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FREDHOLM-VOLTERRA INTEGRAL EQUATION OF THE FIRST KIND WITH POTENTIAL KERNEL

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A series method is used to separate the variables of position and time for the Fredholm-Volterra integral equation of the first kind and the solution of the system in $L_2[0, 1] \times C[0, T]$, $0 \le t \le T < \infty$ is obtained, the Fredholm integral equation is discussed using Krein's method. The kernel is written in a Legendre polynomial form. Some important relations are also, established and discussed.

1. Introduction.

We consider an integral equation of Fredholm-Volterra integral equation of the first kind, where the Fredholm integral term is measured with respect to position while Volterra is measured with respect to time. The solution will be obtained in the space $L_2[0, 1] \times C[0, T]$, $0 \le t \le T < \infty$.

Consider the general kernel form of the Fredholm integral equation

$$\begin{cases} k^{\mu,\lambda}_{n,m}(x, y) = \dfrac{x^{\lambda}}{y^{\epsilon+\lambda-1}} W^{\mu}_{n,m}(x, y), \\ W^{\nu}_{n,m}(x, y) = \displaystyle\int_0^{\infty} J_n(tx) J_m(ty) t^{\nu}\, dt, \end{cases} \tag{1.1}$$

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where $J_n(z)$ is the Bessel function of the first kind.

Many problems of mathematical physics, theory of elasticity, viscodynamic fluids and mixed problems of mechanics of continuous media reduce to Fredholm integral equation with a kernel that takes a special form of equation (1.1) (see [1], [2], [3], [4]).

The monographs [5], [14] give many of spectral relationships in terms of orthogonal polynomials for the integral operators frequently encountered in mathematical physics, and describe a method of orthogonal polynomials based on them. Spectral relations and their applications to the mixed problems of the theory of elasticity are given in [1], [4]. Mkhitarian [11] used the generalized potential theory method to obtain the spectral relationships for a Fredholm integral equation of the first kind with Carleman kernel ($k(x, y) = |x-y|^{-\nu}, 0 \leq \nu < 1, \epsilon = 0$).

$$(1.2) \qquad k(x, y) = \sqrt{x\, y} \int_0^\infty t^\nu\, J_{\pm\frac{1}{2}}(t\, x) J_{\pm\frac{1}{2}}(t\, y)\, dt, \quad ((x, y) \in [-1, 1])$$

(for symmetric and skew - symmetric, respectively.)

The two papers [12], [13] of Mkhitarian and Abdou, respectively, using Krein's method, obtained the spectral relationships for the integral operator containing Carleman's kernel and a logarithmic kernel ($k(x, y) = -\ln |x - y|, \ \epsilon = 0$)

$$(1.3) \qquad k(x, y) = \sqrt{x\, y} \int_0^\infty J_{\pm\frac{1}{2}}(t\, x)\, J_{\pm\frac{1}{2}}(t\, y)\, dt, \quad ((x, y) \in [-1, 1])$$

(for symmetric and skew - symmetric, respectively.)

Kovalenko [10] developed the Fredholm integral equation of the first kind for the mechanics mixed problems of continuous media and obtained an approximate solution for the Fredholm integral equation of the first kind with an elliptic kernel $k(x, y) = \frac{1}{x+y}\, E\left(\frac{\sqrt{2\, x\, y}}{x+y}\right)$

$$(1.4) \qquad k(x, y) = \int_0^\infty J_0(x\, t)\, J_0(y\, t)\, dt, \quad ((x, y) \in [-1, 1])$$

The goal here is to obtain the solution of Fredholm-Volterra integral equation of the first kind in the space $L_2[0, 1] \times C[0, T]$. The method used starts with a series form to separate the variables of position and time, secondly we solve the Fredholm integral equation of the first kind, using Krein's method for solving the integral equation of the first kind with potential kernel.

2. Basic equations.

Consider the integral equation

$$\iint_{\Omega} k(x-\xi, y-\eta)\, p(\xi,\eta,t)\, d\xi d\eta \;+\; \int_0^t F(\tau)\, p(x,y,\tau)\, d\tau = \pi\,[\gamma(t) \;+\; \beta(t)\, x \;-\; f(x,y)], \tag{2.1}$$

under the conditions

$$\begin{cases} N_1(t) = \iint_{\Omega} p(x,y,t)\, dx dy, \\ N_2(t) = \iint_{\Omega} x\; y\; p(x,y,t)\, dx dy. \end{cases} \tag{2.2}$$

Here $p(x,y,t)$ is the unknown potential function, Ω is called the domain of integration, the kernel $k(x-\xi, y-\eta)$ is considered in the potential function form as

$$k(x-\xi, y-\eta) = \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}, \quad (\, k(x-\xi, y-\eta) \in C([\Omega] \times [\Omega]\,),$$

the given function $f(x,y) \in L_2(\Omega)$, while $\gamma(t)$, $\beta(t)$, $F(t)$, $N_1(t)$ and $N_2(t)$ are given positive and continuous functions belong to $C(0,T)$, $(0 < T < \infty)$.

The problem is investigated from the three dimensional contact problem of frictionless impression of a rigid surface (G, ν) having an elastic material occupying the domain Ω, where $f(x,y) \in L_2(\Omega)$ describing the surface of stamp. This stamp is impressed into an elastic layer surface (plane) by a variable known force $N_1(t)$, whose eccentricity of application $e(t)$, and a variable known momentum $N_2(t)$, $(0 \le t \le T)$, that case rigid displacements $\gamma(t)$ and $\beta(t)\, x$, respectively. Here G is the displacement magnitude, ν is Poisson's coefficient and $F(t)$ represents the characterized resistance function of the material.

