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TOEPLITZ MATRIX AND PRODUCT NYSTROM METHODS FOR SOLVING THE SINGULAR INTEGRAL EQUATION

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The Toeplitz matrix and the product Nystrom methods are applied to an integral equation of the second kind. We consider two cases: logarithmic kernel and Hilbert kernel. The two methods are applied to two integral equations with known exact solutions. The error in each case is calculated.

1. Introduction.

Singular integral equations arise in many problems of mathematical physics, such as the theory of elasticity, viscoelasticity, hydrodynamics, biological problems, population genetics and others.

Over the past thirty years substantial progress has been made in developing innovative approximate analytical and purely numerical solution techniques to a large class of singular integral equations. The solution of these problems may be obtained analytically, using the theory developed by Muskhelishvili [12]. The books authored by Green [9], Hochstadt [10], and Tricomi [13] contain different methods to solve certain integral equations analytically. The books authored by Golberg [6] and [7] contain extensive literature surveys on both approximate analytical and purely numerical techniques. The interested reader should consult the fine expositions by Atkinson [4], Delves and Mohamed [5] and Linz [11] for numerical solution.

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In the present work, we consider a Fredholm singular integral equation of the second kind. We consider two types of kernels, logarithmic kernel with a finite power and Hilbert form kernel. The continuity and boundedness of the integral operator are discussed in section 1. In section 2 we apply the Toeplitz matrix method to integral equations. Also, the product Nystrom method is applied to the same integral equations in section 3.

Consider the integral equation

$$\phi(x) - \lambda \int_a^b k(x, y)\phi(y)dy = f(x), \tag{1.1}$$

here $f(x)$ is a given function belongs to the class $C[a, b]$ of all continuous functions. The kernel $k(x, y)$ is a known function and has singular term while $\phi(x)$ represents the unknown function to be found and λ is a numerical parameter.

In order to guarantee the existence of a solution of Eq. (1.1), we assume throughout this work the following conditions:

1. The kernel $k(x, y)$ satisfies the following relation

$$\left\{ \int_a^b \int_a^b k^2(x, y)\, dx dy \right\}^{\frac{1}{2}} = C < \infty. \tag{1.2}$$

2. The unknown function $\phi(x) \in L_2[a, b]$ satisfies Hölder condition, i.e.

$$|\phi(x_1) - \phi(x_2)| \leq A|x_1 - x_2|^{\alpha}$$

where A is a positive constant, $x_1, x_2 \in [a, b]$, and $0 < \alpha \leq 1$.

Now, we prove the continuity and the boundedness of the integral operator

$$K\phi(x) = \lambda \int_a^b k(x, y)\phi(y)dy \tag{1.3}$$

in $L_2[a, b]$.

Taking $x_1, x_2 \in [a, b]$ gives

$$| \lambda | \left| \int_a^b k(x_1, y)\phi(y)dy - \int_a^b k(x_2, y)\phi(y)dy \right| \tag{1.4}$$

$$\leq |\lambda| \left(\int_a^b \phi^2(y) dy \right) g(x_1, x_2)$$
$$= | \lambda | \, \|\phi\| \;\; g(x_1, x_2),$$

where $\| \cdot \|$ denotes the normality of $\phi(x)$ and

$$g^2(x_1, x_2) = \int_a^b \left[k(x_1, y) - k(x_2, y) \right]^2 dy. \tag{1.5}$$

The formula (1.5) shows that $g(x_1, x_2) \to 0$ as $x_1 \to x_2$, i.e. the integral operator of Eq. (1.3) maps the set $C[a, b]$ into itself.

From Eq. (1.3) and condition (1.2) we have

$$\|K\phi\| = |\lambda| \left[\int_a^b dx \left(\int_a^b k(x, y) \phi(y) dy \right)^2 \right]^{\frac{1}{2}}$$
$$\leq |\lambda| \left[\int_a^b dx \int_a^b \phi^2(y) dy \int_a^b k^2(x, y) dy \right]^{\frac{1}{2}}$$
$$= |\lambda| C \|\phi\|.$$

Then

$$\|K\| \leq |\lambda| C, \quad C \text{ is a constant.}$$

Therefore, the integral operator (1.3) is bounded in $L_2[a, b]$, which implies the operator K is continuous.

2. The Toeplitz matrix method.

In this section, we discuss the Toeplitz matrix method [1], [2], [3] to obtain the numerical solution of integral equation of the second kind with singular kernel. The idea of this method is to obtain a system of $2N + 1$ linear algebraic equations, where $2N + 1$ is the number of discretization points used. The coefficients matrix is expressed as sum of two matrices, one of them is the Toeplitz matrix and the other is a matrix with zero elements except the first and last columns.

