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SOME RESULTS ASSOCIATED WITH A GENERALIZED BASIC HYPERGEOMETRIC FUNCTION

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In this paper, we define a q-extension of the new generalized hypergeometric function given by Saxena et al. in [13], and have investigated the properties of the above new function such as q-differentiation and q-integral representation. The results presented are of general character and the results given earlier by Saxena and Kalla in [14], Virchenko, Kalla and Al-Zamel in [15], Al-Musallam and Kalla in [2, 3], Kobayashi in [7, 8], Saxena et al. in [13], Kumbhat et al. in [11] follow as special cases.

1. Introduction

From the point of view of statistical distributions, the generalized form of the hypergeometric function $2F_1(a,b;c;z)$ has been investigated by Dotsenko [4], Malovichko [12] and others. Recently, Kumbhat, Gupta and Surana [11] introduced a new generalized basic hypergeometric function in the following form:

$$ {}_2\Phi_1^{\tau}(a,b;c;\tau;q,z) = \frac{\Gamma_q(c)}{\Gamma_q(b)} \sum_{n=0}^{\infty} \frac{(a;q)_n \Gamma_q(b+\tau n) z^n}{\Gamma_q(c+\tau n)(q;q)_n} \tag{1} $$

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where a,b and c are real or complex numbers, $\tau \in \mathbb{R}$, $\tau > 0$, $c \neq 0,-1,-2,\ldots$, $|z| < 1$, $|q| < 1$, $\Gamma_q(b+\tau n)$ and $\Gamma_q(c+\tau n)$ are finite for integer n.

Its q-integral representation is given by :

$$ {}_2\Phi_1^{\tau}(a,b;c;\tau;q,z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} \frac{(q^a z t^{\tau}, tq;q)_\infty}{(zt^{\tau}, tq^{c-b};q)_\infty} d_q t \tag{2} $$

where $Re(c) > Re(b) > 0$, $\tau \in \mathbb{R}$, $\tau > 0$, $|z| < 1$, $|q| < 1$.

As $q \to 1$ in (1) and (2), we get the generalized hypergeometric function (cf. Virchenko et al. [15]).

The generalized basic hypergeometric series ${}_r\Phi_s(.)$, (cf. Gasper and Rahman [5]) is :

$$ {}_r\Phi_s\left[\begin{array}{ccc} a_1,\ldots a_r & & \\ & ; & q,x \\ b_1,\ldots b_s & & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1,\ldots a_r;q)_n}{(q,b_1,\ldots b_s;q)_n} \left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^{1+s-r} .x^n \tag{3} $$

where the convergence of the series (3) is true for $|q| < 1$, for all x if $r \leq s$, and for $|x| < 1$ if $r = s+1$.

For real or complex a, $|q| < 1$, the q-shifted factorial is defined as

$$ (a;q)_n = \begin{cases} 1 & ; \; if \;\; n=0 \\ (1-q^a)\left(1-q^{a+1}\right)\ldots\left(1-q^{a+n-1}\right) & ; \; if \;\; n \in N \end{cases} \tag{4} $$

In terms of the q-Gamma function, (4) can be expressed as :

$$ (a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n > 0 \tag{5} $$

where

$$ \Gamma_q(\alpha) = \frac{(q;q)_\infty}{(q^\alpha;q)_\infty (1-q)^{\alpha-1}} \tag{6} $$

Indeed, it is easy to verify that

$$ \underset{q\to 1^-}{Lim} \Gamma_q(a) = \Gamma(a) \tag{7} $$

and

$$ \underset{q\to 1^-}{Lim} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n \tag{8} $$

where

$$(a)_n = a(a+1)(a+2)\dots(a+n-1), \; n \geq 1 \tag{9}$$

The q-binomial theorem is

$${}_1\Phi_0(a;-;q,x) = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1 \tag{10}$$

The fractional q-derivative of arbitrary order $\lambda > 0$, for a function $f(x) = x\mu - 1$, is given by :

$$D^{\lambda}_{q,x}\left(x^{\mu-1}\right) = \frac{\Gamma_q(\mu)\, x^{\mu-\lambda-1}}{\Gamma_q(\mu-\lambda)} \tag{11}$$

where $\mu \neq 0, -1, -2, \dots$.