For $t = 0$, the integral equation (2.1) becomes

$$\iint_{\Omega} k(x-\xi, y-\eta)\, p(\xi,\eta,0)\, d\xi d\eta = \pi\,[\,\gamma(0) + \beta(0)\, x - f(x,y)\,],$$

which can be written as

$$\iint_{\Omega} k(x-\xi, y-\eta)\, \phi(\xi,\eta)\, d\xi d\eta = g(x), \tag{2.3}$$

where $\phi(\xi, \eta) = p(\xi, \eta, 0)$, and $g(x) = \pi\ [\ \gamma(0)\ +\ \beta(0)\, x\ -\ f(x, y)\ ]$. Equation (2.3) represents an integral equation with respect to position, while if $k(x, y) = 0$ in (2.1), then we have

$$\int_0^t F(\tau)\, p(x, y, \tau)\, d\tau = \pi\ [\ \gamma(t)\ +\ \beta(t)\, x\ -\ f(x, y)\ ],$$

which represents a Volterra integral equation with respect to time.

3. Method of separation of variables.

Consider the solution of the integral equation (2.1) in the form

$$p(x, y, t) = \sum_{j=0}^{\infty} p_j(x, y, t),$$

that can be approximated as

$$p(x, y, t) \simeq p_0(x, y, t)\ +\ p_1(x, y, t), \tag{3.1}$$

where $p_0(x, y, t)$, $p_1(x, y, t)$ are called the complementary and the particular solution of the integral equation (2.1), respectively.

Using (3.1) in (2.1), we have

$$\begin{aligned} \iint_\Omega k(x-\xi, y-\eta)\, p_j(\xi, \eta, t)\, d\xi\, d\eta\ +\ \int_0^t F(\tau)\, p_j(x, y, \tau)\, d\tau \\ = \pi\ \delta_j\ [\ \gamma(t)\ +\ \beta(t)\, x\ -\ f(x, y)\ ] \end{aligned} \tag{3.2}$$

where

$$\delta_j = \begin{cases} 1, & j = 0 \\ 0, & j = 1. \end{cases} \tag{3.3}$$

Let $t = 0$ in the formula (3.2), we have

$$\iint_\Omega k(x-\xi, y-\eta)\, p_j(\xi, \eta, 0)\, d\xi\, d\eta = \pi\ \delta_j\ [\gamma(0) + \beta(0)\, x - f(x, y)]. \tag{3.4}$$

Subtracting the two equations (3.2) and (3.4), we get

$$\iint_\Omega k(x-\xi, y-\eta)\ [\ p_j(\xi, \eta, t)\ -\ p_j(\xi, \eta, 0)\ ]\, d\xi d\eta \tag{3.5}$$

$$+\int_0^t F(\tau)\, p_j(x, y, \tau)\, d\tau = \pi\, \delta_j\, [\, \gamma(t) - \gamma(0) + (\beta(t) - \beta(0))\, x\,].$$

Assume the approximate solution of (3.1) in the following series expression form

$$p_j(x, y, t) = \sum_{k=1}^{\infty} [\, A_{2k}^j(t)\, p_{2k}(x, y) + A_{2k-1}^j(t)\, p_{2k-1}(x, y)\,]. \tag{3.6}$$

Here, we represent the solution of equation (3.5) in the form of even and odd terms in position and time, hence equation (3.5) becomes

$$\begin{aligned} &\sum_{k=1}^{\infty} \{ \int\!\!\int_{\Omega} [(A_{2k}^j(t) - A_{2k}^j(0))\, p_{2k}(\xi, \eta) + (A_{2k-1}^j(t) - \\ &\quad -A_{2k-1}^j(0))\, p_{2k-1}(\xi, \eta)\,].\ k(x-\xi, y-\eta)\, d\xi d\eta \} + \\ &+ \sum_{k=1}^{\infty} [\, \int_0^t [\, A_{2k}^j(\tau)\, p_{2k}(x, y) + A_{2k-1}^j(\tau)\, p_{2k-1}(x, y)]\, F(\tau)\, d\tau] \\ &\qquad = \pi\, \delta_j\, [\, \gamma(t) - \gamma(0) + (\beta(t) - \beta(0))\, x\,]. \end{aligned} \tag{3.7}$$

Assume in (3.7) the following notations

$$\begin{cases} A_{2k}^j(t) - A_{2k}^j(0) = B_{2k}^j(t), \\ A_{2k-1}^j(t) - A_{2k-1}^j(0) = B_{2k-1}^j(t), \end{cases} \tag{3.8}$$

we obtain

$$\begin{aligned} &\sum_{k=1}^{\infty} \{ \int\!\!\int_{\Omega} [B_{2k}^j(t)\, p_{2k}(\xi, \eta) + B_{2k-1}^j(t)\, p_{2k-1}(\xi, \eta)]\, k(x-\xi, y-\eta)\, d\xi\, d\eta \} \\ &+ \sum_{k=1}^{\infty} \{ \int_0^t [\, A_{2k}^j(\tau)\, p_{2k}(x, y) + A_{2k-1}^j(\tau)\, p_{2k-1}(x, y)\,]\, F(\tau)\, d\tau\, \} \\ &\qquad = \pi\, \delta_j\, [\, \gamma(t) - \gamma(0) + (\beta(t) - \beta(0))\, x\,]. \end{aligned} \tag{3.9}$$

Firstly, let $j = 1$ in (3.9), we obtain

$$A_k^1(t) + N_k \int_0^t A_k^1(\tau)\, F(\tau)\, d\tau = A_k^1(0), \quad (N_k = (\lambda_k)^{-1}), \tag{3.10}$$

where, we use the spectral relation theorem [4]

$$\iint_{\Omega} p_k(\xi,\eta)\,k(x-\xi,\,y-\eta)\,d\xi d\eta = \lambda_k\; p_k(x,y), \quad (k \geq 1). \tag{3.11}$$

