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SEMIREGULAR FACTORIZATION OF SIMPLE GRAPHS

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A graph G is a $(d, d+s)$-*graph* if the degree of each vertex of G lies in the interval $[d, d+s]$. A $(d, d+1)$-graph is said to be *semiregular*. An $(r, r+1)$-*factorization* of a graph is a decomposition of the graph into edge-disjoint $(r, r+1)$-factors.

We discuss here the state of knowledge about $(r, r+1)$-factorizations of d -regular graphs and of $(d, d+1)$-graphs.

For $r, s \geq 0$, let $\phi(r, s)$ be the least integer such that, if $d \geq \phi(r, s)$ and G is any simple $[d, d+s]$-graph, then G has an $(r, r+1)$-factorization. Akiyama and Kano (when r is even) and Cai (when r is odd) showed that $\phi(r, s)$ exists for all r, s. We show that, for $s \geq 2$, $\phi(r, s) = r(r+s+1)+1$. Earlier $\phi(r, 0)$ was determined by Egawa and Era, and $\phi(r, 1)$ was determined by Hilton.

1. Introduction.

We call a graph *simple* if it has no loops or multiple edges. In this paper, *multigraphs* are graphs in which multiple edges may occur, but not loops. If multiple edges and loops may occur we use the term *pseudograph*.

An $(r, r+1)$-*pseudograph* is a pseudograph whose degrees are all either r or $r+1$; in a pseudograph, a loop counts two towards the degree of the vertex it is on. An $(r, r+1)$-*factor* of a pseudograph G is an $(r, r+1)$-subpseudograph which spans G. An $(r, r+1)$-*factorization* of a pseudograph G is a decomposition of G into edge-disjoint $(r, r+1)$ -factors of G.

Let $\mathbb{N}$ be the set of non-negative integers. Given $d, s \in \mathbb{N}$ and a pseudograph G, we say that G is a $(d, d+s)$-*graph* if the degree of any vertex of G is in the interval $[d, d+s]$. Let $\phi, \psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be functions defined as follows. Given $r, s \in \mathbb{N}$, let $\phi(r, s)$ be the smallest integer such that $d \geq \phi(r, s)$ implies that any simple $(d, d+s)$-graph has an $(r, r+1)$-factorization. Similarly, let $\psi(r, s)$ be the smallest integer such that $d \geq \psi(r, s)$ implies that any $(d, d+s)$-multigraph has an $(r, r+1)$-factorization.

It is not clear at first sight that $\phi(r, s)$ and $\psi(r, s)$ exist for all values of r, s, and indeed the corresponding function for $(d, d+s)$-pseudographs does not exist for all values of r and s (see [6]). But specializations of a result of Akiyama and Kano [2] when r is even and of Cai [3] when r is odd yield the following result.

Theorem 1. *For $r, s \in \mathbb{N}$,*

$$\psi(r, s) \leq \begin{cases} (3r+1)(r+s-1) & \text{if } r \text{ is even}, \\ (3r+1)(r+s) & \text{if } r \text{ is odd}. \end{cases}$$

It is clear that $\phi(r, s) \leq \psi(r, s)$ always.

Returning to $\phi(r, s)$, the value of $\phi(r, 0)$ was determined by Era [4] and Egawa [5]. A different proof of the Era-Egawa result was given in [6] where $\phi(r, 1)$ was also determined.

Theorem 2. *For $r, s \in \mathbb{N}$, $s \in \{0, 1\}$,*

$$\phi(r, s) = \begin{cases} r(r+s) & \text{if } r \text{ is even}, \\ r(r+s)+1 & \text{if } r \text{ is odd}. \end{cases}$$

Less precise results are known for $\psi(r, s)$ when $s = 0$ or 1. In [6] it is shown that the following result holds.

Theorem 3. *If $r \in \mathbb{N}$ and $r \geq 1$, then*

$$\frac{3}{2}r^2 - r \leq \psi(r, 0) \leq 2r^2 - 3r$$

if $r \geq 4$ is even, and

$$\psi(r, 0) = r^2 + 1$$

if r is odd.

Thus $\psi(r, 0) \neq \phi(r, 0)$ if r is even, but $\psi(r, 0) = \phi(r, 0)$ if r is odd.

In [6] bounds are also obtained for $\psi(r, 1)$. In this paper we first describe in more detail what is known about $(r, r+1)$-factorizations of d-regular simple graphs and simple $(d, d+1)$-graphs, with particular emphasis on the number of factors in a factorization. We then prove the following theorem on the value of $\phi(r, s)$ when $s \geq 2$. This result stands in unexpected contrast to Theorem 2.

