=m-j}^{m-1} |u|^p_{i,p;\Omega_{m-j}} \le M_j^p |u|^p_{m,p;\Omega,G} \tag{4.5}$$

with $M_j^p > 0$. For $j = 1$, $D^\beta u \in L^p(\Omega_{m-1})$ by Lemma 4.2 and since $|u|_{m,p;\Omega} \le |u|_{m,p;\Omega,G}$ estimate (4.5) follows with $M_1 = (D_{1,k} + D_{2,k})^{\frac{1}{p}}$. If the assertion is true for some $1 \le j \le m - 1$, then by part i) we see $D^\beta u \in L^p(\Omega_{m-j-1})$ for $|\beta| = m - j - 1$ and (4.4) holds with $k = m - j - 1$. By (4.5) $|u|_{m-j-1,p;\Omega_{m-j-1}} \le M_j |u|^p_{m,p;\Omega,G}$. Last estimate we put in (4.4) (with $k = m - j - 1$) to derive (4.5) with j replaced by $(j + 1)$. For $j = m$ we get (4.2) with $C_R = M_m$. □

Corollary 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^0$ and let $G \subset\subset \Omega$ then $L^{m,p}(\Omega) = W^{m,p}(\Omega)$ for $m \in \mathbb{N}$, $1 \le p < \infty$ and there exists a constant $C = C(n, m, p, G, \Omega) > 0$ such that*

$$||u_0||_{W^{m-1,p}(\Omega)} \le C|u_0|_{m,p;\Omega} \qquad \forall u_0 \in L^{m,p}_G(\Omega) \tag{4.6}$$

and

$$||u||_{W^{m,p}(\Omega)} \le C|u|_{m,p;\Omega,G} \qquad \forall u \in L^{m,p}(\Omega). \tag{4.7}$$

Proof. We choose $R > 0$ so big that $\Omega \subset B_R$. Then (4.6) follows immediately from (4.2) and (3.5) for $u_0 \in L^{m,p}_G(\Omega)$. (4.7) follows from (4.2) with $C = (C_R^p + 1)^{\frac{1}{p}}$. □

Estimate (4.6) is a Poincaré inequality for the *whole* bounded domain Ω. We should compare this with (3.9). By (3.4) $u \in L^{m,p}(\Omega)$ may be written $u = u_0 + P_u$ with $u_0 \in L^{m,p}_G(\Omega)$ and $P_u \in \mathcal{P}(m-1)$. We write $\frac{1}{p'} := \frac{p-1}{p}$ for $1 \le p < \infty$. By (3.1) and Hölder's inequality (in case $1 < p < \infty$) using (2.2) we see

$$\begin{aligned} |P_u|_{m-1,G} &\le \sum_{|\beta|\le m-1} ||D^\beta P_u||_{p,G} |G|^{\frac{1}{p'}} \le \\ &\le c(n,m)|G|^{\frac{1}{p'}} \left(\sum_{|\beta|\le m-1} ||D^\beta P_u||^p_{p,G} \right)^{\frac{1}{p}} \le \\ &\le c(n,m)|G|^{\frac{1}{p'}} C||u||_{W^{m-1,p}(G)} =: K||u||_{W^{m-1,p}(G)}. \end{aligned}$$

where $c(n, m)$ denotes the number of all multi-indices $\beta = (\beta_1, \ldots, \beta_n)$ with $|\beta| \leq m - 1$. Then

$$|u|_{m,p;\Omega,G} = |u_0|_{m,p;\Omega} + |P_u|_{m-1,G} \equiv$$
$$\equiv |u|_{m,p;\Omega} + |P_u|_{m-1,G} \leq (1 + K)||u||_{W^{m,p}(G)}.$$

Together with (4.7) this proves equivalence of norms on $W^{m,p}(\Omega)$ in the case of a *bounded domain* Ω with $\partial\Omega \in C^0$. This result we find e.g. in Nečas [11], Chap. 2, Théorème 7.6. Contrary to that case, for *general unbounded domains* Ω with $\partial\Omega \in C^0$, the result of Theorem 4.3 seems to be best possible because of $\mathcal{P}(m-1) \subset L^{m,p}(\Omega)$. Clearly the function $u(x) := 1$ for $x \in \Omega$ satisfies $u \in \mathcal{P}(m-1)$ for all $m \geq 1$, but $u \in W^{m,p}(\Omega)$ if and only if $|\Omega| < \infty$. Therefore $W^{m,p}(\Omega) \underset{\neq}{\subset} L^{m,p}(\Omega)$ if $|\Omega| = \infty$.

If $\Omega \subset \mathbb{R}^n$ is a *bounded domain*, $m \in \mathbb{N}$ and $1 \leq p < \infty$, then it is called e.g. by Nečas ([11], Chapt. 2, Sect. 7.3) a (m, p) - *Nikodym-domain*, if $W^{m,p}(\Omega) = L^{m,p}(\Omega)$ (due to our Theorem 3.1 we see that the spaces $V_p^m(\Omega)$ defined in [11] satisfy $V_p^{(m)}(\Omega) = L^{m,p}(\Omega)$). By the remark above, a (m, p) - *Nikodym-domain* satisfies necessarily $|\Omega| < \infty$. Even in case $m = 1$ and $p = 2$ there exist bounded domains $\Omega \subset \mathbb{R}^n$ so that $W^{1,2}(\Omega) \underset{\neq}{\subset} L^{1,2}(\Omega)$, as was proved by Nykodym [12] (see e.g. [7], Sect 1.1.4. Similar examples had been given later by Courant-Hilbert [3], p. 521. Compare our Appendix 1). With respect to Corollary 4.4, those domains must have a "bad" boundary $\partial\Omega$.

For our next considerations we restrict ourselves to the case $m = 1$ and $1 \leq p < \infty$. If $\Omega \subset \mathbb{R}^n$ is an arbitrary domain then for $u \in W^{1,p}(\Omega)$ by (compare (1.8))

$$(4.8) \qquad |u|_{1,p;\Omega} \equiv ||\nabla u||_{p;\Omega} = \left(\sum_{i=1}^{n} ||\partial_i u||_p^p \right)^{\frac{1}{p}}$$

a semi-norm is defined. In case $|\Omega| < \infty$ the constant functions belong to $W^{1,p}(\Omega)$. We rule them out if we choose an open $\emptyset \neq G \subseteq \Omega$ and consider

$$(4.9) \qquad W_G^{1,p}(\Omega) := \left\{ u \in W^{1,p}(\Omega) : \int_G u(x)\, dx = 0 \right\}.$$

This linear space is even well defined for an arbitrary domain $\Omega \subseteq \mathbb{R}^n$ if $|G| < \infty$. Because of Theorem B by $|.|_{1,p;\Omega}$ even a norm is defined on $W_G^{1,p}(\Omega)$.