For $\lambda = 1$, the equation (11) reduces to

$$D_{q,x}\left(x^{\mu-1}\right) = \frac{\Gamma_q(\mu)\, x^{\mu-2}}{\Gamma_q(\mu-1)} = \frac{\left(1-q^{\mu-1}\right)x^{\mu-2}}{(1-q)} \tag{12}$$

The q-Beta function (cf. Gasper and Rahman [5]) is

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)} \tag{13}$$

$$= \int_0^1 t^{x-1} \frac{(tq;q)_\infty}{(tq^y;q)_\infty} d_q t,$$

where $Re(x) > 0, Re(y) > 0$.

The paper will be organized as follows. In section 2, we define a q-extension of the new generalized hypergeometric function and derive its q-integral representations. Certain analytic properties of this special function are described in section 3.

2. Definition and q-integral representations of ${}_3\Phi_2^{\tau}(.)$.

A new generalized basic hypergeometric function is defined in terms of series representation in the form:

$${}_3\Phi_2^{\tau}(\lambda,a,b;c,d;\tau;q,z) = \frac{\Gamma_q(c)\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(b)} \sum_{n=0}^{\infty} \frac{(\lambda;q)_n \Gamma_q(a+\tau n)\Gamma_q(b+\tau n)\, z^n}{\Gamma_q(c+\tau n)\Gamma_q(d+\tau n)\,(q;q)_n} \tag{14}$$

where $|z| < 1$, $\tau \in \mathbb{R}$, $\tau > 0$, $|q| < 1$, and

$$ {}_3\Phi_2^{\tau}(\lambda,a,b;c,d;\tau;q,z) \tag{15} $$

$$ = \frac{\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(d-a)} \int_0^1 t^{a-1} \frac{(tq;q)_\infty}{(tq^{d-a};q)_\infty} {}_2\Phi_1^{\tau}(\lambda,b;c;\tau;q,zt)\, d_q t $$

where $Re(d) > Re(a) > 0$, $Re(c) > Re(b) > 0$, $\tau \in \mathbb{R}$, $\tau > 0$, $|z| < 1$, $|q| < 1$.

It is interesting to note that for $b = d$, (14) reduces to the generalized basic hypergeometric function studied by Kumbhat et al. [11].

If we let $q \to 1^-$ and make use of the limit formula (8), (14) reduces to the generalized hypergeometric function studied by Saxena et al. [13].

Proof of (15) **:**

To prove (15) we have :

$$ {}_3\Phi_2^{\tau}(\lambda,a;b;c,d;\tau;q,z) = \frac{\Gamma_q(c)\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(b)} \sum_{n=0}^{\infty} \frac{(\lambda;q)_n \Gamma_q(a+\tau n)\Gamma_q(b+\tau n) z^n}{\Gamma_q(c+\tau n)\Gamma_q(d+\tau n)(q;q)_n} $$

$$ = \frac{\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(d-a)} \sum_{n=0}^{\infty} \frac{(\lambda;q)_n \Gamma_q(c)\Gamma_q(b+\tau n) z^n}{(q;q)_n \Gamma_q(b)\Gamma_q(c+\tau n)} B_q(a+\tau n, d-n) $$

Applying the property of integral of q-Beta function (13), we obtain :

$$ \frac{\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(d-a)} \int_0^1 t^{a+\tau n-1} \frac{(tq;q)_\infty}{(tq^{d-a};q)_\infty} \cdot \frac{\Gamma_q(c)\Gamma_q(b+\tau n)(\lambda;q)_n z^n}{\Gamma_q(b)\Gamma_q(c+\tau n)(q;q)_n} d_q t $$

Interchanging the order of integration and summation; we get

$$ \frac{\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(d-a)} \int_0^1 t^{a-1} \frac{(tq;q)_\infty}{(tq^{d-a};q)_\infty} \sum_{n=0}^{\infty} \frac{(\lambda;q)_n \Gamma_q(c)\Gamma_q(b+\tau n) z^n}{\Gamma_q(b)\Gamma_q(c+\tau n)(q;q)_n} \cdot (zt^{\tau})^n\, d_q t $$

After some simplification, we get :

$$ \frac{\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(d-a)} \int_0^1 t^{a-1} \frac{(tq;q)_\infty}{(tq^{d-a};q)_\infty} {}_2\Phi_1^{\tau}(\lambda,b;c;\tau;q,zt^{\tau})\, d_q t $$

which completes the proof of (15).

