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BOUNDARY ESTIMATES FOR SOLUTIONS TO SINGULAR ELLIPTIC EQUATIONS

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We deal with the Dirichlet problem $\Delta u + u^{-\gamma} + g(u) = 0$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ with $u = 0$ on the boundary $\partial\Omega$, where $\gamma > 1$ and g is a smooth function. Assuming $|g(t)|$ grows like $t^{-\nu}$, $1 < \nu < \gamma$, as $t \to 0$, we find optimal estimates of $u(x)$ in terms of the distance of x from the boundary $\partial\Omega$.

1. Introduction.

We investigate the Dirichlet problem

$$\Delta u + f(u) = 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ and $f(t)$ is a smooth, positive, decreasing function which tends to infinity as t tends to zero. Existence and uniqueness for this problem have been discussed by Crandall, Rabinowitz and Tartar in [5]. In particular, in [5] it is proved that there is a classical solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and that, near the boundary $\partial\Omega$, it satisfies

$$\lambda\Phi(\delta(x)) \le u(x) \le \Lambda\Phi(\delta(x)), \tag{1.2}$$

where $\Phi = \Phi(s)$ is a positive solution of the problem

$$\Phi'' + f(\Phi) = 0, \quad \Phi(0) = 0,$$

$\delta(x)$ is the distance from x to $\partial\Omega$, and λ, Λ are two suitable positive constants.

If $f(t) = t^{-\gamma}$, $\gamma > 1$, we can take $\Phi = \phi$, with

$$\phi(s) = \Big(\frac{\gamma+1}{\sqrt{2(\gamma-1)}}s\Big)^{\frac{2}{\gamma+1}}. \tag{1.3}$$

In this special case, the estimate (1.2) has been improved in [4], where it is shown that there is a constant B such that

$$\big|u(x) - \phi(\delta(x))\big| \le B\delta(x). \tag{1.4}$$

For $\gamma > 3$, the last inequality has been made more precise in [13], where the estimate

$$\big|u(x) - \phi(\delta(x))\big| \le B\big(\delta(x)\big)^{\frac{\gamma+3}{\gamma+1}}$$

has been proved. For $1 < \gamma < 3$, in [3] it is shown that

$$\big|u(x) - \phi(\delta(x))\big| \le B\big(\delta(x)\big)^{\frac{2\gamma}{\gamma+1}}.$$

The existence of a solution of the homogeneous Dirichlet problem for more general singular elliptic equations has been recently investigated in [6], [8], [12], [14], [15]. Also the behaviour of the solution near the boundary has been investigated. However, as far as we know, there are not results concerning the precise evaluation of the second order term in the boundary approximation of the solution.

In the present paper we investigate problem (1.1) in case $f(t) = t^{-\gamma} + g(t)$, where $\gamma > 1$ and $g(t)$ is a smooth function such that $t^{-\gamma} + g(t)$ is positive and decreasing for $t > 0$. First we assume that $|g(t)|$ grows like the function $t^{-\nu}$ with

$$\max\Big[\frac{\gamma-1}{2}, 1\Big] < \nu < \gamma,$$

and prove that

$$u(x) = \phi(\delta(x))\big(1 + O(1)\delta^{\beta}\big),$$

where

$$\beta = \frac{2(\gamma-\nu)}{\gamma+1} \tag{1.5}$$

and $O(1)$ is a bounded quantity. Note that $0 < \beta < 1$.

Next we take $\gamma > 3$, suppose that $|g(t)|$ grows like $t^{-\nu}$ with $1 < \nu \leq (\gamma - 1)/2$, and we prove that

$$u(x) = \phi(\delta(x))\Big[1 + \frac{(N-1)K}{3-\gamma}\delta + O(1)\delta^{\beta}\Big],$$

where $K = K(x)$ denotes the mean curvature of the surface $\{x \in \Omega : \delta(x) = constant\}$. The exponent β is defined as in (1.5), but now we have $1 < \beta < 2$ for $1 < \nu < (\gamma - 1)/2$, and $\beta = 1$ for $\nu = (\gamma - 1)/2$.

