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SOME FAMILIES OF MIXED GENERATING FUNCTIONS AND GENERALIZED POLYNOMIALS

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The main object of this paper is to show that combined use of the Lagrange expansion and certain operational techniques allows to derive mixed generating functions of various families of generalized polynomials in a straightforward manner. Relevant connections with many other recent works on this subject are also discussed.

1. Introduction.

The use of operational techniques has provided a fairly powerful tool for the study of generating functions and Burchnall type identities [3] for multivariable and multiindex Hermite polynomials [6].

In this paper we will show that the same method can be applied to extend mixed generating functions, of the type discussed by Carlitz[4] and Srivastava[13], to the case of generalized polynomials. We, therefore recall a few identities which will be exploited in the forthcoming sections (cf. [5] und [8]).

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1.1 Decoupling identities.

Weyl identity:

$$e^{A+B} = e^A \cdot e^B \cdot e^{-K/2}, \tag{1.1}$$

where the operators A and B do not commute and

$$[A, B] = AB - BA = k, [k, A] = [k, B] = 0. \tag{1.2}$$

Berry identity:

$$e^{A+B} = e^{\frac{\mu^2}{12}} \cdot e^{-\frac{\mu}{2} \cdot A^{1/2} + A} \cdot e^B, \tag{1.3}$$

where the operators A and B satisfy the relation

$$[A, B] = \mu \cdot A^{1/2}, \tag{1.4}$$

μ being an arbitrary complex number.

1.2 Crofton-type identities:.

$$e^{\alpha \cdot \left(\frac{d}{dx}\right)^m} f(x) = f\Big(x + m \cdot \alpha \cdot \left(\frac{d}{dx}\right)^{m-1}\Big) \cdot e^{\alpha \cdot (\frac{d}{dx})^m} \tag{1.5}$$

Henceforth we will omit the exponential operator on the r.h.s., because it is assumed to act on unity; an extension to the multivariable case is given by

$$e^{\frac{1}{2}\partial_Z^T M \partial_x} f\big(x, y) = f(x + (a \cdot \partial_x + b \cdot \partial_y), y + (b \cdot \partial_x + c \cdot \partial_y)\Big), \tag{1.6}$$

$$\partial_z = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}, M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

According to the above identities, the generating function of the Kampé de Fériet polynomials [7] can be obtained from that of the ordinary polynomials; indeed we have

$$e^{y \cdot \partial_x^2} e^{x \cdot t} = e^{(x + 2 \cdot y \cdot \partial_x) \cdot t} = e^{x \cdot t + y \cdot t^2}, \tag{1.7}$$

which follows from Eqs. (1.5) and (1.1). The Kampé de Fériet polynomials are defined by the relations

$$(1.8)\qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot H_n(x,y) = e^{xt+yt^2},\ H_n(x,y) = n! \sum_{s=0}^{[n/2]} \frac{y^s \cdot x^{n-2\cdot s}}{s!(n-2\cdot s)!}$$
$$H_n(x,y) = e^{y\partial_x^2}(x^n),$$

the last of which is often overlooked. The $H_n(x, y)$ reduce to the Hermite polynomials in the following particular cases: $H_n(x) = H_n(2x, -1)$ and $He_n(x) = H_n(x, -\frac{1}{2})$.

We introduce the negative derivative operator D_x^{-1} whose action on monomials is just specified by

$$(1.9)\qquad D_x^{-n} \frac{x^m}{m!} = \frac{x^{n+m}}{(n+m)!},$$

while that on a generic function of x can be expressed in terms of the Cauchy integral (within the present formalism the lower limit of integration is taken to be zero)

$$(1.10)\qquad D_x^{-n} f(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) \cdot dt.$$

It should be remarked in passing that, for n a complex number with $\Re(n) > O$, (1.10) defines the familiar Riemann-Liouville fractional integral of order n (see,for example [1], Chapter 5). The following identity, which immediately follows from (9),

$$(1.11)\qquad e^{-yD_x^{-1}}\left(\frac{x^n}{n!}\right) = x^n \cdot C_n(yx),$$

will be exploited in our investigation. The function $C_n(x)$, usually called the Tricomi function, is of the Bessel type and is defined by the series:

$$(1.12)\qquad C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot x^r}{r!(n+r)!}.$$

The paper is organized as follows. In Section 2 we will exploit operational methods to derive mixed generating functions for multivariable and multiindex polynomials.In Section 3, also devoted to concluding remarks, we will comment on the use of the operational methods, to obtain further general families of generating functions valid for multivariable Hermite polynomials.