If $|\Omega| = \infty$ and if $u \in W^{1,p}(\Omega)$ satisfies $\nabla u = 0$, then $u(x) = c \in \mathbb{R}$ a.e. and because of $u \in L^p(\Omega)$ we see $c = 0$. Therefore in case $|\Omega| = \infty$ by $|.|_{1,p;\Omega}$ a norm is defined on $W^{1,p}(\Omega)$. In both cases we study the question whether these normed spaces are complete or not.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain, let $G \subseteq \Omega$ be an open set with $0 < |G| < \infty$ and let $1 \le p < \infty$. Then, $(W^{1,p}_G(\Omega), |.|_{1,p;\Omega})$ is complete if and only if there exists a constant $C > 0$ so that*

$$||u||_{p;\Omega} \le C|u|_{1,p;\Omega} \qquad \forall u \in W^{1,p}_G(\Omega) \tag{4.10}$$

(Poincaré's inequality).

Proof. a) Assume (4.10) to hold. Then

$$||u||_{W^{1,p}(\Omega)} \le (1 + C^p)^{\frac{1}{p}} |u|_{1,p;\Omega} \qquad \forall u \in W^{1,p}_G(\Omega). \tag{4.11}$$

If $(u_j) \subset W^{1,p}_G(\Omega)$ with $|u_j - u_k|_{1,p;\Omega} \to 0$ $(j, k \to \infty)$, then by (4.11) and because of completeness of $W^{1,p}(\Omega)$ there exists $u \in W^{1,p}(\Omega)$ with $||u - u_j||_{W^{1,p}(\Omega)} \to 0$. Since $|G| < \infty$ we see

$$\int_G u\,dx = \lim_{j\to\infty} \int_G u_j\,dx = 0 \quad \text{and therefore} \quad u \in W^{1,p}_G(\Omega),\ |u - u_j|_{1,p;\Omega} \to 0.$$

b) i) Assume that $(W^{1,p}_G, |.|_{1,p;\Omega})$ is complete. Let

$$J : W^{1,p}_G(\Omega) \to L^p_G(\Omega), \qquad Ju := u,$$

$$\text{(where} \quad L^p_G(\Omega) := \{v \in L^p(\Omega) : \int_G v\,dx = 0\}).$$

Let $(u_j) \subset W^{1,p}_G(\Omega)$, $u \in W^{1,p}_G(\Omega)$ so that $|u - u_j|_{1,p;\Omega} \to 0$ and let $v \in L^p_G(\Omega)$ with $||v - u_j||_{p;\Omega} \equiv ||v - Ju_j||_{p;\Omega} \to 0$. Then for $\varphi \in C^\infty_c(\Omega)$ and $i = 1, \dots, n$ we see

$$\int_\Omega v \partial_i \varphi = \lim_{j\to\infty} \int_\Omega u_j \partial_i \varphi = -\lim_{j\to\infty} \int_\Omega \partial_i u \varphi.$$

Therefore v has the weak ∂_i-derivative $\partial_i u \in L^p(\Omega)$, $i = 1, \dots, n$.

ii) Since $v \in L^p_G(\Omega)$ we get $v \in W^{1,p}_G(\Omega)$. Because of $|u - v|_{1,p;\Omega} = 0$ we conclude $v = u$ and the closedness of J. By means of Banach's closed graph theorem the operator J is bounded and (4.10) holds with $C > 0$. □

Remark 4.6. Let Ω, G and p be as in Theorem 4.5. Let $m \in \mathbb{N}$ and let $W_G^{m,p}(\Omega)$ be defined analogously to (3.3). By means of a conclusion completely analogous to part b) of proof of Theorem 4.5, we see directly that completeness of $(W_G^{m,p}(\Omega), |.|_{m,p;\Omega})$ is equivalent to $(C_m > 0)$

$$||u||_{W^{m-1,p}(\Omega)} \leq C_m |u|_{m,p;\Omega} \qquad \forall u \in W_G^{m,p}(\Omega) \tag{4.12}$$

Clearly, if (4.10) holds, then (4.12) follows by iterated application of (4.10). But we didn't succeed to prove conversely that the validity of (4.12) for some $m \geq 2$ implies (4.10). Similiarly if (4.10) holds for a p with $1 \leq p < \infty$, we could not prove that it holds for other $1 \leq s < \infty$ too.

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $G \subseteq \Omega$ be an open set with $0 < |G| < \infty$. If with a constant $C > 0$ the Poincaré-inequality (4.10) holds for all $u \in W_G^{1,p}(\Omega)$, then for every open $G' \subseteq \Omega$ with $0 < |G'| < \infty$ there exists a constant $C_{G'} > 0$ so that*

$$||u||_{p;\Omega} \leq C_{G'} |u|_{1,p;\Omega} \qquad \forall u \in W_{G'}^{1,p}(\Omega).$$

Proof. Suppose that $\emptyset \neq G' \subseteq \Omega$ is open, $|G'| < \infty$ and that the Poincaré-inequality does not apply to $W_{G'}^{1,p}(\Omega)$. Then there is a sequence $(u_k) \subset W_{G'}^{1,p}(\Omega)$ so that $||u_k||_{p;\Omega} = 1$ and $|u_k|_{1,p;\Omega} \to 0$. Since $\emptyset \neq G$ is open, there is a ball $B = B_\varrho(x_0) \subset G$ $(x_0 \in G,\ \varrho > 0)$ and $0 \leq \varphi \in C_c^\infty(B)$ with $A := \int_B \varphi(y)dy > 0$. We set $c_k := \frac{1}{A}\int_G u_k dy$ and $\upsilon_k := u_k - c_k\varphi$. Then

$$\int_G \upsilon_k dy = \int_G u_k dy - c_k \int_G \varphi dy = \int_G u_k dy - c_k A = 0.$$