Theorem 4. *For $r, s \in \mathbb{N}$, $s \geq 2$*

$$\phi(r, s) = r(r + s + 1) + 1.$$

For good references on factorizations of graphs, see [1] and [8].

2. $(r, r+1)$-factorizations of simple graphs.

In the cases when $s = 0$ and $s = 1$, $\phi(r, s)$ was evaluated by a novel method in [6]. A fundamental result of Hilton and de Werra [7] provided the key to this novel method. Let us give some terminology and then explain this fundamental result.

An *edge-colouring* of a pseudograph G is a map $\lambda : E(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set of colours (loops being counted as edges). An edge-colouring is *equitable* if for each vertex v of G and any two colours $C_1, C_2 \in \mathcal{C}$, the number of edges incident to v and coloured C_1 differs by at most one from the corresponding number of edges coloured C_2; here a loop on v coloured C_i counts as two edges on v. For k an integer, $k \geq 2$, the *k-core* of a pseudograph G is the subpseudograph induced by the vertices of G whose degree is divisible by k. The theorem of Hilton and de Werra is:

Theorem 5. *Let k be an integer, $k \geq 2$, and let G be a simple graph. If the k-core of G contains no edges, then G has an equitable colouring with k colours.*

Using this theorem, the first author [6] proved the following result about $(r, r+1)$-factorizations of d-regular simple graphs. Theorem 5 was used to prove the 'hard' part, namely part 1.

Theorem 6. *Let G be a simple d-regular graph, and let x and r be integers with $r \geq 1$.*

1. *G has an $(r, r+1)$-factorization with exactly x $(r, r+1)$-factors if*

$$d/(r + 1) < x < d/r,$$

or if r is odd and $x = d/(r+1)$, or if r is even and $x = d/r$.

2. *If r is even and $(r+1)|d$, then there are d-regular simple graphs G which are, and d-regular simple graphs G which are not $(r, r+1)$-factorizable into $x = d/(r+1)$ $(r, r+1)$-factors; if r is odd and $r|d$, then there are d-regular simple graphs which are, and d-regular simple graphs which are not $(r, r+1)$-factorizable into $x = d/r$ $(r, r+1)$-factors.*
3. *If $x \notin [d/(r+1), d/r]$, then no d-regular simple graph is $(r, r+1)$-factorizable into x $(r, r+1)$-factors.*

For simple $(d, d+1)$-graphs the following similar theorem was also proved in [6].

Theorem 7. *Let x, d and r be integers with $d \geq r \geq 1$.*

1. *If*

$$(d+1)/(r+1) < x < d/r$$

or

$$x = \begin{cases} d/r & \text{if } r \text{ is even,} \\ (d+1)/(r+1) & \text{if } r \text{ is odd,} \end{cases}.$$

then any simple $(d, d+1)$-graph G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors.

2. *If $x \geq 2$ and*

$$x = \begin{cases} d/r & \text{if } r \text{ is odd,} \\ (d+1)/(r+1) & \text{if } r \text{ is even,} \end{cases}.$$

then some simple $(d, d+1)$-graphs do and some do not have an $(r, r+1)$-factorization into x $(r, r+1)$-factors.

3. *If $x \notin [(d+1)/(r+1), d/r]$, then the only simple $(d, d+1)$-graphs G having an $(r, r+1)$-factorization into x $(r, r+1)$-factors occur when*

$$\begin{cases} x = d/(r+1) & \text{and } G \text{ is } d\text{-regular,} \\ x = (d+1)/r & \text{and } G \text{ is } (d+1)\text{-regular.} \end{cases}.$$

Moreover, when these conditions pertain, some but not all such graphs have an $(r, r+1)$-factorization.

Using Theorems 5, 6, and 7, it is a fairly simple matter to deduce Theorem2 (which includes the Era-Egawa theorem).

3. $(r, r+1)$-factorization of simple $(d, d+s)$-graphs.

In this section we prove Theorem 4 which says that if $r, s \in \mathbb{N}$, $s \geq 2$, then $\phi(r, s) = r(r+s+1)+1$.

