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APPROXIMATION THEOREMS FOR MODIFIED SZASZ-MIRAKJAN OPERATORS IN POLYNOMIAL WEIGHT SPACES

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In this paper we will study properties of Szasz-Mirakjan type operators A_n^ν, B_n^ν defined by modified Bessel function I_ν. We shall present theorems giving a degree of approximation for these operators.

1. Introduction.

Let us denote a set of all real-valued function continuous in $\mathbb{R}_0 := [0, +\infty)$ by $C(\mathbb{R}_0)$ and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Similary as in [2], define a polynomial weight function

$$w_p(x) = \begin{cases} 1 & p = 0, \\ \dfrac{1}{1 + x^p} & p \in \mathbb{N} \end{cases} \tag{1}$$

for $x \in \mathbb{R}_0$, and denote a polynomial weight space by C_p

$$C_p := \{f \in C(\mathbb{R}_0) : w_p f \text{ is uniformly continuous and bounded in } \mathbb{R}_0\}. \tag{2}$$

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It can be proved that the formula

$$\|f\|_{C_p} := \sup_{x\in\mathbb{R}_0} w_p(x)|f(x)| \tag{3}$$

for $f \in C_p$ is a well-define norm in the space C_p. Let $\omega(f, C_p; t)$ be the modulus of continuity, defined by the formula

$$\omega(f, C_p; t) := \sup_{h\in[0,t]} \|\Delta_h f\|_{C_p}, \tag{4}$$

where $f \in C_p$, $t \in \mathbb{R}_0$ and

$$\Delta_h f(x) := f(x+h) - f(x)$$

for $x, h \in \mathbb{R}_0$.

The approximation problem conected with Szasz-Mirakjan operators was studied in [1], [2], [3]. In papers [1], [3] the following Szasz-Mirakjan operators were investigated

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f(\frac{k}{n}),$$

$$K_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt,$$

$n \in \mathbb{N}$, $x \in \mathbb{R}_0$ for functions $f \in C_p$.
Note [2] was inspired by the results given in [1], [3] and operators of Szasz-Mirakjan type were defined

$$A_n(f; x) := \frac{1}{1 + sh(nx)} \Big\{ f(0) + \sum_{k=0}^{\infty} \frac{(nx)^{2k+1}}{(2k+1)!} f(\frac{2k+1}{n}) \Big\}, \tag{5}$$

$$B_n(f; x) := \frac{1}{1 + sh(nx)} \Big\{ f(0) + \sum_{k=0}^{\infty} \frac{(nx)^{2k+1}}{(2k+1)!} \frac{n}{2} \int_{\frac{2k+1}{n}}^{\frac{2k+3}{n}} f(t)\, dt \Big\} \tag{6}$$

for $f \in C_p$ $(p \in \mathbb{N}_0)$, $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$ where sh is the elementary hyperbolic function.

In this note we introduce in the space C_p $(p \in \mathbb{N}_0)$ a new modification of Szasz-Mirakjan operators as follows

$$A_n^\nu(f; x) := \begin{cases} \dfrac{1}{I_\nu(nx)} \displaystyle\sum_{k=0}^{\infty} \frac{(\frac{nx}{2})^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)} f(\frac{2k}{n}), & x > 0, \\ f(0), & x = 0, \end{cases} \tag{7}$$

$$(8)\qquad B_n^\nu(f;x) := \begin{cases} \dfrac{1}{I_\nu(nx)} \displaystyle\sum_{k=0}^{\infty} \dfrac{(\frac{nx}{2})^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)} \dfrac{n}{2} \int_{\frac{2k}{n}}^{\frac{2k+2}{n}} f(t)\,dt, & \\ & x > 0 \\ \dfrac{n}{2} \displaystyle\int_0^{\frac{2}{n}} f(t)\,dt, \qquad x = 0, & \end{cases}$$

for $f \in C_p$ $(p \in \mathbb{N}_0)$, $n \in \mathbb{N}$, $\nu \in \mathbb{R}_0$, $x \in \mathbb{R}_0$ where Γ is the Γ-Euler function and I_ν a modified Bessel function defined by the formula ([4], p. 77)

$$(9)\qquad I_\nu(z) := \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)}.$$

Among other things we shall prove that A_n^ν, B_n^ν are well-defined, linear and positive operators for all $f \in C_p$ with every $p \in \mathbb{N}_0$. Moreover, we shall prove that these operators are bounded and transform the space C_p into C_p.

