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FIXED POINT THEOREMS FOR ASYMPTOTICALLY CONTRACTIVE MAPPINGS

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In this short paper, we prove fixed point theorems for nonexpansive mappings whose domains are unbounded subsets of Banach spaces. These theorems are generalizations of Penot's result in [4].

1. Introduction.

Let C be a closed convex subset of a Banach space E, and let T be a *nonexpansive mapping* on C, i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We know that T has a fixed point in the case that E is uniformly convex and C is bounded; see Browder [1] and Göhde [2]. Kirk [3] extended these result to the case that C is weakly compact and has normal structure. We note that such domain C of T is a bounded subset. Recently, Penot proved the following in [4]: T has a fixed point in the case that E is uniformly convex, C is unbounded, and T is *asymptotically contractive*, i.e.,

$$\limsup_{\substack{y \in C \\ \|y\| \to \infty}} \frac{\|Tx_0 - Ty\|}{\|x_0 - y\|} < 1$$

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for some $x_0 \in C$.

In this paper, we prove fixed point theorems for nonexpansive mappings whose domains are unbounded subsets of Banach spaces. These theorems are generalizations of Penot's result in [4].

2. Conditions for Mappings.

In this section, let T be a nonexpansive mapping on a nonempty closed convex subset C of a Banach space E. We discuss the following conditions for T:

(C1) There exists $r \in (0, 1)$ such that for every $x_1 \in C$, there exists $\eta > 0$ satisfying
$$\|Tx_1 - Ty\| \le r\ \|x_1 - y\|$$
for all $y \in C$ with $\|y\| > \eta$;

(C2) there exist $r \in (0, 1)$, $x_0 \in C$ and $\eta > 0$ such that
$$\|Tx_0 - Ty\| \le r\ \|x_0 - y\|$$
for all $y \in C$ with $\|y\| > \eta$;

(C3) for each $\lambda > 0$ and for each $x_1 \in C$, there exists $\eta > 0$ satisfying
$$\|Tx_1 - Ty\| \le \|x_1 - y\| - \lambda$$
for all $y \in C$ with $\|y\| > \eta$;

(C4) there exists $x_0 \in C$ for each $\lambda > 0$, there exists $\eta > 0$ satisfying
$$\|Tx_0 - Ty\| \le \|x_0 - y\| - \lambda$$
for all $y \in C$ with $\|y\| > \eta$;

(C5) there exists $\lambda > 0$ such that for each $x_1 \in C$, there exists $\eta > 0$ satisfying
$$\|Tx_1 - Ty\| \le \|x_1 - y\| - \lambda$$
for all $y \in C$ with $\|y\| > \eta$;

(C6) there exist $x_0 \in C$ and $\eta > 0$ such that
$$\|Tx_0 - Ty\| \le \|x_0 - y\| - \|Tx_0 - x_0\|$$
for all $y \in C$ with $\|y\| > \eta$.

We obtain the following.

Proposition 1. $(C1) \Leftrightarrow (C2) \Rightarrow (C3) \Leftrightarrow (C4) \Rightarrow (C5) \Rightarrow (C6)$ *holds.*

Proof. It is obvious that (C1) ⇒ (C2), (C3) ⇒ (C4), and (C3) ⇒ (C5). We first prove (C2) ⇒ (C1). We assume (C2), i.e., there exist $r' \in (0, 1)$, $x_0 \in C$ and $\eta' > 0$ such that $\|Tx_0 - Ty\| \leq r' \ \|x_0 - y\|$ for all $y \in C$ with $\|y\| > \eta'$. Put $r = (1 + r')/2$. We let $x_1 \in C$ be fixed and put

$$\eta = \max\left\{\eta', \|x_1\| + \frac{\|x_1 - x_0\| + \|Tx_1 - Tx_0\|}{r - r'}\right\}.$$

Then for $y \in C$ with $\|y\| > \eta$, we have

$$\begin{aligned}\|x_1 - x_0\| + \|Tx_1 - Tx_0\| &\leq (r - r')\,(\eta - \|x_1\|)\\ &\leq (r - r')\,(\|y\| - \|x_1\|)\\ &\leq (r - r')\,\|x_1 - y\|\end{aligned}$$

and hence

$$\begin{aligned}\|Tx_1 - Ty\| &\leq \|Tx_1 - Tx_0\| + \|Tx_0 - Ty\|\\ &\leq \|Tx_1 - Tx_0\| + r'\|x_0 - y\|\\ &\leq \|Tx_1 - Tx_0\| + r'\|x_1 - x_0\| + r'\|x_1 - y\|\\ &\leq \|Tx_1 - Tx_0\| + \|x_1 - x_0\| + r'\|x_1 - y\|\\ &\leq r\|x_1 - y\|.\end{aligned}$$

