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FAMILIES OF N-GONAL CURVES WITH MAXIMAL VARIATION OF MODULI

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1. Introduction.

In 1963 Manin proved the Mordell conjecture for function fields (see [18]): *Let K be a function field and let X be a nonisotrivial curve of genus at least 2 defined over K. Then X has finitely many K-rational points.*

Some years later Parshin (in the case of a complete base, [23]) and Arakelov (in the general case, [1]) proved the Shafarevich conjecture for function fields: *Let B be a nonsingular, projective, complex curve and let S be a finite subset of points of B. Fix an integer $g \geq 2$. Then there exist only finitely many nonisotrivial families of smooth curves of genus g over $B - S$.* Moreover Parshin showed how Mordell conjecture follows from Shafarevich conjecture (this is known as Parshin trick).

Two analogous theorems for number fields were proved by Faltings in 1983 (see [11]). Also in this context, the Parshin trick allows to deduce

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Mordell conjecture from Shafarevich one. Moreover it allows to give some explicit estimations on the number of rational points for fixed number field K, g and the set of points of bad reduction S (see [26]).

A uniform version of Shafarevich and Mordell conjectures for function fields was obtained recently by L. Caporaso ([6]). She proved that, in the case of a one-dimensional base, there is a uniform bound for the Shafarevich and Mordell conjectures depending only on the genus of the fiber, the genus of the base and the cardinality of the set of bad reduction (where the fiber is not smooth). She proved analogous uniform results in higher dimension for "canonically polarized varieties", that is smooth varieties V with ample canonical bundle K_V. In that case she found uniform bounds depending only on the Hilbert polynomial $h(x)$ of the canonical polarization ($h(n) = \chi(K_V^n)$), the canonical degree of the subvariety T of bad reduction and the genus g of the fiber. We mention that uniform results in the number field case are still conjectural (the best result in that direction is contained in [8], where it is shown that this uniformity result would follow from the Lang conjectures).

In a subsequent paper ([7]), L. Caporaso considered smooth irreducible subvarieties V of $\mathbb{P}^r$ obtaining a uniform bound depending on the degree of the subvariety V, on the degree of locus T of bad reduction and on the genus g of the fiber. Also she described an example, due to J. de Jong, which shows that in the case where the place T of bad reduction has codimension 1 the bound should depend on its degree (while she proved this is not the case if T has codimension bigger that 1). But in the last section of this paper, she considered an interesting case where one can obtain a uniform bound independent from the locus of bad reduction, namely the case of families with *maximal variation of moduli*.

A family of smooth curves (or more generally of stable curves) of genus g over a base V is said to have maximal variation of moduli if the image of the modular map $V \to M_g$ (or more generally $V \to \overline{M_g}$) is of maximal dimension, namely $\min\{\dim(V), 3g - 3\}$. This means that the family is a truly varying family of curves (just the opposite of an isotrivial family where the modular map is constant and the fibers don't vary at all).

Of particular interest are the families with maximal varation of moduli over a base of dimension $3g-3$ because then the modular map is generically finite and dominant. In that case (and for $g \geq 24$, when it is known that M_g is of general type) Caporaso proved that the number of families over a fixed base V as well as the number of rational sections of every such family is bounded by a constant that depends only on the base V and on the genus g (see [7, Prop. 4]).

Moreover she proved ([7, Lemma 5]) that if such a family has a rational section then the degree of the modular map (which is called the modular degree) must be a multiple of $2g-2$ and from the proof one deduces that this result is sharp, namely that there exist such families with modular degree exactly $2g-2$.

The aim of this paper is to generalize this lemma to families of n-gonal curves, namely curves that have a n to 1 map to $\mathbb{P}^1$ (or equivalently a base-point free g^1_n). In order to explain the results we obtained and to give the ideas of our proofs, we first review the instructive proof of Caporaso's lemma.

Lemma 1.1 (Caporaso). *Let V be a complex irreducible variety of dimension $3g-3$ and let $\mathcal{F} \to V$ be a family of smooth curves of genus $g \geq 2$ with maximal variation of moduli. If this family has a rational section, then the degree of the modular map $V \to M_g$ (which is generically finite by hypothesis) is a multiple of $2g-2$. Moreover this result is sharp, i.e. there exist such families with a section and with modular degree exactly $2g-2$.*

Proof. Consider the modular map (generically finite by hypothesis) $\phi_{\mathcal{F}} : V \to M_g$ associated to our family $f : \mathcal{F} \to V$. Restricting to an open subset of V we can assume the map to be finite and also with the image contained in M^0_g, which is the open subset in M_g corresponding to curves without automorphisms. Now suppose that the family has a section σ (which we can assume to be regular after restricting the base again) and look at the following diagram:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\widetilde{\phi_{\mathcal{F}}}"] \arrow[d, shift left, "f"] \arrow[dr, phantom, "\square"] & \mathcal{C}^0_g \arrow[d, "\pi"] \\ V \arrow[u, shift left, "\sigma"] \arrow[r, "\phi_{\mathcal{F}}"'] & M^0_g \end{tikzcd}$$

where $\mathcal{C}^0_g$ is the universal family over M^0_g. If we call D the horizontal divisor on $\mathcal{C}^0_g$ defined by $D := \mathrm{Im}(\widetilde{\phi_{\mathcal{F}}} \circ \sigma)$, then the diagram above factors as a composition of two cartesian diagrams as follows:

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d, shift left, "f"] & \mathcal{C}_g^0 \times_{M_g^0} D \arrow[r] \arrow[d, shift left, "\pi'"] & \mathcal{C}_g^0 \arrow[d, "\pi"] \\
V \arrow[r, "\widetilde{\phi_{\mathcal{F}} \circ \sigma}"'] \arrow[u, shift left, "\sigma"] \arrow[ur, phantom, "\square"] & D \arrow[r, "\pi_{|D}"'] \arrow[u, shift left, "\sigma'"] \arrow[ur, phantom, "\square"] & M_g^0
\end{tikzcd}$$

where σ' is the tautological section. This implies that $\deg(\phi_{\mathcal{F}})$ is divisible by $\deg(\pi_{|D})$, which is the relative degree of D with respect to the map $\pi : \mathcal{C}_g^0 \to M_g^0$. But D defines an element of the relative Picard group of $\mathcal{C}_g^0$ over M_g^0 and by the Franchetta conjecture over $\mathbb{C}$ (now a theorem of Harer [14] and Arbarello-Cornalba [3]), this relative Picard group is free of rank 1 generated by the relative dualizing sheaf $\omega_{\mathcal{C}_g^0/M_g^0}$ which has vertical degree $2g-2$. Hence the relative degree of D should be a multiple of $2g-2$ and the same for the degree of the modular map.

Moreover, taking D an effective divisor representing $\omega_{\mathcal{C}_g^0/M_g^0}$ and pulling back the universal family above it, we obtain a family with a section (the tautological one) and with modular degree exactly $2g-2$. □

So the main ingredients in the proof of this lemma are the existence of a universal family over M_g^0 and the fact (Franchetta's conjecture, theorem of Harer-Arbarello-Cornalba) that the relative dualizing sheaf generates the relative Picard group of this family over the base M_g^0. Unfortunately this theorem is known only over the complex numbers because the proof relays heavily on the analytic results of Harer ([14]) and hence the lemma of Caporaso is valid only in characteristic 0.

If one wants to generalize to n-gonal curves, one soon realizes that the *hyperelliptic* case is very different from the higher gonal case ($n \geq 3$).