Therefore $\upsilon_k \in W_G^{1,p}(\Omega)$. Further,

$$|c_k| \leq A^{-1}|G|^{\frac{1}{p'}} ||u_k||_{p;G} \leq A^{-1}|G|^{\frac{1}{p'}}.$$

Then there exists a subsequence (again denoted by c_k) and $c \in \mathbb{R}$ such that $c = \lim_{k\to\infty} c_k$. Further

$$\begin{aligned} |\upsilon_k - \upsilon_j|_{1,p;\Omega} &= |u_k - u_j + (c_k - c_j)\varphi|_{1,p;\Omega} \leq \\ &\leq |u_k - u_j|_{1,p;\Omega} + |c_k - c_j|\, |\varphi|_{1,p;\Omega} \to 0 \end{aligned}$$

as $k, j \to \infty$. By completeness of $W_G^{1,p}(\Omega)$ (Theorem 4.5) there exists $\upsilon \in W_G^{1,p}(\Omega)$ with $|\upsilon - \upsilon_k|_{1,p;\Omega} \to 0$. Because of (4.10) we see $||\upsilon - \upsilon_k||_{p;\Omega} \to 0$. We set $u := \upsilon + c\varphi$. Then $u \in W^{1,p}(\Omega)$ and

$$||u - u_k||_{p,\Omega} \leq ||\upsilon - \upsilon_k||_{p;\Omega} + |c - c_k|||\varphi||_{p;\Omega} \to 0 \qquad (k \to \infty).$$

Then $||u||_{p;\Omega} = \lim_{k\to\infty} ||u_k||_{p;\Omega} = 1$. Further, $\int_{G'} u dx = \lim_{k\to\infty} \int_{G'} u_k dx = 0$, therefore $u \in W_{G'}^{1,p}(\Omega)$. If $\varphi \in C_c^\infty(\Omega)$ then for $i = 1, \ldots, n$

$$\int_\Omega u \partial_i \varphi = \lim_{k\to\infty} \int_\Omega u_k \partial_i \varphi = -\lim_{k\to\infty} \int_\Omega \partial_i u_k \varphi = 0$$

Therefore $\nabla u = 0$ a. e., $u \in W_{G'}^{1,p}(\Omega)$ and so $u = 0$ a. e., contradicting $||u||_{p,\Omega} = 1$. □

A first application of last theorem is the proof of

Lemma 4.8. *Let $\Omega \subset \mathbb{R}^n$ be a domain, let $\emptyset \neq G \subseteq \Omega$ be an open set with $|G| < \infty$ and assume that (4.10) holds for some $1 \leq p < \infty$. Then $|\Omega| < \infty$.*

Proof. i) For $\alpha \in \mathbb{R}$ and $x \neq 0$ we consider $\varphi(x) := e^{\alpha|x|}$. Then $\varphi \in C^\infty(\mathbb{R}^n \backslash \{0\})$ and

$$|\partial_i \varphi(x)| \leq |\alpha| e^{\alpha|x|}, \quad i = 1, \ldots, n. \tag{4.13}$$

ii) Because of Theorem 4.7 without any restriction we may assume $\emptyset \neq G \subset\subset \Omega$. We choose $G' \subset\subset \Omega$ so that $G \subset\subset G' \subset\subset \Omega$, and $\eta \in C^\infty(\mathbb{R}^n)$ with the properties $\eta |_{\Omega\backslash G'} = 1$ and $\eta |_G = 0$. Then $\text{supp}|\nabla\eta| \subset G'$.

iii) Let now $\alpha < 0$ and assume without loss of generality that $0 \in G$. We set $u(x) := \eta(x)e^{\alpha|x|}$ for $x \in \mathbb{R}^n$. Then $u \in C^\infty(\mathbb{R}^n)$, $u \in L^p(\mathbb{R}^n)$ and $\nabla u \in L^p(\mathbb{R}^n)^n$. Therefore the restriction of u to Ω (again denoted by u) satisfies $u \in W_G^{1,p}(\Omega)$. By (4.10), (4.13)

$$||\eta e^{\alpha|.|}||_{p;\Omega} \leq C|\alpha|||\eta e^{\alpha|.|}||_{p;\Omega} + C||e^{\alpha|.|}\nabla\eta||_{p;\Omega} \leq$$
$$\leq C|\alpha|||\eta e^{\alpha|.|}||_{p;\Omega} + C||\nabla\eta||_{\infty;\mathbb{R}^n}||e^{\alpha|.|}||_{p;G'}$$

since $\text{supp}|\nabla\eta| \subset G'$. We choose $\alpha < 0$ with $|\alpha| < \frac{1}{2C}$. Then we see

$$||\eta e^{\alpha|.|}||_{p;\Omega} \leq \frac{C}{1 - C|\alpha|}||\nabla\eta||_{\infty;\mathbb{R}^n}|G'|^{\frac{1}{p}}$$

If we pass to the limit $\alpha \to 0$ wee see by Levi's theorem (or by Fatou's lemma) $||\eta||_{p;\Omega} \leq C||\nabla\eta||_{\infty;\mathbb{R}^n}|G'|^{\frac{1}{p}} < \infty$. Since $||\eta||_{p;\Omega} \geq ||\eta||_{p;\Omega\backslash G'} = |\Omega\backslash G'|^{\frac{1}{p}}$ we finally derive

$$|\Omega| = |\Omega\backslash G'| + |G'| \leq (C^p||\nabla\eta||_{\infty;\mathbb{R}^n}^p + 1)|G'| < \infty. \qquad \square$$

From the proofs of Theorem 4.4 and Lemma 4.7 we easily deduce

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $|\Omega| = \infty$. Then $(W^{1,p}(\Omega), |.|_{1,p;\Omega})$ is a normed linear space for $1 \le p < \infty$, but it is not complete. Furthermore, there is no constant $c > 0$ so that estimate (4.10) holds for all $u \in W^{1,p}(\Omega)$.*

Proof. As we mentioned above, $|.|_{1,p;\Omega}$ is a norm on $W^{1,p}(\Omega)$ if $|\Omega| = \infty$. Suppose now that $(W^{1,p}(\Omega), |.|_{1,p;\Omega})$ would be complete. We proceed as in part b. i) of the proof of Theorem 4.5 (replacing $W_G^{1,p}(\Omega)$ by $W^{1,p}(\Omega)$ and $L_G^p(\Omega)$ by $L^p(\Omega)$). Then we find $v \in W^{1,p}(\Omega)$ with $|\nabla v - \nabla u|_{1,p;\Omega} = 0$ and therefore $v = u$. Again by the closed graph theorem with a constant $C > 0$ estimate (4.10) would hold for *all* $u \in W^{1,p}(\Omega)$. With literally the same arguments used in the proof of Lemma 4.8 we would see $|\Omega| < \infty$. □

