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IRREDUCIBILITY OF HURWITZ SPACES OF COVERINGS OF AN ELLIPTIC CURVE OF PRIME DEGREE WITH ONE POINT OF TOTAL RAMIFICATION

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Let Y be an elliptic curve, p a prime number and $WH_{p,n}(Y)$ the Hurwitz space that parametrizes equivalence classes of p-sheeted branched coverings of Y, with n branch points, $n-1$ of which are points of simple ramification and one of total ramification. In this paper, we prove that $WH_{p,n}(Y)$ is irreducible if $n-1 \geq 2p$.

Introduction.

In this paper we prove the irreducibility of the Hurwitz space $WH_{p,n}(Y)$ which parametrizes the equivalence classes of coverings of an elliptic curve Y, whose degree p is a prime number and which have $n-1 \geq 2p$ points of simple ramification and one point of total ramification.

Most of the results on irreducibility of Hurwitz spaces obtained so far treat the case of coverings of $\mathbb{P}^1$. Hurwitz proved in [6] the irreducibility of $H_{d,n}(\mathbb{P}^1)$, the space which parametrizes simple coverings of degree d. Arbarello proved in [1] the irreducibility of any of the Hurwitz spaces which parametrize coverings of $\mathbb{P}^1$ which have $n-1$ points of simple ramification and one point of total ramification. The case of coverings of $\mathbb{P}^1$ with $n-1$ points of simple ramification and one point of arbitrary ramification was studied by Natanzon

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[9], Kluitmann [7] and Mochizuki [8], who proved the irreducibility of the corresponding Hurwitz spaces. Harris, Graber and Starr studied in [5] the Hurwitz spaces which parametrize simple degree d coverings of a positive genus curve Y whose monodromy group is the group S_d. They proved the irreducibility of these spaces when the number of branch points n satisfies $n \geq 2d$.

1. Preliminaries.

Let Y be an elliptic curve, X be a compact, connected Riemann surface and $f : X \to Y$ be an analytic map onto Y. We recall some standard definitions (see e.g.[4]). A branch point $a \in Y$ is called a point of simple ramification for f if f is ramified at only one point $x \in f^{-1}(a)$ and the ramification index $e(x)$ of f at x is 2. A branch point $a \in Y$ is called a point of total ramification for f if $\sharp f^{-1}(a) = 1$. Two p-sheeted branched coverings $f : X_1 \to Y$ and $g : X_2 \to Y$ are said to be equivalent if there exist a biholomorphic map $\varphi : X_1 \to X_2$ such that $g \circ \varphi = f$. The equivalence class containing f is denoted by $[f]$. Let S_p be the symmetric group on p letters acting on the set $\{1, ..., p\}$. Let us say that two homomorphisms φ and η from $\pi_1(Y \backslash A, y)$ to S_p are equivalent if they differ by a inner automorphism, i.e. there is a $\sigma \in S_p$ such that $\varphi([\alpha]) = \sigma \eta [\alpha] \sigma^{-1}$ for any $[\alpha] \in \pi_1(Y \backslash A, y)$.

Let p be a prime number and let $WH_{p,n}(Y)$ be the Hurwitz space that parametrizes equivalence classes of p-sheeted branched coverings of Y, with n branch points, $n - 1$ of which are points of simple ramification and one of total ramification. Let

$$WH^A_{p,n}(Y) = \{[f] \in WH_{p,n}(Y) : f \text{ has discriminant locus } A = \{a_1, ..., a_n\}\}.$$

By Riemann's existence theorem the equivalence classes $[f] \in WH^A_{p,n}(Y)$ are in one-to-one correspondence with equivalence classes of homomorphisms $\mu : \pi_1(Y \backslash A, y) \to S_p$ whose images are transitive subgroups of S_p. Let $\gamma_1, ..., \gamma_n, \alpha, \beta$ be the generators of $\pi_1(Y \backslash A, y)$ represented in figure 1.

The images via the homomorphisms μ of these generators determine a $(n + 2)$-tuple of permutations of S_p

$$(\mu(\gamma_1), ..., \mu(\gamma_n), \mu(\alpha), \mu(\beta)) = (t_1, ..., t_n, t_\alpha, t_\beta)$$

such that the t_i with $1 \leq i \leq n$ are all transpositions except one that is a p-cycle; t_α, t_β are any two permutations of S_p and $\prod_{i=1}^n t_i = [t_\alpha, t_\beta]$. Since one of t_i is a p-cycle and p is prime then, if $n \geq 2$, $< t_1, ..., t_n >= S_p$.