This implies (C1). We can similarly prove (C2) ⇒ (C4) and (C4) ⇒ (C3). We finally show (C5) ⇒ (C6). We assume (C5), i.e., there exists $\lambda > 0$ such that for each $x_1 \in C$, there exists $\eta > 0$ satisfying $\|Tx_1 - Ty\| \leq \|x_1 - y\| - \lambda$ for all $y \in C$ with $\|y\| > \eta$. We put

$$d = \inf_{x \in C} \|Tx - x\|$$

and assume $d > 0$. Then there exists $x_1 \in C$ such that $\|Tx_1 - x_1\| < d + \lambda/2$. For such x_1, we choose $\eta > 0$ satisfying $\|Tx_1 - Ty\| \leq \|x_1 - y\| - \lambda$ for all $y \in C$ with $\|y\| > \eta$. For each $t \in (0, 1)$, since a mapping $x \mapsto (1 - t)Tx + tx_1$ on C is contractive, there exists $y_t \in C$ such that

$$y_t = (1 - t)Ty_t + tx_1.$$

Since

$$\begin{aligned}d &\leq \|Ty_t - y_t\| = t\|Ty_t - x_1\|\\ &\leq t\,\big(\|Ty_t - Tx_1\| + \|Tx_1 - x_1\|\big)\\ &\leq t\,\big(\|y_t - x_1\| + \|Tx_1 - x_1\|\big)\\ &\leq t\,\big(\|y_t\| + \|x_1\| + \|Tx_1 - x_1\|\big),\end{aligned}$$

we have $\|y_t\| > \eta$ for some small $t > 0$. So, we have

$$\begin{aligned}\|x_1 - y_t\| + \|y_t - Ty_t\| &= \|x_1 - Ty_t\| \\ &\le \|x_1 - Tx_1\| + \|Tx_1 - Ty_t\| \\ &\le \|x_1 - Tx_1\| + \|x_1 - y_t\| - \lambda \\ &\le d + \lambda/2 + \|x_1 - y_t\| - \lambda\end{aligned}$$

and hence

$$\|y_t - Ty_t\| \le d - \lambda/2.$$

This contradicts to the definition of d. Therefore we obtain $d = 0$. We can choose $x_0 \in C$ with $\|Tx_0 - x_0\| < \lambda$. Then there exists $\eta > 0$ such that

$$\begin{aligned}\|Tx_0 - Ty\| &\le \|x_0 - y\| - \eta \\ &< \|x_0 - y\| - \|Tx_0 - x_0\|\end{aligned}$$

for all $y \in C$ with $\|y\| > \eta$. This completes the proof. □

We can easily prove the following.

Proposition 2. *Suppose that C is unbounded. Then the following are equivalent to (C1) and (C2):*

(i) *T is asymptotically contractive;*
(ii) *for every $x_1 \in C$,*

$$\limsup_{\substack{y \in C \\ \|y\| \to \infty}} \frac{\|Tx_1 - Ty\|}{\|x_1 - y\|} < 1$$

holds.

And the following are equivalent to (C3) and (C4):

(i) *there exists $x_0 \in C$ such that*

$$\lim_{\substack{y \in C \\ \|y\| \to \infty}} \Big(\|Tx_0 - Ty\| - \|x_0 - y\| \Big) = -\infty;$$

(ii) *for every $x_1 \in C$,*

$$\lim_{\substack{y \in C \\ \|y\| \to \infty}} \Big(\|Tx_1 - Ty\| - \|x_1 - y\| \Big) = -\infty$$

holds.

3. Sufficient and Necessary Condition.

In this section, we discuss about the sufficient and necessary condition for nonexpansive mappings having a fixed point.