Consider the integral equation

$$\phi(x) - \lambda \int_{-a}^a k(x, y) \phi(y) dy = f(x). \tag{2.1}$$

The method assumes

$$\int_{-a}^{a} k(x, y)\phi(y)dy = \sum_{n=-N}^{N-1} \int_{nh}^{nh+h} k(x, y)\phi(y)dy, \quad h = \frac{a}{N}. \tag{2.2}$$

The integral in the right hand side of Eq. (2.2) can be written as

$$\int_{nh}^{nh+h} k(x, y)\phi(y)dy = A_n(x)\phi(nh) + B_n(x)\phi(nh + h) + R, \tag{2.3}$$

where $A_n(x)$ and $B_n(x)$ are two arbitrary functions to be determined and R is the error term. Putting $\phi(y) = 1, y$ in Eq. (2.3) yield a set of two equations in terms of the two functions $A_n(x)$ and $B_n(x)$. If R is assumed negligible we can clearly solve this set of equations for $A_n(x)$ and $B_n(x)$. Let $x = mh$, $-N \le m \le N$ then we obtain

$$\int_{-a}^{a} k(x, y)\phi(y)dy = \sum_{n=-N}^{N} D_{mn}\phi(nh), \tag{2.4}$$

where

$$D_{mn} = \begin{cases} A_{-N}(mh), & n = -N, \\ A_n(mh) + B_{n-1}(mh), & -N < n < N, \\ B_{N-1}(mh), & n = N. \end{cases} \tag{2.5}$$

Hence Eq. (2.1) becomes

$$\phi(mh) - \lambda \sum_{n=-N}^{N} D_{mn}\phi(nh) = f(mh), \quad -N \le m \le N, \tag{2.6}$$

which represents a system of linear algebraic equations. The matrix D_{mn} may be written as $D_{mn} = G_{mn} - E_{mn}$, where

$$G_{mn} = A_n(mh) + B_{n-1}(mh), \quad -N \le m, n \le N, \tag{2.7}$$

which is a Toeplitz matrix of order $2N + 1$ and

$$E_{mn} = \begin{cases} B_{-N-1}(mh), & n = -N, \\ 0, & -N < n < N, \\ A_N(mh), & n = N, \end{cases} \tag{2.8}$$

which represents a matrix of order $2N+1$ whose elements are zeros except the first and last columns.

However, the integral equation (2.1) is reduced to the following system of linear algebraic equations

$$[I-\lambda(G_{mn}-E_{mn})]\phi(mh)=f(mh),$$

or

$$[I-\lambda(G-E)]\Phi=F.$$

Lemma 2.1. *The formula* (2.6), *when* $N\to\infty$ *is bounded and has a unique solution.*

Proof. Consider $\Re$ is the metric space of real bounded sets, where the distance function ρ can be defined as

$$\rho(x_1,x_2)=\sup_{\ell}\mid x_\ell^{(1)}-x_\ell^{(2)}\mid \tag{2.9}$$

such that

$$x_i=\left\{x_\ell^{(i)}\right\}_{\ell=-\infty}^{\infty}\in\Re,\qquad i=1,2. \tag{2.10}$$

Consider an operator K on $\Re$ such that $y=Kx$ where $x,y\in\Re$,

$$y=\left\{y_\ell\right\}_{\ell=-\infty}^{\infty},\qquad x=\left\{x_\ell\right\}_{\ell=-\infty}^{\infty}. \tag{2.11}$$

Let

$$y_\ell=c_\ell+\lambda\sum_{n=-\infty}^{\infty}K_{\ell,n}x_n \tag{2.12}$$

where

$$c=\left\{c_\ell\right\}_{\ell=-\infty}^{\infty}\in\Re$$

under the condition

$$\sup\mid K_{\ell,n}\mid<\infty. \tag{2.13}$$

The operator K satisfies $K : \Re \to \Re$. Hence we write the system (2.6), when $N \to \infty$ to take the form

$$x_m = f_m + \lambda \sum_{n=-\infty}^{\infty} D_{m,n} x_n. \tag{2.14}$$

Assume

$$S_m = \lambda \sum_{n=-\infty}^{\infty} \mid D_{m,n} \mid . \tag{2.15}$$

Apply Cauchy-Minkoviski inequality, we get

$$S_m \leq \lambda \mid \sum_{n=-\infty}^{\infty} D_{m,n}^2 \mid^{\frac{1}{2}} . \tag{2.16}$$