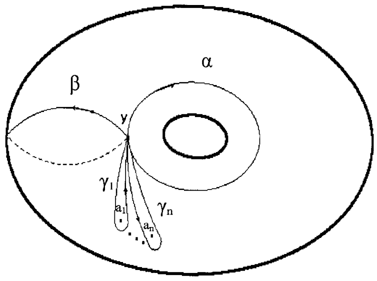

Figure 1.

Let S_p^{n+2} be $(n+2)$-fold product of S_p. Define in S_p^{n+2} an equivalence relation $\sim$ as follows

$$(t_1, ..., t_n, t_{n+1}, t_{n+2}) \sim (\mu_1, ..., \mu_n, \mu_{n+1}, \mu_{n+2})$$

$\Leftrightarrow \mu_i = s t_i s^{-1}$ for some $s \in S_p$ and for all i $(1 \leq i \leq n+2)$.

For the rest of the paper we suppose $n \geq 2$. Let $[t_1, ..., t_{n+2}]$ be the equivalence class containing $(t_1, ..., t_{n+2})$ and let

$A_{p,n+2} = \{[t_1, ..., t_n, t_\alpha, t_\beta] : t_i (i = 1, ..., n)$ are all transpositions except one that is a p-cycle, $\prod_{i=1}^{n} t_i = [t_\alpha, t_\beta]\}$.

By Riemann's existence theorem it is possible to identify $WH_{p,n}^A(Y)$ with $A_{p,n+2}$ via the one-to-one map

$$\omega : WH_{p,n}^A(Y) \rightarrow A_{p,n+2}$$

defined by

$$\omega([f]) = [\mu(\gamma_1), ..., \mu(\gamma_n), \mu(\alpha), \mu(\beta)].$$

Let $Y^{(n)}$ be the symmetric product of Y with itself n times and let Δ be the codimension 1 locus of $Y^{(n)}$ consisting of non simple divisors. Let $\delta : WH_{p,n}(Y) \rightarrow Y^{(n)} \backslash \Delta$ be the map which assigns to each $[f] \in WH_{p,n}(Y)$ its discriminant locus.

It is well known (see [4]) that it is possible to define a topology on $WH_{p,n}(Y)$ in such a way that δ becomes a topological covering map. So the braid group $\pi_1(Y^{(n)} \backslash \Delta, A)$ acts on the fiber $\delta^{-1}(A) = WH_{p,n}^A(Y)$. Our aim is to prove that the action of $\pi_1(Y^{(n)} \backslash \Delta, A)$ on this fiber is transitive. This would imply $WH_{p,n}(Y)$ is connected. In order to prove that $\pi_1(Y^{(n)}\backslash\Delta, A)$ acts transitively on $A_{p,n+2}$, i.e. on $WH^A_{p,n}(Y)$, it is sufficient to prove that it is possible, acting successively by the elements of a system of generators of $\pi_1(Y^{(n)}\backslash\Delta, A)$, to bring every $[t_1, ..., t_n, t_\alpha, t_\beta] \in WH^A_{p,n}(Y)$ to the normal form

$$[(12...p), (12), ..., (12), (23), ..., (p-1\ p), id, id] \tag{1}$$

where the transpositions (12) are in odd number and each transposition $(i\ i+1)$ with $i \neq 1$ is only present one time.

Remark. It is well known (see [2, 3]) that the generators of $\pi_1(Y^{(n)}\backslash\Delta, A)$ are the elementary braids σ_i $(i = 1, ..., n-1)$ and the braid moves ρ_j, τ_j $(j = 1, ..., n)$ relative respectively to the loops α and β. The elementary braids σ_i act on $A_{p,n+2}$ (see [6]) bringing the class

$$[t_1, ..., t_{i-1}, t_i, t_{i+1}, ..., t_\alpha, t_\beta]$$

to

$$[t_1, ..., t_{i-1}, t_i t_{i+1} t_i^{-1}, t_i, ..., t_n, t_\alpha, t_\beta].$$

The actions of ρ_j and τ_j were studied in [5]. The action of the generators τ_j $(j = 1, ..., n)$ changes the loops α and γ_j while it leaves unchanged the loops γ_i (for every $i \neq j$) and β. When t_n is a transposition τ_n transforms t_α into t'_α where

$$t'_\alpha = t_\alpha t_n. \tag{2}$$

Analogously the action of ρ_j $(j = 1, ..., n)$ changes γ_j and β, leaving unchanged the γ_i for every $i \neq j$ $(i = 1, ..., n)$ and α. When t_1 is a transposition ρ_1 transforms t_β into t'_β where