3. q-Differentiation Formulas of ${}_3\Phi_2^{\tau}(\cdot)$

$$D_{q,z}\left[{}_3\Phi_2^{\tau}(\lambda,a,b;c,d;\tau;q,z)\right]=\frac{(\lambda;q)\Gamma_q(d)\Gamma_q(c)\Gamma_q(a+\tau)\Gamma_q(b+\tau)}{\Gamma_q(a)\Gamma_q(b)\Gamma_q(c+\tau)\Gamma_q(d+\tau)}\cdot \tag{16}$$

$$\cdot{}_3\Phi_2^{\tau}(\lambda q,a+\tau,b+\tau;c+\tau,d+\tau;\tau;q,z)$$

$$D_{q,z}\left[z^{\lambda}{}_3\Phi_2^{\tau}(\lambda,a,b;c,d;\tau;q,z)\right]=(\lambda;q)\,z^{\lambda-1}{}_3\Phi_2^{\tau}(\lambda q,a,b;c,d;\tau;q,z) \tag{17}$$

$$D_{q,z}^{n}\left[z^{\lambda+n-1}{}_3\Phi_2^{\tau}(\lambda,a,b;c,d;\tau;q,z)\right]=(\lambda;q)\,z^{\lambda-1}{}_3\Phi_2^{\tau}(\lambda q^{n},a,b;c,d;\tau;q,z) \tag{18}$$

$$D_{q,z}^{n}\left[{}_3\Phi_2^{\tau}(\lambda,a,b;c,d;\tau;q,z)\right]=\frac{\Gamma_q(c)\Gamma_q(d)\Gamma_q(a+\tau n)\Gamma_q(b+\tau n)(\lambda;q)_n}{\Gamma_q(a)\Gamma_q(b)\Gamma_q(c+\tau n)\Gamma_q(d+\tau n)}\cdot \tag{19}$$

$$\cdot{}_3\Phi_2^{\tau}(\lambda q^{n},a+\tau n,b+\tau n;c+\tau n,d+\tau n;\tau;q,z)$$

Proof of (16) **:**
To prove (16) we see that the left hand side of (16) can be written as :

$$\frac{\Gamma_q(c)\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(b)}\sum_{n=0}^{\infty}\frac{(\lambda;q)_n\Gamma_q(a+\tau n)\Gamma_q(b+\tau n)}{\Gamma_q(c+\tau n)\Gamma_q(d+\tau n)(q;q)_n}\cdot D_{q,z}(z^{n})$$

$$=\frac{\Gamma_q(c)\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(b)}\sum_{n=1}^{\infty}\frac{(\lambda;q)_n\Gamma_q(a+\tau n)\Gamma_q(b+\tau n)}{\Gamma_q(c+\tau n)\Gamma_q(d+\tau n)(q;q)_n}\cdot\frac{(1-q^{n})}{(1-q)}z^{n-1}$$

Replacing n by $n+1$; we get :

$$=\frac{\Gamma_q(c)\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(b)}\sum_{n=0}^{\infty}\frac{(\lambda;q)_{n+1}\Gamma_q(a+\tau n+\tau)\Gamma_q(b+\tau n+\tau)}{\Gamma_q(c+\tau n+\tau)\Gamma_q(d+\tau n+\tau)(q;q)_{n+1}}\cdot\frac{\left(1-q^{n+1}\right)}{(1-q)}z^{n}$$

After some simplification; we get :

$$=\frac{(\lambda;q)\Gamma_q(c)\Gamma_q(d)}{\Gamma_q(a)\Gamma_q(b)}\sum_{n=0}^{\infty}\frac{(\lambda;q)_n\Gamma_q(a+\tau n+\tau)\Gamma_q(b+\tau n+\tau)}{\Gamma_q(c+\tau n+\tau)\Gamma_q(d+\tau n+\tau)}\cdot\frac{z^{n}}{(q;q)_n}$$

Using (14), the right hand side of (16) is obtained.
The proof of (17) to (19) are straight forward and hence are omitted.