2. Auxiliary results.

In this section we show some preliminary properties of the operators A_n^ν, B_n^ν.

All proofs of properties for A_n^ν and B_n^ν are analogous so we prove only for the operator A_n^ν. By definitions (7) and (8) we obtain the following

Lemma 1. *For each $n \in \mathbb{N}$, $\nu \in \mathbb{R}_0$ and $x \in \mathbb{R}_0$*

$$A_n^\nu(1;x) = 1, \quad B_n^\nu(1;x) = 1,$$

$$A_n^\nu(t;x) = x\frac{I_{\nu+1}(nx)}{I_\nu(nx)}, \quad B_n^\nu(t;x) = A_n^\nu(t;x) + \frac{1}{n} = x\frac{I_{\nu+1}(nx)}{I_\nu(nx)} + \frac{1}{n},$$

$$A_n^\nu(t^2;x) = x^2\frac{I_{\nu+2}(nx)}{I_\nu(nx)} + x\frac{2}{n}\frac{I_{\nu+1}(nx)}{I_\nu(nx)},$$

$$B_n^\nu(t^2;x) = A_n^\nu(t^2;x) + \frac{2}{n}A_n^\nu(t;x) + \frac{1}{3}(\frac{2}{n})^2 =$$

$$x^2\frac{I_{\nu+2}(nx)}{I_\nu(nx)} + x\frac{4}{n}\frac{I_{\nu+1}(nx)}{I_\nu(nx)} + \frac{1}{3}(\frac{2}{n})^2.$$

Remark. In Lemma 1 as well as in the rest part of this paper the equalities for $x = 0$ are to be understood in the asymptotic meaning with help of the equality

$$\lim_{z\to 0} \frac{I_\nu(z)}{(\frac{z}{2})^\nu} = \frac{1}{\Gamma(\nu+1)}.$$

Using Lemma 1 and basic properties of A_n^ν and B_n^ν we have

Lemma 2. *For each $n \in \mathbb{N}$, $\nu \in \mathbb{R}_0$ and $x \in \mathbb{R}_0$*

$$A_n^\nu(t-x; x) = x(\frac{I_{\nu+1}(nx)}{I_\nu(nx)} - 1), \quad B_n^\nu(t-x; x) = x(\frac{I_{\nu+1}(nx)}{I_\nu(nx)} - 1) + \frac{1}{n},$$

$$A_n^\nu((t-x)^2; x) = x^2(\frac{I_{\nu+2}(nx)}{I_\nu(nx)} - 2\frac{I_{\nu+1}(nx)}{I_\nu(nx)} + 1) + x\frac{2}{n}\frac{I_{\nu+1}(nx)}{I_\nu(nx)},$$

$$B_n^\nu((t-x)^2; x) = x^2(\frac{I_{\nu+2}(nx)}{I_\nu(nx)} - 2\frac{I_{\nu+1}(nx)}{I_\nu(nx)} + 1) + x\frac{2}{n}(2\frac{I_{\nu+1}(nx)}{I_\nu(nx)} - 1) + \frac{1}{3}(\frac{2}{n})^2.$$

Lemma 3. *For all $\nu \in \mathbb{R}_0$ there exists a positive constant M_ν depending only on ν such that*

$$\left|\frac{I_{\nu+1}(z)}{I_\nu(z)}\right| \le M_\nu, \tag{10}$$

$$z\left|\frac{I_{\nu+1}(z)}{I_\nu(z)} - 1\right| \le M_\nu \tag{11}$$

for all $z \in \mathbb{R}_0$.

Proof. First we will prove inequality (10). For $z \in (0; 1)$ by definition (9) there exist $C_1(\nu)$, $C_2(\nu)$ positive constants such that

$$C_1(\nu)z^\nu \le I_\nu(z) \le C_2(\nu)z^\nu. \tag{12}$$

From these we obtain

$$A_\nu z \le \frac{I_{\nu+1}(z)}{I_\nu(z)} \le B_\nu z, \qquad z \in (0; 1)$$

where $A_\nu = \frac{C_1(\nu+1)}{C_2(\nu)}$, $B_\nu = \frac{C_2(\nu+1)}{C_1(\nu)}$. For that reason the quotient $\frac{I_{\nu+1}(z)}{I_\nu(z)}$ is bounded for $z \in (0; 1)$.