$$t'_\beta = t_\beta t_1. \tag{3}$$

2. Irreducibility of $WH_{p,n}(Y)$.

In this section we will prove that $WH_{p,n}(Y)$ is irreducible for $n-1 \geq 2p$. Since $WH_{p,n}(Y)$ is smooth it suffices to prove that $WH_{p,n}(Y)$ is connected. Let

$$[t_1, ..., t_n, t_\alpha, t_\beta] \tag{4}$$

be an element of $\delta^{-1}(A) = WH^A_{p,n}(Y) \cong A_{p,n+2}$. To prove that (4) is in the orbit of (1) under the action of $\pi_1(Y^{(n)}\backslash\Delta, A)$, it is sufficient to prove that there are braid moves transforming (4) into $[t_1^{'}, ..., t_n^{'}, id, id]$ where the $t_i^{'}$ are all transpositions except one that is a p-cycle, $\prod_{i=1}^n t_i = id$ and $< t_1^{'}, ..., t_n^{'} >= S_p$. In fact, once this is proved we observe that the equivalence class of $(t_1^{'}, ..., t_n^{'})$ can be thought as the Hurwitz-system relative to a branched covering of $\mathbb{P}^1$ and utilizing the Arbarello's result [1] we obtain that $[t_1^{'}, ..., t_n^{'}, id, id]$ is in the orbit of (1) under the action of $\pi_1(Y^{(n)}\backslash\Delta, A)$. At first we will prove that (4) can be transformed, via the action of suitable σ_i and σ_i^{-1}, into $[t_1^{'}, ..., t_{n-2}^{'}, \tau, \tau, t_\alpha, t_\beta]$ where τ is a transposition of S_p. After we will prove that there are braid moves transforming $[t_1^{'}, ..., t_{n-2}^{'}, \tau, \tau, t_\alpha, t_\beta]$ into $[t_1^{'}, ..., t_{n-2}^{'}, \tau^{'}, \tau^{'}, t_\alpha, t_\beta]$ with $\tau^{'}$ arbitrary transposition of S_p. Once this is proved it is sufficient to act with suitable ρ_i and τ_j to conclude.

Lemma 1. *Let* $[t_1, ..., t_n, t_\alpha, t_\beta]$ *be an element of* $WH^A_{p,n}(Y)$. *Suppose* $n - 1 \geq 2p$. *Then there are braid moves transforming*

$$[t_1, ..., t_n, t_\alpha, t_\beta] \;\; into \;\; [t_1^{'}, ..., t_{n-2}^{'}, \tau, \tau, t_\alpha, t_\beta],$$

where τ *is a transposition of* S_p.

Proof. Acting with elementary braids it is possible to bring (4) to $[\bar{t}_1, \bar{t}_2, ..., \bar{t}_n, t_\alpha, t_\beta]$ where $\bar{t}_1$ is a p-cycle. Let G be the group generated by the transpositions $\bar{t}_2, ..., \bar{t}_n$ and let $D_1, ..., D_r$ be the domains of transitivity of G. Then

$$G = S_{D_1} \times ... \times S_{D_r}.$$

We observe that if $\bar{t}_j$ and $\bar{t}_{j+1}$ $(2 \leq j \leq n-1)$ are such that $\bar{t}_j \in S_{D_h}$ and $\bar{t}_{j+1} \in S_{D_k}$ with $h \neq k$ and $1 \leq h, k \leq r$, then operating with σ_j we obtain

$$[..., \bar{t}_j, \bar{t}_{j+1}, ...] \to [..., \bar{t}_j\bar{t}_{j+1}\bar{t}_j^{-1}, \bar{t}_j, ...]$$

where $\bar{t}_j\bar{t}_{j+1}\bar{t}_j^{-1} = \bar{t}_j\bar{t}_j^{-1}\bar{t}_{j+1} = \bar{t}_{j+1}$ because the D_i $(i = 1, \ldots, r)$ are disjoint. So acting with elementary braids on transpositions in different domains of transitivity, the result is to interchange place. Then acting with appropriate σ_i and σ_i^{-1} it is possible to replace the sequence $\bar{t}_2, ..., \bar{t}_n$ with a new one in which, for every j, all transpositions moving elements of a D_j stay together. The assumption $n - 1 \geq 2p$ assures that the number of t_i belonging to S_{D_j} is greater or equal to $2|D_j|$, for at least one D_j $(1 \leq j \leq r)$. Once this is achieved the proof is the same as the proof of Proposition 3.1 in [5].