Theorem 4.10. *Let $\Omega \subset \mathbb{R}^n$ be a domain, let $\emptyset \neq G \subseteq \Omega$ be an open set with $|G| < \infty$ and let $1 \le p < \infty$. If $(W_G^{1,p}(\Omega), |.|_{1,p;\Omega})$ is complete (or equivalently, if (4.10) holds), then there exists $\beta_0 > 0$ so that $e^{\beta_0|.|} \in L^p(\Omega)$ and in addition there is $D > 0$ so that*

$$|\Omega \cap (\mathbb{R}^n \backslash B_R)| \le D \cdot e^{-p\beta_0 R} \quad \textit{for} \quad R \ge R_0. \tag{4.14}$$

Proof. Because of Theorem 4.7 we may assume $G \subset\subset \Omega$. Like in part ii) of proof of Lemma 4.8 we choose G' with $G \subset\subset G' \subset\subset \Omega$ and $\eta \in C^\infty(\mathbb{R}^n)$. For $r > 0$ and $\beta > 0$ we set

$$\left(e^{\beta|x|}\right)_r := \begin{cases} e^{\beta|x|} & \text{for} \quad |x| \le r \\ e^{\beta r} & \text{for} \quad |x| > r \end{cases}$$

and $u_r := \eta \cdot (e^{\beta|.|})_r$. By Lemma 4.8 we know $|\Omega| < \infty$ and therefore $u_r \in L^p(\Omega)$ for $1 \le p < \infty$. As is readily seen, u_r has weak derivatives $\partial_i u_r \in L^p(\Omega)$ $(i = 1, \dots, n)$ given by

$$\partial_i u_r(x) = (\partial_i \eta)(x)(e^{\beta|x|})_r + \eta(x) \begin{cases} \beta e^{\beta|x|} \frac{x_i}{|x|} & \text{if} \quad |x| \le r \\ 0 & \text{else} \end{cases} \tag{4.15}$$

Then $u_r \in W_G^{1,p}(\Omega)$. Let $d_\beta := ||\nabla \eta||_\infty \sup\left\{e^{\beta|x|} : x \in G'\right\}$.
Since $\text{supp}|\nabla\eta| \subset G'$ we see $||\nabla\eta(e^{\beta|.|})_r||_{p,\Omega} \le d_\beta |G'|^{\frac{1}{p}}$. Then, by (4.10), (4.15) we get

$$||\eta(e^{\beta|.|})_r||_{p,\Omega} \le C\beta ||\eta e^{\beta|.|}||_{p,\Omega\cap B_r} + C d_\beta |G'|^{\frac{1}{p}}.$$

Clearly,

$$||\eta e^{\beta|.|}||_{p,\Omega\cap B_r} \leq ||\eta(e^{\beta|.|})_r||_{p,\Omega}.$$

We choose $\beta = \beta_0 := \frac{1}{2C}$. Then

$$||\eta(e^{\beta_0|.|})_r||_{p,\Omega} \leq 2Cd_{\beta_0}|G'|^{\frac{1}{p}} \qquad \text{for all } r > 0.$$

By Levi's theorem we may pass to the limit $r \to \infty$ to see $\eta e^{\beta_0|.|} \in L^p(\Omega)$ and therefore $e^{\beta_0|.|} \in L^p(\Omega)$. Then

$$D^p := ||e^{\beta_0|.|}||^p_{p,\Omega} \geq \int_{\Omega\cap(\mathbb{R}^n\setminus B_R)} e^{p\beta_0|x|}\,dx \geq e^{p\beta_0 R}|\Omega \cap (\mathbb{R}^n\setminus B_R)|. \qquad \square$$

Remark 4.11. The result of Theorem 4.10 is best possible in the sense that Ω *needs not* to be bounded. Let $\alpha > 0$ and consider ($n \geq 2$)

$$\Omega := \left\{(x', x_n) \in \mathbb{R}^n : |x'| < e^{-\alpha x_n}, \quad 1 < x_n < \infty\right\}.$$

It is easy to see that Ω supports the Poincaré-estimate (4.10) for $1 \leq p < \infty$ and that (4.14) holds: If $G \subset\subset \Omega$ and $u \in W^{1,p}_G(\Omega)$ one has only to write $u(x', x_n) = u_0(x', x_n) + h(x_n)$ where

$$h(x_n) := \frac{1}{|B'_{x_n}|}\int_{B'_{x_n}} u(y', x_n)dy'$$

(where $B'_{x_n} := \{y' \in \mathbb{R}^{n-1} : |y'| < e^{-\alpha x_n}\}$) and $u_0 := u - h$. For every x_n u_0 has vanishing mean value over the cross-section B'_{x_n}. Since for any fixed $1 < x_n < \infty$ the $(n-1)$ dimensional Poincaré inequality holds, the desired estimate for u_0 follows by means of Fubini's theorem. For h one has to apply a Hardy-typed estimate. $\square$

Theorem 4.12. *Let $\Omega \subset \mathbb{R}^n$ be a domain, let $\emptyset \neq G \subset\subset \Omega$ and let $1 \leq p < \infty$. Suppose that the Poincaré inequality (4.10) holds with some $C > 0$ for all $u \in W^{1,p}_G(\Omega)$. Then $L^{1,p}_G(\Omega) = W^{1,p}_G(\Omega)$.*

Proof. Clearly, $W^{1,p}_G(\Omega) \subseteq L^{1,p}_G(\Omega)$ and it remains to prove the converse inclusion. For $k \in \mathbb{N}$ let $\Phi_k : \mathbb{R} \to \mathbb{R}$,

$$\Phi_k(x) := \begin{cases} t & \text{for} \quad |t| \leq k \\ k\dfrac{t}{|t|} & \text{for} \quad |t| > k. \end{cases}$$