In fact since every hyperelliptic curve has a non-trivial automorphism, namely the hyperelliptic involution (and for the generic hyperelliptic curve this is the only non-trivial automorphism), there doesn't exist a universal family over any open subset of the moduli space H_g of hyperelliptic curves (see [17] for a detailed study of hyperelliptic families and their relation with the coarse moduli space H_g). Thus there is no hope to generalize the method of the proof of Caporaso's lemma to hyperelliptic curves. Nevertheless we realized that the problem of the existence of a rational section for hyperelliptic families is closely related to another important problem, that is the existence of a global g_2^1 for such families, namely of a line bundle on the family (defined uniquely up to the pull-back of a line bundle coming from the base) that restricts on every fiber to the unique g^1_2 of the hyperelliptic curve. Although the unicity of such a g^1_2 on a hyperelliptic curve could make one think that it should extend to a family, actually this is not the case for g odd (in general), while it holds for g even! Moreover in [17] we proved that the existence of such a g^1_2 is equivalent to the Zariski local-triviality of the family of $\mathbb{P}^1$ for which the initial family of hyperelliptic curves is a double cover (in fact it's true that every hyperelliptic family is a double cover of a family of $\mathbb{P}^1$, or in other words it's true that the hyperelliptic involution extends to families but it's not true that it is associated to a global g^1_2!). And then having reduced the problem to the Zariski local-triviality of this family of $\mathbb{P}^1$, we use the existence of a universal family of $\mathbb{P}^1$ over H_g (non locally-trivial!) to deduce a condition on the divisibility of the modular map. The result we obtain in section 2 is the following.

Theorem 1.2. *Let V be an irreducible variety of dimension $2g-1$ over an algebraically closed field of characteristic different from 2 and let $\mathcal{F} \to V$ be a family of smooth hyperelliptic curves of genus $g \geq 2$ with maximal variation of moduli. If this family has a rational section then the degree of the modular map $V \to H_g$ (which is generically finite by hypothesis) is a multiple of 2 and this is sharp (namely there exist such families with modular degree exactly 2 for any g).*

Note that this result is valid in any characteristic (different from 2, to avoid problems in the construction of double covers) and the proof is completely algebraic.

The situation is quite different in the *higher gonal* case. In fact for this case it is known (see section 3) that the generic n-gonal curve (with $n \geq 3$) doesn't have any non-trivial automorphism and hence over the moduli space $(M_{g,n-gon})^0$ of n-gonal curves without automorphisms there exists a universal family $\mathcal{C}_{g,n-can}$, namely the restriction of the universal family over M^0_g. Hence to imitate the proof of Caporaso's lemma it remains to determine the relative Picard group of this universal family over the base.

To do this we use a classical construction of Maroni (see [19], [20]) which permits to embed a canonical n-gonal curve inside an $(n-1)$-rational normal scroll. Moreover it is known (see [5]) that for a generic canonical n-gonal curve the rational normal scrolls obtained in this way are all isomorphic and in fact they are the "generic" scrolls, namely the ones that specialize to all the others. Hence if we fix one of such scrolls X and consider the Hilbert scheme Hilb^X_{n-can} of canonical n-gonal curves inside it, then we have a dominant map of Hilb^X_{n-can} to the locus $M_{g,n-can}$ of n-gonal curves. Now using that the fibers of this map are unirational (they are precisely $\mathrm{Aut}(X)^0$) we prove that the relative Picard groups of the two universal families $\mathcal{C}^X_{n-can} \to \mathrm{Hilb}^X_{n-can}$ and $\mathcal{C}_{g,n-gon} \to (M_{g,n-can})^0$ are isomorphic. Moreover it is possible to deduce from this construction that there is an effective divisor on $\mathcal{C}_{g,n-gon}$ representing a relative G^1_n.

Now observe that on the universal family $\mathcal{C}^X_{n-can}$ over Hilb^X_{n-can} there are two natural line bundles induced by cutting with the hyperplane section D and with the fiber f of the ruling, and this two line bundles restrict, on each fiber of the universal family, to the canonical sheaf and the unique g^1_n correspondingly.

So we are naturally lead to the following

Conjecture (1). The relative Picard group of $\mathcal{C}^X_{n-can} \to \mathrm{Hilb}^X_{n-can}$ is generated by D and f.

From what we said before, this conjecture is equivalent to the following

Conjecture (1'). On the universal family $\mathcal{C}_{g,n-gon}$ over $(M_{g,n-gon})^0$ there is a line bundle G^1_n (that restricts to the unique g^1_n on the generic n-gonal curve) such that the relative Picard group $\mathcal{R}(\mathcal{C}_{g,n-gon})$ is generated by G^1_n and the relative canonical sheaf ω.

In particular it would follow from this second conjecture, just imitating the proof of Caporaso's lemma, the following weaker

Conjecture (2). Let V be an irreducible variety of dimension $2g + 2n - 5$ (with $4 \leq 2n - 2 < g$) and let $\mathcal{F} \to V$ be a family of smooth n-gonal curves of genus g with maximal variation of moduli. If this family has a rational section then the degree of the modular map $V \to M_{g,n-gon}$ is a multiple of $\gcd\{n, 2g-2\}$. Moreover this number is sharp, namely there is no other natural number d being a nontrivial multiple of $\gcd\{n, 2g-2\}$ such that for any family with maximal variation of moduli and with a rational section its modular degree should be a multiple of d.

In the last part of this article, we prove conjecture 1 and hence conjecture 2 in the case of *trigonal* curves over an arbitrary algebraically closed field. Unfortunately our argument seems to work only for families of curves lying on a surface (as in the trigonal case). We don't know yet how to attack this problem in general.

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2. Families of hyperelliptic curves.

In this section we work over an algebraically closed field of characteristic different from 2. Recall that the moduli scheme H_g parametrizing isomorphism classes of hyperelliptic curves is an integral subscheme of M_g of dimension $2g - 1$ and can be described as

$$H_g = (\mathrm{Bin}(2, 2g + 2) - \Delta)/PGL(2)$$

where $\mathrm{Bin}(2, 2g+2)$ is the projective space of binary forms in two variables of degree $2g + 2$, Δ is the closed subset over which the discriminant vanishes and $PGL(2)$ acts naturally on $\mathrm{Bin}(2, 2g+2)$ preserving the locus Δ (see [22, Chap. IV, Section 1]).

We want to study families $\mathcal{F} \to V$ of smooth hyperelliptic curves of genus $g \geq 2$ (with V irreducible) such that the modular map $\phi_{\mathcal{F}} : V \to H_g$ is dominant and generically finite, or equivalently families over a base V of dimension $2g - 1$ with maximal variation of moduli. We want a ”sharp” condition on the degree on the modular map assuming the existence of a rational section for our family. It turns out that this problem is very closely related to the following

Problem. Given a family of hyperelliptic curves $\mathcal{F} \to V$, does there exist a line bundle on $\mathcal{F}$ that restricts to the g_2^1 of every fiber? In other words can the g_2^1 be defined globally on a family of hyperelliptic curves?