Finally, we get

$$\lambda \leq 1 \tag{2.17}$$

which represents the condition to have a unique solution, and the value of $\mid x_m \mid$ satisfies the inequality

$$\mid x_m \mid \leq \frac{\mid f_m \mid}{1-\lambda}. \tag{2.18}$$

Definition 2.1. The Toeplitz matrix method is said to be convergent of order r in $[-a, a]$, if and only if for N sufficiently large, there exist a constant $D > 0$ independent on N such that

$$\|\phi(x) - \phi_N(x)\| \leq DN^{-r}. \tag{2.19}$$

The error term R can be given

$$R = |\phi - \phi_N| \tag{2.20}$$

where $R \to 0$ as $N \to \infty$. So, the error estimate is determined from equation (2.3) by letting $\phi(x) = x^2$ to get

$$R = \left| \int_{nh}^{nh+h} y^2 k(x, y) dy - (nh)^2 A_n(x) - (nh+h)^2 B_n(x) \right| \leq \beta h^3. \tag{2.21}$$

where β is a constant.

In the present work, we apply this method to solve the integral equation (2.1) taking into account a logarithmic kernel with a finite power as well as a Hilbert kernel which is often occurs in the theory of elasticity, contact problems, and other sciences.

Case 1.

We consider the integral equation

$$\phi(x) - \lambda \int_{-1}^{1} \Big(\ln \mid y - x \mid \Big)^q \phi(y)dy = f(x), \quad q = 1, 2, \dots \ . \tag{2.22}$$

In this case we obtain

$$A_n(x) = \frac{1}{h(q+1)} \sum_{j=0}^{q} (-1)^j (q+1)q(q-1)\dots(q-j+1) \tag{2.23}$$
$$\times \Big\{ (nh+h-x)^2 \Big(\ln \mid nh+h-x \mid \Big)^{q-j} \Big[1 - \frac{1}{2^{j+1}} \Big]$$
$$-(nh-x)\Big(\ln \mid nh-x \mid \Big)^{q-j} \Big[(nh+h-x) - \frac{nh-x}{2^{j+1}} \Big] \Big\},$$

and

$$B_n(x) = \frac{1}{h(q+1)} \sum_{j=0}^{q} (-1)^j (q+1)q(q-1)\dots(q-j+1) \tag{2.24}$$
$$\times \Big\{ (nh+h-x)\Big(\ln \mid nh+h-x \mid \Big)^{q-j} \Big[(x-nh) + \frac{nh+h-x}{2^{j+1}} \Big]$$
$$+(nh-x)^2 \Big(\ln \mid nh-x \mid \Big)^{q-j} \Big[1 - \frac{1}{2^{j+1}} \Big] \Big\}.$$

The elements of the Toeplitz matrix G_{mn} are given by

$$G_{m,n} = A_n(mh) + B_{n-1}(mh) \tag{2.25}$$
$$= \frac{h}{q+1} \sum_{j=0}^{q} (-1)^j (q+1)q(q-1)\dots(q-j+1) \Big[1 - \frac{1}{2^{j+1}} \Big]$$
$$\times \Big\{ (n-m+1)^2 \Big(\ln \mid n-m+1 \ \mid h \Big)^{q-j} - 2(n-m)^2 \Big(\ln \mid n-m \mid h \Big)^{q-j}$$
$$+(n-m-1)^2 \Big(\ln \mid n-m-1 \ \mid h \Big)^{q-j} \Big\},$$

the elements of the first column of the matrix E_{mn} are given by

$$E_{m,-N} = B_{-N-1}(mh) \tag{2.26}$$

$$= \frac{h}{q+1}\sum_{j=0}^{q}(-1)^j(q+1)q(q-1)\dots(q-j+1)$$

$$\times\left\{-(m+N)\Big(\ln|\, m+N \;|\, h\Big)^{q-j}\left[m+N+1-\frac{m+N}{2^{j+1}}\right]\right.$$

$$\left.+(m+N+1)^2\Big(\ln|\, m+N+1 \;|\, h\Big)^{q-j}\left[1-\frac{1}{2^{j+1}}\right]\right\},$$

while the elements of the last column of the matrix E_{mn} are given by

$$E_{m,N} = A_N(mh) \tag{2.27}$$

$$= \frac{h}{q+1}\sum_{j=0}^{q}(-1)^j(q+1)q(q-1)\dots(q-j+1)$$

$$\times\left\{(N-m+1)^2\Big(\ln|\, N-m+1 \;|\, h\Big)^{q-j}\left[1-\frac{1}{2^{j+1}}\right]\right.$$

$$\left.-(N-m)\Big(\ln|\, N-m \,|\, h\Big)^{q-j}\left[N-m+1-\frac{N-m}{2^{j+1}}\right]\right\},$$

where $-N \le m, n \le N$.

Case 2.