Then Φ_k is Lipschitz and even $\Phi_k \in C^\infty(\mathbb{R}\backslash\{-k,k\})$. By Lemma 4.8 we know $|\Omega| < \infty$. If $u \in L^{1,p}_G(\Omega)$, then $\Phi_k(u) := \Phi_k \circ u$ is measurable, $\Phi_k(u) \in L^\infty(\Omega)$ and therefore $\Phi_k(u) \in L^p(\Omega)$. Since for any $\Omega' \subset\subset \Omega$ we have $u\mid_{\Omega'} \in W^{1,p}(\Omega')$, by the chain rule for the spaces $W^{1,p}(\Omega')$ (see e.g. [4], Section 7.4) we see $\Phi_k(u)\mid_{\Omega'} \in W^{1,p}(\Omega')$ and for $x \in \Omega'$

$$(4.16)\qquad (\partial_i \Phi_k(u))(x) \underset{\text{a.e.}}{=} \begin{cases} \partial_i u(x) & \text{for} \quad |u(x)| \le k \\ 0 & \text{for} \quad |u(x)| > k \end{cases} \qquad i = 1, \ldots, n.$$

The functions at the right hand side belong (for every $k \in \mathbb{N}$) even to $L^p(\Omega)$. Since $\Omega' \subset\subset \Omega$ was arbitrary we finally see $\Phi_k(u) \in W^{1,p}(\Omega)$. We see $|\Phi_k(u)| \le |u|$ and $\Phi_k(u) \to u \in \Omega$. Since $u\mid_{\Omega'} \in L^p(\Omega')$ for each $\Omega' \subset\subset \Omega$ we see

$$(4.17)\qquad ||u - \Phi_k(u)||_{p;\Omega'} \to 0 \qquad (k \to \infty)$$

Let $c_k := |G|^{-1} \int_G \Phi_k(u)dy$. Then from (4.17) with $\Omega' = G$ we derive $c_k \to |G|^{-1} \int_G u dy = 0$. Now $(\Phi_k(u) - c_k) \in W^{1,p}_G(\Omega)$ and by (4.10), (4.16)

$$|| (\Phi_k(u) - c_k) - \left(\Phi_j(u) - c_j\right) ||_{p;\Omega} \le C|\Phi_k(u) - \Phi_j(u)|_{1,p;\Omega} \to 0 \text{ as } k, j \to 0$$

Then

$$||\Phi_k(u) - \Phi_j(u)||_{p;\Omega} \le || (\Phi_k(u) - c_k) - \left(\Phi_j(u) - c_j\right) ||_{p;\Omega} + |c_k - c_j||\Omega|^{\frac{1}{p}} \to 0.$$

Then there is $\upsilon \in L^p(\Omega)$ so that $||\upsilon - \Phi_k(u)||_{p;\Omega} \to 0$. On the other hand, by (4.17) $u\mid_{\Omega'} = \upsilon\mid_{\Omega'}$ a.e. in $\Omega' \subset\subset \Omega$. If we use a sequence (Ω_j) with $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$ for all $j \in \mathbb{N}$, $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$ then we get finally $u = \upsilon$ a. e. in Ω and therefore $u \in L^p(\Omega)$. Then $u \in W^{1,p}_G(\Omega)$. □

Theorem 4.13. *Let $\Omega \subset \mathbb{R}^n$ be a domain. Then the following statements are equivalent*

1. $L^{1,p}(\Omega) = W^{1,p}(\Omega)$ *(as vector spaces)* ($\Rightarrow |\Omega| < \infty$).
2. $L^{1,p}_G(\Omega) = W^{1,p}_G(\Omega)$ *for all* $\emptyset \ne G \subset\subset \Omega$.
3. $\left(W^{1,p}_G(\Omega), |.|_{1,p;\Omega}\right)$ *is complete for all* $\emptyset \ne G \subset\subset \Omega$.
4. *The Poincaré-estimate* (4.10) *holds for all* $u \in W^{1,p}_G(\Omega)$ *and for all* $\emptyset \ne G \subseteq \Omega$.
5. *For every* $\emptyset \ne G \subset\subset \Omega$ *the norms* $||.||_{W^{1,p}(\Omega)}$ *and* $|.|_{1,p;\Omega,G}$ *are equivalent on* $W^{1,p}(\Omega)$.

Proof. 1. $\Rightarrow$ 2. $u(x) := 1$ for $x \in \Omega$, $u \in L^{1,p}(\Omega) = W^{1,p}(\Omega)$, therefore $u \in L^p(\Omega)$ and necessarily $|\Omega| < \infty$. If $u \in L_G^{1,p}(\Omega) \subset L^{1,p}(\Omega) = W^{1,p}(\Omega)$, then clearly $u \in W_G^{1,p}(\Omega)$. Because trivially $W_G^{1,p}(\Omega) \subset L_G^{1,p}(\Omega)$ 2. follows.

2. $\Rightarrow$ 3. If $(u_k) \subset W_G^{1,p}(\Omega)$ with $|u_k - u_j|_{1,p;\Omega} \to 0$ (as $k, j \to \infty$), then by completeness of $L_G^{1,p}(\Omega)$ with respect to $|.|_{1,p;\Omega}$-norm (Theorem 3.4), there exists $u \in L_G^{1,p}(\Omega) = W_G^{1,p}(\Omega)$ with $|u - u_k|_{1,p;\Omega} \to 0$.

3. $\Rightarrow$ 4. Theorem 4.5.

4. $\Rightarrow$ 1. By Theorem 4.12 $L_G^{1,p}(\Omega) = W_G^{1,p}(\Omega)$. Further by Lemma 4.8 $|\Omega| < \infty$. Then $\mathscr{P}(0) = \mathbb{R} \subset W^{1,p}(\Omega)$. Because of (3.4)

$$L^{1,p}(\Omega) = L_G^{1,p}(\Omega) \oplus \mathbb{R} = W_G^{1,p}(\Omega) \oplus \mathbb{R} = W^{1,p}(\Omega).$$

4. $\Rightarrow$ 5. By Hölder's inequality and (3.2) for $u \in W^{1,p}(\Omega)$

$$\begin{aligned} |u|_{1,p;\Omega,G} &= |u|_{1,p;\Omega} + |u|_{o;G} \leq \\ &\leq |u|_{1,p;\Omega} + ||u||_{p,G}|G|^{\frac{1}{p'}} \leq \\ &\leq (1 + |G|)^{\frac{1}{p'}} ||u||_{W^{1,p}(\Omega)}. \end{aligned} \tag{4.18}$$