Lemma 2. *Let* $[t_1, \dots, t_{n-2}, \tau, \tau, t_\alpha, t_\beta]$ *be an element of* $WH^A_{p,n}(Y)$*, where* t_1 *is a* $p-cycle$ *and* τ *is a transposition of* S_p*. Then there are braid moves transforming*

$$[t_1, \dots, t_{n-2}, \tau, \tau, t_\alpha, t_\beta] \; into \; [t_1, \dots, t_{n-2}, \tau^{'}, \tau^{'}, t_\alpha, t_\beta]$$

where $\tau^{'}$ *is an arbitrary transposition of* S_p*.*

Proof. Let $H =< t_1, \dots, t_{n-2} >$, let $h \in H$ and let $h = h_1 \cdots h_s$ where h_i or h_i^{-1} for $i = 1, \dots, s$ lies in the set $\{t_1, \dots, t_{n-2}\}$. Define $\tau^h = h^{-1}\tau h$. We will prove that acting with braid moves and their inverses it is possible to bring $[t_1, \dots, t_{n-2}, \tau, \tau, t_\alpha, t_\beta]$ to $[t_1, \dots, t_{n-2}, \tau^h, \tau^h, t_\alpha, t_\beta]$.

We distinguish two cases. If h_1 is equal to t_i for some $i = 1, \dots, n-2$, acting with suitable inverses of elementary braids move the pair (τ, τ) to the left of t_i. Applying σ_i^{-1} and σ_{i+1}^{-1} we bring $[t_1, \dots, t_{i-1}, \tau, \tau, t_i, \dots, t_{n-2}, t_\alpha, t_\beta]$ to $[t_1, \dots, t_{i-1}, t_i, \tau^{t_i}, \tau^{t_i}, t_{i+1}, \dots, t_{n-2}, t_\alpha, t_\beta]$. Now acting with the appropriate σ_j move (τ^{t_i}, τ^{t_i}) to the $(n-1)-th$ and $n-th$ place.

If h_1 is equal to t_i^{-1} for some $i = 1, \dots, n-2$, we move (τ, τ) to the right of t_i and applying σ_i and σ_{i+1} we bring

$$[t_1, \dots, t_{i-1}, t_i, \tau, \tau, t_{i+1}, \dots, t_{n-2}, t_\alpha, t_\beta]$$

to

$$[t_1, \dots, t_{i-1}, \tau^{h_1}, \tau^{h_1}, t_i, \dots, t_{n-2}, t_\alpha, t_\beta].$$

Now move (τ^{h_1}, τ^{h_1}) to the $(n-1)-th$ and $n-th$ place. Proceeding in this way successively for every h_i $(i = 2, \dots, s)$ we conclude.

So Lemma 1 and Lemma 2 assure that choosing h appropriately we may obtain among the first n permutations of (4) an arbitrary transposition of S_p.

Now we are ready to prove the following theorem.

Theorem 1. $WH_{p,n}(Y)$ *is connected for* $(n-1) \geq 2p$.

Proof. Let $[t_1, \dots, t_n, t_\alpha, t_\beta] \in WH^A_{p,n}(Y)$. Let $t_\alpha = \lambda_1\lambda_2 \cdots \lambda_s$ be a factorization of t_α as product of disjoint cycles such that $\sharp\lambda_1 \geq \sharp\lambda_2 \geq \dots \geq \sharp\lambda_s$ and let $t_\beta = \mu_1\mu_2 \cdots \mu_t$ be a factorization of t_β in the product of disjoint cycles such that $\sharp\mu_1 \geq \sharp\mu_2 \geq \dots \geq \sharp\mu_t$. (Note that λ_i and μ_j may also be trivial).

Define the norm of t_α and t_β as follows

$$|t_\alpha| := \sum_{i=1}^{s}(\sharp\lambda_i - 1) \;\text{ and }\; |t_\beta| := \sum_{j=1}^{t}(\sharp\mu_j - 1)$$

We will prove the transitivity of the action of $\pi_1(Y^{(n)}\backslash\Delta, A)$ on $WH^A_{p,n}(Y)$ using induction on $|t_\alpha| + |t_\beta|$.

If (4) is such that $|t_\alpha|+|t_\beta| = 0$ then $t_\alpha = t_\beta = id$, i.e. $[t_1, \dots, t_n, t_\alpha, t_\beta] = [t_1, \dots, t_n, id, id]$. So applying the result of [1] we obtain that (4) is in the orbit of (1) under the action of $\pi_1(Y^{(n)}\backslash\Delta, A)$.