In case of $g \equiv 0$ and $\gamma > 3$ we prove that

$$u(x) = \phi(\delta(x))\Big[1 + \frac{(N-1)K}{3-\gamma}\delta + O(1)\delta^{1+\sigma}\Big], \tag{1.6}$$

with

$$0 < \sigma < \frac{\gamma - 3}{\gamma + 1}.$$

To prove our results we use the estimate

$$\lim_{x \to \partial\Omega} \frac{\phi(\delta(x))}{u(x)} = 1. \tag{1.7}$$

For general f (1.7) follows, for example, by [8]. Indeed, if

$$F(t) = \int_t^{\infty} f(\tau)\, d\tau, \tag{1.8}$$

Theorem 2.6 of [8] assumes that $F(t) \to \infty$ as $t \to 0$ and that, for some $M > 1$, $F(t) < MF(2t)$ for t small. Both these conditions hold in our situation.

We emphasize that when the perturbation $|g(t)|$ grows less than $t^{-\gamma}$ then, concerning the boundary behaviour of the solution, it does not make effects at the first level. Furthermore, when $|g(t)|$ grows less than $t^{-\frac{\gamma-1}{2}}$ then it does not make effects at the first two levels.

Estimate (1.6) with $o(\delta)$ in place of $O(1)\delta^{1+\sigma}$ has been announced in [2]. The influence of the domain geometry in boundary blow-up problems has been discussed in [1], [7].

2. Boundary estimates.

Let $\gamma > 1$ be a real number and let $g(t)$ be a smooth function satisfying

$$g(t) = O(1)t^{-\nu}, \quad 1 < \nu < \gamma, \tag{2.1}$$

where $O(1)$ is a bounded quantity. We also suppose

$$t^{-\gamma} + g(t) > 0 \quad \text{and} \quad -\gamma t^{-\gamma-1} + g'(t) < 0 \quad \forall t > 0. \tag{2.2}$$

Under conditions (2.1) and (2.2), we investigate the Dirichlet problem

$$\Delta u + u^{-\gamma} + g(u) = 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{2.3}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$. The integral function $F(t)$ defined as in (1.8) with $f(t) = t^{-\gamma} + g(t)$ behaves like

$$F(t) = \frac{t^{1-\gamma}}{\gamma - 1} + O(1)t^{1-\nu}.$$

Since $F(t) \to \infty$ as $t \to 0$ and since we have, for some $M > 1$ and for t small $F(t) < M F(2t)$, we can use the estimate (1.7) for the solution u to problem (2.3). We have the following

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded smooth domain, let $\gamma > 1$ be a real number, and let $g(t)$ be a smooth function satisfying* (2.2) *and* (2.1) *with*

$$\max\left[\frac{\gamma - 1}{2}, 1\right] < \nu < \gamma.$$

If $u(x)$ is a solution to problem (2.3) *then*

$$u(x) = \phi(\delta(x))\left[1 + O(1)\delta^{\beta}\right], \quad \beta = \frac{2(\gamma - \nu)}{\gamma + 1},$$

where ϕ is defined in (1.3)*, and $O(1)$ is a bounded quantity.*

Proof. We look for a super-solution of the form

$$w(x) = \phi(\delta)\left(1 + \alpha\delta^{\beta}\right),$$

where α is a positive real number to be determined. We have

$$w_{x_i} = \phi'\delta_{x_i}\left(1 + \alpha\delta^{\beta}\right) + \alpha\beta\phi\delta^{\beta-1}\delta_{x_i}.$$

Recall that [9]

$$\sum_{i=1}^{N} \delta_{x_i}\delta_{x_i} = 1, \quad \sum_{i=1}^{N} \delta_{x_i x_i} = -(N-1)K,$$

where $K = K(x)$ is the mean curvature of the surface $\{x \in \Omega : \delta(x) = constant\}$. Putting $(N-1)K = H$ we find

$$\Delta w = \phi''\big(1+\alpha\delta^\beta\big) - \phi' H\big(1+\alpha\delta^\beta\big) + 2\alpha\beta\delta^{\beta-1}\phi' + \\ + \phi\alpha\beta(\beta-1)\delta^{\beta-2} - \alpha\beta\phi H\delta^{\beta-1}. \tag{2.4}$$

The function $\phi = \phi(s)$ satisfies the equations

$$\phi'' = -\phi^{-\gamma}, \ \ \phi' = \phi^{-\gamma}\frac{\gamma+1}{\gamma-1}s, \quad \phi = \phi^{-\gamma}\frac{(\gamma+1)^2}{2(\gamma-1)}s^2. \tag{2.5}$$