Secondly, we let $j = 0$ in the formula (3.9), we have

$$\begin{cases} A^0_{2k}(t) \;+\; N_{2k} \displaystyle\int_0^t A^0_{2k}(\tau)\,F(\tau)\,d\tau = \\ \qquad = \pi\; N_{2k}\; C_{2k}(x,y)\,[\,\gamma(t)-\gamma(0)\,], \\ A^0_{2k-1}(t) + N_{2k-1} \displaystyle\int_0^t A^0_{2k-1}(\tau)\,F(\tau)\,d\tau = \\ \qquad = \pi\, N_{2k-1}\; C_{2k-1}(x,y)\,[\,\beta(t)-\beta(0)\,], \end{cases} \tag{3.12}$$

where

$$\sum_{k=1}^{\infty} C_{2k}(x,y)\,p_{2k}(x,y) = 1, \quad \sum_{k=1}^{\infty} C_{2k-1}(x,y)\,p_{2k-1}(x,y) = x. \tag{3.13}$$

and $A^0_{2k}(0) = A^0_{2k-1}(0) = 0$. From the two equations (3.11) and (3.12), we can assume that the solution $p(x, y, t)$ takes the form

$$p(x,y,t) = \sum_{k=1}^{\infty} [\; A^0_k(t) \;+\; A^1_k(t)\;]\; p_k(x,y), \tag{3.14}$$

under the condition of convergence

$$|\; A^0_k(t) \;+\; A^1_k(t)\;|^2 \;<\; \epsilon, \quad (\epsilon < 1).$$

Equation (3.10) and the two formulae of equation (3.12) represent a Volterra integral equation of the second kind with continuous kernel, i.e. the three integral equations are equivalent to the following formula, on noting the difference notation

$$\phi(t) \;+\; \lambda \int_0^t k(\tau)\,\phi(\tau)\,d\tau = f(t). \tag{3.15}$$

To solve the integral equation (3.15), many different methods are stated and used in [9], [15], one of these methods is changing the integral equation to differential equation. So, differentiate (3.15) with respect to t, one has

$$\frac{d\phi}{dt} \;+\; \lambda\; k(t)\,\phi(t) = f'(t),$$

which represents a linear differential equation of the first kind and its solution is given by using the integrating factor. Hence, we have

$$\phi(t) = e^{-\lambda \int k(t)\, dt} \left[\int e^{\lambda \int k(t)\, dt} f'(t)\, dt \; + \; C \right],$$

where C is the constant of integration, which can be determined from the boundary conditions.

Now, we are going to solve the Fredholm integral equation of the first kind which obtained from the mixed problem by using Krein's method when the contact domain Ω takes the form

$$\Omega = \left\{ (x, y) \in \Omega \; : \; \sqrt{x^2 + y^2} \; \le \; a, \quad z = 0 \right\}.$$

4. Krein's method for solving Fredholm integral equation of the first kind.

Many different methods can be used to obtain the solution of Fredholm integral equation of the first kind with singular kernel. Krein's method is considered as one of the best methods in the theory of elasticity, for solving the singular integral equations, where the singularity disappears and the integral equations can be solved directly without singularity.

Principal of Krein's method [1].

The principal of Krein's method is : to solve the integral equation

$$\int_a^b k(x, y)\, \phi(y)\, dy \; = \; \pi \; f(x), \tag{4.1}$$

we must find a function $q(x, a)$ that satisfies the integral equation

$$\int_a^b k(x, y)\, q(y, a)\, dy \; = \; 1,$$

and, in this case, equation (4.1) has a unique solution in the form

$$\phi(x) \; = \; \frac{1}{2M'(a)} \left[\frac{d}{du} \int_{-a}^{a} q(y, a)\, f(y)\, dy \right] q(x, a)$$

$$- \; \frac{1}{2} \int_{|x|}^{a} q(x, u) \frac{d}{du} \left[\frac{1}{M'(u)} \frac{d}{du} \int_{-u}^{u} q(y, u)\, f(y)\, dy \right] du$$

$$-\frac{1}{2}\frac{d}{dx}\int_{|x|}^{a}\frac{q(x,u)}{M'(u)}\left[\int_{-u}^{u}q(y,u)\,df(y)\right]du, \quad (|\ x\ |< a),$$

where

$$M(u)=\int_{0}^{a}q(y,u)\,dy,$$

$$M'(u)=\frac{d}{du}\,M(u).$$

Here, the Fredholm integral equation of the first kind, with a kernel takes one form of Weber - Sonin integral, will established from the integral equation with a potential kernel. Also, Krein's method is used to solve the Fredholm integral equation.

Let us consider the integral equation

$$\iint_{\Omega}\frac{p(\xi,\eta)\,d\xi d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}=\pi\ g^{*}(x,y),\ (g^{*}(x,y)=\theta[\delta-f(x,y)])$$

$$\{\Omega=(x,y,z)\in\Omega:\sqrt{x^2+y^2}\le a,\ z=0\},$$

under the static condition

$$\iint_{\Omega}p(x,y)\,dxdy=P<\infty.$$

Using the polar coordinates

$$x=r\cos\theta,\quad y=r\sin\theta,$$

$$\xi=\rho\cos\phi,\quad \eta=\rho\sin\phi,$$

we have

$$\int_{0}^{a}\int_{-\pi}^{\pi}\frac{p(\rho,\phi)\rho\,d\rho d\phi}{\sqrt{r^2+\rho^2-2r\rho\cos(\theta-\phi)}}=g(r,\theta), \tag{4.2}$$

$$\int_{0}^{a}\int_{-\pi}^{\pi}p(\rho,\phi)\rho\,d\rho d\phi=P, \tag{4.3}$$

where $g(r,\theta)=2\pi\theta[\delta-g^{*}(r,\theta)]$.

