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PROJECTIVELY COHEN-MACAULY SURFACES OF SMALL DEGREE IN $\mathbb{P}^5$

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In this paper we consider the nondegenerate projectively Cohen-Macaulay (p.C.M.) surfaces of small degree in $\mathbb{P}^5$. We determine those of degree $d \leq 9$ and all candidate rational surfaces as p.C.M. surfaces.

Introduction.

The problem of describing smooth embedded surfaces having particular properties, such as, for example, being projectively normal or projectively Cohen-Macaulay (p.C.M. for short), has been considered by many authors in the past (recall that such surface is p.C.M. if its homogeneous coordinate ring is Cohen-Macaulay).

Our aim in this paper is to determine all the nondegenerate p.C.M. surfaces of degree $d \leq 9$ in $\mathbb{P}^5_{\mathbb{C}} = \mathbb{P}^5$.

From our previous work (see [16]), we know that, if $g(H) = g$ is the sectional genus of a nondegenerate p.C.M. surface $X \subset \mathbb{P}^N$, then $N = d - g + 1 + p_a - h^2(\mathcal{O}_X(1)) = d - g + 1 + h^1(\mathcal{O}_H(1))$, where p_a denotes the arithmetic genus of X, and that for the degree d of X we have the bounds

$$N - 1 \leq d \leq \binom{N}{2} + h^1(\mathcal{O}_H(1)).$$

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In particular, when $p_a = 0$, we have $h^1(\mathcal{O}_H(1)) = p_a - h^2(\mathcal{O}_X(1)) = 0$, hence

$$N - 1 \leq d = N - g + 1 \leq \binom{N}{2}.$$

We also recall that the irregularity $q(X) = p_g - p_a$, of any p.C.M. surface is zero (e.g. see[16]).

So, in $\mathbb{P}^5$, we only have to consider surfaces of degree $d \leq 10 + h^1(\mathcal{O}_H(1))$ and sectional genus $g = d - 4 + h^1(\mathcal{O}_H(1))$.

All the nondegenerate p.C.M. surfaces $X \subset \mathbb{P}^5$ of degree $d \leq 9$ can be determined. Our results are summarized in Table 1.

TABLE 1. Projectively C.M. surfaces in $\mathbb{P}^5$ of degree ≤ 9

d	g	p_a	*Structure of X*	$\mathcal{O}_X(H) = \mathcal{O}_X(1)$
4	0	0	Veronese Surface	
	0	0	Rational Normal Scroll	
5	1	0	Del Pezzo Surface, X_4	$\mathcal{O}_X(3E_0 - E_1 - \ldots - E_4)$
6	2	0	Castelnuovo Surface, X_7	$\mathcal{O}_X(4E_0 - 2E_1 - E_2 - \ldots - E_7)$
7	3	0	Bordiga-White Surface, X_{10}	$\mathcal{O}_X(5E_0 - 3E_1 - E_2 - \ldots - E_{10})$
	3	0	$\mathbb{F}_e^9,\ e = 2, 3$	$\mathcal{O}_X(2C_0 + (4+e)f - E_1 - \ldots - E_9)$
	3	0	Veronesean Surface, X_9	$\mathcal{O}_X(4E_0 - E_1 - \ldots - E_9)$
	3	0	X_8	$\mathcal{O}_X(6E_0 - 2E_1 - \ldots - 2E_7 - E_8)$
8	5	1	K3 Surface	
	4	0	X_9	$\mathcal{O}_X(9E_0 - 3E_1 - \ldots - 2E_{10})$
	4	0	Bordiga-White Surface, X_{11}	$\mathcal{O}_X(5E_0 - 2E_1 - 2E_2 - E_3 - \ldots - E_{11})$
	4	0	X_{10}	$\mathcal{O}_X(6E_0 - 2E_1 - \ldots - 2E_6 - E_7 - \ldots - E_{10})$
9	5	0	X_{10}	$\mathcal{O}_X(7E_0 - 2E_1 - \ldots - E_{10})$
	5	0	X_{12}	$\mathcal{O}_X(6E_0 - 2E_1 - \ldots - 2E_5 - E_6 - \ldots - E_{12})$
	5	0	$\mathbb{F}_e^{10},\ 0 \leq e \leq 2$	$\mathcal{O}_X(4C_0 + (2e+5)f - 2E_1 - \ldots - 2E_7 - E_8 - \ldots - E_{10})$
	6	1	Y_1	
	7	2	Elliptic Surface	

We use the following notations:

$-d = degX$, $g(H) = g$ sectional genus of X

$-X_s$: blowing-up of $\mathbb{P}^2$ at s generic points

$-\mathbb{F}_e^s$: blowing-up of the rational ruled surface $\mathbb{F}_e$ at s generic points

$-Y_s$: blowing-up a K3 surface Y at s generic points.

Since the maximum degree of a rational p.C.M. surface X in $\mathbb{P}^N$ is $d = \binom{N}{2}$, in order to complete the description of the rational p.C.M. surfaces in $\mathbb{P}^5$ it remains to consider the case $d = 10$.

Table 2, in Section 6, shows all the possible candidates as rational p.C.M. surfaces of degree $d = 10$; it still an open problem to check if all of them actually exist and which of them are p.C.M.

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1. Background on the p.C.M. embeddings of blowing-ups of $\mathbb{P}^2$ at a finite set of distinct points.