2. Mixed generating functions and general families of generalized polynomials.

According to Carlitz [4] and to successive investigations by Srivastava [13], the Lagrange expansions [11]

$$\frac{f(z)}{1-w\cdot\phi'(z)}=\sum_{n=0}^{\infty}\frac{w^n}{n!}D_\lambda^n\Big\{f(\lambda)\cdot[\phi(\lambda)]^n\Big\}_{\lambda=z_0}, \tag{2.1}$$
$$z=z_0+w\cdot\phi(z);\ D_\lambda=\frac{d}{d\lambda},$$

can be used as a flexible tool to derive entirely new classes of generating functions [14].

To give an idea df how the identity (2.1) works, we consider the infinite sum

$$S(x,y;t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\cdot(x+n\cdot y)^n, \tag{2.2}$$

where x, y and t are independent of n. By rearranging the r.h.s. of (2.2) as

$$S(x,y;t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}D_\lambda^n\Big\{e^{x\cdot\lambda}\cdot(e^{y\cdot\lambda})^n\Big\}_{\lambda=0}, \tag{2.3}$$

we can use the identity (2.1) and derive the result [12]:

$$S(x,y;t)=\frac{e^{\xi\cdot x}}{1-y\cdot\xi},\ \Big(\xi=t\cdot e^{y\cdot\xi}\Big) \tag{2.4}$$

We can now exploit the last of the identities (1.8) to get

$$e^{\eta\cdot\partial_x^2}S(x,y;t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\cdot H_n(x+n\cdot y,\eta). \tag{2.5}$$

Therefore, by combining (2.4) and (1.7), we finally obtain the mixed generating function:

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\cdot H_n(x+n\cdot y,\eta)=\frac{e^{\xi\cdot x+\xi^2\cdot\eta}}{1-y\cdot\xi},\ (\xi=te^{y\xi}) \tag{2.6}$$

which is well known (see Refs.[4] and [13]), but derived earlier within the framework of a different procedure.

The two-variable and two-index polynomials [10]:

$$h_{m,n}(x,t;\tau)=m!n!\sum_{s=0}^{\min(m,n)}\frac{\tau^s x^{m-s}y^{n-s}}{s!(m-s)!(n-s)!} \tag{2.7}$$

are defined by the generating function

$$\sum_{m,n=0}^{\infty}\frac{u^m}{m!}\cdot\frac{v^n}{n!}\cdot h_{m,n}(x,y;\tau)=e^{u\cdot x+v\cdot y+\tau\cdot u\cdot v}. \tag{2.8}$$

According to the Crofton identities (1.5) we can rewrite the r.h.s. of (2.8) as follows

$$e^{u\cdot x+v\cdot y+\tau\cdot u\cdot v}=e^{\tau\cdot\partial^2_{x\cdot y}}e^{u\cdot x}e^{v\cdot y}. \tag{2.9}$$

This last relation (2.9) can be exploited, along with (2.4), to derive the mixed generating function:

$$\sum_{m,n=0}^{\infty}\frac{u^m}{m!}\cdot\frac{v^n}{n!}\cdot h_{m,n}(x+m\cdot\xi,\,y+n\cdot\eta;\,\tau)=\frac{e^{x\cdot w_1+y\cdot w_2+\tau w_1\cdot w_2}}{(1-\xi\cdot w_1)(1-\eta\cdot w_2)}, \tag{2.10}$$

where $w_1=u\cdot e^{\xi\cdot w_1}$ and $w_2=v\cdot e^{\eta\cdot w_2}$.

A more general class of polynomials with two variables and two indices introduced by Hermite himself (see Refs.[1] and [10]), can be specified by a set of identities which are a direct generalization of the identities (1.8) and (2.4), namely

$$\sum_{m,n=0}^{\infty}\frac{u^m}{m!}\cdot\frac{v^n}{n!}\cdot H_{m,n}(x,y)=e^{w^TMz-\frac{1}{2}w^TMw},\ w=\begin{pmatrix}u\\v\end{pmatrix},\ z=\begin{pmatrix}x\\y\end{pmatrix} \tag{2.11}$$

$$(a,c)>0,\ \Delta=a\cdot c-b^2>0,$$

and [7]

$$H_{m,n}(x,y)=H_{m,n}\big(ax+by,-\frac{1}{2}a;bx+cy,-\frac{1}{2}c|-b)\big)= \tag{2.12}$$

$$=m!n!\sum_{s=0}^{\min(m,n)}\frac{(-b)^s H_{m-s}\big(ax+by,-\frac{1}{2}a\big)\cdot H_{n-s}\big(bx+cy,-\frac{1}{2}c\big)}{s!(m-s)!(n-s)!},$$

$$H_{m,n}(x,y)=e^{-\frac{1}{2}\partial_z^T M^{-1}\partial_z}[(ax+by)^m(bx+cy)^n].$$