Let $z \in (1; +\infty)$. According to paper [4], p. 203, we have the following property for modified Bessel function

$$\lim_{z \to +\infty} \frac{I_\nu(z)}{\frac{e^z}{(2\pi z)^{\frac{1}{2}}}} = 1, \qquad \nu \in \mathbb{R}_0.$$

Hence

$$\lim_{z \to +\infty} \frac{I_{\nu+1}(z)}{I_\nu(z)} = 1.$$

So, there exists a number $a > 1$ such that

$$\Big| \frac{I_{\nu+1}(z)}{I_\nu(z)} - 1 \Big| < 1, \qquad z > a.$$

Therefore, the quotient $\frac{I_{\nu+1}(z)}{I_\nu(z)}$ is bounded in the interval $(a, +\infty)$.

For $z \in [1; a]$ the existence of constant M_ν such that (10) holds is obvious. The proof of (10) is completed.

The proof of inequality (11) is similiar to that of (10). If $z \in (0; 1)$ we have estimations (12) and from these we obtain

$$z(A_\nu z - 1) \leq z(\frac{I_{\nu+1}(z)}{I_\nu(z)} - 1) \leq z(B_\nu z - 1), \qquad z \in (0; 1).$$

Concluding we have

$$z\Big| \frac{I_{\nu+1}(z)}{I_\nu(z)} - 1 \Big| \leq M_\nu, \qquad z \in (0; 1).$$

Let $z \in (1; +\infty)$. According to paper [4], p. 203, we obtain an approximation of modified Bessel function

$$I_\nu(z) = \frac{e^z}{(2\pi z)^{\frac{1}{2}}} (\sum_{k=0}^{n} \frac{(-1)^k (\nu, k)}{(2z)^k} + O(\frac{1}{z^{n+1}})) \tag{13}$$

for $n \in \mathbb{N}_0$, $\nu \in \mathbb{R}_0$ and $z > 0$ where

$$\begin{cases} (\nu, 0) := 1, \\ (\nu, k) := \dfrac{\Gamma(\nu + \frac{1}{2} + k)}{k! \Gamma(\nu + \frac{1}{2} - k)}, \qquad k = 1, 2, 3... \quad . \end{cases}$$

If we use formula (13) for $n = 0$ and $z > 1$ we get

$$z\Big|\frac{I_{\nu+1}(z)}{I_\nu(z)} - 1\Big| = \frac{|h(z) - g(z)|}{|1 + \frac{g(z)}{z}|}$$

where h, g are bounded functions. Hence, there exist constants C_1, C_2 such that

$$|h(z)| < C_1, \quad |g(z)| < C_2, \qquad z > 1.$$

Let $a > \max(1, 2C_2)$ be a fixed real number. For $z > a$ we have

$$\frac{|g(z)|}{z} < \frac{1}{2}.$$

Now we will consider $z \in (a; +\infty)$. By the above remark we can write

$$z\Big|\frac{I_{\nu+1}(z)}{I_\nu(z)} - 1\Big| \leq 2(C_1 + C_2) = M.$$

For $z \in [1; a]$ inequality (11) is obvious. Therefore, the proof of inequality (11) is completed. □

Lemma 4. *For all $\nu \in \mathbb{R}_0$ there exists a positive constant M_ν depending only on ν such that*

$$|A_n^\nu(t - x; x)| \leq \frac{M_\nu}{n}, \quad |B_n^\nu(t - x; x)| \leq \frac{M_\nu}{n}, \tag{14}$$

$$|A_n^\nu((t - x)^2; x)| \leq M_\nu \frac{x + 1}{n}, \quad |B_n^\nu((t - x)^2; x)| \leq M_\nu \frac{x + 1}{n}, \tag{15}$$

for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

Proof. By Lemma 2 we have

$$|A_n^\nu(t - x; x)| = x\Big|\frac{I_{\nu+1}(nx)}{I_\nu(nx)} - 1\Big|, \qquad n \in \mathbb{N}, \quad x \in \mathbb{R}_0.$$

We will try to prove that there exists a positive constant M_ν such that

$$nx\Big|\frac{I_{\nu+1}(nx)}{I_\nu(nx)} - 1\Big| \leq M_\nu. \tag{16}$$

Let us substitute $nx = z$, $z > 0$. Hence inequality (11) in Lemma 3 implies (16), so the proof of (14) is ended.