If we consider the integral equation (2.1) in the form

$$\phi(x) - \lambda\int_{-\pi}^{\pi}\cot\left(\frac{y-x}{2}\right)\phi(y)dy = f(x), \tag{2.28}$$

with the condition $\phi(\pm\pi) = 0$, knowing that, (see [8] p. 192)

$$\int x^p \cot x dx = \sum_{j=0}^{\infty}\frac{(-1)^j 2^{2j} B_{2j}}{(p+2j)(2j)!} x p + 2j, \quad [p \ge 1, |\, x \,| < \pi], \tag{2.29}$$

where B_{2j} are Bernoulli numbers, then one obtains

$$(2.30)\quad A_n(x) = \frac{1}{h}\Bigg\{2(nh+h-x)\ln\left|\sin\frac{nh+h-x}{2}\right| - 2(nh+h-x)\ln\left|\sin\frac{nh-x}{2}\right|$$

$$-4\sum_{j=0}^{\infty}\frac{(-1)^j B_{2j}}{2(1+2j)(2j)!}\Big[(nh+h-x)^{1+2j}-(nh-x)^{1+2j}\Big]\Bigg\},$$

and

$$(2.31)\quad B_n(x) = \frac{1}{h}\Bigg\{-2(nh-x)\ln\left|\sin\frac{nh+h-x}{2}\right| + 2(nh-x)\ln\left|\sin\frac{nh-x}{2}\right|$$

$$+4\sum_{j=0}^{\infty}\frac{(-1)^j B_{2j}}{2(1+2j)(2j)!}\Big[(nh+h-x)^{1+2j}-(nh-x)^{1+2j}\Big]\Bigg\}.$$

The elements of the Toeplitz matrix G_{mn} are given by

$$(2.32)\qquad G_{m,n} = 2(n-m+1)\ln\left|\sin\frac{h(n-m+1)}{2}\right|$$

$$-4(n-m)\ln\left|\sin\frac{h(n-m)}{2}\right| + 2(n-m-1)\ln\left|\sin\frac{h(n-m-1)}{2}\right|$$

$$-4\sum_{j=0}^{\infty}\frac{(-1)^j h^{2j} B_{2j}}{2(1+2j)(2j)!}\Big[(n-m+1)^{1+2j}-2(n-m)^{1+2j}+(n-m-1)^{1+2j}\Big]$$

$$(\,-N+1 \le m,n \le N-1\,).$$

The condition $\phi(\pm\pi) = 0$ reduces the matrix E_{mn} to $2N-1$ zero matrix and the Toeplitz matrix G_{mn} to $2N-1$ matrix.

3. The product Nystrom method.

In this section, we discus the product Nystrom method [5]. Consider the integral equation

$$(3.1)\qquad \phi(x) - \lambda\int_a^b p(x,y)\overline{k}(x,y)\phi(y)dy = f(x)$$

where p and $\overline{k}$ are respectively 'badly behaved' and 'well behaved' functions of their arguments, and $f(x)$ is a given function, while $\phi(x)$ is the unknown function.

Eq. (3.1) can be written in the form

$$\phi(x_i) - \lambda \sum_{j=0}^{N} w_{ij}\overline{k}(x_i, y_j)\phi(y_j) = f(x_i), \tag{3.2}$$

where $x_i = y_i = a + ih, i = 0, 1, \dots, N$ with $h = \frac{b-a}{N}$ and N even. The weights w_{ij} are constructed by insisting that the rule in (3.2) be exact when $\overline{k}(x_i, y)\phi(y)$ is a polynomial of degree $\leq r$ say. We illustrate the method by approximating the integral term of formula (3.1) by a product integration form of Simpson's rule. It is obvious that the procedure can be easily extended to give product integration formulas of higher order. Finally, the approximate solution of Eq. (3.1) takes the form

$$\phi_N(x_i) = f(x_i) + \lambda \sum_{j=0}^{N} w_{ij}\overline{k}(x_i, y_j)\phi(y_j), \tag{3.3}$$

where w_{ij} are determined completely in [5].

The formula (3.1) has a unique solution $\phi \in C[a, b]$ that may be expected to have unbounded derivatives at the endpoints $x = a, x = b$. The method is said to be convergent of order r in [a,b] if and only if for N sufficiently large there exists a constant $C > 0$ independent of N such that

$$\|\phi(x) - \phi_N(x)\|_\infty \leq CN^{-r}. \tag{3.4}$$

The uniform convergence of the approximate solution ϕ_N to the exact solution can be examined if we write

$$\phi(x) - \phi_N(x) = \sum_{j=0}^{N} w_j(k, x)\Big[\phi(x_j) - \phi_N(x_j)\Big] + t_N(k, \phi, x), \tag{3.5}$$

where

$$w_j(k, x) = \int_a^b k(x, s)\ell_{N,j}(s)\,ds,$$

$\ell_{N,j}(s)$ is the Lagrange interpolation polynomial and $t_N(k, \phi, x)$ is the local truncation error defined by

$$t_N(k, \phi, x) = \int_a^b k(x, y)\phi(y)dy - \sum_{j=0}^{N} w_j(k, x)\phi(x_j). \tag{3.6}$$