By combining the above identities with the previously developed concepts, we end up with the mixed generating function:

$$(2.13)\quad \sum_{m,n=0}^{\infty} \frac{u^m}{m!} \cdot \frac{v^n}{n!} \cdot H_{m,n}\big(x + \frac{1}{\triangle} \cdot (cmp - bnq),\, y + \frac{1}{\triangle} \cdot (anq - bmp)\big) = \\ = \frac{e^{\psi^T M z - \frac{1}{2}\psi^T M \psi}}{(1 - p\psi_1)(1 - q\psi_2)},$$

where

$$(2.14)\qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},\ \psi_1 = u \cdot e^{p\psi 1}, \quad \text{and} \quad \psi_2 = u \cdot e^{q\psi_2}$$

An analogous relation can also be obtained for the associated polynomials $G_{m,n}(x, y)$ defined by

$$(2.15)\qquad G_{m,n}(x, y) = e^{-\frac{1}{2}\partial_z^T M^{-1}\partial_z}[x^m \cdot y^n]$$

The operational method can be extended to other classes of polynomials as e.g. the two-variable Laguerre polynomials introduced in [9], according to the relation

$$(2.16)\qquad L_n(x, y) = (y - D_x^{-1})^n,$$

which yields

$$(2.17)\qquad L_n(x, y) = \sum_{s=0}^{n} \frac{n!(-1)^s \cdot y^{n-s} \cdot x^s}{(s!)^2 (n-s)!},$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot L_n(x, y) = e^{y \cdot t} C_0(x \cdot t).$$

By exploiting the Eqs.(2.16), (2.3), (2.4) and the identity (1.11) we find that

$$(2.18)\quad \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x, y + n \cdot z) = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_\lambda^n \big[e^{\lambda \cdot (y - D_x^{-1})} (e^{\lambda \cdot z})^n\big]_{\lambda=0} = \\ = \frac{e^{w \cdot y} \cdot C_0(x \cdot w)}{1 - z \cdot w},\ w = t \cdot e^{z \cdot w}$$

This last relation (2.18) will be examined more closely in the following section.

3. Concluding remarks.

Before discussing more general cases and commenting on the results obtained in the preceding sections, let us consider the following infinite sum:

$$S_n(x, y) = \sum_{m=0}^{\infty} \frac{t^m}{m!}(x + m \cdot y)^n. \tag{3.1}$$

In analogy with (2.2), we rewrite the r.h.s of (3.1) as

$$\begin{gathered}\sum_{m=0}^{\infty} \frac{t^m}{m!}(x + m \cdot y)^n = \sum_{m=0}^{\infty} \frac{t^m}{m!} D_\lambda^n \left[e^{\lambda\cdot(x+m\cdot y)}\right]_{\lambda=0} = \\ = D_\lambda^n \left[e^{\lambda\cdot x} \cdot e^{t e^{y\lambda}}\right]_{\lambda=0} = \sum_{s=0}^{\infty} \binom{n}{s} x^{n-s} \cdot F_s(t, y), \\ F_s(t, y) = D_\lambda^s \left[e^{t\cdot e^{y\lambda}}\right]_{\lambda=0}\end{gathered} \tag{3.2}$$

By applying the above elementary results and the operational rules concerning the generalized Hermite polynomials, we get

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} \cdot H_n(x + m \cdot y, \eta) = \sum_{s=0}^{n} \binom{n}{s} H_{n-s}(x, \eta) \cdot F_s(t, y). \tag{3.3}$$

The above result (3.3) can be combined with Lagrange's expansion (2.1) to get mixed generating functions of the type:

$$\sum_{m,n=0}^{\infty} \frac{u^n}{n!} \cdot \frac{t^m}{m!} H_n(x+n\cdot y, z+m\cdot q) = \frac{e^{w\cdot x+z\cdot w^2} \cdot e^{t\cdot e^{q\cdot w^2}}}{1 - y \cdot w}, \left(w = u\cdot e^{y\cdot w}\right). \tag{3.4}$$

The relevant proof can be given by noting that

$$\begin{gathered}\sum_{m,n=0}^{\infty} \frac{u^n}{n!} \cdot \frac{t^m}{m!} H_n(x + n \cdot y, z + m \cdot q) = \\ = \sum_{m,n=0}^{\infty} \frac{u^n}{n!} \cdot \frac{t^m}{m!} D_\lambda^n [e^{\lambda\cdot(x+n\cdot y)+\lambda^2(z+mq)}]_{\lambda=0} = \\ = \sum_{n=0}^{\infty} \frac{u^n}{n!} D_\lambda^n [e^{\lambda\cdot x+\lambda^2\cdot z+t\cdot e^{q\cdot\lambda^2}} \cdot (e^{\lambda\cdot y})^n]_{\lambda=0}.\end{gathered} \tag{3.5}$$

The last identity (3.5) and (2.4) yield (3.4).