Let $Z = (P_1, \dots, P_s; m_1, \dots, m_s)$ be, with $m_1 \geq \dots \geq m_s$, the 0-dimensional subscheme of $\mathbb{P}^2_{\mathbb{C}} = \mathbb{P}^2$ associated to the homogeneous ideal $I_Z = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s} \subset \mathbb{C}[x_0, x_1, x_2]$, where each $\wp_i$ is a homogeneous prime ideal which corresponds to a point P_i of $\mathbb{P}^2$, $i = 1, \dots, s$.

If X_s is the blowing-up $\mathbb{P}^2$ at the distinct points $P_1, \dots, P_s$ of the support of Z, we denote by $E_1, \dots, E_s$ the divisor classes on X_s which contain the exceptional divisor and by E_0 the divisor class on X_s of the strict transform of generic line of $\mathbb{P}^2$. It is well known that $Pic\, X_s \cong \mathbb{Z}^{s+1}$ is freely generated by $E_0, E_1, \dots, E_s$ and that, if C is a plane curve of degree t with a singularity at P_i of multiplicity $= m_i$, $i = 1, \dots, s$, then the strict transform of C on X_s is an effective divisor in the divisor class of $tE_0 - m_1E_1 - \dots - m_sE_s$ (e.g. see [11]).

Now, let $H_Z(t)$ be the Hilbert function of Z; let $\sigma(Z) = min\{t/\Delta H_Z(t) = 0\}$, where $\Delta H_Z(t) = H_Z(t) - H_Z(t-1)$ is the first difference of $H_Z(t)$, then we have:

$$\sigma(Z) - 1 = \tau(Z) = \min\{t/h^0(\mathcal{I}_Z(t)) \cdot h^1(\mathcal{I}_Z(t)) = 0\},$$

where $\mathcal{I}_Z \subseteq \mathcal{O}_{\mathbb{P}^2}$ denotes the ideal sheaf of Z. Namely, $\tau(Z)$ is the smallest integer t for which the linear system of all the plane curves of degree t passing through each P_i, $i = 1, \dots, s$, with multiplicity al least m_i is regular (e.g. see [7]).

If $D_t = tE_0 - m_1E_1 - \dots - m_sE_s$ is a divisor on X_s associated to the scheme $Z \subset \mathbb{P}^2$, then we have the following results (see [6]):

Proposition 1.1. *D_t is very ample on X_s for every $t \geq \sigma(Z)$ if, and only if, no line of $\mathbb{P}^2$ has intersection of degree $\geq \sigma(Z)$ with Z.*

Proposition 1.2. *The very ample linear system $|D_t|$ embeds X_s as a projectively Cohen-Macaulay surface for every $t \geq \sigma(Z)$.*

We also know (e.g. see [16]) that a necessary condition so that $|D_t|$ embeds X_s as a projectively Cohen-Macaulay surface is that $h^1(\mathcal{O}_{X_s}(D_t)) = 0$ and so that $t \geq \tau(Z)$.

When D_t is very ample, we denote by $V_{t,Z}$ the image of the embedding $\varphi_{t,Z} : X_s \to \mathbb{P}^N$, where $N + 1 = h^0(\mathcal{O}_{X_s}(D_t)) = \binom{t+2}{2} - degZ$ (e.g. see[6]), which is determined by $|D_t|$ on X_s.

On the homogeneous ideal of the surface $V_{t,Z} \subseteq \mathbb{P}^N$ know what follows:

Proposition 1.3. *(See [6]). Let $t \geq \sigma(Z) + 1$, then the homogeneous ideal of $V_{t,Z} \subseteq \mathbb{P}^N$ is generated by forms of degree ≤ 3.*

Proposition 1.4. *(See [5]). Let $t \geq \sigma(Z) + 1$, then the homogeneous ideal of $V_{t,Z} \subseteq \mathbb{P}^N$ is generated by quadrics.*

On the defining ideal of certain surfaces $V_{t,Z} \subseteq \mathbb{P}^N$ we have more detailed information, namely we know that their generators can be given as minors of suitable matrices. In particular:

a) $t = d$, $Z = (P_1, \dots, P_s)$, $s = \binom{d+1}{2}$: for every $d \geq 3$, the surface $V_{d,Z}$ is called a *White Surface* in $\mathbb{P}^d$. It has degree $\binom{d}{2}$, sectional genus $\binom{d-1}{2}$ and its ideal is generated by the 3×3 minors of a $3 \times d$ matrix of linear forms (see [8]);

b) $t = d$, $Z = (P_1, \dots, P_s)$, $s = \binom{d+1}{2}$: for every $d \geq 3$, the surface $V_{d+1,Z}$ is called a *Room Surface* in $\mathbb{P}^{2d+2}$. It has degree $\binom{d+2}{2}$ and sectional genus $\binom{d}{2}$. Its ideal is generated by the 2×2 minors of a $3 \times (d + 1)$ matrix of linear forms (see [5]);

c) $t = d+1$, $Z = (P_1, \dots, P_s)$, $s = \binom{d+1}{2}+k$, with $0 < k < d+1$: for every $d \geq 3$, the surface $V_{d+1,Z}$ is called a *Veronesean Surface* in $\mathbb{P}^{2d-k+2}$. It has degree $\binom{d+2}{2} - k$ and sectional genus $\binom{d}{2}$. Its ideal is given as follows: its generators are the entries of the matrix $A \cdot B$, the 2×2 minors of B and the $3 \times d$ minors of A, where B and A are two matrices of linear forms of order, respectively, $3 \times (d - k + 1)$ and $k \times 3$ (see [10]);