To separate the variables, we assume

$$p(r,\theta) = p_m(r)\begin{cases}\cos m\theta \\ \sin m\theta\end{cases}, \qquad g(r,\theta) = g_m(r)\begin{cases}\cos m\theta \\ \sin m\theta\end{cases}, \tag{4.4}$$

substituting from (4.4) in (4.2), we get

$$\int_0^a \int_{-\pi}^{\pi} \frac{\cos m\phi \, d\phi}{\sqrt{r^2 + \rho^2 - 2\, r\, \rho\, \cos(\theta - \phi)}}\, p_m(\rho)\, \rho\, d\rho = g_m(r)\, \cos m\theta.$$

Using the substitution $\gamma = \theta - \phi$, we obtain

$$\int_0^a k_m(r,\rho)\, p_m(\rho)\, d\rho = g_m(r), \tag{4.5}$$

where

$$k_m(r,\rho) = \int_{-\pi}^{\pi} \frac{\cos m\phi \, d\phi}{\sqrt{r^2 + \rho^2 - 2r\rho\cos\phi}}. \tag{4.6}$$

Also, the boundary condition (4.3) takes the form

$$\int_0^a p_m(\rho)\, d\rho = \begin{cases}\frac{P}{2\pi} & , \quad m = 0 \\ 0 & , \quad m \geq 1\end{cases} \tag{4.7}$$

To write the kernel (4.6) in one forms of equation (1.1), firstly, we use the following relation [6]

$$\int_0^{2\pi} \frac{\cos m\phi \, d\phi}{[1 - 2\, z\, \cos\phi + z^2]^{\alpha}} = \frac{2\,\pi\, (\alpha)_m\, z^m}{m!}\, F(\alpha,\; m+\alpha;\; m+1;\; z^2), \tag{4.8}$$

and

$$F(\alpha,\; \alpha + \frac{1}{2} - \beta;\; \beta + \frac{1}{2};\; z^2) = (1+z)^{-2\alpha}\, F(\alpha,\; \beta;\; 2\,\beta;\; \frac{4z}{1+z^2}), \tag{4.9}$$

$$(\,|\, z\,| < 1, \quad Re(\alpha) > 0; \quad (\alpha)_m = \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)}\,),$$

where $(\alpha)_m$ is called the Pochhmmer symbol, and $F(a,\; b;\; c;\; z)$ is the Gauss hypergeometric function. Hence, equation (4.6) can be written as

$$k_m(r,\rho) = \frac{2\,\sqrt{\pi}\,\Gamma(m+\frac{1}{2})\, r^m}{rho^m\; m!}\, F(m + \frac{1}{2},\; \frac{1}{2};\; m+1;\; \frac{\rho^2}{r^2}), \;\; (\,\rho < r\,),$$

one can, also, proves that

$$k_m(r,\rho) \;=\; \frac{2\sqrt{\pi}\;\Gamma(m+\frac{1}{2})\,r^m}{\rho^m\;m!}\;F(m+\frac{1}{2},\;\frac{1}{2};\;m+1;\;\frac{r^2}{\rho^2}),\quad (\,r\;<\;\rho\,).$$

Using the formula (4.9), we have

$$k_m(r,\rho) = \frac{2\sqrt{\pi}\,\Gamma(m+\frac{1}{2})(r\;\rho)^m}{(r+\rho)^{2m+1}\;m!}\;F(m+\frac{1}{2},\;m+\frac{1}{2};\;2m+1;\;\frac{4\,r\,\rho}{r^2+\rho^2}). \tag{4.10}$$

The formula (4.10) tells us that the value of $k_m(r,\rho)$ is independent of $r<\rho$ or $\rho<r$.

Secondly, using the famous relation [7]

$$\int_0^\infty J_\alpha(ax)\;J_\alpha(bx)\,x^{-\beta}\,dx \tag{4.11}$$

$$= \frac{a^\alpha\;b^\alpha\;2^{-\beta}\;\Gamma(\alpha+\frac{1-\beta}{2})}{(a+b)^{2\alpha-\beta+1}\Gamma(\alpha+1)\Gamma(\frac{1+\beta}{2})}F(\alpha+\frac{1-\beta}{2},\alpha+\frac{1}{2};2\alpha+1;\;\frac{4ab}{(a+b)^2}),$$

if $\alpha=m,\;\beta=0,\;a=r,\;b=\rho$, in equation (4.11), we get

$$\int_0^\infty J_m(rx)J_m(\rho x)\,dx = \frac{\Gamma(m+\frac{1}{2})(r\rho)^m}{\sqrt{\pi}(r+\rho)^{2m+1}m!} \tag{4.12}$$

$$F(m+\frac{1}{2},\;m+\frac{1}{2};2m+1;\;\frac{4\,r\,\rho}{(r+\rho)^2}).$$

Hence, using equation (4.12) in (4.10), we obtain

$$k_m(r,\rho)\;=\;2\,\pi\int_0^\infty J_m(rt)\;J_m(\rho t)\,dt. \tag{4.13}$$

The kernel of equation (4.13) represents as one form of Weber - Sonin integral formula.