Using the first part of the proof we get

$$(nx)^2\left|\frac{I_{\nu+2}(nx)}{I_{\nu+1}(nx)} - 1\right| \le nxM_{\nu+1},$$

$$(nx)^2\left|\frac{I_{\nu+1}(nx)}{I_{\nu}(nx)} - 1\right| \le nxM_{\nu}, \qquad x \in \mathbb{R}_0, \quad n \in \mathbb{N}.$$

Above inequalities, Lemma 2 and (10) imply the following estimation

$$\begin{aligned}|A_n^\nu((t-x)^2;x)| &= \left|x^2\frac{I_{\nu+2}(nx)}{I_\nu(nx)} + x\frac{2}{n}\frac{I_{\nu+1}(nx)}{I_\nu(nx)} - 2x^2\frac{I_{\nu+1}(nx)}{I_\nu(nx)} + x^2\right| \\ &\le x^2\left|\frac{I_{\nu+2}(nx)}{I_{\nu+1}(nx)} - 1\right|\frac{I_{\nu+1}(nx)}{I_\nu(nx)} + x^2\left|\frac{I_{\nu+1}(nx)}{I_\nu(nx)} - 1\right| + x\frac{2}{n}\frac{I_{\nu+1}(nx)}{I_\nu(nx)} \\ &\le M_\nu\frac{x}{n} \le M_\nu\frac{x+1}{n}\end{aligned}$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$. Lemma 4 has been proved. $\square$

Lemma 5. *For every fixed $p \in \mathbb{N}$ there exist positive numbers $a_{p,i}$, $b_{p,i}$ depending only on p, i, $0 \le i \le p$ such that $a_{p,p} = 1$, $b_{p,p} = 1$, $b_{p,0} = \frac{1}{p+1}$ and for all $n \in \mathbb{N}$, $x \in \mathbb{R}_0$, $\nu \in \mathbb{R}_0$*

$$A_n^\nu(t^p;x) = \frac{1}{I_\nu(nx)}(\frac{2}{n})^p\sum_{i=1}^{p} a_{p,i}(\frac{nx}{2})^i I_{\nu+i}(nx), \tag{17}$$

$$B_n^\nu(t^p;x) = \frac{1}{I_\nu(nx)}(\frac{2}{n})^p\sum_{i=0}^{p} b_{p,i}(\frac{nx}{2})^i I_{\nu+i}(nx) \tag{18}$$

hold.

Proof. In order to prove conection (17) we use the mathematical induction for $p \in \mathbb{N}$. If $p = 1, 2$ it is Lemma 1. Assuming (17) for $f(t) = t^j$, $j \in \mathbb{N}$ and $j \le p$, we get from definition (7)

$$\begin{aligned}A_n^\nu(t^{p+1};x) &= \frac{1}{I_\nu(nx)}\sum_{k=0}^{+\infty}\frac{(\frac{nx}{2})^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)}(\frac{2k}{n})^{p+1} \\ &= \frac{1}{I_\nu(nx)}(\frac{2}{n})^{p+1}\sum_{k=1}^{\infty}\frac{(\frac{nx}{2})^{2k+\nu}}{\Gamma(k)\Gamma(k+\nu+1)}k^p\end{aligned}$$

$$= \frac{1}{I_\nu(nx)}(\frac{2}{n})^{p+1}\sum_{k=0}^{\infty}\frac{(\frac{nx}{2})^{2k+\nu+2}}{\Gamma(k+1)\Gamma(k+\nu+2)}(k+1)^p$$

$$= \frac{1}{I_\nu(nx)}(\frac{2}{n})^{p+1}\sum_{s=0}^{p}\binom{p}{s}\sum_{k=0}^{\infty}\frac{(\frac{nx}{2})^{2k+\nu+2}}{\Gamma(k+1)\Gamma(k+\nu+2)}k^s$$

$$= \frac{1}{I_\nu(nx)}(\frac{2}{n})^{p+1}\frac{nx}{2}I_{\nu+1}(nx)$$

$$+ \frac{1}{I_\nu(nx)}(\frac{2}{n})^{p+1}\frac{nx}{2}\sum_{s=1}^{p}\binom{p}{s}\sum_{k=0}^{\infty}\frac{(\frac{nx}{2})^{2k+\nu+1}}{\Gamma(k+1)\Gamma(k+\nu+2)}k^s.$$