By (3.4) we write $u = u_0 + c$, $u_0 \in W_G^{1,p}(\Omega)$, $c \in \mathbb{R}$. Then by (4.10)

$$\begin{aligned} ||u||_{W^{1,p}(\Omega)} &\leq ||u_0||_{W^{1,p}(\Omega)} + ||c||_{W^{1,p}(\Omega)} \leq \\ &\leq (C^p + 1)^{\frac{1}{p}} ||u_0||_{1,p;\Omega} + |c||\Omega|^{\frac{1}{p}} \\ &\leq (C^p + 1)^{\frac{1}{p}} ||u_0||_{1,p;\Omega} + |G|^{-1}|\Omega|^{\frac{1}{p}} \left|\int_G u dx\right| \\ &\leq \max\left\{(C^p + 1)^{\frac{1}{p}}, |G|^{-1}|\Omega|^{\frac{1}{p}}\right\} |u|_{1,p;\Omega,G}. \end{aligned}$$

5. $\Rightarrow$ 4. Let with a constant $D > 0$

$$||u||^p_{W^{1,p}(\Omega)} = ||u||^p_{p;\Omega} + |u|^p_{1,p;\Omega} \leq D^p \left(|u|_{1,p;\Omega} + |u|_{0;G}\right)^p$$

for all $u \in W^{1,p}(\Omega)$ (see (3.1), (3.2)). If $u_0 \in W_G^{1,p}(\Omega)$, $u_0 \neq 0$, then we see $D > 1$ and

$$||u_0||_{p,\Omega} \leq (D^p - 1)^{\frac{1}{p}} |u_0|_{1,p;\Omega}$$

for all $u_0 \in W_G^{1,p}(\Omega)$.

Remark 4.14. a) Once we know that either 1. or 2. hold, then because of $|\Omega| < \infty$ we may allow in 2. – 5. even $\emptyset \neq G \subseteq \Omega$.

b) Suppose that $\Omega \subset \mathbb{R}^n$ is a domain, that $1 \leq p < \infty$, $\emptyset \neq G \subset\subset \Omega$ and that $L_G^{1,p}(\Omega) = W_G^{1,p}(\Omega)$ holds, then from (4.10) by iterated application we see (4.12) for $u \in W_G^{m,p}(\Omega)$ ($m \in \mathbb{N}$ arbitrary). If $m \geq 2$ and $u \in W_G^{m,p}(\Omega)$, then for $|\beta| = m-1$ we see $D^\beta u \in L_G^{1,p}(\Omega) = W_G^{1,p}(\Omega)$ and therefore $D^\beta u \in L^p(\Omega)$. Iterating this argument, we see finally $u \in W_G^{m,p}(\Omega)$. By Theorem 4.10 there is $\beta_0 > 0$ so that $e^{\beta_0|.|} \in L^p(\Omega)$. If $P \in \mathcal{P}(m-1)$ then there is constant $K_P = K(P, \beta) > 0$ so that

$$|P(x)| \leq K_P e^{\beta_0|x|} \quad \text{for } x \in \Omega.$$

Then we see $D^\beta P \in L^p(\Omega)$ for $|\beta| \leq m$ and for all $P \in \mathcal{P}(m-1)$. Therefore $\mathcal{P}(m-1) \subset W^{m,p}(G)$. Then, $L^{m,p}(G) = W^{m,p}(G)$. This proves the inclusion "$L^{1,p}(\Omega) = W^{1,p}(\Omega) \Rightarrow L^{m,p}(\Omega) = W^{m,p}(\Omega) \quad \forall m \in \mathbb{N}$". $\square$

Appendix 1: Example of a bounded domain $\Omega \subset \mathbb{R}^2$ with $W^{1,p}(\Omega) \underset{\neq}{\subset} L^{1,p}(\Omega)$.

For $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ let $a_k := \sum_{j=0}^{k} 2^{-j}$. Then $a_k - a_{k-1} = 2^{-k}$ for $k \in \mathbb{N}$. If $1 \leq p < \infty$ let

$$\begin{aligned} Q_0 &:= \big\{(x_1, x_2) \in \mathbb{R}^2 : |x_i| < 1, i = 1, 2\big\} \\ H_k^{(p)} &:= \big\{(x_1, x_2) \in \mathbb{R}^2 : a_{2k} \leq x_1 \leq a_{2k+1}, |x_2| < 2^{-3pk}\big\} \quad \text{for } k \in \mathbb{N}_0 \\ Q_k &:= \big\{(x_1, x_2) \in \mathbb{R}^2 : a_{2k-1} < x_1 < a_{2k}, |x_2| < 1\big\} \quad \text{for } k \in \mathbb{N}. \end{aligned}$$

Let

$$\begin{aligned} M_+^{(p)} &:= H_0^{(p)} \cup \bigcup_{k=1}^{\infty} (Q_k \cup H_k^{(p)}) \\ M_-^{(p)} &:= \{(x_1, x_2) \in \mathbb{R}^2 : (-x_1, x_2) \in M_+^{(p)}\}. \end{aligned}$$

Then $\Omega^{(p)} := Q_0 \cup M_+^{(p)} \cup M_-^{(p)}$ is a domain with $\Omega^{(p)} \subset \{x \in \mathbb{R}^2 : |x_1| <$

2, $|x_2| < 1\}$. Let $f : \Omega \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 0 & \text{in} \quad Q_0 \\ 2^{\frac{2}{p}+1}(x_1 - a_0) & \text{for} \quad x \in H_0^{(p)} \\ 2^{\frac{2}{p}\cdot k} & \text{for} \quad x \in Q_k, k \in \mathbb{N} \\ (2^{\frac{2}{p}} - 1)2^{2k+1+\frac{2}{p}\cdot k}(x_1 - a_{2k}) + 2^{\frac{2}{p}\cdot k} & \text{for} \quad x \in H_k^{(p)}, k \in \mathbb{N} \\ -f(-x_1, x_2) & \text{for} \quad x \in M_-^{(p)}. \end{cases}$$

For $n \in \mathbb{N}$ we set

$$\Omega_n := \Omega \cap \{x \in \mathbb{R}^2 : |x_1| < a_{2n}\}.$$

Then we define

$$f_n(x) := \begin{cases} f(x) & \text{for} \quad x \in \Omega_n \\ 2^{\frac{2}{p}\cdot k} \operatorname{sgn} f(x) & \text{for} \quad x \in \Omega \backslash \Omega_n. \end{cases}$$