Method of solution

Using Krein's method [1], the general solution of the Fredholm integral equation

$$K\,\phi = \int_0^a k(t,s)s\phi(s)\,ds = f(t),$$

with a kernel in the form of equation (4.13) and under the static condition (4.7) can be written in the form

$$\phi(s) = \frac{\gamma}{\pi^2\sqrt{1-s^2}} - \frac{1}{\pi^2}\int_0^1 \frac{du}{\sqrt{u^2-s^2}}\frac{d^2}{du^2}\int_0^u \frac{t\,f(t)\,dt}{\sqrt{u^2-t^2}}, \tag{4.14}$$

where

$$\gamma = \left[\frac{d}{du}\int_0^u \frac{tf(t)\,dt}{\sqrt{u^2-t^2}}\right]_{u=1}, \quad (a = 1). \tag{4.15}$$

The solution of the problem can be derived in the following theorem.

Theorem 1.1. *The eigenfunctions of equation* (4.14)*, when the known function takes a Legendre polynomial form, has the form*

$$\phi(s) = \frac{(-1)^n\, A_n\, 2^{2n}\,(2n+1)\,(n!)^2}{\pi^2\sqrt{1-s^2}\,(2n)!}\, P_{2n}(\sqrt{1-s^2}), \tag{4.16}$$

where $P_{2n}(y)$ *is the Legendre polynomial.*

The proof of this theorem depends on the following two lemmas.

Lemma 1.1. *For all integers* $(n > 0)$*, the value of the integro - differential term*

$$L_n(u) = \frac{d^2}{du^2}\int_0^u \frac{t\,P_{2n}(\sqrt{1-t^2})}{\sqrt{u^2-t^2}}\,dt, \tag{4.17}$$

takes the form

$$L_n(u) = (2n+1)\,u\,A_n\left[3P_{n-1}^{(0,\,\frac{3}{2})}(2u^2-1) + (2n+3)u^2\,P_{n-2}^{(1,\,\frac{5}{2})}(2u^2-1)\right], \tag{4.18}$$

where

$$A_n = \frac{(-1)^n\sqrt{\pi}\,\Gamma(n+1)}{2\,\Gamma(n+\frac{3}{2})}, \tag{4.19}$$

and $P_n^{(\alpha,\,\beta)}(x)$ *is the Jacobi polynomial of order* n $(n \geq 0)$, $(P_n^{(\alpha,\,\beta)}(x) = 0,$ $n < 0)$.

Proof. To prove this lemma, let us assume the following parameters

$$\xi = \sqrt{1 - u^2}\ , \ \eta = \sqrt{1 - t^2},$$

in the integral form

$$\overline{L}_n(u) = \int_0^u \frac{t\ P_{2n}(\sqrt{1 - t^2})}{\sqrt{u^2 - t^2}}\, dt, \tag{4.20}$$

to obtain

$$\overline{L}_n(u) = \overline{L}_n(\sqrt{1 - \xi^2}) = \int_\xi^1 \frac{\eta\ P_{2n}(\eta)}{\sqrt{\eta^2 - \xi^2}}\, d\eta. \tag{4.21}$$

Using the following relations (see [7])

$$P_{2n}(x) = C_{2n}^{\frac{1}{2}}(x) = P_n^{(0,\ -\frac{1}{2})}(2x^2 - 1),$$

we have

$$\overline{L}_n(u) = \int_\xi^1 \frac{P_n^{(0,\ -\frac{1}{2})}(2\,\eta^2 - 1)\,\eta}{\sqrt{\eta^2 - \xi^2}}\, d\eta, \tag{4.22}$$

where $C_n^\lambda(x)$ is the Gegenbauer polynomial. Putting $t = 2\,\xi^2 - 1,\ \ v = 2\,\eta^2 - 1,$ in equation (4.22), we obtain

$$\overline{L}_n(u) = 2^{-\frac{3}{2}} \int_t^1 P_n^{(0,\ -\frac{1}{2})}(v)\, dv \sqrt{v - t}\ . \tag{4.23}$$

Taking in (4.23) the transformation $v = 1 - (1 - t)\,\tau$, we have

$$\overline{L}_n(u) = 2^{-\frac{3}{2}} \int_0^1 (1 - t)^{\frac{1}{2}}\ (1 - \tau)^{\frac{-1}{2}}\ P_n^{(0,\ -\frac{1}{2})}[1 - (1 - t)\,\tau]\, d\tau. \tag{4.24}$$

Using the integral relation (see Eq. (7. 392), pp. 856 of [8])

$$\int_0^1 t^{\lambda - 1}\ (1 - t)^{\mu - 1}\ P_n^{(\alpha,\ \beta)}(1 - \gamma\ t)\, dt = \tag{4.25}$$

$$= \frac{\Gamma(n + \alpha + 1)\ \Gamma(\lambda)\ \Gamma(\mu)}{\Gamma(1 + \alpha)\ \Gamma(\lambda + \mu)\ n!} \cdot {}_3F_2(-n,\ n + \alpha + \beta + 1,\ \lambda;\ \alpha + 1,\ \lambda + \mu;\ \frac{\gamma}{2}),$$

and the following relation (see [7])

$$P_n^{(\alpha,\,\beta)}(y) = \binom{n+\alpha}{n} F(-n,\; n+\alpha+\beta+1;\; \alpha+1;\; \frac{1-y}{2}). \tag{4.26}$$

The integral formula (4.24) becomes

$$\overline{L}_n(u) \;=\; 2^{-\frac{1}{2}}\,\sqrt{1-t}\;\frac{\Gamma(\frac{3}{2})\,\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\; P_n^{(\frac{1}{2},\,-1)}(t).$$