The last problem is birational on the base, namely if there exists a global g_2^1 defined on an open subset of the base V, then it extends in a unique way to a global g_2^1 on the whole V (see [17, prop. 3.4]). Let's begin with the following

Proposition 2.1. *If a family of smooth hyperelliptic curves $\mathcal{F} \to V$ has a rational section then it has also a globally defined g^1_2.*

Proof. Every family of smooth hyperelliptic curves is a $2:1$ cover of a family $\mathcal{P}$ of $\mathbb{P}^1$ (see [17, theo. 3.1])

$$\begin{tikzcd}
\mathcal{F} \arrow[dd, "f"'] \arrow[dr, "\pi"', "2:1"] & \\
& \mathcal{P} \arrow[dl, "p", "\mathbb{P}^1"'] \\
V &
\end{tikzcd}$$

Now if the family $\mathcal{F} \to V$ has a rational section σ then also the family $\mathcal{P} \to V$ has a rational section given by the composition $\pi \circ \sigma$. This implies that the family of $\mathbb{P}^1$ is Zariski locally trivial (see [17, prop. 2.1]) and hence it has a line bundle $\mathcal{O}_{\mathcal{P}}(1)$ of vertical degree 1. Now the line bundle $\pi^*(\mathcal{O}_{\mathcal{P}}(1))$ is the required globally defined g^1_2.

Explicitly, as a representative of a global g^1_2 we can take the divisor $\overline{\sigma(V)}+\overline{i(\sigma(V))}$, where $\sigma : V \to \mathcal{F}$ is the section, i stays for the involution $i : \mathcal{F} \to \mathcal{F}$ corresponding to $\pi : \mathcal{F} \to \mathcal{P}$, and the bar denotes Zariski closure. □

The converse of this is false, namely to have a globally defined g^1_2 is strictly stronger than having a rational section. Nevertheless the following is true.

Proposition 2.2. *If a family of smooth hyperelliptic curves $\mathcal{F} \to V$ has a globally defined g^1_2 then, up to restricting to an open subset of the base, we can find another family $\mathcal{F}' \to V$ with the same modular map and admitting a rational section.*

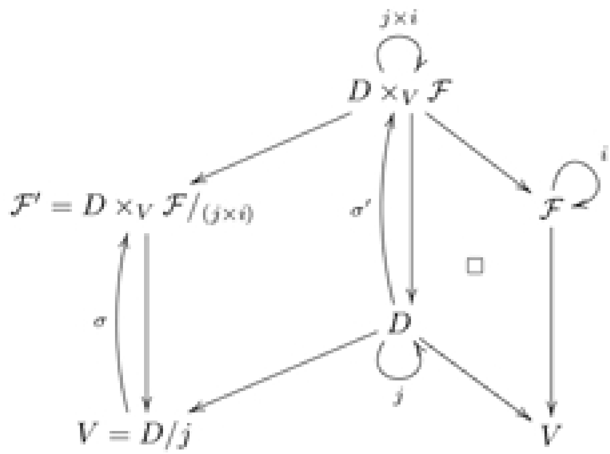
We will give two proofs of this proposition.

Proof. [I Proof] In [17, prop. 3.4] we proved that the existence of a global g^1_2 is equivalent to the Zariski local triviality of the underlying family $\mathcal{P}$ of $\mathbb{P}^1$ and actually the global g^1_2 is the pullback of the line bundle $\mathcal{O}_{\mathcal{P}}(1)$. This shows that we can choose an effective divisor D on $\mathcal{F}$ that represents the global g^1_2. Also, taking D to be the pull-back of a general rational section of $\mathcal{P} \to V$, we can suppose D is not entirely contained in the

Weierstrass divisor (the ramification divisor of the map $\mathcal{F} \to \mathcal{P}$). So, after possibly restricting to an open subset of the base V, we can assume that the projection $D \to V$ is an $2:1$ étale cover (and let's call j the involution on D that exchanges the two sheets). Also on the family $\mathcal{F}$ there is a natural involution i that on every fiber is the hyperelliptic involution. Now consider the diagram

$$\begin{tikzcd}
D \times_V \mathcal{F} \arrow[loop above, "j\times i"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{F} \arrow[loop right, "i"] \arrow[d] \\
D \arrow[u, bend left, "\sigma'"] \arrow[loop below, "j"] \arrow[r] & V
\end{tikzcd}$$

where σ' is the tautological section. Since all the involutions commute with the map, we can form the quotients obtaining



where the tautological section σ', being compatible with the involutions, gives rise to a section σ of the new family $\mathcal{F}' \to V$ that, by construction, has also the same modular map of the original family, q.e.d. $\square$

Proof. [II Proof] This proof is done by passing to the generic point $\eta = Spec(k(V))$ of V. In [17, prop. 2.1 and 3.4] it is shown that the existence of a global g^1_2 is equivalent to the isomorphism $\mathcal{P}_\eta \cong \mathbb{P}^1_\eta$, where $\mathcal{P}$ as before is the family of $\mathbb{P}^1$ underlying $\mathcal{F}$. In this case we showed also that $\mathcal{F}_\eta$ is a hyperelliptic curve over $k(V)$ whose affine part is given in $\mathbb{A}^2$ by an equation of the form $ay^2 = f(x)$ where $a \in k(V)^*/(k(V)^*)^2$ and $f(x)$ is a homogeneous polynomial of degree $2g+2$ whose roots in $\overline{k(V)}$ define a point in H_g corresponding to the image of η under the modular map.

Now the existence of a rational section of the family $\mathcal{F} \to V$ is equivalent to the existence of a rational point of $\mathcal{F}_\eta$ over $k(V)$. But this is achieved very easily by varying our a without modifying the polynomial $f(x)$ (and thus the modular map): for example if we just take $a' := f(x_0) \neq 0$ with $x_0 \in k(V)$ then the new hyperelliptic curve $\mathcal{F}'_\eta$ given by the equation $a'y^2 = f(x)$ will have an evident rational solution $(x, y) = (x_0, 1)$. $\square$

Thus studying families of hyperelliptic curves having a rational section from the point of view of the degree of their modular map is equivalent to the studying of families of hyperelliptic curves with a globally defined g^1_2. The last problem is solved in the last section of [17] as an application of the theory developed there.

Let us briefly review certain results from [17] and describe how do they provide an answer to the problem in question.

RESULTS from [17]

(i) There exists a family $\mathcal{P}_g \to H^0_g$ of $\mathbb{P}^1$ and a horizontal flat and relatively smooth divisor D_{2g+2} of vertical degree $2g+2$ that is universal, in the sense that every other family $\mathcal{P} \to V$ of $\mathbb{P}^1$ endowed with a horizontal flat and relatively smooth divisor D of vertical degree $2g+2$ is the pull-back of this one by a unique map from the base V to H_g. In particular given a family $\mathcal{F} \to V$ of smooth hyperelliptic curves, the underlying family $\mathcal{P}$ of $\mathbb{P}^1$ together with the branch divisor D of the $2:1$ cover $\mathcal{F} \to \mathcal{P}$ fulfills this properties and hence it is the pull-back of the couple $(\mathcal{P}_g, D_{2g+2})$ by mean of the modular map $V \to H^0_g$. Moreover the universal $\mathbb{P}^1$-family $\mathcal{P}_g \to H^0_g$ is not Zariski locally trivial (see [17, theo. 6.5]).

(ii) A family $\mathcal{F} \to V$ of smooth hyperelliptic curves has a globally defined g^1_2 if and only if the underlying family $\mathcal{P} \to V$ of $\mathbb{P}^1$ is Zariski locally trivial (see [17, prop. 3.4]).

(iii) Given a Zariski locally trivial family $\mathcal{P}$ of $\mathbb{P}^1$ over the base V, there exists an open subset $U \subset V$ such that $\mathcal{P}|_U$ corresponds to a family $\mathcal{F} \to U$ of hyperelliptic curves (see [17, theo. 3.5]).

For a discussion of the existence of a global g^1_2 for families of hyperelliptic curves, see also the last section of [21].

Now we can use these results to answer the initial problem.