Then $f_n, f : \Omega^{(p)} \to \mathbb{R}$ are continuous, $f_n \to f (n \to \infty)$ pointwise. It is easy to see that f is even piecewise continuously differentiable with respect to x_1, therefore weakly differentiable, and that $\partial_2 f(x) = 0$ for all $x \in \Omega$. Further

$$\partial_1 f(x_1, x_2) = \begin{cases} 0 & \text{in} \quad Q_0 \\ 2^{\frac{2}{p}+1} & \text{in} \quad H_0^{(p)} \\ 0 & \text{in } Q_k \\ (2^{\frac{2}{p}} - 1)2^{2k+1+\frac{2}{p}\cdot k} & \text{in} \quad H_k^{(p)} \end{cases}$$

For $n \in \mathbb{N}$ we see

$$\int_\Omega |f_n(x)|^p \, dx \geq 2 \sum_{k=1}^{n} 2^{2k} |Q_k| = 2 \sum_{k=1}^{n} 2^{2k-2k+1} = 4n \to \infty \ (n \to \infty)$$

$$\int_\Omega |\partial_1 f_n(x)|^p dx \leq 2 \cdot 2^{2+p} |H_0^{(p)}| + 2 \cdot (2^{\frac{2}{p}} - 1)^p \sum_{k=1}^{\infty} 2^{p(2k+1)+2k} |H_k^{(p)}|$$

$$\leq 2^{3+p} + 2^{p+1}(2^{\frac{2}{p}} - 1)^p \sum_{k=1}^{\infty} 2^{-2pk} < \infty \text{ for all } n \in \mathbb{N}.$$

Since $|f_n(x)| \leq |f(x)|$ and $||f_n||_{p;\Omega} \to \infty$ we see $f \notin L^p(\Omega)$, but clearly $f \in L^1_{\text{loc}}(\Omega)$. Therefore $f \in L^{1,p}(\Omega)$ and $f_n \in W^{1,p}(\Omega)$ for all $n \in \mathbb{N}$. If we choose any open $G \subset\subset Q_0$ then we even have $f_n \in W^{1,p}_G(\Omega)$ for all $n \in \mathbb{N}$. But a Poincaré-type inequality cannot hold true for Ω and $W^{1,p}_G(\Omega)$ since $||f_n||_{p;\Omega} \to \infty$ but $||\nabla f_n||_{p;\Omega} \leq C < \infty$ for all $n \in \mathbb{N}$. In addition, the embedding $J : W^{1,p}(\Omega) \to L^p(\Omega)$ cannot be compact, because otherwise a Poincaré-type estimate (4.10) has to hold for $W^{1,p}_G(\Omega)$.

Appendix 2: A simple proof of Poincaré's inequality in a cube.

Theorem. *For $a > 0$ let $I_a :=]-a, a[$. Let $1 \leq p < \infty$ and $\frac{1}{p'} := \frac{p-1}{p}$. Then for $n \in \mathbb{N}$*

$$||u||_{p;I_a^n} \leq a n^{\frac{1}{p'}} ||\nabla u||_{p;I_a^n} \tag{A.1}$$

holds for all $u \in W^{1,p}(I_a^n)$ with $\int_{I_a^n} u(x)\,dx = 0$. (Here, $||\nabla u||_{p;I_a^n} := (\sum_{i=1}^{n} ||\partial_i u||^p_{p;I_a^n})^{\frac{1}{p}}$)

Proof. (A) i) Let $n = 1$ and let $u \in C^1(\bar{I}_a)$ satisfy $\int_{I_a} u(x)\,dx = 0$. Then for $x, y \in \bar{I}_a$ we see $u(y) - u(x) = \int_x^y u'(t)dt$.

ii) Therefore

$$2au(y) - \int_{I_a} u(x)\,dx = \int_{-a}^{a} \left(\int_x^y u'(t)dt \right) dx. \tag{A.2}$$

Then

$$2au(y) = \int_{-a}^{a} \left(\int_x^y u'(t)dt \right) dx - \int_y^a \left(\int_y^x u'(t)dt \right) dx$$

and

$$2a|u(y)| \leq \int_{-a}^{y} \left(\int_x^y |u'(t)|dt \right) dx + \int_y^a \left(\int_y^x |u'(t)|dt \right) dx.$$

After a partial integration we see

$$2a|u(y)| \leq \int_{-a}^{y} (a+x)|u'(x)|\,dx + \int_{y}^{a} (a-x)|u'(x)|\,dx. \tag{A.3}$$

For $a > 0$ fixed we set $\Psi(x, y) := a + x \cdot \operatorname{sgn}(y - x)$. Then

$$\Psi(x, y) \geq a - |x| \geq 0 \qquad \text{for } x, y \in \bar{I}_a \tag{A.4}$$

and (A.3) may be rewritten as

$$2a|u(y)| \leq \int_{-a}^{+a} \Psi(x, y)|u'(x)|\,dx. \tag{A.5}$$

We observe

$$\int_{-a}^{+a} \Psi(x, y)dy = 2(a^2 - x^2) \leq 2a^2 \qquad \text{for } x \in \bar{I}_a \tag{A.6}$$

$$\int_{-a}^{+a} \Psi(x, y)\,dx = a^2 + y^2 \leq 2a^2 \qquad \text{for } y \in \bar{I}_a. \tag{A.7}$$

Integrating (A.5) with respect to y yields after interchanging the order of integration because of (A.6)

$$2a \int_{-a}^{+a} |u(y)|dy \leq \int_{-a}^{+a} |u'(x)| \left(\int_{-a}^{+a} \Psi(x, y)dy \right) dx \leq 2a^2 \int_{-a}^{+a} |u'(x)|\,dx$$

and therefore (A.1). In case of $1 < p < \infty$ we see because of (A.4), (A.6,7) by means of Hölder's inequality

$$2a|u(y)| \leq \int_{-a}^{+a} \Psi(x, y)^{\frac{1}{p'}} \Psi(x; y)^{\frac{1}{p}} |u'(x)|\,dx \leq$$

$$\leq (2a^2)^{\frac{1}{p'}} \left(\int_{-a}^{+a} \Psi(x, y)|u'(x)|^p\,dx \right)^{\frac{1}{p}}$$

and

$$(2a)^p \int_{-a}^{+a} |u(y)|^p dy \leq (2a^2)^{\frac{p}{p'}} \int_{-a}^{+a} |u'(x)|^p \left(\int_{-a}^{+a} \Psi(x, y) dy \right) dx$$

$$\leq (2a^2)^p \int_{-a}^{a} |u'(x)|^p \, dx.$$

But this is (A.1) for $n = 1$, $1 < p < \infty$.