By (2.4) and (2.5) we obtain

$$-\Delta w = \phi^{-\gamma}\Big[1+\alpha\delta^\beta + \frac{\gamma+1}{\gamma-1}\delta H(1+\alpha\delta^\beta) - 2\alpha\beta\frac{\gamma+1}{\gamma-1}\delta^\beta \\ - \alpha\beta(\beta-1)\frac{(\gamma+1)^2}{2(\gamma-1)}\delta^\beta + \alpha\beta\frac{(\gamma+1)^2}{2(\gamma-1)}\delta^{\beta+1}H\Big]. \tag{2.6}$$

By (2.6) we get

$$-\Delta w > \phi^{-\gamma}\Big[1+\alpha\delta^\beta\Big(1-2\frac{\gamma+1}{\gamma-1}\beta-\frac{(\gamma+1)^2}{2(\gamma-1)}\beta(\beta-1)-C_1\delta\Big)-C_2\delta\Big], \tag{2.7}$$

where C_i, $i = 1, 2, \cdots$, here and in the sequel, denote suitable positive constants independent of α. We will take α and $\delta_0 > 0$ such that

$$\alpha\delta_0^\beta < \frac{1}{2}, \tag{2.8}$$

and consider points $x \in \Omega$ such that $\delta(x) < \delta_0$. Using condition (2.1) and Taylor's expansion we find

$$w^{-\gamma} + g(w) = \phi^{-\gamma}\big[(1+\alpha\delta^\beta)^{-\gamma} + O(1)\phi^{\gamma-\nu}\big] \\ < \phi^{-\gamma}\big[1 - \gamma\alpha\delta^\beta + C_3(\alpha\delta^\beta)^2 + C_4\delta^\beta\big]. \tag{2.9}$$

By (2.7) and (2.9) we find that

$$-\Delta w > w^{-\gamma} + g(w) \tag{2.10}$$

provided

$$1 + \alpha\delta^{\beta}\Big(1 - 2\frac{\gamma+1}{\gamma-1}\beta - \frac{(\gamma+1)^2}{2(\gamma-1)}\beta(\beta-1) - C_1\delta\Big) - C_2\delta$$
$$> 1 - \gamma\alpha\delta^{\beta} + C_3(\alpha\delta^{\beta})^2 + C_4\delta^{\beta}.$$

After simplification we find

$$C_4 + C_2\delta^{1-\beta} < \alpha\Big[\frac{(\gamma+1)^2}{2(\gamma-1)}(\beta+1)\Big(\frac{2(\gamma-1)}{\gamma+1} - \beta\Big) - C_1\delta - C_3\alpha\delta^{\beta}\Big]. \tag{2.11}$$

Since $\beta < 2(\gamma-1)/(\gamma+1)$, we can take α and δ_0 so that (in addition to (2.8))

$$C_1\delta_0 + C_3\alpha\delta_0^{\beta} < \frac{(\gamma+1)^2}{2(\gamma-1)}(\beta+1)\Big(\frac{2(\gamma-1)}{\gamma+1} - \beta\Big).$$

Since $(\gamma-1)/2 < \nu$, we have $\beta < 1$. Increase α (and decrease δ_0) until (2.11) holds for $\delta < \delta_0$.

Let us show now that we can choose δ_0 and α so that $u(x) < w(x)$ for $\delta(x) = \delta_0$. Let $q = \alpha\delta_0^{\beta}$, where α and δ_0 are as in above. Since by (1.7)

$$\lim_{x\to\partial\Omega} \frac{\phi(\delta(x))}{u(x)} = 1,$$

we can decrease δ_0 (increasing α according to $\alpha\delta_0^{\beta} = q$) until

$$\frac{\phi(\delta(x))}{u(x)} > \frac{1}{1+q}$$

for $\delta(x) \le \delta_0$. Multiplying by $\big(1 + \alpha\delta^{\beta}\big)$ we find

$$\frac{w(x)}{u(x)} > \frac{1}{1+q}\big(1 + \alpha\delta^{\beta}\big).$$

Then $w(x) > u(x)$ for $\delta(x) = \delta_0$. On $\partial\Omega$ we have $w(x) = u(x) = 0$. Therefore, since $t^{-\gamma} + g(t)$ is decreasing, by (2.10) we find that $w(x) \ge u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$.