Using the substitution $t = 2\xi^2 - 1,\;\; \xi = \sqrt{1-u^2}$, we get

$$\overline{L}_n(u) = u\; A_n\; P_n^{(-1,\,\frac{1}{2})}(2u^2-1), \tag{4.27}$$

where

$$A_n = \frac{(-1)^n\,\sqrt{\pi}\;\Gamma(n+1)}{2\;\Gamma(n+\frac{3}{2})}.$$

In the view of the Jacobi differential relation (see [7])

$$D\;P_n^{(\alpha,\,\beta)}(x) = \frac{n+\alpha+\beta+1}{2}\;P_{n-1}^{(\alpha+1,\,\beta+1)}(x), \tag{4.28}$$

the first derivative of equation (4.27) takes the form

$$\frac{d}{du}\,\overline{L}_n(u) = A_n\left[\;P_n^{(-1,\,\frac{1}{2})}(2u^2-1) + (2n+1)\,u^2\;P_{n-1}^{(0,\,\frac{3}{2})}(2u^2-1)\;\right]. \tag{4.29}$$

The required result of equation (4.17) is obtained, after differentiating equation (4.29) with respect to u, and using equation (4.28), again.

Hence, the lemma can be proved. The value of the constant γ is obtained by putting $u = 1$ in equation (4.29), to obtain

$$\gamma \;=\; (2n+1)\;A_n\;. \tag{4.30}$$

Lemma 1.2. *With the aid of the two equations* (4.25) *and* (4.26)*, we can obtain the following two relations*

$$\begin{aligned} & z\;.\;{}_3F_2(-n+2,\;n+\frac{5}{2},\;1;\;2,\;\frac{5}{2};\;z) \\ &= \frac{3}{(2n+3)\,(n-1)} - \frac{3\,\sqrt{\pi}\;(n-2)!}{2\,(2n+3)\;\Gamma(n+\frac{1}{2})}\;P_{n-1}^{(\frac{1}{2},\,1)}(1-2z), \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} & z\;.\;{}_3F_2(-n+2,\;n+\frac{5}{2},\;1;\;2,\;\frac{3}{2};\;z) \\ &= \frac{1}{(2n+3)\,(n-1)} \;-\; \frac{\sqrt{\pi}\;(n-2)!}{(2n+3)\;\Gamma(n-\frac{1}{2})}\;P_{n-1}^{(-\frac{1}{2},\,2)}(1-2\,z), \end{aligned} \tag{4.32}$$

where ${}_3F_2(\alpha_1,\;\alpha_2,\;\alpha_3;\;\beta_1,\;\beta_2;\;z)$ *is the generalized hypergeometric series.*

Proof. Using the relation (4.25), we get

$$\int_0^1 (1-t)^{\frac{1}{2}} P_{n-2}^{(1,\frac{5}{2})}[1-(1-\xi)t]\,dt = \frac{2(n-1)}{3}.{}_3F_2(-n+2, n+\frac{5}{2},\ 1;\ 2,\ \frac{5}{2}; z),$$

where

$${}_3F_2(-n+2,\ n+\frac{5}{2},\ 1;\ 2,\ \frac{5}{2};\ z) = \sum_{m=0}^{\infty} \frac{(-n+2)_m\ (n+\frac{5}{2})_m\ (1)_m}{m!\ (2)_m\ (\frac{5}{2})_m}\, z^m .$$

Putting

$$f(z) \ = \ z\ .\ {}_3F_2(-n+2,\ n+\frac{5}{2},\ 1;\ 2,\ \frac{5}{2};\ z),$$

and differentiating it, we obtain

$$\frac{df(z)}{dz} \ = \ F(-n+2,\ n+\frac{5}{2};\ \frac{5}{2};\ z),$$

hence, using equation (4.26), we get

$$\frac{df(z)}{dz} \ = \ \frac{3\sqrt{\pi}\ (n-2)!}{4\,\Gamma(n+\frac{1}{2})}\ P_{n-2}^{(\frac{3}{2},\ 2)}(1-2z).$$

Finally, integrating the above equation with respect to z and using the formula (4.28) with the boundary condition $f(0) = 0$, we have

$$f(z) = \frac{3}{(2n+3)\ (n-1)} \ - \ \frac{3\sqrt{\pi}\ (n-2)!}{2\ (2n+3)\ \Gamma(n+\frac{1}{2})}\ P_{n-1}^{(\frac{1}{2},\ 1)}(1-2z). \tag{4.33}$$

Similarly, for the equation (4.32), we put

$$g(z) \ = \ z\ .\ {}_3F_2(-n+2,\ n+\frac{5}{2},\ 1;\ 2,\ \frac{3}{2};\ z),$$

and we find

$$g(z) = \frac{1}{(2n+3)(n-1)} - \frac{\sqrt{\pi}(n-2)!}{(2n+3)\Gamma(n-\frac{1}{2})}\ P_{n-1}^{(-\frac{1}{2},\ 2)}(1-2z), \tag{4.34}$$

which completes the proof.