Lemma 1. *Let C be a closed convex subset of a Banach space E and let T be a nonexpansive mapping on C. Suppose that (C6), i.e., there exist $x_0 \in C$ and $\eta > 0$ such that*

$$\|Tx_0 - Ty\| \leq \|x_0 - y\| - \|Tx_0 - x_0\|$$

for all $y \in C$ with $\|y\| > \eta$. Then there exists $\rho > 0$ such that $T(D) \subset D$, where

$$D = \{y \in C : \|y - x_0\| \leq \rho\}.$$

Proof. We put

$$\rho = \eta + \|x_0\| + \|Tx_0 - x_0\| > 0.$$

Then in the case of $y \in D$ and $\|y\| \leq \eta$, we have

$$\begin{aligned} \|Ty - x_0\| &\leq \|Ty - Tx_0\| + \|Tx_0 - x_0\| \\ &\leq \|y - x_0\| + \|Tx_0 - x_0\| \\ &\leq \|y\| + \|x_0\| + \|Tx_0 - x_0\| \\ &\leq \eta + \|x_0\| + \|Tx_0 - x_0\| \\ &= \rho. \end{aligned}$$

In the case of $y \in D$ and $\|y\| > \eta$, we have

$$\begin{aligned} \|Ty - x_0\| &\leq \|Ty - Tx_0\| + \|Tx_0 - x_0\| \\ &\leq \|y - x_0\| \\ &\leq \rho. \end{aligned}$$

Therefore we obtain the desired result. □

A closed convex subset C of a Banach space E is said to have the *fixed point property* for nonexpansive mappings (*FPP*, for short) if for every bounded closed convex subset D of C, every nonexpansive mapping on D has a fixed point. Similarly, C is said to have the *weak fixed point property* for nonexpansive mappings (*WFPP*, for short) if for every weakly compact convex subset D of C, every nonexpansive mapping on D has a fixed point. Let E^* be the dual of E. Then a closed convex subset C of E^* is said to have the *weak* fixed point property* (with respect to E) for nonexpansive mappings (*W*FPP*, for short) if for every weakly* compact convex subset D of C, every nonexpansive mapping on D has a fixed point. So, by the results of Browder [1] and Göhde [2], every uniformly convex Banach space has FPP. Also, by Kirk's result [3], every Banach space with normal structure has WFPP. We recall that a closed convex subset C of a Banach space E is *locally weakly compact* if and only if every bounded closed convex subset of C is weakly compact. So, every closed convex subset of a reflexive Banach space is locally weakly compact.

Using Lemma 1, we obtain the following propositions.

Proposition 3. *Let C be a closed convex subset of a Banach space E. Assume that C has FPP. Let T be a nonexpansive mapping on C. Then the following are equivalent:*

(i) *T has a fixed point in C;*
(ii) *T satisfies (C6).*

Proof. We first show (ii) implies (i). We suppose that (ii), i.e., there exist $x_0 \in C$ and $\eta > 0$ such that

$$\|Tx_0 - Ty\| \le \|x_0 - y\| - \|Tx_0 - x_0\|$$

for all $y \in C$ with $\|y\| > \eta$. By Lemma 1, there exists $\rho > 0$ such that $T(D) \subset D$, where

$$D = \{x \in C : \|x - x_0\| \le \rho\}.$$

So, by the assumption, there exists $z_0 \in D$ such that $Tz_0 = z_0$. Conversely, let us prove that (i) implies (ii). Let x_0 be a fixed point of T. Since T is nonexpansive, we have

$$\|Tx_0 - Ty\| \le \|x_0 - y\| = \|x_0 - y\| - \|Tx_0 - x_0\|$$

for all $y \in C$. This implies (C6). This completes the proof. □

Proposition 4. *Let C be a closed convex subset of a Banach space E. Assume that C is locally weakly compact and has WFPP. Let T be a nonexpansive mapping on C. Then T has a fixed point in C if and only if T satisfies (C6).*

Proposition 5. *Let E be a Banach space and let E^* be the dual of E. Let C be a weakly* closed convex subset of E^*. Assume that C has W*FPP. Let T be a nonexpansive mapping on C. Then T has a fixed point in C if and only if T satisfies (C6).*

As a direct consequence, we have the following.

Theorem 1. *Let E be a Banach space and let E^* be the dual of E. Assume that either of the following:*

(i) *C is a closed convex subset of E and has FPP.*
(ii) *C is a closed convex subset of E, which is locally weakly compact and has WFPP.*
(iii) *C is A weakly* closed convex subset of E^* and has W*FPP.*

Let T be a nonexpansive mapping on C. Suppose that C is unbounded, and T is asymptotically contractive. Then T has a fixed point.

Remark. (ii) implies (i).

Theorem 2. (Penot [4]) *Let C be a unbounded closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive mapping on C. Suppose that T is asymptotically contractive. Then T has a fixed point.*

4. Examples.

In Proposition 1, we prove (C1) $\Rightarrow$ (C3) $\Rightarrow$ (C5) $\Rightarrow$ (C6). In this section, we give three examples which show that the inverse of the above implications do not hold in general.