d) $t = d$, $Z = (P_1, \dots, P_s; d - 2, 1, \dots, 1)$, $s = 2d$: for every $d \geq 4$, the surface $V_{d,Z}$ is called a *Bordiga-White Surface* in $\mathbb{P}^d$. It has degree $2d-3$, sectional genus $d-2$ and its ideal is generated by the 2×2 minors of a matrix of type

$$\begin{pmatrix} y_{1,1} & Y_{1,2} & \dots & Y_{1,d-2} & Q_1 \\ y_{2,1} & Y_{2,2} & \dots & Y_{2,d-2} & Q_2 \end{pmatrix},$$

where the $Y_{a,b}$ are linear forms, while Q_1 and Q_2 are quadratic forms (see [6]).

2. Some results on rational p.C.M. surfaces.

Let X be a rational p.C.M. surface in $\mathbb{P}^N$ of degree d, with $n-1 \leq d \leq \binom{N}{2}$; then X has sectional genus $g = d+1-N+h^1(\mathcal{O}_H(1)) = d+1-N$ (see [16]). We recall that, in terms of the coomology of the ideal sheaf of X, $\mathcal{I}_X$, the fact that X is p.C.M. in $\mathbb{P}^N$ can be expressed by the condition $h^i(\mathcal{I}_X(m)) = 0$, for $i = 1, 2$ and for all $m \geq 0$.

In our previous work (see [16]) we showed that a rational surface $X \subseteq \mathbb{P}^N$ of degree $\binom{N}{2}$, sectional genus $\binom{N-1}{2}$ and with $h^1(\mathcal{O}_X(1)) = 0$ is p.C.M. if, and only if, it is projectively normal.

Now we want to extend this result, namely we have:

Proposition 2.1. *Let $X \subseteq \mathbb{P}^N$ be a smooth surface of degree $d = N+g-1$, sectional genus g and irregularity $q = h^1(\mathcal{O}_X = 0$. If $h^1(\mathcal{O}_X(1)) = 0$, then X is p.C.M. if, and only if, it is projectively normal.*

Proof. Let us suppose that $X \subseteq \mathbb{P}^N$ is projectively normal, hence that $h^1(\mathcal{I}_X(m)) = 0$, for all $m \geq 0$.

Since $h^i(\mathcal{O}_{\mathbb{P}^n}(m)) = 0$, for all $0 < i < N$ and $m \geq 0$, from the exact sequence

$$0 \to \mathcal{I}_X(m) \to \mathcal{O}_{\mathbb{P}^N}(m) \to \mathcal{O}_X(m) \to 0$$

we deduce that $h^1(\mathcal{O}_X(m)) = h^2(\mathcal{I}_X(m))$. We want to show that $h^1(\mathcal{O}_X(m)) = 0, \forall m \geq 0$. Consider the exact sequence

$$0 \to \mathcal{O}_X(m-1) \to \mathcal{O}_X(m) \to \mathcal{O}_H(m) \to 0,$$

where H is a smooth hyperplane section of X.

Since $md = m(N+g-1) > 2g-2, \forall m \geq 2$, we have $h^1(\mathcal{O}_H(m)) = 0, \forall m \geq 2$. Thus, by the above exact sequence, $h^1(\mathcal{O}_X(m-1)) = 0$ implies $h^1(\mathcal{O}_X(m)) = 0$ for all $m \geq 2$, and so, since $h^1(\mathcal{O}_X(1)) = 0$ by hypothesis, we get what wanted. $\square$

Proposition 2.2. *The homogeneous ideal of a rational p.C.M. surface* $X \subseteq \mathbb{P}^N$ *can always be generated by forms of degree* ≤ 3 *and* $h^0(\mathcal{I}_X(2)) \neq 0$ *except when* X *has maximum degree* $d = \binom{N}{2}$.

Proof. From [16] we know that the ideal I_X of a rational p.C.M. surface $X \subseteq \mathbb{P}^N$ can always be generated by forms of degree ≤ 3, and tha only generators are cubics in the case in which X has maximum degree $\binom{N}{2}$. So it remains to prove that, when X has not maximum degree $\binom{N}{2}$, I_X always contains quadratic forms.

Let H be a smooth hyperplane section of X and consider the exact sequence

$$0 \to \mathcal{O}_X(1) \to \mathcal{O}_X(2) \to \mathcal{O}_H(2) \to 0.$$

Since

$$h^0(\mathcal{O}_X(2)) = h^0(\mathcal{O}_X(1)) + h^0(\mathcal{O}_X(2)) = N + 1 + 2d + 1 - g = 3N + g,$$

we have:

$$0 = h^1(\mathcal{I}_X(2)) = h^0(\mathcal{I}_X(2)) - h^0(\mathcal{O}_{\mathbb{P}^N}(2)) + h^0(\mathcal{O}_X(2)) =$$

$$= h^0(\mathcal{I}_X(2)) - \binom{N-1}{2} + g.$$

Clearly, $h^0(\mathcal{I}_X(2)) = 0$ only when $g = \binom{N-1}{2}$, and this terminates the proof. □

3. Projectively p.C.M. surfaces of degree ≤ 8.

THe smooth surfaces of degree $d \leq 8$ in $\mathbb{P}^5$ have been completely described (see [12], [4]). In this section we determine which of them are p.C.M. (Table 1 in the introduction summarizes our results).