Now, we are in a position to evaluate the integral

$$B_n(t) = \frac{1}{\pi^2} \int_t^1 \frac{du}{\sqrt{u^2-t^2}} \frac{d^2}{du^2} \int_0^u \frac{s\, f(s)\, ds}{\sqrt{u^2-s^2}} . \tag{4.35}$$

Introducing (4.18) in (4.35) to get

$$B_n(t) = \frac{(2n+1)\, A_n}{\pi^2} \Big[3 \int_t^1 \frac{u\, P_{n-1}^{(0,\, \frac{3}{2})}(2u^2-1)\, du}{\sqrt{u^2-t^2}}$$

$$+\ (2n+3) \int_t^1 \frac{u^3\ P_{n-2}^{(1,\, \frac{5}{2})}(2u^2-1)\, du}{\sqrt{u^2-t^2}} \Big],$$

then using the substitution $\xi = 2t^2-1,\ \eta = 2u^2-1$, we obtain

$$B_n(t) = \frac{(2n+1)\, A_n}{2\sqrt{2}\,\pi^2} \Big[3 \int_\xi^1 (\eta-\xi)^{-\frac{1}{2}}\ P_{n-1}^{(0,\, \frac{3}{2})}(\eta)\, d\eta + \frac{(2n+3)}{2} \tag{4.36}$$

$$\int_\xi^1 (\eta-\xi)^{\frac{1}{2}}\ P_{n-2}^{(1, \frac{5}{2})}(\eta)\, d\eta + \frac{(2n+3)}{2}\,(1+\xi) \int_\xi^1 (\eta-\xi)^{-\frac{1}{2}}\ P_{n-2}^{(1,\, \frac{5}{2})}(\eta)\, d\eta \Big].$$

Taking the parameter $\eta = 1-(1-\xi)\,\tau,\ (0 < \tau < 1)$, the formula (4.36) can be written in the form

$$B_n(t) = \frac{(2n+1)\, A_n}{2^{\frac{3}{2}}\ \pi^2} \sqrt{1-\xi} \Big[3G_n(\xi) + \frac{1}{2}\,(2n+3)\,(1-\xi)\ Q_n(\xi) \tag{4.37}$$

$$+\frac{1}{2}\,(2n+3)\,(1+\xi)\ H_n(\xi) \Big],\ (\xi\ =\ 2t^2-1),$$

where

$$G_n(\xi) = \int_0^1 (1-\tau)^{-\frac{1}{2}}\ P_{n-1}^{(0,\, \frac{3}{2})}[1-(1-\xi)\,\tau]\, d\tau,$$

$$Q_n(\xi) = \int_0^1 (1-\tau)^{\frac{1}{2}}\ P_{n-2}^{(1,\, \frac{5}{2})}[1-(1-\xi)\,\tau]\, d\tau,$$

and

$$H_n(\xi) = \int_0^1 (1-\tau)^{-\frac{1}{2}}\ P_{n-2}^{(1,\, \frac{5}{2})}[1-(1-\xi)\,\tau]\, d\tau.$$

If we use the famous integral relation of (4.25), we get

$$G_n(\xi)\ =\ \frac{\sqrt{\pi}\,(n-1)!}{\Gamma(n+\frac{1}{2})}\ P_{n-1}^{(\frac{1}{2},\, 1)}(\xi). \tag{4.38}$$

Similarly, for $Q_n(\xi)$, by following the same previous steps, we obtain

$$Q_n(\xi) = \frac{2}{(2n+3)\,(1-\xi)} \left[\, 2 - \frac{\sqrt{\pi}\,(n-1)!}{\Gamma(n+\frac{1}{2})}\; P_{n-1}^{(\frac{1}{2},\,1)}(\xi) \,\right]. \tag{4.39}$$

Also, for $H_n(\xi)$, we have

$$H_n(\xi) = \frac{4}{(2n+3)\,(1-\xi)} \left[\, 1 - \frac{\sqrt{\pi}\,(n-1)!}{\Gamma(n-\frac{1}{2})}\; P_{n-1}^{(-\frac{1}{2},\,2)}(\xi) \,\right]. \tag{4.40}$$

Finally, substituting from equations (4.38), (4.39) and (4.40) in equation (4.37), we get

$$B_n(t) = (2n+1)\; A_n 2^{\frac{3}{2}}\; \sqrt{1-\xi}\; \pi^2 \left[\; \frac{2\sqrt{\pi}\,(n-1)!}{\Gamma(n-\frac{1}{2})} \left[\frac{1-\xi}{n-\frac{1}{2}}\; P_{n-1}^{(\frac{1}{2},\,1)}(\xi) - (1+\xi)\; P_{n-1}^{(-\frac{1}{2},\,2)}(\xi)\right] + 4 \;\right], \tag{4.41}$$

where $\xi = 2t^2 - 1$.

We write the following relations

$$L_n^{(1)}(\xi) = \frac{1-\xi}{n-\frac{1}{2}}\; P_{n-1}^{(\frac{1}{2},\,1)}(\xi),$$

$$L_n^{(2)}(\xi) = (1+\xi)\; P_{n-1}^{(-\frac{1}{2},\,2)}(\xi),$$

to represent the Jacobi polynomial in the Legendre polynomial form, we write

$$P_{n-1}^{(\frac{1}{2},\,1)}(\xi) \;=\; \frac{(-1)^n}{(2n+1)\,t}\; \frac{d}{dt} \left[\; P_{2n}(\sqrt{1-t^2}) \;\right],$$

hence, $L_n^{(1)}(\xi)$ becomes

$$L_n^{(1)}(\xi) = \frac{4\,(-1)^{n+1}}{(4n^2-1)}\; \frac{\sqrt{1-t^2}}{t^2} \left[\; (1-y^2)\; P'_{2n}(y) \;\right], \quad (y = \sqrt{1-t^2}). \tag{4.42}$$