4. Application. The q-distribution

Using the result given by Saxena et al. [13], the q-Laplace type integral

$$I = \int_0^\infty e_q \{-\gamma (1-q) z\} z^{\beta-1} \xi (z) \, d (z;q)$$

Where $\xi (z) = {}_3\Phi_2^\tau (\lambda, a, b; c, d; \tau; q, kz^\tau)$
On using the definition (14), it yields

$$\int_0^\infty e_q \{-\gamma (1-q) z\} z^{\beta-1} \frac{\Gamma_q (c) \; \Gamma_q (d)}{\Gamma_q (a) \; \Gamma_q (b)}$$

$$\cdot \sum_{n=0}^{\infty} \frac{(\lambda;q)_n \, \Gamma_q (a+\tau n) \, \Gamma_q (b+\tau n) \, (kz^\tau)^n}{\Gamma_q (c+\tau n) \, \Gamma_q (d+\tau n) \, (q;q)_n} \, d (z;q)$$

On interchanging the order of integration and summation and then using the equation namely [cf. Abdi [1]],

$$z^{\beta-1} \underset{Q}{\supset} (q;q)_{\beta-1} \, q^{\frac{-\beta(\beta-1)}{2}} s^{-\beta} Re (\beta) > 0$$

$$\underset{Q}{\supset} \frac{(q;q)_{\beta-1} \left(-sq^\beta : q\right)_\infty \left(-\frac{q}{s}^{1-\beta} : q\right)_\infty}{(-s : q)_\infty \left(-\frac{q}{s} : q\right)_\infty}$$

It reduces to

$$q^{\frac{\left(\beta-\beta^2\right)}{2}} \frac{\Gamma_q (\beta)}{\gamma^\beta} \cdot {}_4\Phi_2^\tau \left(\lambda, a, b, \beta; c, d; \tau; q, \frac{kq^{\frac{-\tau(2\beta+\tau n-1)}{2}}}{\gamma^\tau}\right)$$

Thus the function

$$f (z; \beta, \gamma, \lambda, a, b, c, d, \tau, k, q)$$

$$= \frac{e_q \{-\gamma(1-q)z\} z^{\beta-1} \, {}_3\Phi_2^\tau (\lambda, a, b; c, d; \tau; q, kz^\tau)}{q^{\frac{\left(\beta-\beta^2\right)}{2}} \gamma^{-\beta} \Gamma_q (\beta) \; {}_4\Phi_2^\tau \left(\lambda, a, b; \beta; c, d; \tau; q, \frac{kq^{\frac{-\tau(2\beta+\tau n-1)}{2}}}{\gamma^\tau}\right)}$$

Where

$$Re (\beta) > 0, Re (\gamma) > 0, Re (\gamma) > Re (k), \; \tau > 0, \; 0 < |q| < 1, \; 0 < z < \infty,$$

$$c, d \neq 0, -1, -2, \ldots, \qquad = 0 \text{ elsewhere}$$

provides a basic analogue of probability density function.

The generalized basic hypergeometric function ${}_3\Phi_2^{\tau}(\lambda, a, b; c, \ d\,; \tau; q, kz^{\tau})$ can be reduced to basic hypergeometic function ${}_2\Phi_2\,(a, b, c, \ d\,; q, z)$ by taking $\tau = 1$ and replacing k by $\frac{k}{\lambda}$ then taking the limit $\lambda \to \infty$. The basic analogue of probability density function changes to

$$f\,(z\,; \beta, \gamma, \ a, \ b, \ c, d, \ 1, \ k, q) \tag{20}$$

$$= \frac{e_q\{-\gamma(1-q)z\}\, z^{\beta-1}\ {}_2\Phi_2\,(a, b; c, d\,;\, q, kz)}{\Gamma_q(\beta)\ q^{\frac{(\beta-\beta^2)}{2}}\ \gamma^{-\beta}\ {}_3\Phi_2\left(a, b\,, \beta\,; c, d\,; q, \frac{kq^{-2\beta}}{\gamma}\right)}$$