Hence, we have

$$\|\phi - \phi_N\|_\infty = \|(I - A_N)^{-1}\| \, \| t_N \|_\infty, \tag{3.7}$$

where A_N is the linear operator defined by

$$A_N : C[a,b] \to C[a,b], \tag{3.8}$$

$$A_N g(x) = \sum_{j=0}^{N} w_j(k,x) g(x_j), \quad g \in C[a,b], x \in [a,b].$$

According to the product Nystrom method [5] we approximate the integral term in (3.1) when $x = x_i$ by a product integration form such as Simpson's rule, therefore we may write

$$\int_a^b p(x_i, y)\overline{k}(x_i, y)\phi(y)dy = \sum_{j=0}^{\frac{N-2}{2}} \int_{y_{2j}}^{y_{2j+2}} p(x_i, y)\overline{k}(x_i, y)\phi(y)dy. \tag{3.9}$$

Now if we approximate the nonsingular part of the integrand over each interval $[y_{2j}, y_{2j+2}]$ by the second degree Lagrange interpolation polynomial which interpolates it at the points $y_{2j}, y_{2j+1}, y_{2j+2}$ we obtain

$$\int_a^b p(y_i, y)\overline{k}(y_i, y)\phi(y)dy \approx \sum_{j=0}^{N} w_{ij}\overline{k}(y_i, y_j)\phi(y_j), \tag{3.10}$$

where

$$\begin{aligned} w_{i,0} &= \beta_1(y_i), \\ w_{i,2j+1} &= 2\gamma_{j+1}(y_i), \\ w_{i,2j} &= \alpha_j(y_i) + \beta_{j+1}(y_i), \\ w_{i,N} &= \alpha_{\frac{N}{2}}(y_i), \end{aligned} \tag{3.11}$$

such that

$$\alpha_j(y_i) = \frac{1}{2h^2} \int_{y_{2j-2}}^{y_{2j}} p(y_i, y)(y - y_{2j-2})(y - y_{2j-1})dy, \tag{3.12}$$

$$\beta_j(y_i) = \frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} p(y_i, y)(y_{2j-1} - y)(y_{2j} - y)dy,$$

$$\gamma_j(y_i) = \frac{1}{2h^2}\int_{y_{2j-2}}^{y_{2j}} p(y_i, y)(y - y_{2j-2})(y_{2j} - y)dy.$$

Therefore, the integral equation (3.1) is reduced to the following system of linear algebraic equations

$$\phi(x_i) - \lambda\sum_{j=0}^{N} w_{ij}\overline{k}(y_i, y_j)\phi(y_j) = f(x_i), \quad i = 0, 1, \ldots, N,$$

or

$$(I - \lambda W)\Phi = F.$$

This method is applied to solve the integral equation (3.1) taking into account a logarithmic kernel with a finite power as well as a Hilbert kernel.

Case 1.

Consider the integral equation (2.22), and introduce the change of variable $y = y_{2j-2} + \mu h, \quad 0 \le \mu \le 2$ in Eqs. (3.12), and if we define

$$\psi_i(z) = \int_0^2 \mu^i\Big((\ln \mid \mu - z \mid h\Big)^q d\mu, \quad i = 0, 1, 2, \tag{3.13}$$

where $z = i - 2j + 2$, then the weights w_{ij} in Eqs. (3.11) becomes

$$w_{i,0} = \frac{h}{2}\Big[2\psi_0(z) - 3\psi_1(z) + \psi_2(z)\Big], \quad z = i, \tag{3.14}$$

$$w_{i,2j+1} = h\Big[2\psi_1(z) - \psi_2(z)\Big], \quad z = i - 2j,$$

$$w_{i,2j} = \frac{h}{2}\Big[\psi_2(z) - \psi_1(z) + 2\psi_0(z-2) - 3\psi_1(z-2) + \psi_2(z-2)\Big], \quad z = i - 2j + 2,$$

$$w_{i,N} = \frac{h}{2}\Big[\psi_2(z) - \psi_1(z)\Big], \quad z = i - N + 2.$$

The values of $\psi_i(k)$, $i = 0, 1, 2$ are given by

$$\psi_0(z) = \frac{1}{q+1}\sum_{j=0}^{q}(-1)^j(q+1)q(q-1)\ldots(q-j+1) \tag{3.15}$$