(B) Suppose now that $n \geq 1$ and (A.1) holds true for all $u \in C^1(\bar{I}_a^n)$ with $\int_{I_a^n} u(x)\, dx = 0$. Let $\upsilon \in C^1(\bar{I}_a^{n+1})$ and let $x \in \bar{I}_a^n$, $t \in \bar{I}_a$ and $y := (x, t) \in \bar{I}_a^{n+1}$. Let

$$h(t) := |I_a^n|^{-1} \int_{I_a^n} \upsilon(z, t) dz \qquad \text{for } t \in \bar{I}_a \tag{A.8}$$

and let $u(x, t) := \upsilon(x, t) - h(t)$ for $(x, t) \in \bar{I}_a^{n+1}$. For fixed $t \in \bar{I}_a$ we see $u(., t) \in C^1(\bar{I}_a^n)$ and $\int_{I_a^n} u(x, t)\, dx = 0$. By induction hypothesis for $t \in \bar{I}_a$

$$\int_{I_a^n} |u(x; t)|^p \, dx \leq a^p n^{p-1} \int_{I_a^n} |\nabla_x u(x, t)|^{p\, dx}.$$

Integrating with respect to $t \in \bar{I}_a$ yields

$$||u||_{p; I_a^{n+1}} \leq a n^{\frac{1}{p'}} ||\nabla_x u||_{p; I_a^{n+1}} = a n^{\frac{1}{p'}} ||\nabla_x \upsilon||_{p; I_a^{n+1}}. \tag{A.9}$$

Further

$$||u||_{p; I_a^{n+1}} \geq ||\upsilon||_{p; I_a^{n+1}} - ||h||_{p; I_a^{n+1}}. \tag{A.10}$$

Since $\upsilon \in C^1(\bar{I}_a^{n+1})$ we see

$$h'(t) = |I_a^n|^{-1} \int_{I_a^n} \partial_t \upsilon(z, t) dz$$

and therefore (in case $1 < p < \infty$ by Hölder's inequality)

$$|h'(t)| \le |I_a^n|^{-1+\frac{1}{p'}} ||\partial_t v(.,t)||_{p;I_a^n}. \tag{A.11}$$

By means of (A.1) for $n = 1$ we see with the help of (A.11)

$$||h||^p_{p;I_a^{n+1}} = \int\limits_{I_a^n} \left(\int\limits_{I_a} |h(t)|^p dt \right) dx \le |I_a^n| a^p \int\limits_{I_a} |h'(t)|^p dt \le$$

$$\le |I_a^n| a^p |I_a^n|^{-p+\frac{p}{p'}} \int\limits_{I_a} \left(\int\limits_{I_a^n} |\partial_t v(z,t)|^p dz \right) dt.$$

If we combine last estimate with (A.9), (A.10) we derive

$$||v||_{p;I_a^{n+1}} \le a n^{\frac{1}{p'}} ||\nabla_x v||_{p;I_a^{n+1}} + a||\partial_t v||_{v;I_a^{n+1}}.$$

In case $p = 1$ (that is $\frac{1}{p'} = 0$) we see (A.1) for $(n+1)$. In case $1 < p < \infty$ we apply Hölder's inequality (for vectors) to get

$$||v||_{p;I_a^{n+1}} \le a \left(n^{\frac{p'}{p'}} + 1 \right)^{\frac{1}{p'}} \left(\sum_{i=1}^n ||\partial_i v||^p_{p,I_a^{n+1}} + ||\partial_t v||^p_{p,I_a^{n+1}} \right)^{\frac{1}{p}}$$

$$\le a(n+1)^{\frac{1}{p'}} ||\nabla_{n+1} v||_{p;I_a^{n+1}}.$$

This is (A.1) for $n + 1$ and smooth functions.

(C) If $u \in W^{1,p}(I_a^n)$ with $\int\limits_{I_a^n} u(x)\,dx = 0$, then we choose $0 < a' < a$ and for $0 < \rho < a - a'$ we regard the mollified function u_ρ (using a standard mollifier kernel). Then $u_\rho|_{I_{a'}^n} \in C^\infty(\bar{I}_{a'}^n)$ and for $x \in \bar{I}_{a'}^n$ we see $\partial_i[u_\rho(x)] = (\partial_i u)_\rho(x)$, $i = 1, \ldots, n$. Therefore

$$||u - u_\rho||_{p;I_{a'}^n} \to 0, \quad ||\nabla u - \nabla u_\rho||_{p;I_{a'}^n} \to 0 (\rho \to 0).$$

Let $c_{a'} := |I_{a'}^n|^{-1} \int\limits_{I_{a'}^n} u(y)\,dy$ and

$$c_{a',\rho} := |I_{a'}^n|^{-1} \int\limits_{I_{a'}^n} u_\rho(y)\,dy, \quad \text{then } c_{a',\rho} \to c_{a'} (\rho \to 0).$$

We apply (A.1) to $(u_\rho - c_{a',\rho})|_{\bar{I}^n_{a'}}$ $(0 < \rho < a - a')$. After passing to the limit $\rho \to 0$ we see

$$||u - c_{a'}||_{p;I^n_{a'}} \leq a' n^{\frac{1}{p'}} ||\nabla u||_{p;I^n_{a'}}.$$

For $a' \to a$ we get $c_{a'} \to 0$ and by means of Lebesgue's theorem in last estimate we may pass to the limit $a' \to a$ to get (A.1) for u. □

Remark. Part.A. ii) of proof could be replaced by a much shorter argument, but for the price of a bigger constant. From (A.2) it follows

$$2a|u(y)| \leq \int_{-a}^{a} \left(\int_{-a}^{a} |u'(t)|dt \right) dx = 2a \int_{-a}^{+a} |u'(t)|dt.$$

This gives for $1 \leq p \leq \infty$ and $n = 1$

$$||u||_{p;I_a} \leq 2a||u'||_{p;I_a}$$

and finally

$$||u||_{p;I^n_a} \leq 2an^{\frac{1}{p'}} ||\nabla u||_{p;I^n_a}.$$

But that constant is twice the constant from (A.1)! □

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