Theorem 2.3. *If a family of smooth hyperelliptic curves $\mathcal{F} \to V$ with dominant and generically finite modular map has a globally defined g_2^1, then the degree of the modular map is a multiple of* 2 *and this is sharp in the sense that there exist families with that property.*

Proof. Note that, after restricting V to an open subset, we can assume that the image of the modular map is contained in H_g^0. By (ii) above the existence of a global g_2^1 is equivalent to the Zariski local-triviality of the underlying family of $\mathbb{P}^1$. But by (i) the $\mathbb{P}^1$-family $\mathcal{P} \to V$ is the pull-back via the modular (finite) map of the universal family $\mathcal{P}_g \to H_g^0$, and since the last one is not Zariski locally trivial a necessary condition to become trivial is the parity of the degree of the modular map (see the last section of [17]).

On the other hand, there exists a map $V \to H_g^0$ of degree 2 such that the pull-back of the universal family $\mathcal{P}_g$ is Zariski locally trivial. Hence by (iii) this map is modular on a suitable open subset $U \subset V$, having hence a globally defined g_2^1 by (ii), and this concludes the proof. □

Now combining theorem 2.3 with propositions 2.1 and 2.2, we get theorem 1.2.

3. Families of n-gonal curves ($n \geq 3$).

In this section we study the case of higher n-gonal curves. Before doing this we want to recall (for the convenience of the reader) some known classical facts on gonal curves.

Let $g \geq 2$ and $n \geq 2$ be integers. Inside the moduli space M_g of curves of genus g let us denote with $M_{g,n-gon}$ the subset corresponding to curves carrying a g_n^1 (i.e. a linear system of degree n and dimension 1). It is known that:

(1) $M_{g,n-gon}$ is a closed irreducible subvariety of M_g (see [12]).

(2) The dimension of $M_{g,n-gon}$ is $\min\{3g-3, 2n+2g-5\}$. In particular every curve of genus g has a g_n^1 for $2n-2 \geq g$ (see [25]).

(3) The generic n-gonal curve with $n > 2$ doesn't have non-trivial automorphisms; the generic 2-gonal curve (i.e. hyperelliptic curve) has only the hyperelliptic involution as non-trivial automorphism. Hence for $n \geq 3$

there exists a universal family over $(M_{g,n-gon})^0$ (simply the restriction of the universal family over the open subset of M_g of curves without non-trivial automorphisms).

(4) As for the number of g^1_n carried by a generic n-gonal curve, we have that:

(i) If $2n - 2 < g$, then a generic n-gonal curve has only one g^1_n (see [2]).

(ii) If $2n - 2 = g$, then the generic n-gonal curve has only a finite number of g^1_n and this number is equal to $\dfrac{(2n-2)!}{n!(n-1)!}$ (see [16, pag. 359]).

(iii) If $2n - 2 > g$, then the dimension of the space of all the g^1_n is equal to $2n - 2 - g$ (Brill-Noether theory).

As explained in the introduction, we will use a classical construction (due to Maroni [19], [20]) that allows to embed a canonical n-gonal curve (after having chosed a base-point-free g^1_n) in a rational normal scroll. The construction is based on the observation that, by the geometric version of the Riemann-Roch theorem (see [4, pag. 12]), on the canonical curve a g^1_n is given by effective divisors of degree n lying on a $(n-2)$-plane. Thus we obtain a ruling of $(n-2)$-planes parametrized by $\mathbb{P}^1$. It's a classical result of B. Segre (see [25]) that, since $2n - 2 < g$, this ruling is made of non-intersecting planes and so they sweep a non-singular rational normal scroll. Observe also that this construction is canonical for the generic curve, since it has only one g^1_n. From now on, we always assume that $2n - 2 < g$.

Thus we have embedded our n-gonal canonical curve C inside a non-singular rational normal scroll X of dimension $n-1$ inside $\mathbb{P}^{g-1}$ (and hence of degree $g-n+1$). It's well known that the Chow ring of a rational normal scroll is generated by the hyperplane section D and the fiber f of the ruling (see [13, Chap. 3, Sect. 3]):

$$CH(X) = \mathbb{Z}[D, f]/(f^2, D^{n-1} - (g - n + 1)D^{n-2} \cdot f). \tag{3.1}$$

By construction it follows that D cuts on our curve the canonical divisor while f cuts the linear system g^1_n.

Theorem 3.1. *In the construction above the curve C inside X is rationally equivalent to $n \cdot D^{n-2} + (n-2)(n-g+1)D^{n-3} \cdot f$.*

Proof. Since the Chow ring of X is generated by D and f, our curve C is rationally equivalent to $aD^{n-2} + bD^{n-3} \cdot f$. Intersecting with the fiber f

we get

$$n = f \cdot C = f \cdot (aD^{n-2} + bD^{n-3} \cdot f) = a$$

while intersecting with D we get

$$2g - 2 = D \cdot C = D \cdot (aD^{n-2} + bD^{n-3} \cdot f) = a(g - n + 1) + b$$

since $D^{n-1} = \deg(X)D^{n-2} \cdot f = \deg(X)\{pt\}$ and the degree of X is $g - n + 1$. Solving the two equations we get the desired result. $\square$

Now observe that a $(n-1)$-rational normal scroll is isomorphic abstractly to the projectivization of a vector bundle E of rank $n-1$ over $\mathbb{P}^1$ and it's well known that every vector bundle on $\mathbb{P}^1$ splits as direct sum of line bundles and, multiplying by a line bundle, we can normalize it as $E = \bigoplus_{i=1}^{n-1} \mathcal{O}(-r_i)$ with $0 = r_1 \leq \cdots \leq r_{n-1}$. Further a necessary and sufficient condition for $\mathbb{P}(E)$ to be embedded as a non-singular rational normal scroll inside $\mathbb{P}^{g-1}$ (of degree $g - n + 1$) is that, set $N := \sum_{i=1}^{n-1} r_i$, it holds $N < g - n + 1$ and $N \equiv g \pmod{n-1}$. In fact the embedding is the map associated to the very ample divisor $c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \left[\dfrac{g-N}{n-1} - 1\right] f$.

The invariants $0 = r_1 \leq r_2 \leq \cdots \leq r_{n-1}$ we obtain via the Maroni construction are also related to the dimension of the multiples of the g_n^1 we start with (see [24, section 2]). Precisely, if we put $\eta := \dfrac{g-N}{n-1} - 1$, it holds that

$$h^0(C, kg_n^1) = \begin{cases} k+1 & \text{if } 0 \leq k < \eta \\ (j+1)k + 1 - j\eta - \displaystyle\sum_{t=1}^{j} r_t & \text{if } \eta + r_j \leq k < \eta + r_{j+1} \\ & \quad \text{for } j = 1, \cdots, n-2 \\ nk + 1 - g & \text{if } \eta + r_{n-1} \leq k. \end{cases}$$

Note that there is a finite number of isomorphic classes of rational normal scrolls inside $\mathbb{P}^{g-1}$. Hence, since the locus of the n-gonal curves inside M_g is irreducible, the rational normal scrolls canonically associated to the generic n-gonal curves will be isomorphic. The next theorem of Ballico (see [5]) says that a general n-gonal canonical curve lies inside the generic rational normal scroll.