3.1. ***Surfaces of degree d* = 4.** In $\mathbb{P}^5$ the p.C.M. surfaces of degree $d = 4$ are either *Veronese Surfaces* or *rational scrolls*, which are well known to be p.C.M.

3.2. ***Surfaces of degree d* = 5.** The only p.C.M. surfaces of degree 5 in $\mathbb{P}^5$ are the *Del Pezzo Surfaces*.

3.3. ***Surfaces of degree d = 6.*** The possibilities for a smooth surface of degree 6 in $\mathbb{P}^5$ are described in [12] and are the following:

(i) *An elliptic, scroll white* $e = 0$ *and* $g = 1$;

(ii) *A Castelnuovo Surface, with* $g = 2$, *defined by the embedding of* X_7 *in* $\mathbb{P}^5$ *via the very ample linear system* $|D_t| = |4E_0 - 2E_1 - E_2 - \ldots - E_7|$.

Since $5 = N \neq d - g + 1 + h^1(\mathcal{O}_H(1)) = 6 + H^1(\mathcal{O}_H(1))$ (see [16]), the unique p.C.M. surface of degree $d = 6$ in $\mathbb{P}^5$ is the *Castelnuovo Surface* (see also [12]).

3.4. ***Surfaces of degree d = 7.*** The smooth surfaces of degree $d = 7$ in $\mathbb{P}^5$ are classified by Ionescu in [12] and they are described as follows.

If $X \subseteq \mathbb{P}^5$ *is a smooth surface of degree* $d = 7$, *then it has sectional genus* $g(H) = 3$ *and it is one of the following rational surfaces*:

(i) *A blowing-up* π *of* $\mathbb{F}_e$, $e = 0, 1, 2, 3$, *with center* 9 *points;* $H = \pi^*(H_e) - E_1 - \ldots - E_9$, *where* $H_e = 2C_0 + (4 + e)f$;

(ii) *A blowing-up* π *of* $\mathbb{P}^2$ *with center* 9 *points,* $H = \pi^*(4L) - E_1 - \ldots - E_9$;

(iii) *A blowing-up of a point on a Del Pezzo double plane S, i.e. on a double covering of* $\mathbb{P}^2$ *ramified along a smooth quartic,* $H = \pi^*(H_s) - E$.

The surface $X \subseteq \mathbb{P}^5$ of the case 3.4 (*ii*) is a p.C.M. surface, called (*Veronesean Surface* (see Section 1). Hence we have to consider the surfaces of the cases (*i*) and (*iii*).

A) The case 3.4 (i).
Let us denote by $\mathbb{F}_e^9$ the blowing-up of $\mathbb{F}_e$ at 9 generic points and let us consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1) \to \mathcal{O}_X(1) \to 0.$$

By [1], Theorem 4.1, the smooth rational surface $X \subseteq \mathbb{P}^5$ is projectively normal. Moreover, since $d = 7 > 2g(H) - 2 = 4$, we have $h^1(\mathcal{O}_H(1)) = 0$ and so $h^1(\mathcal{O}_X(1)) = 0$. By Proposition 2.1, this is enough to conclude that X is p.C.M. in $\mathbb{P}^5$.

In particular, when $e = 0, 1$, another description of the surface X can be given, using a plane model, as follows.

a) e = 0: $\mathbb{F}_0^9$ is isomorphic to X_{10}, the blowing-up of $\mathbb{P}^2$ at 10 generic points.
In fact, $\mathbb{F}_0$ is isomorphic to the Quadric Surface $Q \subseteq \mathbb{P}^3$ and Q is obtained from X_2 via the complete (not very ample) linear system $|2E_0 - E_1 - E_2|$.

Since the line $E_0 - E_1 - E_2$ on X_2 is contracted to a point $P \in Q$, we have that to blow-up Q at the point P and at other 8 generic points is equivalent to blow-up $\mathbb{P}^2$ at 10 generic points, as we said.

Taking $C_0 = E_0 - E_1$ and $f = E_0 - E_2$, to the very ample divisor $H_0 = 2C_0 + 4f - E_1 - \ldots - E_9$ on $\mathbb{F}_0^9$ corresponds the very ample divisor $D_5 = 2(E_0 - E_1 + 4(E_0 - E_2) - (E_0 - E_1 - E_2) - E_3 - \ldots - E_{10} = 5E_0 - E_1 - 3E_2 - E_3 - \ldots - E_{10}$ on X_{10}.

The embedding of X_{10} in $\mathbb{P}^5$ via the complete linear system $|D_5|$ is a *Bordiga-White Surface* (see Section 1).

***b) e* = 1:** Since $\mathbb{F}_1$ is isomorphic to X_1, we have that $\mathbb{F}_1^9$ is isomorphic to X_{10}.

Let us determine the very ample divisor $D_t = tE_0 - mE_1 - E_2 - \ldots - E_{10}$ on X_{10} which corresponds to the divisor $2C_0 + 5f - E_1 - \ldots - E_9$ on $\mathbb{F}_1^9$. The integers $t, m > 0$ are such that

$$\begin{cases} \binom{t+2}{2} - \binom{m+1}{2} - 10 = 5 \\ t^2 - m^2 - 9 = 7, \end{cases}$$

from which we get:

$$\begin{cases} t^2 + 3t + 2 - m^2 - m = 30 \\ t^2 = m^2 + 16. \end{cases}$$

Solving the equations we find $t = 5$ and $m = 3$.