Proof of Theorem 4 We first show that if

$$d \geq r(r+s+1)+1,$$

then any simple $(d, d+s)$-graph has an $(r, r+1)$-factorization. Note that

$$\frac{d}{r} - \frac{d+s}{r+1} \geq \frac{r^2+r+1}{r(r+1)} > 1,$$

so there is an integer x with

$$\frac{d+s}{r+1} < x < \frac{d}{r}.$$

By Theorem 5, G has an equitable coloring with x colors. Let v be a vertex of G. Since $rx < d$, there is a color class with at least $r+1$ edges incident to v. Since $(r+1)x > d+s$, there is a color class with at most r edges incident to v. Since the coloring is equitable the number of vertices incident to v in each color class is r or $r+1$. Thus the color classes give us an $(r, r+1)$-factorization of G.

Next we show that if

$$d = r(r+s+1),$$

then there is a simple $(d, d+s)$-graph without an $(r, r+1)$-factorization. Note that

$$d+s = (r+1)(r+s),$$

so any $(r, r+1)$-factorization of a $(d, d+s)$-graph contains either $r+s$ or $r+s+1$ factors. We are going to consider four cases depending on the parity of r and s. In all cases G will be a disjoint union of graphs G_1 and G_2 such that G_1 has no $(r, r+1)$-factorization into $r+s+1$ factors and G_2 has no $(r, r+1)$-factorization into $r+s$ factors.

In each case, the argument will be that if G_1 or G_2 did have such an $(r, r+1)$-factorization, then some $(r, r+1)$-factor would have to have an odd number of vertices of odd degree, which is impossible.

Assume first that r is even. Then d is even. Let G_1 be a graph with one vertex of degree $d+2$ and the remaining vertices of degree d. Some factor of an $(r, r+1)$-factorization of such a G_1 into $r+s+1$ factors would have exactly one vertex of degree $r+1$ which is impossible. For example, a suitable G_1 can be obtained by taking K_{d+2}, removing a Hamiltonian cycle, and adding a new vertex adjacent to all the other vertices. To construct G_2 we consider two cases.

If s is odd, let G_2 be a graph in which each vertex has degree $d+s$ except for one which has degree $d+s-1$. Since $d+s$ is odd, G_2 has odd order. In any $(r, r+1)$-factorization of G into $r+s$ factors, all but one of the factors of G_2 would have to be regular of degree $r+1$ which is odd. But since the order of G_2 is also odd, this is impossible. A suitable graph G_2 can be obtained by taking K_{d+s+2} and removing a spanning subgraph with $(d+s+1)/2$ components, where $(d+s-1)/2$ components are P_2's and one is a P_3.

If s is even, let G_2 be a regular graph of degree $d+s$ of odd order. In any $(r, r+1)$-factorization of G_2 into $r+s$ factors, all the $(r, r+1)$-factors would be $(r+1)$-regular, which is impossible since $r+1$ is odd and G_2 has odd order.

Now assume that r is odd (so $d+s$ is even). Let G_2 be a graph which has one vertex of degree $d+s-2$, the remainder having degree $d+s$. Some factor of an $(r, r+1)$-factorization of such a G_2 into $r+s$ factors would have exactly one vertex of degree r, the remaining vertices having degree $r+1$. This is impossible as r is odd. An example of a suitable G_2 may be obtained by taking K_{d+s+2}, marking two of its vertices as u and v, removing a 1-factor from $K_{d+s+2} - \{u, v\}$, and removing a path of length 2 with endpoints u and v. To construct G_1 we consider two cases.

If s is even (so that d is even), let G_1 be a d-regular graph of odd order. If s is odd (so that d is odd), let G_1 be a graph with one vertex of degree $d+1$ and the remaining vertices of degree d. An example of such G_1 can be obtained from K_{d+2} by removing $(d+1)/2$ independent edges.

4. $(r, r+1)$-factorizations of multigraphs.

Recall that we have defined multigraphs as having no loops.

Theorem 3 shows that the upper bounds for $\psi(r, s)$ given in Theorem 1 are not best possible, at least in the case when $s = 0$. This is also true if $s = 1$, as in [6] the following is proved.

Theorem 8. *If $r \in \mathbb{N}$, $r \geq 1$, then*

$$\frac{3r^2}{2} - r \leq \psi(r, 1) \leq 2r^2 + r - 1$$

if r is even, and

$$r(r+1) + 1 \leq \psi(r, 1) \leq 2r^2 + 3r - 1$$

if r is odd.

Thus to determine $\psi(r, s)$ remains an open problem.

Theorem 3 also seems to suggest the surprising possibility that $\phi(r, s) = \psi(r, s)$ holds for every $r, s \in \mathbb{N}$ with r odd. However, the question if that is really true requires more evidence.

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