$$\times\left\{(2-z)\left(\ln\ |\ 2-z\ |\ h\right)^{q-j}+z\left(\ln\ |\ z\ |\ h\right)^{q-j}\right\},$$

$$\psi_1(z)=\frac{1}{q+1}\sum_{j=0}^{q}(-1)^j(q+1)q(q-1)\dots(q-j+1)$$

$$\times\left\{(2-z)\left(\ln\ |\ 2-z\ |\ h\right)^{q-j}\left[z+\frac{2-z}{2^{j+1}}\right]\right.$$

$$\left.+z^2\left(\ln\ |\ z\ |\ h\right)^{q-j}\left[1-\frac{1}{2^{j+1}}\right]\right\},$$

$$\psi_2(z)=\frac{1}{q+1}\sum_{j=0}^{q}(-1)^j(q+1)q(q-1)\dots(q-j+1)$$

$$\times\left\{(2-z)\left(\ln\ |\ 2-z\ |\ h\right)^{q-j}\left[z^2+\frac{2z(2-z)}{2^{j+1}}+\frac{(2-z)^2}{3^{j+1}}\right]\right.$$

$$\left.+z^3\left(\ln\ |\ z\ |\ h\right)^{q-j}\left[1-\frac{2}{2^{j+1}}+\frac{1}{3^{j+1}}\right]\right\}.$$

Thus we obtain

$$w_{i,0}=\frac{h}{2(q+1)}\sum_{j=0}^{q}(-1)^j(q+1)q(q-1)\dots(q-j+1)$$

$$\times\left\{(2-z)\left(\ln\ |\ 2-z\ |\ h\right)^{q-j}\left[(2-z)(1-z)+\frac{(2-z)(2z-3)}{2^{j+1}}+\frac{(2-z)^2}{3^{j+1}}\right]\right.$$

$$\left.+z\left(\ln\ |\ z\ |\ h\right)^{q-j}\left[(2-z)(1-z)+\frac{z(3-2z)}{2^{j+1}}+\frac{z^2}{3^{j+1}}\right]\right\},$$

$$w_{i,2j+1}=\frac{h}{q+1}\sum_{j=0}^{q}(-1)^j(q+1)q(q-1)\dots(q-j+1)$$

$$\times\left\{(2-z)\left(\ln\ |\ 2-z\ |\ h\right)^{q-j}\left[z(2-z)+\frac{2(2-z)(1-z)}{2^{j+1}}-\frac{(2-z)^2}{3^{j+1}}\right]\right.$$

$$\left.+z^2\left(\ln\ |\ z\ |\ h\right)^{q-j}\left[2-z+\frac{2z-2}{2^{j+1}}-\frac{z}{3^{j+1}}\right]\right\},$$

$$w_{i,2j} = \frac{h}{2(q+1)} \sum_{j=0}^{q} (-1)^j (q+1)q(q-1)\dots(q-j+1)$$

$$\times \left\{ (2-z)\left(\ln \mid 2-z \mid h \right)^{q-j} \left[-6(2-z) + \frac{6(2-z)}{2^{j+1}} \right] \right.$$

$$-z^2 \left(\ln :\mid z \mid h \right)^{q-j} \left[1 - z + \frac{2z-1}{2^{j+1}} - \frac{z}{3^{j+1}} \right]$$

$$\left. + (4-z)\left(\ln \mid 4-z \mid h \right)^{q-j} \left[(z-4)(z-3) - \frac{(4-z)(7-2z)}{2^{j+1}} + \frac{(4-z)^2}{3^{j+1}} \right] \right\},$$

and

$$w_{i,N} = \frac{h}{2(q+1)} \sum_{j=0}^{q} (-1)^j (q+1)q(q-1)\dots(q-j+1)$$

$$\times \left\{ (2-z)\left(\ln \mid 2-z \mid h \right)^{q-j} \left[z^2 - z + \frac{(2-z)(2z-1)}{2^{j+1}} + \frac{(2-z)^2}{3^{j+1}} \right] \right.$$

$$\left. -z^2 \left(\ln \mid z \mid h \right)^{q-j} \left[1 - z + \frac{2z-1}{2^{j+1}} - \frac{z}{3^{j+1}} \right] \right\},$$

where $0 \leq i, j \leq N$.

Case 2.