Hence $D_t = D_5 = 5E_0 - 3E_1 - E_2 - \ldots - E_10$, which is the same divisor we found in a).

B) The case 3.4 (iii).

A Del Pezzo double plane S is defined by the embedding of the blowing-up X_7 of $\mathbb{P}^2$ at 7 generic points via the very ample linear system $|6E_0 - 2E_1 - \ldots - 2E_7|$; it is a smooth surface of degree 8 in $\mathbb{P}^6$.

Hence our surface X is determined by the very ample linear system $|6E_0 - 2E_1 - \ldots - 2E_7 - E_s|$ on X_8 (see also [15]).

From the work of Alzati, Bertolini and Besana (see[1]) we know that the surface X is projectively normal in $\mathbb{P}^5$ thus, since $h^1(\mathcal{O}_X(6E_0 - 2E_1 - \ldots - 2E_7 - E_8)) = 0$, X is p.C.M. in $\mathbb{P}^5$, by Proposition 2.1.

We summarize the above results as follows:

Proposition 3.1. *If $X \subseteq \mathbb{P}^5$ is a p.C.M. surface of degree $d = 7$, then it has sectional genus $g = 3$ and it is one of the following rational surfaces:*

1) *A Bordiga-White Surface, obtained embedding* X_{10} *in* $\mathbb{P}^5$ *via the linear system* $|5E_0 - 3E_1 - E_2 - \ldots - E_{10}|$;
2) *The embedding of* $\mathbb{F}_e^9$, $e = 2, 3$, *in* $\mathbb{P}^5$ *via the linear system*
$$|2C_0 + (4+e)f - E_1 - \ldots - E_9|;$$
3) *A Veronesean Surface, defined by the embedding of* X_9 *in* $\mathbb{P}^5$ *via the linear system* $|4E_0 - E_1 - \ldots - E_9|$;
4) *A blowing-up of a point on a Del Pezzo double plane S, i.e.* X_8 *embedded in* $\mathbb{P}^5$ *via the linear system* $|6E_0 - 2E_1 - \ldots - 2E_7 - E_8|$.

3.5. *Surfaces of degree* $d = 8$. Since a p.C.M. surface $X \subseteq \mathbb{P}^5$ of degree $d = 8$ has sectional genus $g = 4 + h^1(\mathcal{O}_H(1))$ and since $g \leq 5$, by Castelnuovo's bound, it is enough to consider the smooth surfaces of sectional genus $g = 4, 5$.

Their classification is known (see[4], [14] and [15]) and it is the following:

If $X \subseteq \mathbb{P}^5$ *is a smooth surface of degree* $d = 8$ *and sectional genus* $4 \leq g \leq 5$, *then it is either a* $K3$ *Surface of sectional genus* $g = 5$ *or it is one of the following rational surfaces of sectional genus* $g = 4$:

(i) *A blowing-up* π *of the quadric surface* $Q \subseteq \mathbb{P}^3$ *with center* 10 *generic points,* $H = \pi^*(3H_Q) - E_1 - \ldots - E_{10}$;

(ii) *A blowing-up* π *of a cubic surface* $S \subseteq \mathbb{P}^3$ *with center* 4 *generic points,* $H = \pi^*(2H_S) - E_1 - \ldots - E_4$;

(iii) *A blowing-up* π *of a Hirzebruch surface* $\mathbb{F}_e$, $e \leq 4$, *with center* 12 *generic points,* $H = \pi^*(2C_0 + (5+e)f) - E_1 - \ldots - E_{12}$.

K3 Surfaces in $\mathbb{P}^5$ of degree $d = 8$ and of sectional genus $g = 5$ are well know and are p.C.M. Hence it remains to prove that the surfaces $X \subseteq \mathbb{P}^5$ in cases $(i), \ldots, (iii)$ are p.C.M.

A) The case 3.5(i).
We recall that the quadric surface $Q \subseteq \mathbb{P}^3$ can be defined as the image of the morphism $X_2 \to \mathbb{P}^3$, where X_2 is the blowing-up $\mathbb{P}^2$ at 2 points, determined by the complete linear system $|2E_0 - E_1 - E_2|$ (see 3.4 $(i)a)$).

Thus to blow-up Q at 10 generic points is equivalent to blow-up $\mathbb{P}^2$ at 11 generic points.

So $H_Q = 2E_0 - E_1 - E_2$, while $H = \pi^*(3H_Q) - (E_0 - E_1 - E_2 - E_3 - \ldots - E_{11} = (6E_0 - 3E_1 - 3E_2) - (E_0 - E_1 - E_2) - E_3 - \ldots - E_{11} = 5E_0 - 2E_1 - 2E_2 - E_3 - \ldots - E_{11}$ is a very ample divisor on X_{11} which defines a *Bordiga-White Surface* in $\mathbb{P}^5$ which is p.C.M. (see Section 1).

B) The case 3.5(ii).
The cubic surface $S \subseteq \mathbb{P}^3$ is defined by the embedding of X_6 in $\mathbb{P}^3$ via the very ample linear system $|3E_0 - E_1 - \ldots - E_6|$.