Using the inductive assumption, we obtain

$$A_n^\nu(t^{p+1};x) = \frac{1}{I_\nu(nx)}(\frac{2}{n})^{p+1}\frac{nx}{2}I_{\nu+1}(nx)$$

$$+ \frac{1}{I_\nu(nx)}(\frac{2}{n})^{p+1}\frac{nx}{2}\sum_{s=1}^{p}\binom{p}{s}\sum_{i=1}^{s}a_{s,i}(\frac{nx}{2})^i I_{\nu+1+i}(nx)$$

$$= \frac{1}{I_\nu(nx)}(\frac{2}{n})^{p+1}\{\frac{nx}{2}I_{\nu+1}(nx) + \sum_{s=1}^{p}\binom{p}{s}\sum_{k=2}^{s+1}a_{s,k-1}(\frac{nx}{2})^k I_{\nu+k}(nx)\},$$

where $a_{s,s} = 1$.

Hence we have

$$A_n^\nu(t^{p+1};x) = \frac{1}{I_\nu(nx)}(\frac{2}{n})^{p+1}\sum_{i=1}^{p+1}a_{p+1,i}(\frac{nx}{2})^i I_{\nu+i}(nx)$$

and $a_{p+1,p+1} = 1$ for $p \in \mathbb{N}$.
Thus, by the mathematical induction, Lemma 5 is proved. □

Lemma 6. *For every fixed $p \in \mathbb{N}_0$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M_{p,\nu}$ such that*

$$\left\| A_n^\nu(\frac{1}{w_p(t)};.)\right\|_{C_p} \le M_{p,\nu}, \tag{19}$$

$$\left\| B_n^\nu(\frac{1}{w_p(t)};.)\right\|_{C_p} \le M_{p,\nu} \tag{20}$$

for all $n \in \mathbb{N}$.

Proof. From (1), (3) and Lemma 1 we immediately obtain (19) for $p = 0$ and $p = 1$. Let $2 \leq p \in \mathbb{N}$ be a fixed integer. Then, by (1) and Lemma 5, we have for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$

$$w_p(x)A_n^\nu(\frac{1}{w_p(t)}; x) = w_p(x)\{A_n^\nu(1; x) + A_n^\nu(t^p; x)\}$$
$$= \frac{1}{1+x^p} + \sum_{i=1}^{p} a_{p,i}(\frac{2}{n})^p(\frac{n}{2})^i \frac{x^i}{1+x^p}\frac{I_{\nu+i}(nx)}{I_\nu(nx)}.$$

By Lemma 3 the quotient $\frac{I_{\nu+i}(nx)}{I_\nu(nx)}$ is bounded for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ so we get

$$0 \leq w_p(x)A_n^\nu(\frac{1}{w_p(t)}; x) \leq M_{p,\nu},$$

where $M_{p,\nu}$ is a positive constant depending on p and ν. From these and by (3) we obtain (19). □

Theorem 1. *For every fixed $p \in \mathbb{N}_0$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M_{p,\nu}$ such that for every $f \in C_p$ and $n \in \mathbb{N}$*

(21) $$\|A_n^\nu(f; .)\|_{C_p} \leq M_{p,\nu}\|f\|_{C_p},$$

(22) $$\|B_n^\nu(f; .)\|_{C_p} \leq M_{p,\nu}\|f\|_{C_p}$$

hold.