If we use the recurrence relation (see [7])

$$(1-x^2)\; P'_n(x) \;=\; (n+1)\,[\; x\; P_n(x) - P_{n+1}(x)\;],$$

then, the formula (4.42) becomes :

$$L_n^{(1)}(\xi) \;=\; \frac{4\,(-1)^{n-1}}{(2n-1)}\;\frac{y}{1-y^2}\;[\;y\;P_{2n}(y) - P_{2n+1}(y)\;], \tag{4.43}$$

$$(y \;=\; \sqrt{1-t^2},\quad t \;=\; \sqrt{\frac{1+\xi}{2}}).$$

Also, by following the same way for $L_n^{(2)}(\xi)$, we have

$$L_n^{(2)}(\xi) \;=\; 2\,t^2\;P_{n-1}^{(-\frac{1}{2},\,2)}(2t^2-1).$$

Using the formula (see [6])

$$C_{2n}^{\lambda}(x) \;=\; (\lambda)_n(\frac{1}{2})_n\;P_n^{(\lambda-\frac{1}{2},\,-\frac{1}{2})}(2\,x^2-1),$$

when $\lambda = \frac{5}{2}$ and n is replaced by $n\;-\;1$, we get

$$C_{2n-2}^{\frac{5}{2}}(x) \;=\; (\frac{5}{2})_{n-1}(\frac{1}{2})_{n-1}\;P_{n-1}^{(2,\,-\frac{1}{2})}(2\,x^2-1),$$

hence,

$$P_{n-1}^{(-\frac{1}{2},\,2)}(2\,x^2-1) \;=\; \frac{3\,(-1)^{n-1}}{(2n+1)\,(2n-1)}\;C_{2n-2}^{\frac{5}{2}}(x).$$

Finally, we have

$$L_n^{(2)}(\xi) \;=\; \frac{6\,t^2\,(-1)^{n-1}}{(2n+1)\,(2n-1)}\;C_{2n-2}^{\frac{5}{2}}(x).$$

Using the following relation (see [7])

$$D^m\;C_n^{\lambda}(x) \;=\; 2^m\,(\lambda)_m\;C_{n-m}^{\lambda+m}(x);\quad (\;D^m \;=\; \frac{d^m}{dx^m},\;\; m=1,\;2,\;\ldots\;,\;n\;),$$

the value of $L_n^{(2)}(\xi)$ takes the form

$$L_n^{(2)}(\xi) \;=\; \frac{2\,(-1)^{n-1}\,(1-y^2)}{(4\,n^2-1)}\;d^2dy^2\;P_{2n}(y).$$

From the two Legendre differential equations (see [6])

$$(1\;-\;y'')\;P_{2n}''(y)\;-\;2\;y\;P_{2n}'(y)\;+\;2n\;(2n+1)\;P_{2n}(y)\;=\;0.$$

$$P'_{2n}(y) = \frac{2n + 1}{1 - y^2} [\, y\, P_{2n}(y) - P_{2n+1}(y)\,],$$

we get

$$(1-y^2)\, P''_{2n}(y) = \frac{2\, y\, (2n+1)}{1-y^2} [\, y\, P_{2n}(y) - P_{2n+1}(y)\,] - 2\, n\, (2n+1)\, P_{2n}(y),$$

then, $L_n^{(2)}(\xi)$ takes the form

$$L_n^{(2)}(\xi) = \frac{4\,(-1)^{n-1}}{2n-1} \Big[\frac{y^2\, P_{2n}(y) - y\, P_{2n+1}(y)}{1-y^2} - n\, P_{2n}(y) \Big]. \tag{4.44}$$

Hence, we obtain the values of $L_n^{(1)}(\xi)$, $L_n^{(2)}(\xi)$ in the Legendre polynomial form. If we write

$$L_n(\xi) = L_n^{(1)}(\xi) - L_n^{(2)}(\xi),$$

then substituting from equations (4.43) and (4.44) in the equation of $L_n(\xi)$, we can easily, show that

$$L_n(\xi) = \frac{4\, n\, (-1)^{n-1}}{2n-1}\, P_{2n}(y), \tag{4.45}$$

and hence, substitute from equation (4.45) in equation (4.41), we get

$$B_n(t) = \frac{(2n+1)\, A_n}{\pi^2 \sqrt{1-t^2}} \Big[\, 1 + \frac{(-1)^{n-1} \sqrt{\pi}\, n!}{\Gamma(n+\frac{1}{2})}\, P_{2n}(\sqrt{1-t^2})\, \Big]. \tag{4.46}$$

Introduce equations (4.30) and (4.46) in (4.14), then the theorem can be proved.

5. Conclusions.

The following interested cases can be discussed :

(1) The potential function kernel reduces to one formula of the Weber - Sonin integral forms :

$$k(u, v) = 2\,\pi \int_0^\infty J_n(tu)\, J_n(tv)\, dt.$$

(2) The spectral relations for the integral operator $K\phi$ contain the elliptic kernel

$$k(x, y) = \frac{1}{\pi\,(x+y)}\, E(\frac{2\sqrt{x\, y}}{x+y}),$$

included as a special case of our work. Kovalenko [10], developed the Fredholm integral equation of the first kind for the mechanics mixed problems of continuous media and obtained the eigenfunctions of the problem when the kernel is in the form of elliptic function.

(3) The integral equation with logarithmic kernel is contained as a special case of the potential kernel, also, the Carleman kernel is contained too.

(4) The eigenfunctions for the contact problem of zero harmonic symmetric kernel of the potential function are included as special case $m = 0$. Also, the eigenfunctions of the contact problem of the first and higher order ($m \geq 1$) harmonic are included as special cases (see equations (4.5) and (4.13)).

(5) The value of the kernel (4.13) can be represented in the Legendre polynomial as follows

$$k_m(u, v) = \pi\, (u\, v)^m \sum_{n=0}^{\infty} \frac{\Gamma^2(n + m + \frac{1}{2})\, P_n^m(u)\, P_n^m(v)}{\Gamma^2(n + m + 1)\, .\, (m + 2n + \frac{1}{2})^{-1}}\, .$$

(6) Krein's method is considered one of the best methods for solving the Fredholm integral equation of the first kind for the contact problems in the theory of elasticity depending on the known function avoiding the singular point through the work.

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