Thus a smooth hyperplane section of the surface $X \subseteq \mathbb{P}^5$ is a divisor of type $H = \pi^*(2H_S) - E_7 - \ldots - E_{10} = 6E_0 - 2E_1 - \ldots - 2E_6 - E_7 - \ldots - E_{10} = 6E_0 - E$, where $E = 2E_1 + \ldots + 2E_6 + E_7 + \ldots + E_{10}$.

Hence we can denote by X_{10} the blowing-up of the cubic surface $S \subseteq \mathbb{P}^3$ at 4 generic points.

Since $h^1(\mathcal{O}_X(1)) = h^1(\mathcal{O}_{X_{10}}(6E_0 - E)) = 0$ and the surface X is projectively normal in $\mathbb{P}^5$ (see [1]), then X is p.C.M., by Proposition 2.1.

C) The case 3.5(iii).
The embedding X of $\mathbb{F}_e^{12}$, the blowing-up of $\mathbb{F}_e$ at 12 generic points, in $\mathbb{P}^5$ via the very ample linear system $|2C_0 + (5+e)f - E_1 - \ldots - E_{12}|$, with $e \leq 4$, is not projectively normal, by [1; Theorem 5.4]. Thus, clearly, it is not p.C.M. too.

The following proposition summarizes what we have seen above.

Proposition 3.2. *$X \subseteq \mathbb{P}^5$ is a p.C.M. surface of degree $d = 8$, then X is either a $K3$ Surface of sectional genus $g = 5$ or a rational surface of sectional genus $g = 4$. In this case X is one of the following:*

1) *A Bordiga-White Surface, defined by the embedding of X_{11} in $\mathbb{P}^5$ via the linear system $|5E_0 - 2E_1 - 2E_2 - E_3 - \ldots - E_{11}|$;*
2) *The embedding of X_{10}, the blowing-up a cubic surface $S \subseteq \mathbb{P}^3$ at 4 generic points, in $\mathbb{P}^5$ via the linear system*

$$|6E_0 - 2E_1 - \ldots - 2E_6 - E_7 - \ldots - E_{10}|.$$

4. Rational p.C.M. surfaces of degree 9.

In order to complete the description of the rational p.C.M. surfaces in $\mathbb{P}^5$, it would remain to consider the rational surfaces of degree $d = 9, 10$. Here we consider the case $d = 9$.

Let X be a rational p.C.M. surface in $\mathbb{P}^5$ of degree $d = 9$, then its sectional genus has to be $g(H) = g = 5$ (see the introduction).

On the other hand, if $X \subseteq \mathbb{P}^5$ ia a smooth rational surface of degree $d = 9$ and sectional genus $g = 5$, consider the exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1) \to \mathcal{O}_X(1) \to 0.$$

Since $d = 9 > 2g - 2 = 8$, we have $h^1(\mathcal{O}_H(1)) = 0$, hence $h^1(\mathcal{O}_H(1)) = 0$. This implies, by Proposition 2.1, that the surface $X \subseteq \mathbb{P}^5$ is p.C.M. if, and only if, it is projectively normal.

On the projective normality of smooth surfaces of degree 9 and sectional genus 5 in $\mathbb{P}^5$ we have the following result:

Theorem 4.1. *(See [2; Theorem 1.1]). Let S be a smooth surface embedded by the complete linear system associated with a very ample line bundle L as a surface of degree $d = 9$ and sectional genus $g = 5$ in $\mathbb{P}^5$. Assume (S, L) is not a scroll over a curve. Then (S, L) fails to be projectively normal if and only if it is a rational conic bundle such as (S, L) = $(\mathbb{F}_e^{15}, 2C_0+(6+e)f-E_1-\ldots-E_15)$, $0 \leq e \leq 5$.*

Note that, if (S, L) ia a scroll over a curve, in order to be p.C.M. it must be a rational scroll in $\mathbb{P}^5$ (see [16]). But there exist no values of $b > e > 0$ such that the very ample linear system $|C_0 + bf|$ determines an embedding of $\mathbb{F}_e$ in $\mathbb{P}^N = \mathbb{P}^5$ of degree $d = 9$.

In fact $d = -e + 2b = 9$, while $N = 2(b + 1) - e - 1 = 5$, from which we get the equations $-e + 2b = 9$ and $-e + 2b = 4$, which give no solutions.

Thus any smooth rational surface $X \subseteq \mathbb{P}^5$ with $d = 9$ and $g = 5$, different from a rational conic bundle as in Theorem 4.1, is projectively normal and so p.C.M.. Such surfaces have been classified by E. L. Livorni in [15] and we list them as follows:

Let X be a smooth rational surface and L a very ample line bundle on X such that $L^2 = 9, h^0(L) = 6,\ \ g(X, L) = g = 5$. Then X is one of the following:

(i) $(X_{10}, 7E_0 - 2E_1 - \ldots - 2E_{10})$;
(ii) $(X_{12}, 6E_0 - 2E_1 - \ldots - 2E_5 - E_6 - \ldots - E_{12})$;
(iii) $(\mathbb{F}_e^{15}, 2C_0 + (6 + e)f - E_1 \ldots - E_{15}), 0 \leq e \leq 5$;
(iv) $(\mathbb{F}_e^{10}, 4C_0 + (2e + 5)f - 2E_1 \ldots - 2E_7 - E_8 - \ldots - E_{10}), 0 \leq e \leq 2$;
(v) $(\mathbb{F}_1^{12}, 3C_0 + 5f - E_1 \ldots - E_{12})$.