Theorem 3.2 (Ballico). *Let C be a generic n-gonal curve of genus g (with $2n-2<g$) and let g_n^1 be the unique linear system of dimension 1 and degree n. Then*

$$h^0(C,kg_n^1)=\begin{cases} k+1 & \text{if } k<\dfrac{g}{n-1},\\ nk-g+1 & \text{if } k\geq\dfrac{g}{n-1}.\end{cases}$$

Corollary 3.3. *Let r be the integer such that $0\leq r<n-1$ and $r\equiv g \mod n-1$. The rational normal scroll associated via the Maroni construction to the generic n-gonal curve of genus g is abstractely isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{n-1-r}\oplus\mathcal{O}_{\mathbb{P}^1}(-1)^r)$.*

Since all the embeddings inside a projective space are conjugated by a projective automorphism, from now on we can fix a generic rational normal scroll $X_{g-n+1}^{n-1}\subset\mathbb{P}^{g-1}$ (of dimension $n-1$ and degree $g-n+1$). We will consider the locally closed subset of the Hilbert scheme of canonical curves inside X consisting of the curves rationally equivalent to $n\cdot D^{n-2}+(n-2)(n-g+1)D^{n-3}\cdot f$ and we will call it Hilb_{n-can}^X. What the preceding theorem tells us is that the canonical map from Hilb_{n-can}^X to $M_{g,n-gon}$ is dominant. We want to look more closely to the fibers of this map as well as to the variety Hilb_{n-can}^X.

First we need two results about the canonical class of a rational normal scroll and its automorphism group.

Lemma 3.4. *The canonical class of a rational normal scroll $X_{g-n+1}^{n-1}\subset\mathbb{P}^{g-1}$ is $K_X=-(n-1)D+(g-n-1)f$.*

Proof. Recall that X is isomorphic to $\mathbb{P}(E)$ (with E a vector bundle over $\mathbb{P}^1$ of rank $n-1$ and $c_1(E)=-N$) embedded via the map associated to the very ample divisor $D=c_1(\mathcal{O}_{\mathbb{P}(E)}(1))+\left[\dfrac{g-N}{n-1}-1\right]f$. On $\mathbb{P}(E)$ we have the two exact sequences (let's denote with $T_{\mathbb{P}(E)/\mathbb{P}^1}$ the vertical tangent bundle with respect to the fibration π over $\mathbb{P}^1$):

$$0\to\mathcal{O}_{\mathbb{P}(E)}\to\pi^*E\otimes\mathcal{O}_{\mathbb{P}(E)}(1)\to T_{\mathbb{P}(E)/\mathbb{P}^1}\to 0$$

and

$$0\to T_{\mathbb{P}(E)/\mathbb{P}^1}\to T_{\mathbb{P}(E)}\to\pi^*T_{\mathbb{P}^1}\to 0$$

or putting them togheter

$$0\to\mathcal{O}_{\mathbb{P}(E)}\to\pi^*E\otimes\mathcal{O}_{\mathbb{P}(E)}(1)\to T_{\mathbb{P}(E)}\to\pi^*T_{\mathbb{P}^1}\to 0. \tag{3.2}$$

Taking the first Chern classes in the last exact sequence, we get

$$c_1(T_{\mathbb{P}(E)}) = c_1(\pi^* T_{\mathbb{P}^1}) + c_1(\pi^* E \otimes \mathcal{O}_{\mathbb{P}(E)}(1)) =$$
$$\pi^*(c_1(T_{\mathbb{P}^1})) + (n-1)c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \pi^*(c_1(E)) =$$
$$(n-1)c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + (2-N)f.$$

Hence $K_X = -(n-1)c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + (N-2)f$ and substituting $D = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) + \left[\frac{g-N}{n-1} - 1\right] f$ we get the conlusion. □

Proposition 3.5. *Let* $X^{n-1}_{g-n+1} \subset \mathbb{P}^{g-1}$ *be a* generic *rational normal scroll. Then:*

(i) $\mathrm{Aut}(X)$ *has dimension* $n^2 - 2n + 3$.

(ii) $\mathrm{Aut}(X)$ *is connected with the exception of the case when* $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ *in which case it has two connected component (according to whether an automorphism exchanges or not the two components of* $\mathbb{P}^1$*).*

(iii) $\mathrm{Aut}(X)$ *is rational.*

Proof. Let's first suppose that $X \cong \mathbb{P}(E) \ncong \mathbb{P}^1 \times \mathbb{P}^1$ In this case an automorphism of $\mathbb{P}(E)$ must respect the fibration $\pi : \mathbb{P}(E) \to \mathbb{P}^1$ (since the only subvarieties of the scroll isomorphic to $\mathbb{P}^{n-2}$ are the fibers of the map π), so that we have a map $\mathrm{Aut}(\mathbb{P}(E)) \to \mathrm{Aut}(\mathbb{P}^1)$. The kernel of this map is the group $\mathrm{Aut}(\mathbb{P}(E)_{\mathbb{P}^1})$ of *vertical* automorphisms, i.e. automorphisms of the scroll that induce the identity on the base of the fibration $\mathbb{P}(E) \to \mathbb{P}^1$. So we have an exact sequence:

$$0 \to \mathrm{Aut}(\mathbb{P}(E)_{\mathbb{P}^1}) \to \mathrm{Aut}(\mathbb{P}(E)) \to \mathrm{Aut}(\mathbb{P}^1). \tag{3.3}$$

The last map is surjective. In fact given an automorphism ϕ of $\mathbb{P}^1$, from the fact that ϕ doesn't change the linear class of divisors on $\mathbb{P}^1$ and that on $\mathbb{P}^1$ every vector bundle is split, we have that $\phi^*(E) \cong E$. Therefore there exists an isomorphism $\mathbb{P}(\phi^*(E)) \cong \mathbb{P}(E)$ commuting with ϕ on the base.

Moreover, $\mathrm{Aut}(\mathbb{P}(E)_{\mathbb{P}^1}) = \mathbb{P}(\mathrm{Aut}(E))$. In general the subgroup of vertical automorphisms of the projectivized bundle coming from the automorphism of the vector bundle can be identified with the subgroup of the vertical automorphisms that preserve the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ (use the fact that $\pi_*(\mathcal{O}_{\mathbb{P}(E)}(-1)) = E^*$). However using the fact that the base of the fibration is $\mathbb{P}^1$, we can prove that the two groups are the same. In fact an element $\psi \in \mathrm{Aut}(\mathbb{P}(E)_{\mathbb{P}^1})$ should preserve the relative canonical sheaf which is

$$K_{\mathbb{P}(E)/\mathbb{P}^1} = \pi^*(c_1(E)^{-1}) \otimes \mathcal{O}_{\mathbb{P}(E)}(1-n)$$

(since the relative tangent is

$$T_{\mathbb{P}(E)/\mathbb{P}^1} = \mathrm{Hom}(\mathcal{O}_{\mathbb{P}(E)}(-1), \pi^*(E)/\mathcal{O}_{\mathbb{P}(E)}(-1))).$$

Since ψ commutes with π, then $\psi^*(\mathcal{O}_{\mathbb{P}(E)}(1-n)) = \mathcal{O}_{\mathbb{P}(E)}(1-n)$ from which we get the claim since the Picard group of $\mathbb{P}^1$ doesn't contain elements of $(1-n)$-torsion.