Now we seek for a sub-solution of the type

$$v(x) = \phi(\delta)\big(1 - \alpha\delta^{\beta}\big).$$

We find an equation similar to (2.6) with $-\alpha$ in place of α, that is

$$-\Delta v = \phi^{-\gamma}\Big[1 - \alpha\delta^{\beta} + \frac{\gamma+1}{\gamma-1}\delta H(1-\alpha\delta^{\beta}) + 2\alpha\beta\frac{\gamma+1}{\gamma-1}\delta^{\beta} + \alpha\beta(\beta-1)\frac{(\gamma+1)^2}{2(\gamma-1)}\delta^{\beta} - \alpha\beta\frac{(\gamma+1)^2}{2(\gamma-1)}\delta^{\beta+1}H\Big].$$

From this equation we find

$$-\Delta v < \phi^{-\gamma}\Big[1-\alpha\delta^{\beta}\Big(1-2\frac{\gamma+1}{\gamma-1}\beta-\frac{(\gamma+1)^2}{2(\gamma-1)}\beta(\beta-1)-C_5\delta\Big)+C_6\delta\Big]. \tag{2.12}$$

Let δ_0 and α satisfy (2.8). For $\delta(x) < \delta_0$, using condition (2.1) and Taylor's expansion we find

$$\begin{aligned} v^{-\gamma} + g(v) &= \phi^{-\gamma}\big[(1-\alpha\delta^{\beta})^{-\gamma} + O(1)\phi^{\gamma-\nu}\big] \\ &> \phi^{-\gamma}\big[1 - \gamma\alpha\delta^{\beta} - C_7\delta^{\beta}\big]. \end{aligned} \tag{2.13}$$

By (2.12) and (2.13) we find that

$$-\Delta v < v^{-\gamma} + g(v) \tag{2.14}$$

provided

$$1-\alpha\delta^{\beta}\Big(1-2\frac{\gamma+1}{\gamma-1}\beta-\frac{(\gamma+1)^2}{2(\gamma-1)}\beta(\beta-1)-C_5\delta\Big)+C_6\delta < 1-\gamma\alpha\delta^{\beta}-C_7\delta^{\beta}.$$

After simplification we find

$$C_7 + C_6\delta^{1-\beta} < \alpha\Big[\frac{(\gamma+1)^2}{2(\gamma-1)}(\beta+1)\Big(\frac{2(\gamma-1)}{\gamma+1} - \beta\Big) - C_5\delta\Big]. \tag{2.15}$$

Inequality (2.15) (which is similar to (2.11) but easier) holds for $\delta < \delta_0$ with δ_0 small enough and α large enough.

Let us show that we can choose δ_0 and α so that $v(x) < u(x)$ for $\delta(x) = \delta_0$. Let $q = \alpha\delta_0^\beta$, with α and δ_0 as in above. Since we have

$$\lim_{x \to \partial\Omega} \frac{\phi(\delta(x))}{u(x)} = 1,$$

we can decrease δ_0 (increasing α according to $\alpha\delta_0^\beta = q$) until

$$\frac{\phi(\delta(x))}{u(x)} < \frac{1}{1-q}$$

for $\delta(x) \le \delta_0$. As a consequence we have

$$\frac{v(x)}{u(x)} < \frac{1}{1-q}\big(1 - \alpha\delta^\beta\big).$$

Then $v(x) < u(x)$ for $\delta(x) = \delta_0$. By (2.14) it follows that $v(x) \le u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$.

We have proved that for x near $\partial\Omega$ and α large enough we have

$$\phi(\delta)\big(1 - \alpha\delta^\beta\big) \le u(x) \le \phi(\delta)\big(1 + \alpha\delta^\beta\big).$$

Equivalently we have

$$u(x) = \phi(\delta)\big[1 + O(1)\delta^\beta\big].$$

The theorem is proved. □

Recall that $\Delta\delta = -(N-1)K$, where K is the mean curvature of the surface $\{x \in \Omega : \delta(x) = constant\}$. We state now our main result.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded smooth domain, let $\gamma > 3$ be a real number, and let $g(t)$ be a smooth function satisfying* (2.2) *and* (2.1) *with*

$$1 < \nu \le \frac{\gamma - 1}{2}.$$

If $u(x)$ is a solution to problem (2.3) *then*

$$u(x) = \phi(\delta(x))\Big[1 + \frac{(N-1)K}{3-\gamma}\delta + O(1)\delta^\beta\Big], \quad \beta = \frac{2(\gamma - \nu)}{\gamma + 1},$$

where ϕ is defined in (1.3), *and $O(1)$ is a bounded quantity. Furthermore, if $g(t) \equiv 0$ then we have*

$$u(x) = \phi(\delta(x))\Big[1 + \frac{(N-1)K}{3-\gamma}\delta + O(1)\delta^{1+\sigma}\Big],$$

where σ is any positive real number such that $\sigma < (\gamma - 3)/(\gamma + 1)$.