Let us now consider a further generalization of the Hermite polynomials, which can be obtained means of the identity:

$$e^{x_2 \cdot \partial_{x1}^2 + x3 \cdot \partial_{x1}^3} \cdot e^{x1 \cdot t} = e^{x_1 \cdot t + x_2 \cdot t^2 + x_3 \cdot t^3} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot H_n(x_1, x_2, x_3), \tag{3.6}$$

where

$$H_n(x_1, x_2, x_3) = n! \sum_{s=0}^{[n/3]} \frac{x_3^s \cdot H_{n-3s}(x_1, x_2)}{s!(n - 3 \cdot s)!}. \tag{3.7}$$

The first of the identities (3.6) can be derived by using Eqs. (1.5), (1.1) and (1.3). It is worth noting that they satisfy the differential equation:

$$\Big(\sum_{m=1}^{3} m \cdot x_m \cdot D_{x1}^m \Big) H_n(x_1, x_2, x_3) = n \cdot H_n(x_1, x_2, x_3), \tag{3.8}$$

and the generating function:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot H_{n+l}(x_1, x_2, x_3) = e^{P_3(t)} \cdot H_l\Big(P_3'(t), \frac{1}{2!} P_3''(t), \frac{1}{3} P_3'''(t)\Big), \tag{3.9}$$
$$P_3(t) = x_1 \cdot t + x_2 \cdot t^2 + x_3 \cdot t^3.$$

where the primes denote derivatives with respect to t.

The extension of (3.4) to the polynomials (3.7) is quite straightforward and indeed we get

$$\sum_{n_1, n_2, n_3 = 0}^{\infty} \Pi_{j=1}^3 \frac{u_j^{n_j}}{n_j!} \cdot H_{n_1}(x_1 + n_1 \cdot y_1, x_2 + n_2 \cdot y_2, x_3 + n_3 \cdot y_3) = \tag{3.10}$$
$$= \frac{e^{P_3(w)}}{1 - y_1 \cdot w} \cdot \Pi_{j=2}^3 e^{u_j \cdot e^{y_j \cdot w^j}}, \ \Big(w = u_1 e^{y_1 w} \Big).$$

It is evident that the extension to more variables is always possible. In this case the $H_n(x_1, x_2, \cdots, x_N)$, are essentially the Bell polynomials [2] and the identities (3.6) to (3.10) can be suitably generalized.

Before concluding this paper, we mention a further class of polynomials, known as Appell polynomials, denoted by $f_n(x)$ and satisfying the relations:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot f_n(x) = A(t) \cdot e^{x \cdot t}, \ A(0) \neq 0 \tag{3.11}$$
$$D_x f_n(x) = n \cdot f_{n-1}(x),$$

where A(t) is a generic function of t. From the operational point of view, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot f_n(x) = A(D_x) \cdot e^{x \cdot t}. \tag{3.12}$$

The $f_n(x)$ can be expressed in the form

$$f_n(x) = A(D_x) \cdot (x^n) \tag{3.13}$$

and the following identity can also be shown fairly easily:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot f_n(x + n \cdot y) = A(w) \cdot \frac{e^{x \cdot w}}{1 - y \cdot w}, \ w = t \cdot e^{y \cdot w} \tag{3.14}$$

This general result (3.14) was also proved by Carlitz [4] in a different manner.

It is finally worth noting that, within this respect (2.18), cannot be considered to be new. The polynomials $L_s(x, y)$ indeed exhibit a two-fold nature. As to the y-variable, they should be viewed as belonging to the Appell class (see the second of (2.17)),while as to the x-variable, they belong to a further class of polynomials satisfying the relations:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot g_n(x) = B(t) \cdot C_0(x \cdot t), -D_x x D_X(g_n(x)) = n \cdot g_{n-1}(x) \tag{3.15}$$

Further comments on this last point will be presented elsewhere.

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