There is a cohomological interepretation of these ideas. Consider the sheaf of non-commutative groups $\underline{SL}(E)$ consisting of the automorphisms of E of determinant 1 (indeed, the determinant for automorphisms of vector bundles is well-defined since it doesn't change under the conjugation). Also there is a non-commutative sheaf $\underline{\mathrm{Aut}}(\mathbb{P}(E)_{\mathbb{P}^1})$ and the exact sequence in the étale topology (we really need the étale topology since we have to extract root of degree $1-n$ from the determinant):

$$1 \to \mu_{1-n} \to \underline{SL}(E) \to \underline{\mathrm{Aut}}(\mathbb{P}(E)_{\mathbb{P}^1}) \to 1,$$

where μ_{1-n} denotes the sheaf of roots of unity of degree $1-n$. From the long cohomological sequence we get the inclusion $\mathrm{Coker}(SL(E) \to \mathrm{Aut}(\mathbb{P}(E)_{\mathbb{P}^1})) \subset H^1_{\acute{e}t}(\mathbb{P}^1, \mu_{1-n}) = \mathrm{Pic}(\mathbb{P}^1)_{1-n} = 0$. The last map may be also defined as follows: it associates to $h \in \mathrm{Aut}(\mathbb{P}(E)_{\mathbb{P}^1})$ the invertible sheaf $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}^1)$ such that $h^*\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \mathcal{O}_{\mathbb{P}(E)}(1)^{-1} \cong \pi^*(\mathcal{L})$. As it was discussed above the sheaf $\mathcal{O}_{\mathbb{P}(E)}(1-n)$ must be preserved by h, so $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}^1)_{1-n}$.

Finally let us mention that we could reformulate this fact in a more explicit way by passing from the étale topology to the Zariski one, replacing $\underline{SL}(E)$ by $\underline{GL}(E) = \underline{\mathrm{Aut}}(E)$, μ_{1-n} by $\mathcal{O}^*_{\mathbb{P}^1}$, and using the inclusion $H^1_{\acute{e}t}(\mathbb{P}^1, \mu_{1-n}) = \mathrm{Pic}(\mathbb{P}^1)_{1-n} \subset \mathrm{Pic}(\mathbb{P}^1) = H^1_{Zar}(\mathbb{P}^1, \mathcal{O}^*_{\mathbb{P}^1})$. Namely let's take $h \in \mathrm{Aut}(\mathbb{P}(E)_{\mathbb{P}^1})$. Consider a Zariski open covering of the base by the open subsets U_i over which E is trivial, and over which there exist elements $g_i \in \mathrm{Aut}(E|_{U_i})$ coinciding with h on $\mathbb{P}(E|_{U_i})$. Let $A_{ij} \in GL(U_i \cap U_j)$ be the transition functions for E. Then on the intersection $U_i \cap U_j$ we have the equality

$$\lambda_{ij} g_i = A_{ij} g_j A_{ij}^{-1}$$

for some $\lambda_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. As for determinants we get the equality $\lambda_{ij}^{1-n} \det(g_i) = \det(g_j)$ so the cocyle λ_{ij}^{1-n} is trivial, while the cocycle λ_{ij} defines an element in $\mathrm{Pic}(\mathbb{P}^1)_{1-n}$ whose triviality implies the existence of $g \in \mathrm{Aut}(E)$ whose action on $\mathbb{P}(E)$ coincides with h.

To compute the dimension of $\mathrm{Aut}(\mathbb{P}(E))$, note that $\mathbb{P}(\mathrm{Aut}(E))$ is the open subset of $\mathbb{P}(\mathrm{End}(E)) = \mathbb{P}(H^0(E \otimes E^*))$ over which the determinant doesn't vanish and therefore, since $E = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{n-1-r} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^r)$, we have

$$\dim \mathrm{Aut}(\mathbb{P}(E)_{\mathbb{P}^1}) = h^0(\mathbb{P}^1, E \otimes E^*) - 1 = (n-1)^2 - 1$$

from which one get part (i) using the exact sequence 3.3.

The connectedeness and the rationality of $\mathrm{Aut}(\mathbb{P}(E))$ follows from the exact sequence 3.3 since both the first and the last group is connected and rational.

Finally, in the case where $X \cong \mathbb{P}(E) \cong \mathbb{P}^1 \times \mathbb{P}^1$, an automorphism may also exchange the two fibrations of $\mathbb{P}^1$, so that we have an exact sequence:

$$0 \to \mathrm{Aut}(\mathbb{P}(E) \to \mathbb{P}^1) \to \mathrm{Aut}(\mathbb{P}(E)) \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

and for $\mathrm{Aut}(\mathbb{P}(E) \to \mathbb{P}^1)$ we can repeat the same argument of above which will conclude our proof. $\square$

Theorem 3.6. *For a generic rational normal scroll $X^{n-1} \subset \mathbb{P}^{g-1}$, the scheme Hilb^X_{n-can} is smooth and irreducible of dimension $2g + n^2 - 2$ and the natural (dominant) map $\mathrm{Hilb}^X_{n-can} \to M_{g,n-gon}$ has generic fiber isomorphic to the algebraic subgroup $\mathrm{Aut}(X)^0$ (the connected component of the unity inside $\mathrm{Aut}(X)$).*

Proof. From the theory of deformations, we known that the tangent space to the semiversal space for the embedded deformations of a curve C inside X has dimension $h^0(C, N_{C/X})$ while the space of obstructions sits inside $H^1(C, N_{C/X})$. Hence at the point $[C] \in \mathrm{Hilb}^X_{n-can}$, it holds:

$$\begin{aligned} h^0(C, N_{C/X}) - h^1(C, N_{C/X}) &\leq \dim_{[C]}(\mathrm{Hilb}^X_{n-can}) \\ &\leq \dim T_{[C]}(\mathrm{Hilb}^X_{n-can}) \leq h^0(C, N_{C/X}). \end{aligned}$$

We will prove that for every $[C] \in \mathrm{Hilb}^X_{n-can}$, it holds:

$$\chi(C, N_{C/X}) = 2g + n^2 - 2 \tag{3.4}$$

$$h^1(C, N_{C/X}) = 0 \tag{3.5}$$

from which it follows that Hilb^X_{n-can} is smooth of dimension $2g + n^2 - 2$.

From the exact sequence

$$0 \to T_C \to T_{X|C} \to N_{C/X} \to 0$$

it follows that

$$\chi(C, N_{C/X}) = \chi(C, T_{X|C}) - \chi(C, T_C) = \chi(C, T_X|C) - (3 - 3g).$$

To compute $\chi(T_{X|C})$, we apply the Riemann-Roch theorem for fiber bundles, using the fact that, in the Chow ring of X, the class of C is $nD^{n-2}+(n-2)(n-g+1)D^{n-3}\cdot f$ (by lemma 3.1) and the canonical class of X is $-(n-1)D+(g-n-1)f$ (by lemma 3.4):

$$\begin{aligned}\chi(C,T_{X|C}) &= \deg(c_1(T_{X|C})) + \mathrm{rk}(T_{X|C})\cdot(1-g) = \\ &= -K_X\cdot C + (n-1)(1-g) = n^2+1-g\end{aligned} \tag{3.6}$$

from which it follows formula 3.4.

Now consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & T_C \arrow[r] \arrow[d] & T_{X|C} \arrow[r] \arrow[d] & N_{C/X} \arrow[r] & 0 \\ & p^*T_{\mathbb{P}^1} \arrow[r, "id"] \arrow[d] & (\pi^*T_{\mathbb{P}^1})_{|C} \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd}$$

where $\pi : X \to \mathbb{P}^1$ indicates the projection of the scroll X onto $\mathbb{P}^1$ (or in in other words the map associated to the divisor f), $p : C \to \mathbb{P}^1$ denotes its restriction to C (which is therefore the map associated to the unique g^1_n of C) and the vertical maps are the differential maps of these morphisms.