Proof. Define

$$A = \frac{H}{3-\gamma}, \quad H = (N-1)K. \tag{2.16}$$

We look for a super-solution of the kind

$$w(x) = \phi(\delta)\big(1 + A\delta + \alpha\delta^{1+\sigma}\big),$$

where σ is a positive real number such that $\sigma < (\gamma - 3)/(\gamma + 1)$ and α is a positive real number to be determined. We have

$$w_{x_i} = \phi'\delta_{x_i}\big(1 + A\delta + \alpha\delta^{1+\sigma}\big) + \phi\big(A_{x_i}\delta + A\delta_{x_i} + \alpha(1+\sigma)\delta^{\sigma}\delta_{x_i}\big).$$

Since $\Delta\delta = -H$ we find

$$\begin{aligned}\Delta w = \phi''\big(1 + A\delta + \alpha\delta^{1+\sigma}\big) - \phi' H\big(1 + A\delta + \alpha\delta^{1+\sigma}\big)\\ + 2\phi'\big(\nabla A\cdot\nabla\delta\,\delta + A + \alpha(1+\sigma)\delta^{\sigma}\big)\\ + \phi\big(\Delta A\,\delta + 2\nabla A\cdot\nabla\,\delta - AH + \alpha(1+\sigma)\sigma\delta^{\sigma-1} - \alpha(1+\sigma)H\delta^{\sigma}\big).\end{aligned}$$

Using equations (2.5) we find

$$\begin{aligned}-\Delta w = \phi^{-\gamma}\Big[1 + A\delta + \alpha\delta^{1+\sigma} + \frac{\gamma+1}{\gamma-1}\delta H\big(1 + A\delta + \alpha\delta^{1+\sigma}\big)\\ - 2\frac{\gamma+1}{\gamma-1}\delta\big(\nabla A\cdot\nabla\delta\,\delta + A + \alpha(1+\sigma)\delta^{\sigma}\big)\\ -\frac{(\gamma+1)^2}{2(\gamma-1)}\delta^2\big(\Delta A\,\delta + 2\nabla A\cdot\nabla\,\delta - AH + \alpha(1+\sigma)\sigma\delta^{\sigma-1} - \alpha(1+\sigma)H\delta^{\sigma}\big)\Big].\end{aligned}$$

Denoting with C_i, $i = 1, 2, \cdots$, positive constants independent of α we get

$$\begin{aligned}-\Delta w > \phi^{-\gamma}\Big[1 + \delta\Big(A + \frac{\gamma+1}{\gamma-1}H - 2\frac{\gamma+1}{\gamma-1}A\Big)\\ + \alpha\delta^{1+\sigma}\Big(1 - 2\frac{\gamma+1}{\gamma-1}(1+\sigma) - \frac{(\gamma+1)^2}{2(\gamma-1)}(1+\sigma)\sigma - C_1\delta\Big) - C_2\delta^2\Big].\end{aligned} \tag{2.17}$$

For α fixed, we consider $\delta_0 > 0$ such that for $\delta(x) < \delta_0$ we have

$$-\frac{1}{2} < A\delta + \alpha\delta^{1+\sigma} < 1. \tag{2.18}$$

Using condition (2.1) and Taylor's expansion we have

$$w^{-\gamma} + g(w) = \phi^{-\gamma}\Big[\big(1 + A\delta + \alpha\delta^{1+\sigma}\big)^{-\gamma} + O(1)\phi^{\gamma-\nu}\Big] \tag{2.19}$$

$$< \phi^{-\gamma}\Big[1 - \gamma A\delta - \gamma\alpha\delta^{1+\sigma} + C_3\delta^2 + C_4\delta\alpha\delta^{1+\sigma} + C_5(\alpha\delta^{1+\sigma})^2 + C_6\delta^{\beta}\Big].$$