If we set $\gamma = \mu$ in (20), we obtain q-extension of the function of the type George and Mathai [6].

$$f\,(z; \beta, \gamma, \ a, \ c, \ 1, \ k, q) \tag{21}$$

$$= \frac{e_q\{-\gamma(1-q)z\}\ {}_1\Phi_1\,(a; c\,;\, q, \, kz)}{q^{\frac{(\beta-\beta^2)}{2}}\ \gamma^{-\beta}\ \Gamma_q(\beta)\ \ {}_2\Phi_1\left(a, \ \beta\,; c; q, \frac{kq^{-2\beta}}{\gamma}\right)}$$

If we take $k \to 0$ in (20), we obtain two parameters q-gamma density type function

$$f\,(z; \beta, \gamma, \, q) \tag{22}$$

$$= \frac{e_q\{-\gamma(1-q)z\}\, z^{\beta-1}}{\Gamma_q(\beta)\ q^{\frac{(\beta-\beta^2)}{2}}\ \gamma^{-\beta}}$$

Indeed, if we let $q \to 1^-$ and make use of the relation (8), in (20), we get known results due to Kumbhat and Shanu [10]. On the other hand, if we put $b = d$ and $q \to 1^-$ in (20), we obtain a known result due to George and Mathai [6].

Finally if we take $k \to 0$ or either $a = 0$ or $b = 0$ and $q \to 1^-$, we obtain a known result due to Kumbhat and Shanu [10].

REFERENCES

[1] W. H. Abdi, *On q-Laplace transforms*, Proc. Nat. Acad. Sci. India Sect. A 29 (1960), 389–408.

[2] F. Al-Musallam - S. L. Kalla, *Asymptotic expansions for the generalized gamma and incomplete gamma functions*, Appl. Anal. 66 (1997), 173–187.

[3] F. Al-Musallam - S. L. Kalla, *Further results on a generalized gamma function occurring in diffraction theory*, Integral Transform and Special Functions, 7 (1998), 175–190.

[4] M. Dotsenko, *On some applications of Wright's hypergeometric function*, Comp. Rend. De I Aead. Bnlgare des sci. 44 (1991), 13–16.

[5] G. Gasper - M. Rahman, *Basic Hypergeometric series*, Cambridge University Press, Cambridge, 1990.

[6] A. George - A. M. Mathai, *A generalized distribution for inter five birth intervals*, Sankhya, Vol. 37, series B. (1975), 332–342.

[7] K. Kobayashi, *On generalized gamma functions occurring in diffraction theory*, Journal of the Phys. Soc. of Japan, 60 (1991), 1501–1512.

[8] K. Kobayashi, *Plane wave diffraction by a strip; Exact and asymptotic solutions*, Journal of the Physical Society of Japan, 60 (1991), 1891–1905.

[9] R. K. Kumbhat - S. Sharma, *Some results on a generalized probability density function*, J. Stat. and Appl., Vol. 3 (1) (2008), 49–58.

[10] R. K. Kumbhat - S. Sharma, *Fractional integration of hypergeometric functions of three variables*, Vijnana Parishad Anusandhan Patrika 51 (1) (2008), 105–112.

[11] R. K. Kumbhat - R. K. Gupta - M. Surana, *Some results on generalized basic hypergeometric functions*, South East Asian J. of Math. and Math. Sci., Accepted for publication (2008).

[12] V. Malovichko, *On a generalized hypergeometric function and some integral operators*, Math. Phy. 19 (1976), 99–103.

[13] R. K. Saxena - C. Ram - Naresh, *Some results associated with a generalized hypergeometric function*, Bull. of Pure and Appl. Sci., Vol. 24E (n.2) (2005), 305–316.

[14] R. K. Saxena - S. L. Kalla, *On a generalized gamma function occurring in diffraction theory*, Int. J. Appl. Math.5 (2001), 189–202.

[15] N. Virchenko - S. L. Kalla - A. Al-Zamel, *Some results on a generalized hypergeometric function*, Integral transforms and Special Functions. 12 (2001), 89–100.

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