Therefore suppose that $|t_\alpha| + |t_\beta| > 0$ and suppose, by way of induction, that for each $[t_1, \dots, t_n, t'_\alpha, t'_\beta]$ such that $|t'_\alpha|+|t'_\beta| < |t_\alpha|+|t_\beta|$ it is possible, acting with the braid moves σ_i, ρ_j, τ_h and their inverses, to bring $[t_1, \dots, t_n, t'_\alpha, t'_\beta]$ to $[t'_1, \dots, t'_n, id, id]$.

$|t_\alpha| + |t_\beta| > 0$ implies that either $|t_\alpha| > 0$ or $|t_\beta| > 0$. Suppose first that $|t_\alpha| > 0$. Let us choose a transposition σ such that $\sharp\lambda_1\sigma = \sharp\lambda_1 - 1$. By Lemma 1 and Lemma 2 $[t_1, \dots, t_n, t_\alpha, t_\beta]$ is in the orbit of $[t'_1, \dots, t'_{n-2}, \sigma, \sigma, t_\alpha, t_\beta]$ under the action of $\pi_1(Y^{(n)}\backslash\Delta, A)$. Acting with the braid move τ_n, by (2), we obtain a new class $[t'_1, \dots, t'_{n-2}, \sigma, t'_n, t'_\alpha, t_\beta]$ such that

$$|t'_\alpha| + |t_\beta| < |t_\alpha| + |t_\beta|.$$

By the induction assumption applied to $[t'_1, \dots, t'_{n-2}, \sigma, t'_n, t'_\alpha, t_\beta]$ we conclude that there are braid moves transforming (4) into $[\bar{t}_1, \dots, \bar{t}_n, id, id]$.

If instead it holds $|t_\beta| > 0$ and $|t_\alpha| = 0$, let σ be a transposition of S_p such that $\mu_1\sigma$ is a $(\sharp\mu_1 - 1) - cycle$. By Lemma 1 and Lemma 2 $[t_1, \dots, t_n, id, t_\beta]$ is in the orbit of $[t'_1, \dots, t'_{n-2}, \sigma, \sigma, id, t_\beta]$.

Acting with $\sigma^{-1}_{n-2}, \sigma^{-1}_{n-3}, \dots, \sigma^{-1}_1$ and $\sigma^{-1}_{n-1}, \dots, \sigma^{-1}_2$ we bring $[t_1, \dots, t_n, id, t_\beta]$ to $[\sigma, \sigma, t'_3, \dots, t'_n, id, t_\beta]$. Applying ρ_1, by (3), we have $[t_1, \dots, t_n, id, t_\beta]$ is bringed to $[t'_1, \sigma, t'_3, \dots, t'_n, id, t'_\beta]$, with $|t'_\beta| < |t_\beta|$. By the induction assumption we conclude $[t_1, \dots, t_n, id, t_\beta]$ is in the orbit of $[\bar{t}_1, \dots, \bar{t}_n, id, id]$. In this way it is proved that there are braid moves transforming $[t_1, \dots, t_n, t_\alpha, t_\beta]$ into $[t'_1, \dots, t'_n, id, id]$. To conclude it is sufficient to apply the Arbarello's result [1].

REFERENCES

[1] E. Arbarello, *On subvarieties of the moduli space of curves of genus g defined in terms of Weierstrass points,* Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur., Sez. Ia (8), 15 (1978), pp. 3–20.

[2] J.S. Birman, *Braids, links and mapping class groups,* Annals of Math. Studies, 82, Princeton Univ. Press, Princeton N. J. and Univ. of Tokyo Press, Tokyo, 1974.

[3] J.S. Birman, *On Braid Groups,* Comm. Pure Applied Math., 22 (1968), pp. 41–72.

[4] W. Fulton, *Hurwitz schemes and irreducibility of moduli of algebraic curves,* Annals. of Math., 90 (1969), pp. 542–575.

[5] J. Harris - T. Graber - J. Starr, *A note on Hurwitz schemes of covers of a positive genus curve,* Prepint Math. AG/0205056 (2002).

[6] A. Hurwitz, *Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten,* Math. Annalen, 39 (1891), pp. 1–61.

[7] P. Kluitmann, *Hurwitz action and finite quotients of braid groups,* Braids (Santa Cruz, CA, 1986), pp. 299–325, Contemp. Math. 78, Amer. Math. Soc., Providence, RI, 1988.

[8] S. Mochizuki, *The Geometry of the Compactification of the Hurwitz Scheme,* Publ. RIMS, Kyoto Univ., 31 (1995), pp. 355–441.

[9] S.M. Natanzon, *Topology of 2-dimensional coverings and meromorphic functions on real and complex algebraic curves,* Selected translations., Selecta Math. Soviet., 12 n. 3 (1993), pp. 251–291.

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