Observe that we can take $C_6 = 0$ when $g(t) \leq 0$. By (2.17) and (2.19) we find that

$$-\Delta w > w^{-\gamma} + g(w) \tag{2.20}$$

provided

$$1 + \Big(A + \frac{\gamma+1}{\gamma-1}H - 2\frac{\gamma+1}{\gamma-1}A\Big)\delta \tag{2.21}$$

$$+ \alpha\delta^{1+\sigma}\Big(1 - 2\frac{\gamma+1}{\gamma-1}(1+\sigma) - \frac{(\gamma+1)^2}{2(\gamma-1)}(1+\sigma)\sigma - C_1\delta\Big) - C_2\delta^2$$

$$> 1 - \gamma A\delta - \gamma\alpha\delta^{1+\sigma} + C_3\delta^2 + C_4\delta\alpha\delta^{1+\sigma} + C_5(\alpha\delta^{1+\sigma})^2 + C_6\delta^{\beta}.$$

Using (2.16) we find

$$A + \frac{\gamma+1}{\gamma-1}H - 2\frac{\gamma+1}{\gamma-1}A = -\gamma A. \tag{2.22}$$

Hence, (2.21) can be rewritten as

$$\alpha\delta^{1+\sigma}\Big(1 - 2\frac{\gamma+1}{\gamma-1}(1+\sigma) - \frac{(\gamma+1)^2}{2(\gamma-1)}(1+\sigma)\sigma - C_1\delta\Big) - C_2\delta^2 \tag{2.23}$$

$$> -\gamma\alpha\delta^{1+\sigma} + C_3\delta^2 + C_4\delta\alpha\delta^{1+\sigma} + C_5(\alpha\delta^{1+\sigma})^2 + C_6\delta^{\beta}.$$

Rearranging we have

$$(C_2 + C_3)\delta^2 + C_6\delta^{\beta} \tag{2.24}$$

$$< \alpha\delta^{1+\sigma}\Big[\frac{(\gamma+1)^2}{2(\gamma-1)}\Big(\frac{\gamma-3}{\gamma+1} - \sigma\Big)(\sigma+2) - C_5\alpha\delta^{1+\sigma} - (C_1 + C_4)\delta\Big].$$

If we take $1 + \sigma = \beta$ we find

$$(C_2 + C_3)\delta^{2-\beta} + C_6 \tag{2.25}$$

$$< \alpha\Big[\frac{(\gamma+1)^2}{2(\gamma-1)}(\beta+1)\Big(\frac{2(\gamma-1)}{\gamma+1}-\beta\Big) - C_5\alpha\delta^\beta - (C_1+C_4)\delta\Big].$$

Take α and $\delta_0 > 0$ so that

$$C_5\alpha\delta_0^\beta + (C_1+C_4)\delta_0 < \frac{(\gamma+1)^2}{2(\gamma-1)}(\beta+1)\Big(\frac{2(\gamma-1)}{\gamma+1}-\beta\Big).$$

Decrease δ_0 again and increase α so that (2.25) and (2.18) (with $1+\sigma=\beta$) hold for $\delta(x) < \delta_0$. In this situation, inequality (2.20) holds.

As in the proof of the previous theorem, let us show that we can choose δ_0 and α so that $u(x) < w(x)$ for $\delta(x) = \delta_0$. Let $q = \alpha\delta_0^\beta$, with α and δ_0 as in above. Since we have

$$\lim_{x\to\partial\Omega}\frac{\phi(\delta(x))}{u(x)} = 1,$$

we can decrease δ_0 (increasing α according to $\alpha\delta_0^\beta = q$) until

$$\frac{\phi(\delta(x))}{u(x)} > \frac{2}{2+q}$$

for $\delta(x) \le \delta_0$. As a consequence we have

$$\frac{w(x)}{u(x)} > \frac{2}{2+q}\big(1 + A\delta + \alpha\delta^\beta\big).$$

Decrease δ_0 again and increase α in order to have $\alpha\delta_0^\beta = q$ and $A\delta_0 > -q/2$. Then $w(x) > u(x)$ for $\delta(x) = \delta_0$. Since $w(x) = u(x)$ on $\partial\Omega$, and since $t^{-\gamma} + g(t)$ is decreasing, by (2.20) it follows that $w(x) \ge u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$.