Proof. By (1), (3) and (7) we can get

$$w_p(x)|A_n^\nu(f(t); x)| \leq w_p(x)A_n^\nu(|f(t)|; x)$$
$$= w_p(x)A_n^\nu(w_p(t)|f(t)|\frac{1}{w_p(t)}; x) \leq \|f\|_{C_p} w_p(x)A_n^\nu(\frac{1}{w_p(t)}; x)$$

for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.
Using Lemma 6 we obtain (21). □

Corollary 1. *The operators A_n^ν, B_n^ν are linear and bounded from C_p into C_p.*

Lemma 7. *For every fixed $p \in \mathbb{N}_0$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M_{p,\nu}$ such that for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$*

$$w_p(x)A_n^\nu(\frac{(t-x)^2}{w_p(t)};x) \le M_{p,\nu}\frac{x+1}{n}, \tag{23}$$

$$w_p(x)B_n^\nu(\frac{(t-x)^2}{w_p(t)};x) \le M_{p,\nu}\frac{x+1}{n} \tag{24}$$

hold.

Proof. Inequalities (23) and (24) for $p = 0$ are proved in Lemma 4. For $p \ge 1$ from (1) and the linearity of the operator A_n^ν it follows that

$$A_n^\nu(\frac{(t-x)^2}{w_p(t)};x) = A_n^\nu((t-x)^2;x) + A_n^\nu(t^p(t-x)^2;x), \tag{25}$$

$$A_n^\nu(t^p(t-x)^2;x) = A_n^\nu(t^{p+2};x) - 2xA_n^\nu(t^{p+1};x) + x^2A_n^\nu(t^p;x).$$

According to Lemma 5 we get

$$\begin{aligned}
w_p(x)A_n^\nu(t^p(t-x)^2;x) &= \frac{x^{p+2}}{1+x^p}\left\{\frac{I_{\nu+p+2}(nx)}{I_\nu(nx)} - 2\frac{I_{\nu+p+1}(nx)}{I_\nu(nx)} + \frac{I_{\nu+p}(nx)}{I_\nu(nx)}\right\} \\
&+\frac{x^{p+1}}{1+x^p}\frac{2}{n}\left\{a_{p+2,p+1}\frac{I_{\nu+p+1}(nx)}{I_\nu(nx)} - 2a_{p+1,p}\frac{I_{\nu+p}(nx)}{I_\nu(nx)} + a_{p,p-1}\frac{I_{\nu+p-1}(nx)}{I_\nu(nx)}\right\} \\
&+\sum_{i=1}^{p} a_{p+2,i}(\frac{n}{2})^{i-(p+2)}\frac{x^i}{1+x^p}\frac{I_{\nu+i}(nx)}{I_\nu(nx)} \\
&-\sum_{i=1}^{p-1} 2a_{p+1,i}(\frac{n}{2})^{i-(p+1)}\frac{x^{i+1}}{1+x^p}\frac{I_{\nu+i}(nx)}{I_\nu(nx)} + \sum_{i=1}^{p-2} a_{p,i}(\frac{n}{2})^{i-p}\frac{x^{i+2}}{1+x^p}\frac{I_{\nu+i}(nx)}{I_\nu(nx)} \\
&\le \frac{x^p}{1+x^p}x^2\left|\frac{I_{\nu+p+2}(nx)}{I_{\nu+p+1}(nx)} - 1\right|\frac{I_{\nu+p+1}(nx)}{I_\nu(nx)} \\
&+ \frac{x^p}{1+x^p}x^2\left|1 - \frac{I_{\nu+p+1}(nx)}{I_{\nu+p}(nx)}\right|\frac{I_{\nu+p}(nx)}{I_\nu(nx)} \\
&+ \frac{x^p}{1+x^p}\frac{2}{n}xA_p\left|\frac{I_{\nu+p+1}(nx)}{I_{\nu+p}(nx)} - 1\right|\frac{I_{\nu+p}(nx)}{I_\nu(nx)}
\end{aligned}$$

$$+\frac{x^p}{1+x^p}\frac{2}{n}xB_p\Big|1-\frac{I_{\nu+p}(nx)}{I_{\nu+p-1}(nx)}\Big|\frac{I_{\nu+p-1}(nx)}{I_\nu(nx)}$$

$$+(\frac{2}{n})^2\sum_{i=1}^{p}a_{p+2,i}(\frac{n}{2})^{i-p}\frac{x^i}{1+x^p}\frac{I_{\nu+i}(nx)}{I_\nu(nx)}$$

$$-(\frac{2}{n})^2\sum_{i=2}^{p}2a_{p+1,i-1}(\frac{n}{2})^{i-p}\frac{x^i}{1+x^p}\frac{I_{\nu+i-1}(nx)}{I_\nu(nx)}$$