Passing to the cohomological exact sequences, the vanishing of $H^1(C, N_{C/X})$ will follow once we prove that

$$H^1(C, T_{X|C}) \xrightarrow{\cong} H^1(C, p^*T_{\mathbb{P}^1}).$$

We will prove the last isomorphism by showing that both these groups have the same dimension $g-2n+2$ (since we know that the map between them is surjective).

In fact since $2n-2 < g$ and C is a general n-gonal curve, from theorem 3.2 it follows:

$$\begin{aligned}h^1(C, p^*T_{\mathbb{P}^1}) &= h^1(C, 2g^1_n) = h^0(C, 2g^1_n) - \chi(C, 2g^1_n) \\ &= 3-(2n+1-g) = g-2n+2.\end{aligned}$$

On the other hand it's easy to see that the automorphisms of X come from projectivity of $\mathbb{P}^{g-1}$ and the ones that fix the canonical curve C are a finite number because Aut(C) is finite. Hence by proposition 3.5 (since X is generic)

$$h^0(C, T_{X|C}) = h^0(X, T_X) = \dim\mathrm{Aut}(T_X) = n^2-2n+3,$$

which togheter with formula 3.6 gives $h^1(C, T_{X|C}) = g - 2n + 2$.

Now two generic curves C and C' in Hilb^X_{n-can} are isomorphic if and only if there exists a projectivity of $\mathbb{P}^{g-1}$ sending one into the other. But clearly, since X is canonically attached to the generic C, this projectivity must stabilize X and hence is an automorphism of X that will preserve the rational class of C and hence, in view of proposition 3.5, belongs to $\mathrm{Aut}(X)^0$.

The connectedeness of Hilb^X_{n-can} (and hence its irreducibility because of its smoothness) follows from the fact that it has a dominant map into a connected variety with connected fibers. □

So after this construction, we end up with the following situation

$$\begin{tikzcd} \mathcal{C}^X_{n-can} \arrow[r] \arrow[d] & \mathcal{C}_{g,n-gon} \arrow[d] \\ \mathrm{Hilb}^X_{n-can} \arrow[r, "\phi"'] & M_{g,n-gon} \end{tikzcd}$$

where $\mathcal{C}^X_{n-can} \to \mathrm{Hilb}^X_{n-can}$ is the universal family over the Hilbert scheme, $\mathcal{C}_{g,n-gon} \to M_{g,n-gon}$ is the tautological family over the locus of n-gonal curves (universal over the open subset of curves without automorphisms) and we know that the canonical map ϕ is dominant with generic fiber isomorphic to $\mathrm{Aut}(X)^0$. We want to compute the relative Picard group (i.e. line bundles of the family modulo pull-back of line bundles of the base) of these two families of curves. This is called classically the group of *rationally determined line bundles* (the terminology is due to the fact that it doesn't change if we restrict the family to an open subset of the base, [9, prop.2.2]).

In [9] there are many properties of this group and in particular we found there a very usefull result that allows to compare the groups of rationally determined line bundles for two families between which there is a correspondence which is "generically unirational".

More precisely, let $\mathcal{F} = (\mathcal{C}, S, p)$ and $\mathcal{F}' = (\mathcal{C}', S', p')$ be two families of schemes over an irreducible and smooth base and let's denote with $\mathcal{R}(\mathcal{F})$ and $\mathcal{R}(\mathcal{F}')$ the two relative Picard groups. Let $T \subset S \times S'$ be an algebraic correspondence between S and S' such that the two projections $\pi : T \to S$ and $\pi' : T \to S'$ are dominant. For any point $x \in S$ we will denote by S'_x the closed subset of S' associated to x by this correspondence, namely $S'_x = \pi^{'}(\pi^{-1}(x))$. We use analogous notation for $S_{x'}$ if $x' \in S'$.

Theorem 3.7. [see [9]] *If, for every x ranging in an open dense subset of S, S'_x is unirational then there is a natural monomorphism of groups $\mathcal{R}(\mathcal{F}') \hookrightarrow \mathcal{R}(\mathcal{F})$. If the same hypothesis is true for $S_{x'}$, then we have a natural isomorphism $\mathcal{R}(\mathcal{F}') \cong \mathcal{R}(\mathcal{F})$.*

The hypothesis of the theorem are valid in our case since ϕ : $\mathrm{Hilb}^X_{n-can} \to M_{g,n-gon}$ is dominant between irreducible varieties with generic rational fiber $\mathrm{Aut}(X)^0$, and it gives the isomorphism between the relative Picard groups of our two families.

Now observe that on the universal family $\mathcal{C}^X_{n-can}$ over Hilb^X_{n-can} there are two natural line bundles induced by cutting with the divisors D and f (we will use the same letters for these two line bundles). Since over each fiber of the universal family, D restricts to the canonical sheaf while f restricts to the unique g^1_n, these two line bundles correspond to the relative canonical line bundle (defined everywhere and in a canonical way) and to a globally defined g^1_n (which is well defined only as an element of the relative Picard group). Now we make the following

Conjecture (1). The relative Picard group of $\mathcal{C}^X_{n-can} \to \mathrm{Hilb}^X_{n-can}$ is generated by D and f.

From what we said before, this conjecture is equivalent to the following

Conjecture (1'). On the universal family $\mathcal{C}_{g,n-gon}$ over $(M_{g,n-gon})^0$ there is a line bundle G^1_n (that restricts to the unique g^1_n on the generic n-gonal curve) such that the relative Picard group $\mathcal{R}(\mathcal{C}_{g,n-gon})$ is generated by G^1_n and the relative canonical sheaf ω.

Corollary 3.8. *Any line bundle on the universal family $\mathcal{C}^1_{g,n}$ has relative degree a multiple of* $\gcd\{2g-2, n\}$.

Now imitating words for words the proof of Caporaso's lemma, one proves that this conjecture imply the following answer to the original problem.

Conjecture (2). Let V be an irreducible variety of dimension $2g + 2n - 5$ (with $4 \leq 2n-2 < g$) and let $\mathcal{F} \to V$ be a family of smooth n-gonal curves of genus g with maximal variation of moduli. If this family has a rational section then the degree of the modular map $V \to M_{g,n-gon}$ is a multiple of $\gcd\{n, 2g-2\}$. Moreover this number is sharp, namely there is no other natural number d being a nontrivial multiple of $\gcd\{n, 2g-2\}$ such that for any family with maximal variation of moduli and with a rational section its modular degree should be a multiple of d.

The sharpness may be obtained as follows. On $\mathcal{C}^X_{n-can}$ there exists an effective divisor representing the sheaf f: just fix any fiber on the scroll X and take its intersection with corresponding curves on X. This divisor on $\mathcal{C}^X_{n-can}$ is $\mathrm{Aut}(X)^0$-equivariant and so we get an effective divisor on $\mathcal{C}_{g,n-gon}$ being a section of G^1_n. Thus we get a family with maximal variation of moduli, with a rational section and whose modular degree is equal to n. The sharpness easily follows from this and from the existence of an effective relative canonical divisor on $\mathcal{C}_{g,n-gon}$.

Now we are going to prove conjecture (1) for *trigonal* curves over an arbitrary algebraically closed field.

Theorem 3.9. *Conjecture (1) (and hence (1') and (2)) is true for* $n = 3$.

To prove this theorem first we will establish a certain rather general statement.

Consider a smooth projective surface S, and let L be a linear system of divisors on S. Let's say that the system L is *rather free* if and only if for a generic curve C from the linear system L and for each point $x \in C$ we have an equality $\dim((L|_C)(-x)) = \dim(L|_C) - 1$. Here by the restriction $L|_C$ we mean the image of L under the natural map $H^0(S, \mathcal{O}_S(C)) \to H^0(C, \mathcal{O}_C(C))$.