To finish, we look for a sub-solution of the kind

$$v(x) = \phi(\delta)\big(1 + A\delta - \alpha\delta^{1+\sigma}\big).$$

We find an inequality similar to (2.17) with $-\alpha$ in place of α, that is

$$-\Delta v < \phi^{-\gamma}\Big[1 + \delta\Big(A + \frac{\gamma+1}{\gamma-1}H - 2\frac{\gamma+1}{\gamma-1}A\Big) - \alpha\delta^{1+\sigma}\Big(1 - 2\frac{\gamma+1}{\gamma-1}(1+\sigma) - \frac{(\gamma+1)^2}{2(\gamma-1)}(1+\sigma)\sigma - C_7\delta\Big) + C_8\delta^2\Big]. \tag{2.26}$$

Take α and $\delta_0 > 0$ such that for $\delta(x) < \delta_0$ we have

$$-\frac{1}{2} < A\delta - \alpha\delta^{1+\sigma} < 1. \tag{2.27}$$

Using condition (2.1) and Taylor's expansion we have

$$\begin{aligned} v^{-\gamma} + g(v) &= \phi^{-\gamma}\Big[\big(1 + A\delta - \alpha\delta^{1+\sigma}\big)^{-\gamma} + O(1)\phi^{\gamma-\nu}\Big] \\ &> \phi^{-\gamma}\Big[1 - \gamma A\delta + \gamma\alpha\delta^{1+\sigma} - C_9\delta^{\beta}\Big]. \end{aligned} \tag{2.28}$$

In case of $g(t) \ge 0$ we can take $C_9 = 0$. By (2.26) and (2.28) we find that

$$-\Delta v < v^{-\gamma} + g(v) \tag{2.29}$$

when

$$\begin{aligned} &1 + \delta\Big(A + \frac{\gamma+1}{\gamma-1}H - 2\frac{\gamma+1}{\gamma-1}A\Big) \\ &- \alpha\delta^{1+\sigma}\Big(1 - 2\frac{\gamma+1}{\gamma-1}(1+\sigma) - \frac{(\gamma+1)^2}{2(\gamma-1)}(1+\sigma)\sigma - C_7\delta\Big) + C_8\delta^2 \\ &< 1 - \gamma A\delta + \gamma\alpha\delta^{1+\sigma} - C_9\delta^{\beta}. \end{aligned}$$

Using (2.22), after simplification we find the following inequality (similar to (2.24) but easier):

$$C_8\delta^2 + C_9\delta^{\beta} < \alpha\delta^{1+\sigma}\Big[\frac{(\gamma+1)^2}{2(\gamma-1)}\Big(\frac{\gamma-3}{\gamma+1} - \sigma\Big)(\sigma+2) - C_7\delta\Big]. \tag{2.30}$$

With $1 + \sigma = \beta$ we have

$$C_8\delta^{2-\beta} + C_9 < \alpha\Big[\frac{(\gamma+1)^2}{2(\gamma-1)}(\beta+1)\Big(\frac{2(\gamma-1)}{\gamma+1} - \beta\Big) - C_7\delta\Big]. \tag{2.31}$$

Now let us take α and δ_0 so that (2.31) and (2.27) (with $1 + \sigma = \beta$) hold for $\delta < \delta_0$.

With α and δ_0 as in above, let $q = \alpha\delta_0^{\beta}$. Since we have

$$\lim_{x\to\partial\Omega} \frac{\phi(\delta(x))}{u(x)} = 1,$$

we can decrease δ_0 (increasing α according to $\alpha\delta_0^\beta = q$) until

$$\frac{\phi(\delta(x))}{u(x)} < \frac{2}{2-q}$$

for $\delta(x) \le \delta_0$. As a consequence we have

$$\frac{v(x)}{u(x)} < \frac{2}{2-q}\big(1 + A\delta - \alpha\delta^\beta\big).$$

Decrease δ_0 again and increase α in order to have $\alpha\delta_0^\beta = q$ and $A\delta_0 < q/2$. Then $v(x) < u(x)$ for $\delta(x) = \delta_0$. By (2.29) it follows that $v(x) \le u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$.