$$+(\frac{2}{n})^2\sum_{i=3}^{p}a_{p,i-2}(\frac{n}{2})^{i-p}\frac{x^i}{1+x^p}\frac{I_{\nu+i-2}(nx)}{I_\nu(nx)}$$

for $x\in\mathbb{R}_0$, $n\in\mathbb{N}$, where $a_{r,k}$, A_p, B_p are positive numbers. The quotient $\frac{I_{\nu+i}}{I_\nu}$ is bounded for all $x\in\mathbb{R}_0$, $n\in\mathbb{N}$ and $i\in\mathbb{N}_0$ so, by Lemma 3 we have

$$w_p(x)A_n^\nu(t^p(t-x)^2;x)\le M_{p,\nu}\frac{x+1}{n},\qquad x\in\mathbb{R}_0,\quad n\in\mathbb{N}$$

which proves Lemma 7. □

3. Approximation theorems.

Theorem 2. *Suppose that $p\in\mathbb{N}_0$, $\nu\in\mathbb{R}_0$ are fixed numbers and $g\in C_p^1$, where $C_p^1:=\{f\in C_p : f'\in C_p\}$. Then there exists a positive constant $M^*_{p,\nu}$ such that*

$$w_p(x)|A_n^\nu(g;x)-g(x)|\le M^*_{p,\nu}\|g'\|_{C_p}(\frac{x+1}{n})^{\frac{1}{2}},\tag{26}$$

$$w_p(x)|B_n^\nu(g;x)-g(x)|\le M^*_{p,\nu}\|g'\|_{C_p}(\frac{x+1}{n})^{\frac{1}{2}}\tag{27}$$

for all $x\in\mathbb{R}_0$ and $n\in\mathbb{N}$.

Proof. Let us $x\in\mathbb{R}_0$ be fixed. For $t\in\mathbb{R}_0$ we have

$$g(t)-g(x)=\int_x^t g'(u)\,du.$$

By (7) and Lemma 1 we get

$$A_n^\nu(g(t);x)-g(x)=A_n^\nu(\int_x^t g'(u)\,du;x),\qquad n\in\mathbb{N}.\tag{28}$$

Since

$$\Big|\int_x^t g'(u)\,du\Big| \le \|g'\|_{C_p}\Big|\int_x^t \frac{du}{w_p(u)}\Big| \le \|g'\|_{C_p}\Big(\frac{1}{w_p(x)}+\frac{1}{w_p(t)}\Big)|t-x|$$

we get from (28)

$$w_p(x)|A_n^\nu(g;x)-g(x)| \le \|g'\|_{C_p}\{A_n^\nu(|t-x|;x)+w_p(x)A_n^\nu(\frac{|t-x|}{w_p(t)};x)\}.$$

But (7) and Cauchy's inequality imply

$$A_n^\nu(|t-x|;x) \le \{A_n^\nu((t-x)^2;x)\}^{\frac{1}{2}},$$

$$A_n^\nu(\frac{|t-x|}{w_p(t)};x) \le \{A_n^\nu(\frac{1}{w_p(t)};x)\}^{\frac{1}{2}}\{A_n^\nu(\frac{(t-x)^2}{w_p(t)};x)\}^{\frac{1}{2}}.$$

From (15), Lemma 6 and Lemma 7 it follows that

$$A_n^\nu(|t-x|;x) \le (M_\nu\frac{x+1}{n})^{\frac{1}{2}},$$

$$w_p(x)A_n^\nu(\frac{|t-x|}{w_p(t)};x) \le M_{p,\nu}(\frac{x+1}{n})^{\frac{1}{2}}$$

for $x\in\mathbb{R}_0$, $n\in\mathbb{N}$, $p\in\mathbb{N}_0$, $\nu\in\mathbb{R}_0$.
Combinig these estimations we obtain (26). □

Theorem 3. *Suppose that $f\in C_p$, with fixed $p\in\mathbb{N}_0$ and $\nu\in\mathbb{R}_0$. Then there exists a positive constant $M_{p,\nu}$ such that*

$$w_p(x)|A_n^\nu(f;x)-f(x)| \le M_{p,\nu}\,\omega(f,C_p;(\frac{x+1}{n})^{\frac{1}{2}}), \tag{29}$$

$$w_p(x)|B_n^\nu(f;x)-f(x)| \le M_{p,\nu}\,\omega(f,C_p;(\frac{x+1}{n})^{\frac{1}{2}}) \tag{30}$$

for all $x\in\mathbb{R}_0$, $n\in\mathbb{N}$.

