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A LIOUVILLE-TYPE THEOREM FOR THE HOMOGENEOUS WAVE EQUATION

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In this paper, we characterize those bounded from below solutions of the homogeneous wave equation $\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0$ on $\mathbb{R}^2$ which are constant.

1. Introduction.

Motivated by the classical result of Liouville on harmonic functions, one generically calls Liouville-type theorem any result ensuring that the solutions of a given differential equation, which satisfy a suitable growth condition, are constant.

In this paper, we are interested in the classical homogeneous wave equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0 \tag{1}$$

on $\mathbb{R}^2$.

Our starting point is the observation that there are bounded solutions of (1) which are not constant. For instance, consider $f(x, y) = \sin(x + y)$.

So, it is our aim to characterize those solutions of (1), bounded from below, which are constant. Our result is Theorem 2.2, proved in the next section.

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2. The result.

For the reader’s convenience, let us first recall a result, proved in [1], which is the main tool in proving Theorem 2.2.

Theorem 2.1. *Let X be a real Banach space, $f : X \to \mathbb{R}$ a lower semicontinuous and Gâteaux differentiable function, bounded from below. Assume that*

$$\limsup_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} < +\infty.$$

Then, for every $\epsilon > \limsup_{\|x\|\to+\infty} \frac{f(x)}{\|x\|}$ one has

$$\overline{conv}(\{x \in X : \|f'(x)\|_{X^*} \le \epsilon\}) = X.$$

In the sequel, for a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, we denote by f_x and f_y its first-order partial derivatives and by ∇f its gradient, that is $\nabla f = (f_x, f_y)$. Also, if $(x, y) \in \mathbb{R}^2$, we put $|(x, y)| = \sqrt{x^2 + y^2}$.

Theorem 2.2. *Let $f \in \mathcal{C}^3(\mathbb{R}^2)$ be a solution of equation* (1)*, bounded from below. Assume that*

$$\lim_{|(x,y)|\to+\infty} \frac{f(x, y)}{|(x, y)|} = 0. \tag{2}$$

Then the function f is constant if and only if the inequality

$$G_x(x, y) \ge |G_y(x, y)| \tag{3}$$

holds for every $(x, y) \in \mathbb{R}^2$, where

$$G(x, y) = f_x(x, y) f_{xx}(x, y) + f_y(x, y) f_{xy}(x, y)$$

for every $(x, y) \in \mathbb{R}^2$.

Proof. Obviously, if f is a constant, (3) holds. Now let’s prove the opposite implication. It suffices to prove that

$$\sup_{(x,y)\in\mathbb{R}^2} |\nabla f(x, y)| = 0.$$

Arguing by contradiction, suppose that the above number is strictly greater then zero. Then we can choose ϵ satisfying

$$(4) \qquad 0 = \lim_{|(x,y)|\to+\infty} \frac{f(x, y)}{|(x, y)|} < \epsilon < \sup_{(x,y)\in\mathbb{R}^2} |\nabla f(x, y)|.$$

Therefore Theorem 2.1 assures that

$$\overline{\text{conv}}(\{(x, y) \in \mathbb{R}^2 : |\nabla f(x, y)| \le \epsilon\}) = \mathbb{R}^2$$

and, because of the finite dimensionality, this fact is equivalent (see [2]) to

$$\text{conv}(\{(x, y) \in \mathbb{R}^2 : |\nabla f(x, y)| \le \epsilon\}) = \mathbb{R}^2.$$

Now we show that the set $\{(x, y) \in \mathbb{R}^2 : |\nabla f(x, y)| \le \epsilon\}$ is convex. To this aim, it suffices to prove that the function $|\nabla f|^2$ is convex.

So, put $H = f_x^2 + f_y^2$. Since $H \in C^2(\mathbb{R}^2)$ (recall that $f \in C^3(\mathbb{R}^2)$), by a classical result, to prove that the function H is convex, we have to show that

(i) $H_{xx} \ge 0$;
(ii) $H_{xx}\, H_{yy} - H_{xy}^2 \ge 0$

pointwise in $\mathbb{R}^2$.

In order to prove (i) we make use of (3), obtaining, in particular $G_x(x, y) \ge 0$. In fact one has

$$H_{xx} = 2\frac{\partial}{\partial x}(f_x f_{xx} + f_y f_{xy}) = 2G_x(x, y) \ge 0.$$

Moreover, being $f_{xx} = f_{yy}$, condition (ii) holds; in fact

$$H_{xx}H_{yy} - H_{xy}^2 = 4(f_{xx}^2 + f_{xy}^2 + f_x f_{xxx} + f_y f_{xxy})^2 -$$
$$-4(2f_{xx} f_{xy} + f_x f_{xxy} + f_y f_{xxx})^2 = 4(G_x^2 - G_y^2) \ge 0$$

thanks to (3).

Then one has

$$\{(x, y) \in \mathbb{R}^2 : |\nabla f(x, y)| \le \epsilon\} = \mathbb{R}^2$$

and this is a contradiction with (4).

Hence the function f has to be a constant because its gradient is identically zero in $\mathbb{R}^2$. □

It is worth noticing that condition (2) is not superfluous for the validity of the conclusion of Theorem 2.2. In this connection, consider the function e^{x+y}: it is a non constant, bounded from below solution of equation (1) which satisfies (3).

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