Consider the integral equation (2.28) with Hilbert kernel subject to the condition $\phi(\pm\pi = 0)$, and if we define

$$\psi_i(z) = \int_0^2 \mu^i \cot \frac{(\mu - z)h}{2} d\mu, \quad i = 0, 1, 2, \tag{3.16}$$

where $z = i - 2j + 2$, then we have

$$\psi_0(z) = \frac{2}{h} \ln \left| \sin \frac{h(2-z)}{2} \right| - \frac{2}{h} \ln \left| \sin \frac{zh}{2} \right|,$$

and by using formula (2.29) we obtain

$$\psi_1(z) = \frac{2z}{h} \ln \left| \sin \frac{h(2-z)}{2} \right| - \frac{2z}{h} \ln \left| \sin \frac{zh}{2} \right|$$

$$+\frac{2}{h}\sum_{j=0}^{\infty}\frac{(-1)^j h^{2j} B_{2j}}{(2j)!}\left[\frac{(2-z)^{1+2j}+z^{1+2j}}{1+2j}\right],$$

$$\psi_2(z)=\frac{2z^2}{h}\ln\left|\sin\frac{h(2-z)}{2}\right|-\frac{2z^2}{h}\ln\left|\sin\frac{zh}{2}\right|$$

$$+\frac{4}{h}\sum_{j=0}^{\infty}\frac{(-1)^j h^{2j} B_{2j}}{(2j)!}\left[\frac{z(2-z)^{1+2j}+z^{2+2j}}{1+2j}+\frac{(2-z)^{2+2j}-z^{2+2j}}{2(2+2j)}\right].$$

Substituting $\psi_i(z)$, $i=0,1,2$, into (3.14) we obtain

$$w_{i,2j+1}=2z(2-z)\ln\left|\sin\frac{h(2-z)}{2}\right|-2z(2-z)\ln\left|\sin\frac{hz}{2}\right|$$

$$+4\sum_{j=0}^{\infty}\frac{(-1)^j h^{2j} B_{2j}}{(2j)!}\left[\frac{(2-z)^{1+2j}(1-z)+z^{1+2j}(1-z)}{1+2j}-\frac{(2-z)^{2+2j}-z^{2+2j}}{2(2+2j)}\right],$$

$$w_{i,2j}=6(z-2)\ln\left|\sin\frac{h(2-z)}{2}\right|+(z-3)(z-4)\ln\left|\sin\frac{h(4-z)}{2}\right|-z(z-1)\ln\left|\sin\frac{hz}{2}\right|$$

$$+\sum_{j=0}^{\infty}\frac{(-1)^j h^{2j} B_{2j}}{(2j)!}\left[\frac{6(2-z)^{1+2j}+z^{1+2j}(2z-1)+(4-z)^{1+2j}(2z-7)}{1+2j}\right.$$

$$\left.-\frac{2z^{2+2j}-2(4-z)^{2+2j}}{2(2+2j)}\right],$$

where $1\le i,\ j\le N-1$.

The condition $\phi(\pm\pi)=0$ reduces the matrix W to $(N-1)\times(N-1)$ matrix, i.e. this condition avoids the calculations of $w_{i,0}$ and $w_{i,N}$.

Table 1 displays the exact solution $\phi(x)=x$, the approximate solution $\phi_n^{(T)}$ and the error $E^{(T)}$ of the integral equation (2.22) by using the Toeplitz matrix method with $N=10$, $\lambda=1$, $q=5$. Also it displays the values of the approximate solution $\phi_n^{(N)}$ and the error $E^{(N)}$ at the same points for the same integral equation but by using the product Nystrom method with $N=20$, $\lambda=1$, $q=5$.

Table 2 displays the values of the exact solution $\phi(x)=\sin x$, the approximate solution $\phi_n^{(T)}$ and the error $E^{(T)}$ at the interior points of the integral equation (2.28) by using the Toeplitz matrix method with $N=10$, $\lambda=1$. Also it displays the values of the approximate solution $\phi_n^{(N)}$ and the error $E^{(N)}$ at the same points for the same integral equation but by using the product Nystrom method with $N = 20$, $\lambda = 1$.