Proof. Let f_h be the Stieklov mean of $f \in C_p$, i.e.

$$f_h(x) = \frac{1}{h}\int_0^h f(x+t)\,dt, \qquad x \in \mathbb{R}_0, \quad h \in \mathbb{R}_+,$$

where $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$. We have

$$f_h(x) - f(x) = \frac{1}{h}\int_0^h (f(x+t) - f(x))\,dt,$$
$$f_h'(x) = \frac{1}{h}\{f(x+h) - f(x)\}$$

for $x \in \mathbb{R}_0$, $h \in \mathbb{R}_+$. It is easy to notice that if $f \in C_p$ then $f_h \in C_p^1$ for every fixed $h \in \mathbb{R}_+$. Moreover, for $h \in \mathbb{R}_+$

$$\|f_h - f\|_{C_p} \le \sup_{x\in\mathbb{R}_0}\{\frac{1}{h}\int_0^h w_p(x)|f(x+t) - f(x)|\,dt\} \le \omega(f, C_p; h), \tag{31}$$

$$\|f_h'\|_{C_p} \le \frac{1}{h}\omega(f, C_p; h) \tag{32}$$

hold. Since A_n^ν is a linear operator, we have

$$\begin{aligned} w_p(x)|A_n^\nu(f;x) - f(x)| \le w_p(x)\{&|A_n^\nu(f - f_h;x)| \\ &+ |A_n^\nu(f_h;x) - f_h(x)| + |f_h(x) - f(x)|\} \end{aligned}$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $h \in \mathbb{R}_+$.
Using Theorem 1 and (31), we get

$$w_p(x)|A_n^\nu(f - f_h;x)| \le M_{p,\nu}\|f - f_h\|_{C_p} \le M_{p,\nu}\omega(f, C_p; h).$$

From Theorem 2 and (32) it follows that

$$\begin{aligned} w_p(x)|A_n^\nu(f_h;x) - f_h(x)| &\le M_{p,\nu}\|f_h'\|_{C_p}(\frac{x+1}{n})^{\frac{1}{2}} \\ &\le M_{p,\nu}\omega(f, C_p; h)\frac{1}{h}(\frac{x+1}{n})^{\frac{1}{2}}. \end{aligned}$$

From these and by (31) we obtain

$$w_p(x)|A_n^\nu(f;x) - f(x)| \le M_{p,\nu}\omega(f, C_p; h)\{1 + \frac{1}{h}(\frac{x+1}{n})^{\frac{1}{2}}\} \tag{33}$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $h \in \mathbb{R}_+$. Setting, for every fixed $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$, $h = (\frac{x+1}{n})^{\frac{1}{2}}$ to (33), we get the desired estimation (29) for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. □

Theorem 3 implies the following corollaries:

Corollary 2. *If $f \in C_p$ with some $p \in \mathbb{N}_0$ and $\nu \in \mathbb{R}_0$, then*

$$\lim_{n\to\infty} A_n^\nu(f;x) = f(x), \tag{34}$$

$$\lim_{n\to\infty} B_n^\nu(f;x) = f(x) \tag{35}$$

for all $x \in \mathbb{R}_0$.

Moreover, statements tm (34) and (35) *hold uniformly on every interval* $[0, a]$, $a > 0$.

Corollary 3. *If $f \in Lip(C_p, \alpha) := \{f \in C_p : \omega(f, C_p; t) = 0(t^\alpha), t \to 0^+\}$ with some $p \in \mathbb{N}_0$, $0 < \alpha \leq 1$ and $\nu \in \mathbb{R}_0$, then there exists a positive constant $M_{p,\nu,\alpha}$ such that*

$$w_p(x)|A_n^\nu(f;x) - f(x)| \leq M_{p,\nu,\alpha}(\frac{x+1}{n})^{\frac{\alpha}{2}},$$

$$w_p(x)|B_n^\nu(f;x) - f(x)| \leq M_{p,\nu,\alpha}(\frac{x+1}{n})^{\frac{\alpha}{2}}$$

for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

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