It has been shown that there exist no surfaces as in (v) (e.g. see [2]), while Theorem 4.1 gives us that only case (*iii*) is not p.C.M.. So we can conclude that the only rational p.C.M. surfaces in $\mathbb{P}^5$ of degree $d = 9$ are the ones in Table 1, if they exist. In order to check that they actually do, see [15] for cases (*i*), (*iv*) and [3] for case (*ii*).

5. Nonrational p.C.M. surfaces of degree 9.

Suppose that X is a nonrational p.C.M. surface in $\mathbb{P}^5$ of degree $d = 9$, then $g = 5 + h^1(\mathcal{O}_H(1)) \leq 7$.

All the smooth surfaces of sectional genus $g \leq 7$ whose minimal model is a surface with nonnegative Kodaira dimension have been classified in [15], from where we have the following:

Fact. *Let $X \subseteq \mathbb{P}^5$ be a nonrational smooth surface of degree $d = 9$, sectional genus $5 \leq g \leq 7$, arithmetic genus p_a and geometric genus p_g. Then we have the following cases:*

(i) $g = 6, q = h^1(\mathcal{O}_X) = 0$, $p_a = p_g = 1$, *X is the blowing-up at one point of a $K3$ Surface;*

(ii) $g = 7, q = h^1(\mathcal{O}_X) = 0$, $p_a = p_g = 2$, *X is an Elliptic Surface.*

By [2], Theorem 1.1], the surfaces X listed above are both projectively normal.

Proposition 5.1. *Let $d, g \in \mathbb{Z}$ be such that $d > g-1$. Let $X \subseteq \mathbb{P}^N$ be a smooth surface of degree $d = N + g - 1 - h^1(\mathcal{O}_H(1))$, sectional genus $g(H) = g$ and irregularity $q = h^1(\mathcal{O}_X) = 0$. If $h^1(\mathcal{O}_X(1)) = 0$, then X is p.C.M. if, and only if it is projectively normal.*

Proof. Let $X \subseteq \mathbb{P}^N$ be projectively normal. We want to prove that, when $h^1(\mathcal{O}_X(1)) = 0$, the surface X is P.C.M., i.e. that $h^2(\mathcal{I}_X(m)) = h^1(\mathcal{O}_X(m)) = 0$, for all $m \geq 2$.

Applying the Riemann-Roch Theorem on the smooth hyperplane section H of X, since $d > g - 1$, by hypothesis, we get

$$N - h^1(\mathcal{O}_H(1)) = h^0(\mathcal{O}_H(1)) - h^1(\mathcal{O}_H(1)) = d - g + 1 > 0,$$

hence $N > h^1(\mathcal{O}_H(1))$. So we have:

$$md = m(N+g-1-h^1(\mathcal{O}_H(1))) = m(N-h^1(\mathcal{O}_H(1)))+m(g-1) > 2(g-1),$$

$\forall m \geq 2$, from which we deduce that $h^1(\mathcal{O}_H(m)) = 0, \forall m \geq 2$.

Consider the exact sequence

$$0 \to \mathcal{O}_X(m-1) \to \mathcal{O}_X(m) \to \mathcal{O}_H(m) \to 0.$$

Since $h^1(\mathcal{O}_X(1)) = 0$ and $h^1(\mathcal{O}_H(m)) = 0, \forall m \geq 2$, we have that $h^1(\mathcal{O}_X(m-1)) = 0$ implies $h^1(\mathcal{O}_X(m)) = 0, \forall m \geq 2$, and this is what we required. $\square$

Now, let us consider the two projectively normal surfaces $X \subseteq \mathbb{P}^5$ quoted in the Fact above. Since $d = 9 > g - 1$, by Proposition 5.1, the surfaces X will also be p.C.M. in $\mathbb{P}^5$ if $h^1(\mathcal{O}_X(H)) = h^1(\mathcal{O}_X(1)) = 0$.

This is what we show in the following proposition.

Proposition 5.2. *A nonrational smooth surface $X \subseteq \mathbb{P}^5$ of degree 9 is p.C.M.*

Proof. Let $X \subseteq \mathbb{P}^5$ be as above.

Consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1) \to \mathcal{O}_H(1) \to 0,$$

where H is a smooth hyperplane section of X of genus g.

Since $h^1(\mathcal{O}_X) = 0$, $h^1(\mathcal{O}_H(1)) = h^0(\mathcal{O}_H(1)) = d - 1 + g = g - 5$ and, in our two cases, $g - 5 = p_g = h^2(\mathcal{O}_X)$, we have $h^1(\mathcal{O}_X(1)) = h^2(\mathcal{O}_X(1))$.

By the Serre Duality Theorem, $h^2(\mathcal{O}_X(1)) = h^2(\mathcal{O}_X(H)) = 0$ if, and only if, $h^0(\mathcal{O}_X(K_X - H)) = 0$.

Consider $(K_X - H).H = K_X.H - H^2 = 2g - 2 - 2d = 2g - 20$, which is < 0 when $g = 6$ or 7, hence $h^0(\mathcal{O}_X(K_X - H)) = 0$.