x	$\phi = x$	$\phi_n^{(T)}$	$\phi_n^{(N)}$	$E^{(T)}$	$E^{(N)}$
-1.00	-1.00	$-.10000E+01$	$-.10000E+01$	$.66613E-15$	$.48850E-13$
$-.90$	$-.90$	$-.90000E+00$	$-.90000E+00$	$.16653E-14$	$.10880E-13$
$-.80$	$-.80$	$-.80000E+00$	$-.80000E+00$	$.55511E-15$	$.11435E-13$
$-.70$	$-.70$	$-.70000E+00$	$-.70000E+00$	$.11102E-14$	$.15543E-14$
$-.60$	$-.60$	$-.60000E+00$	$-.60000E+00$	$.77716E-15$	$.87708E-14$
$-.50$	$-.50$	$-.50000E+00$	$-.50000E+00$	$.66613E-15$	$.35527E-14$
$-.40$	$-.40$	$-.40000E+00$	$-.40000E+00$	$.44409E-15$	$.12768E-14$
$-.30$	$-.30$	$-.30000E+00$	$-.30000E+00$	$.44409E-15$	$.28311E-14$
$-.20$	$-.20$	$-.20000E+00$	$-.20000E+00$	$.11102E-15$	$.70777E-14$
$-.10$	$-.10$	$-.10000E+00$	$-.10000E+00$	$.41633E-16$	$.23176E-14$
$.00$	$.00$	$.58966E-16$	$.42828E-15$	$.58966E-16$	$.42828E-15$
$.10$	$.10$	$.10000E+00$	$.10000E+00$	$.97145E-16$	$.62728E-14$
$.20$	$.20$	$.20000E+00$	$.20000E+00$	$.33307E-15$	$.59119E-14$
$.30$	$.30$	$.30000E+00$	$.30000E+00$	$.16653E-15$	$.27756E-15$
$.40$	$.40$	$.40000E+00$	$.40000E+00$	$.22204E-15$	$.55511E-15$
$.50$	$.50$	$.50000E+00$	$.50000E+00$	$.55511E-15$	$.70499E-14$
$.60$	$.60$	$.60000E+00$	$.60000E+00$	$.15543E-14$	$.53291E-14$
$.70$	$.70$	$.70000E+00$	$.70000E+00$	$.77716E-15$	$.15210E-13$
$.80$	$.80$	$.80000E+00$	$.80000E+00$	$.13323E-14$	$.21427E-13$
$.90$	$.90$	$.90000E+00$	$.90000E+00$	$.14433E-14$	$.12434E-13$
1.00	1.00	$.10000E+01$	$.10000E+01$	$.00000E+00$	$.96145E-13$

Table 1: The results for the Eq. (2.22).

x	$\phi = \sin x$	$\phi_n^{(T)}$	$\phi_n^{(N)}$	$E^{(T)}$	$E^{(N)}$
$-.28274E+01$	$-.30902E+00$	$-.30915E+00$	$-.30850E+00$	$.13788E-03$	$.51658E-03$
$-.25133E+01$	$-.58779E+00$	$-.59287E+00$	$-.59109E+00$	$.50892E-02$	$.33080E-02$
$-.21991E+01$	$-.80902E+00$	$-.81486E+00$	$-.80572E+00$	$.58451E-02$	$.32985E-02$
$-.18850E+01$	$-.95106E+00$	$-.95972E+00$	$-.95617E+00$	$.86654E-02$	$.51090E-02$
$-.15708E+01$	$-.10000E+01$	$-.10087E+01$	$-.99678E+00$	$.86809E-02$	$.32193E-02$
$-.12566E+01$	$-.95106E+00$	$-.96041E+00$	$-.95418E+00$	$.93543E-02$	$.31247E-02$
$-.94248E+00$	$-.80902E+00$	$-.81706E+00$	$-.80761E+00$	$.80461E-02$	$.14113E-02$
$-.62832E+00$	$-.58779E+00$	$-.59460E+00$	$-.58628E+00$	$.68143E-02$	$.15008E-02$
$-.31416E+00$	$-.30902E+00$	$-.31336E+00$	$-.31012E+00$	$.43430E-02$	$.11065E-02$
$.00000E+00$	$.00000E+00$	$-.19509E-02$	$.66818E-02$	$.19509E-02$	$.66818E-02$
$.31416E+00$	$.30902E+00$	$.30995E+00$	$.30584E+00$	$.92883E-03$	$.31738E-02$
$.62832E+00$	$.58779E+00$	$.59120E+00$	$.59800E+00$	$.34130E-02$	$.10219E-01$
$.94248E+00$	$.80902E+00$	$.81472E+00$	$.80515E+00$	$.57041E-02$	$.38632E-02$
$.12566E+01$	$.95106E+00$	$.95830E+00$	$.96168E+00$	$.72410E-02$	$.10621E-01$
$.15708E+01$	$.10000E+01$	$.10081E+01$	$.99720E+00$	$.81197E-02$	$.28012E-02$
$.18850E+01$	$.95106E+00$	$.95912E+00$	$.95872E+00$	$.80619E-02$	$.76678E-02$
$.21991E+01$	$.80902E+00$	$.81622E+00$	$.80875E+00$	$.72012E-02$	$.26383E-03$
$.25133E+01$	$.58779E+00$	$.59327E+00$	$.59033E+00$	$.54874E-02$	$.25446E-02$
$.28274E+01$	$.30902E+00$	$.31203E+00$	$.31257E+00$	$.30148E-02$	$.35549E-02$

Table 2: The results for the Eq. (2.28).

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