Therefore, for x near $\partial\Omega$ and α large enough we have

$$\phi(\delta)\big(1 + A\delta - \alpha\delta^\beta\big) \le u(x) \le \phi(\delta)\big(1 + A\delta + \alpha\delta^\beta\big).$$

Equivalently we have

$$u(x) = \phi(\delta)\big(1 + A\delta + O(1)\delta^\beta\big).$$

Recalling the definition of A given in (2.16), the first assertion of the theorem follows.

Now let $g \equiv 0$. Looking for a super-solution of the form

$$w(x) = \phi(\delta)\big(1 + A\delta + \alpha\delta^{1+\sigma}\big),$$

we find that

$$-\Delta w > w^{-\gamma}$$

provided inequality (2.24) with $C_6 = 0$ holds, that is, provided

$$(C_2 + C_3)\delta^{1-\sigma} < \alpha\Big[\frac{(\gamma+1)^2}{2(\gamma-1)}\Big(\frac{\gamma-3}{\gamma+1} - \sigma\Big)(\sigma+2) - C_5\alpha\delta^{1+\sigma} - (C_1 + C_4)\delta\Big]. \tag{2.32}$$

Since $0 < \sigma < (\gamma-3)/(\gamma+1)$, for α fixed (2.32) holds with δ small enough. By the same argument used in the proof of the first part of this theorem, we can increase α and decrease δ_0 so that $w(x) > u(x)$ for $\delta(x) = \delta_0$. It follows that $w(x)$ is a super-solution on $\{x \in \Omega : \delta(x) < \delta_0\}$.

By a similar argument, using (2.30) with $C_9 = 0$, we find that the function

$$v(x) = \phi(\delta)\big(1 + A\delta - \alpha\delta^{1+\sigma}\big),$$

is a sub-solution on $\{x \in \Omega : \delta(x) < \delta_0\}$. The theorem follows. □

REFERENCES

[1] C. Bandle, *Asymptotic behaviour of large solutions of quasilinear elliptic problems,* ZAMP, 54 (2003), pp. 1-8.

[2] C. Bandle - W. Reichel, *Personal communication,* .

[3] S. Berhanu - F. Cuccu - G. Porru, *On the boundary behaviour, including second order effects, of solutions to singular elliptic problems,* In print in Acta Mathematica Sinica, English Series.

[4] S. Berhanu - F. Gladiali - G. Porru, *Qualitative properties of solutions to elliptic singular problems,* J. of Inequal. & Appl., 3 (1999), pp. 313-330.

[5] M.G. Crandall - P.H. Rabinowitz - L. Tartar, *On a Dirichlet problem with a singular nonlinearity,* Comm. Part. Diff. Eq., 2 (1977), pp. 193-222.

[6] Cui Shangbin, *Positive solutions for Dirichlet problems associated to semilinear elliptic equations with singular nonlinearity,* Nonlinear Analysis, 21 (1993), pp. 181-190.

[7] M. del Pino - R. Letelier, *The influence of domain geometry in boundary blow-up elliptic problems,* Nonlinear Anal. TMA, 48 (2002), pp. 897-904.

[8] E. Giarrusso - G. Porru, *Boundary behaviour of solutions to nonlinear elliptic singular problems,* Appl. Mathematics in the Golden Age (J.C. Misra ed.), Narosa, New Delhi, 2003, pp. 163-178.

[9] D. Gilbarg - N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer Verlag, Berlin, 1977.

[10] B. Kawohl, *On a class of singular elliptic equations,* Pitman Res. Notes Math. Ser., 266 (1992), pp. 156-163.

[11] A.C. Lazer - P.J. McKenna, *Asymptotic behaviour of solutions of boundary blow-up problems,* Differential and Integral Equations, 7 (1994), pp. 1001-1019.

[12] A.C. Lazer - P.J. McKenna, *On a singular nonlinear elliptic boundary value problem,* Proc. American. Math. Soc., 111 (1991), pp. 721-730.

[13] P.J. McKenna - W. Reichel, *Sign-changing solutions to singular second-order boundary value problems,* Adv. Differential Equations, 6 (2001), pp. 441-460.

[14] Z. Zhang - J. Yu, *On a singular nonlinear Dirichlet problem with a convection term,* SIAM J. Math. Anal., 32 (2000), pp. 916-927.

[15] Z. Zhang - J. Cheng, *Existence and optimal estimates of solutions for singular nonlinear Dirichlet problems,* Nonlinear Analysis, 57 (2004), pp. 473-484.

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