So $h^1(\mathcal{O}_X(1)) = h^2(\mathcal{O}_X(1)) = 0$ and, by what we have seen above, this is enough to conclude that the surfaces $X \subseteq \mathbb{P}^5$ are p.C.M. □

6. Rational p.C.M. surfaces of degree 10.

In [16] we showed that the maximum degree of a rational p.C.M. surface $X \subseteq \mathbb{P}^N$ is $d = \binom{N}{2}$.

There are known rational p.C.M. surfaces which attain the maximum degree, namely the *White Surfaces* (see Section 1).

In $\mathbb{P}^5$ the candidate rational surfaces as p.C.M. surfaces of maximum degree $d = 10$ are described in the following table (see [15] for a classification of rational surfaces of degree 10).

The existence of the surface is known in case (*vi*) (*White Surface*) and in case (*v*), see [15], while in cases (*i*), (*iii*), (*vii*) and (*viii*) we can consider the following theorem.

Theorem 6.1. (See [3], Theorem 2.1]). *Let $P_1, \dots, P_r, R_1, \dots, R_n$ be general points on $\mathbb{P}^2$, with $r \geq 1$. Define $X_{r,n}$ as the blowing-up of $\mathbb{P}^2$ along these points, $\pi_{r,n}$ the corresponding projection map, and $E_1, \dots, E_r, F_1, \dots, F_n$ the exceptional divisor corresponding resp. to the points $P_1, \dots, P_r, R_1, \dots, R_n$. Let $l_1, \dots, l_r$ be integers, with $l_1 \geq \dots \geq l_r \geq 2$. Suppose m, r and l_i are such that there exists a "good" curve of degree $m - 1$; and either $l_1 \leq 3$ and $4m \geq l_1 + l_2 + \dots + l_r + 9$ or $l_1 > 3$ and $4m \geq 2l_1 + l_2 + \dots + l_r + 10$. Then the sheaf $\mathcal{L} = \pi^*_{r,n}(\mathcal{O}_{\mathbb{P}^2}(m)) \otimes (-l_1E_1 - \dots - l_rE_r - F_1 - \dots - F_n)$ is very ample on $X_{r,n}$ for all $n \leq \frac{m(m+3) - l_1(l_1+1) - \dots - l_r(l_r+1)}{2} - 5$.*

TABLE 2. Rational p.C.M. surfaces in $\mathbb{P}^5$ of degree 10

	$(X_s, D_t) - (\mathbb{F}_e^s, D)$
(i)	$(X_{14}, 6E_0 - 2E_1 - \ldots - 2E_4 - E_5 - \ldots - E_{14})$
(ii)	$(X_{12}, 9E_0 - 3E_1 - \ldots - 3E_7 - 2E_8 - E_9 - \ldots - E_{12})$
(iii)	$(X_{12}, 7E_0 - 2E_1 - \ldots - 2E_9 - E_{10} - \ldots - E_{12})$
(iv)	$(X_{11}, 9E_0 - 3E_1 - \ldots - 3E_6 - 2E_7 - \ldots - 2E_{10} - E_{11})$
(v)	$(X_{10}, 10E_0 - 3E_1 - \ldots - 3E_{10})$
(vi)	$(X_{15}, 5E_0 - E_1 - \ldots - E_{15})$
(vii)	$(X_{15}, 6E_0 - 3E_1 - 2E_2 - E_3 - \ldots - E_{15})(equiv.\ \mathbb{F}_0^{14}, 3C_0 + 4f - E_1 - \ldots - E_{14})$
$(viii)$	$(X_{11}, 8E_0 - 3E_1 - 3E_2 - 2E_3 - \ldots - 2E_{11})(equiv.\ \mathbb{F}_0^{10}, 5C_0 + 5f - 2E_1 - .. - 2E_{10})$
(ix)	$(\mathbb{F}_2^{14}, 3C_0 + 7f - E_1 - \ldots - E_{14})$
(x)	$(\mathbb{F}_e^{12}, 4C_0 + (2e+5)f - 2E_1 - \ldots - 2E_6 - E_7 - \ldots - E_{12}; 0 \leq e \leq 2)$
(xi)	$(\mathbb{F}_e^{11}, 4C_0 + (2e+6)f - 2E_1 - \ldots - 2E_9 - E_{10} - E_{11}; 0 \leq e \leq 2)$

By Theorem 6.1, in order to have that D_t is very ample on X_s it is enough to show that there exists a "good" plane curve of degree $(t-1)$, i.e. a curve having, as its only singularities, r multiple points at the $P_i's$, $i = 1, \ldots, r$, of multiplicity $= l_i$, respectively, and such that its strict transform on X_r is smooth.

Since there are plane curves of degree 5 and 6 with, respectively, 4 and 9 nodes (e.g. see[9; Proposition 1.1]) and there are plane curves of degree 5 (resp. 7) with 1 triple point and 1 node (resp. 2 triple points and 9 nodes) (e.g. see Section 2 in [17]), we deduce that the surfaces in cases (*i*), (*iii*), (*vii*) and (*viii*) really exist, as required.

The problem of the existence of the surfaces in cases (*ii*), (*iv*), (*ix*), (*x*), (*xi*) remains open.

Except for the White Surface (case (*vi*)), it is still unknown if the surfaces in Table 2 are really p.C.M.

We recall that (see [16; Proposition 3.5]) a sufficient condition to have that the surfaces in Table 2 are p.C.M. is that their ideal contains no quadric.

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