Example If $L = H^0(S, \mathcal{L})$ where $\mathcal{L}$ is a very ample sheaf then evidently L is rather free.

Example If $H^1(S, \mathcal{O}_S) = 0$, $L = H^0(S, \mathcal{L})$ and $(\mathcal{L}.\omega_S) \leq -2$ where $\mathcal{L}$ is an invertible sheaf on S and ω_S denotes a canonical sheaf, then L is rather free.

To show this fact first consider the exact sequence of sheaves

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0,$$

associated to a general curve C from L. Together with the condition $H^1(S, \mathcal{O}_S) = 0$ it leads to the equality $L|_C = H^0(C, \mathcal{O}_C(C)) = H^0(C, \mathcal{L}|_C)$ and so $(L|_C)(-x) = H^0(C, \mathcal{L}|_C(-x))$. Now the property of

L to be rather free comes from the inequality $\mathrm{deg_C}(\mathcal{L}|_\mathrm{C}(-\mathrm{x})) > 2\mathrm{g(C)}-2$. Indeed

$$2g(C) - 2 = (\omega_X \otimes \mathcal{L}.\mathcal{L}) < (\mathcal{L}.\mathcal{L}) - 1,$$

and

$$(\mathcal{L}.\mathcal{L}) - 1 = \mathrm{deg_C}(\mathcal{L}|_\mathrm{C}(-\mathrm{x}))).$$

Now consider the universal curve $\mathcal{C}$ inside $\mathbb{P}(L) \times S = \mathbb{P} \times S$. Let's denote by π_1 and π_2 two natural projections to $\mathbb{P}$ and S respectively.

Theorem 3.10. *If L is rather free then the map*

$$\pi_2^*\colon \mathrm{Pic}(S) \to \mathrm{Pic}(\mathcal{C})/\pi_1^*(\mathrm{Pic}(\mathbb{P}))$$

is surjective.

Proof. Step 1. Fix a generic point $p \in \mathbb{P}$ and consider the blow-up $\sigma(\mathbb{P})$ of $\mathbb{P}$ with the center at p. Let π be the natural contraction map from $\sigma(\mathbb{P})$ to $\mathbb{P}$. Then $\pi^*(\mathcal{C}) \to \mathcal{C}$ is also a blow-up with the center at the curve C that is a fiber of $\mathcal{C}$ over p. By a general result (see [9]) the relative Picard group doesn't change so we may concentrate on the new family $\pi^*(\mathcal{C}) \to \sigma(\mathbb{P})$. Indeed if the total space of the family is regular then the relative Picard group of a family is invariant when changing the base by its open subset: the isomorphism between two relative Picard groups is obtained in one direction just by the restriction and in the other direction by taking the Zariski closure (here we use the regularity of the total space).

Step 2. We have the embedding $\pi^*(\mathcal{C}) \subset \sigma(\mathbb{P}) \times S$. Let f denote the composition of this embedding with the natural map $\sigma(\mathbb{P}) \times S \to \mathbb{P}^{N-1} \times S$ where $N+1 = \dim(L)$ and $\mathbb{P}^{N-1}$ corresponds to lines in $\mathbb{P}$ passing through p. Then $f\colon \pi^*(\mathcal{C}) \to \mathbb{P}^{N-1} \times S$ is the blow-up with the center at the subvariety $R \subset \mathbb{P}^{N-1} \times S$ of codimension 2 that may be defined in the following way: a point $(l, x) \in \mathbb{P}^{N-1} \times S$ belongs to R if and only if $x \in C \cap C_q$ where C_q stays for the fiber of $\mathcal{C}$ over a generic point $q \in l$. Let's remark that indeed the set $C \cap C_q$ doesn't depend on $q \in l$ because obviously all the curves in the pencil l are passing through $(C.C)$ points on C obtained by intersecting it with one of the element of this pencil.

Hence $\mathrm{Pic}(\pi^*(\mathcal{C}))$ is generated by $\mathrm{Pic}(\mathbb{P}^{N-1} \times S) = \mathrm{Pic}(\mathbb{P}^{N-1}) \times \mathrm{Pic}(S)$ and the irreducible components of $f^{-1}(R)$. Besides, the image of the generator of $\mathrm{Pic}(\mathbb{P}^{N-1})$ in $\mathrm{Pic}(\pi^*(\mathcal{C}))$ is coming from the base $\sigma(\mathbb{P})$.

Step 3. We claim that R is irreducible. There is an embedding $R \subset \mathbb{P}^{N-1} \times C$. Also we may identify $\mathbb{P}^{N-1}$ with $\mathbb{P}(L|_C)$. Hence a fiber of R

over a point $x \in C$ is exactly $\mathbb{P}(L|_C(-x))$. Since L is rather free, R is just the projectivization of the vector bundle of rank $N-1$ over C whose fiber over $x \in C$ is the vector space $(L|_C)(-x)$ (which is of dimension $N-1$ by hypothesis).

Moreover we may easily compute the class of $f^{-1}(R)$ in the relative Picard group: on the restriction of the family $\pi^*(\mathcal{C})$ over the open subset $\sigma(\mathbb{P})-\mathbb{P}^{N-1}$ the divisor $f^{-1}(R)$ coincides with the pull-back of $\mathbb{P}^{N-1}\times C \subset \mathbb{P}^{N-1} \times S$ so comes from $\mathrm{Pic}(S)$.

Thus we get the initial statement. □

Remark The proof of this theorem is very much inspired by the proof of Theorem 4.2 from [9], essentially by the idea of considering pencils of curves inside a family to obtain the information about the relative Picard group (in [9] this method is said to have been already known to Enriques and Chisini).

Now let's come back to our initial problem and take $S = X$, $L = H^0(X, 3D+(4-g)f)$, where X is a scroll corresponding to trigonal curves of genus g. Since $\omega_S = -2D + (g-4)f$, $f^2 = 0$, $(D.f) = 1$ and $D^2 = g-2$ we have the equality $(\omega_S.(3D+(4-g)f)) = -g-8$. Also $H^1(X, \mathcal{O}_X) = 0$, and so L is rather free by the example before theorem 3.10.

Another way to see that L is rather free is to show that $3D+(4-g)f$ is very ample on X. In fact in the case g even, X will be isomorphic to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with $D = C_0 + \dfrac{g-2}{2} f$ (in the notation of [15b, V, section 2]). Hence $3D+(4-g)f = 3C_0 + \dfrac{g-2}{2} f$ is very ample by [15b, V, cor. 2.18]. In the case g odd, X will be isomorphic to $\mathbb{F}_1 = \mathbb{P}(\mathcal{O} \otimes \mathcal{O}(-1))$ with $D = C_0 + \dfrac{g-1}{2} f$. Hence $3D+(4-g)f = 3C_0 + \dfrac{g+5}{2} f$ is very ample by the same corollary.

Next note that Hilb^X_{n-can} is a Zariski open subset inside $\mathbb{P}(L)$ because the Chow groups of scrolls are discrete: rational equivalency and algebraic equivalency coincide with each other (in the case of divisors it is again the reflection of the fact that $H^1(X, \mathcal{O}_X) = 0$). Moreover $\mathcal{C}^X_{n-can}$ is the restriction of $\mathcal{C}$ to Hilb^X_{n-can}. So applying theorem 3.10 and using the birational invariancy of the relative Picard group we get conjecture 1 for the trigonal case.

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