Example 1. Put $E = \mathbb{R}$ and $C = [1, \infty)$. Define a nonexpansive mapping T on C by

$$Tx = x - \log(x)$$

for all $x \in C$. Then T satisfies (C3) and does not satisfy (C1).

Proof. Since

$$\lim_{\substack{y\in C\\ \|y\|\to\infty}} \frac{\|T1 - Ty\|}{\|1 - y\|} = \lim_{y\to\infty} \frac{y - \log(y) - 1}{y - 1} = 1,$$

T does not satisfy (C1) by Proposition 2. Since

$$\lim_{\substack{y\in C\\ \|y\|\to\infty}} \Big(\|T1 - Ty\| - \|1 - y\|\Big) = \lim_{y\to\infty} \Big(\big(y - \log(y) - 1\big) - \big(y - 1\big)\Big)$$

$$= \lim_{y\to\infty} -\log(y) = -\infty,$$

T satisfies (C3) by Proposition 2. $\square$

Example 2. Let $E = c_0$ be the Banach space consisting of all real sequences converging to 0 with supremum norm. Define a closed convex subset C of E by

$$C = \{x \in E : 0 \leq x(n) \leq n \text{ for all } n \in \mathbb{N}\}.$$

Define a nonexpansive mapping T on C by

$$(Tx)(n) = \max\{0, x(n) - 2\}$$

for $n \in \mathbb{N}$. Then T satisfies (5) and does not satisfy (3).

Proof. Put $\lambda = 3$ and $x_1 = 0 \in C$. It is clear that $Tx_1 = 0$. Fix $\eta > 0$ and choose $n \in \mathbb{N}$ with $\eta < n$ and $2 \leq n$. Put $y \in C$ by

$$y(k) = \begin{cases} n, & \text{if } k = n, \\ 0, & \text{if } k \neq n. \end{cases}$$

Then $\|y\| = n > \eta$ and

$$(Ty)(k) = \begin{cases} n - 2, & \text{if } k = n, \\ 0, & \text{if } k \neq n. \end{cases}$$

So, we have

$$\|Tx_1 - Ty\| = \|Ty\| = n - 2 > n - \lambda = \|y\| - \lambda = \|x_1 - y\| - \lambda.$$

Therefore T does not satisfy (C3). We next put $\lambda = 1$ and fix $x_1 \in C$. Then there exists $n_1 \in \mathbb{N}$ such that $0 \leq x_1(n) < 1$ for all $n \in \mathbb{N}$ with $n \geq n_1$. By the definition of T, $(Tx_1)(n) = 0$ for $n \in \mathbb{N}$ with $n \geq n_1$. Put $\eta = n_1 + 5$, and fix $y \in C$ with $\|y\| > \eta$. We choose $n_2 \in N$ with $y(n_2) = \|y\|$. Then from the definition of C, we have

$$n_1 < n_1 + 5 = \eta < \|y\| = y(n_2) \leq n_2.$$

It is clear that $\|y\| = y(n_2) > 2$. For $n \in \mathbb{N}$ with $n < n_1$, we have

$$|\,(Tx_1)(n) - (Ty)(n)\,| \leq n < n_1 < n_1 + 3 = \eta - 2.$$

On the other hand, for $n \in \mathbb{N}$ with $n \geq n_1$, we have

$$|\,(Tx_1)(n) - (Ty)(n)\,| = |\,(Ty)(n)\,| =$$

$$= \max\{y(n) - 2, 0\} \leq \|y\| - 2 = y(n_2) - 2.$$

Since $n_1 < n_2$, $\eta - 2 < \|y\| - 2 = y(n_2) - 2$, and

$$|\,(Tx_1)(n_2) - (Ty)(n_2)\,| = \max\{y(n_2) - 2, 0\} = y(n_2) - 2,$$

we have

$$\|Tx_1 - Ty\| = y(n_2) - 2.$$

So, we obtain

$$\begin{aligned}\|Tx_1 - Ty\| &= y(n_2) - 2 \\ &\leq y(n_2) - x_1(n_2) - \lambda \\ &\leq \|x_1 - y\| - \lambda.\end{aligned}$$

This implies (C5). This completes the proof. □

Exmple 3. Put $E = \mathbb{R}$ and $C = [1, \infty)$. Define a nonexpansive mapping T on C by

$$Tx = x$$

for all $x \in C$. Then T satisfies (C6) and does not satisfy (C5).

Proof. Since

$$\|Tx - Ty\| = \|x - y\| = \|x - y\| - \|x - Tx\|$$

for all $x, y \in C$, T satisfies (C6). And from the first equality, T does not satisfy (C5). □

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