

**ON CHERN RATIOS FOR SURFACES
WITH AMPLE COTANGENT BUNDLE**

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In this paper we study the problem of density in $(1, 3]$ for the Chern ratio of surfaces with ample cotangent bundle. In particular we prove density in $(1, 2)$ by constructing a family of complete intersection surfaces in a product of varieties with big cotangent bundle. We also analyse the case of complete intersections in a product of curves of genus at least 2.

1. Introduction.

We work over an algebraically closed field.

The Chern numbers of a surface of general type satisfy the well-known inequality $c_1^2 \leq 3c_2$ due to Miyaoka (see [6]). Mathematicians tried to understand what values the ratio c_1^2/c_2 can assume. Sommese in [8] found an answer for this problem by showing that the set of these ratios for minimal surfaces of general type is dense in the interval $[1/5, 3]$. For his proof, he uses covers of a family of surfaces constructed by Hirzebruch in [4].

In this paper we are interested in the analogous question for surfaces with ample cotangent bundle. We need to recall some definitions. Given

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a vector bundle E on a variety X , we define, following Grothendieck, $\mathbb{P}(E) \rightarrow X$ to be the projective bundle of *hyperplanes* in the fibers of E . It is equipped with a line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$.

Definition 1.1. A vector bundle E on a variety X is

- *ample* if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample;
- *big* if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is big.

Varieties with ample cotangent bundle are of general type, but they have much stronger properties: all their subvarieties are of general type and, over the complex numbers, they are analytically hyperbolic (any holomorphic map from \mathbb{C} to such a variety is constant). They are related to the following well-known conjecture.

Conjecture 1.2. (Lang). *A smooth complex projective variety is analytically hyperbolic if and only if all its subvarieties are of general type.*

Surfaces with ample cotangent bundle are minimal surfaces of general type, and they satisfy (see [3]) the supplementary inequality $c_1^2 > c_2$. So the set of possible Chern ratios is now restricted to $(1, 3]$.

As noted by Spurr in [9], the surfaces constructed by Sommese do not have, in general, this property of ampleness which is usually quite difficult to check. In section 4 we prove the following result.

Theorem 1.3. *The set of ratios c_1^2/c_2 for surfaces with ample cotangent bundle is dense in the interval $(1, 2)$. In other words, any number in $(1, 2)$ is a limit of Chern ratios of surfaces with ample cotangent bundle.*

The proof is based on the following theorem, to be proved in section 3, that generalizes a result of Bogomolov.

Theorem 1.4. *Let X_1, \dots, X_m be smooth projective varieties with big cotangent bundle, all of dimension at least $d > 0$. Let Y be a general complete intersection in $X_1 \times \dots \times X_m$. If $\dim Y \leq \frac{d(m+1)+1}{2(d+1)}$, the cotangent bundle of Y is ample.*

In section 5 we explicitly compute the Chern ratio in the special case of complete intersections in a product of curves.

2. Notation and conventions.

For all finite sequences of integers $\mathbf{d} = (d_1, \dots, d_c)$, we set $|\mathbf{d}| = \sum_{i=1}^c d_i$. If V_{d_1}, \dots, V_{d_c} are varieties, we denote by $V_{\mathbf{d}}$ the product $V_{d_1} \times \dots \times V_{d_c}$.

We use \sim for numerical equivalence of divisors, and \cdot for the intersection product of cycles on a variety.

We denote by $c_j(\mathcal{F})$ the j -th Chern class of a vector bundle \mathcal{F} . For any smooth variety V , the class $c_j(V)$ is the j -th Chern class of the tangent bundle \mathcal{T}_V of V .

3. A generalization of Bogomolov’s theorem.

We first prove a generalization of the following theorem of Bogomolov ([2], Proposition 23).

Theorem 3.1 (Bogomolov). *Let X_1, \dots, X_m be smooth projective varieties with big cotangent bundle, all of dimension at least $d > 0$. Let V be a general linear section of $X_1 \times \dots \times X_m$. If $\dim V \leq \frac{d(m+1)+1}{2(d+1)}$, the cotangent bundle of V is ample.*

In order to generalize this result from general linear section to general complete intersections we need the following.

Lemma 3.2. *Let X be a smooth subvariety of a projective space and let B be a subvariety of $\mathbb{P}(\Omega_X)$. A general complete intersection Y in X of dimension at most $\frac{1}{2}\text{codim}B$ satisfies*

$$\mathbb{P}(\Omega_Y) \cap B = \emptyset.$$

Proof. Let \mathbb{P}^n be the ambient projective space and let $V_r = \mathbb{P}^{\binom{n+r}{r}}$ be the Veronese variety that parametrizes hypersurfaces of degree r in \mathbb{P}^n . Let $\mathbf{d} = (d_1, \dots, d_c)$ be a sequence of positive integers. Now consider the variety

$$W := \{(t, x), Y_1, \dots, Y_c\} \in B \times V_{\mathbf{d}} \mid x \in Y, t \in T_{X,x} \cap T_{Y_1,x} \cap \dots \cap T_{Y_c,x}\}$$

where $Y = X \cap Y_1 \cap \dots \cap Y_c$. We have two projections $W \rightarrow B$ and $W \rightarrow V_{\mathbf{d}}$. The fibers of the first projection have codimension $2c$, so the

second projection is not dominant when $2c > \dim B$. But $c = \text{codim}_X Y = \dim X - \dim Y$, so the last condition is equivalent to

$$\begin{aligned} 2(\dim X - \dim Y) - 1 &\geq \dim B = \dim \mathbb{P}(\Omega_X) - \text{codim} B \\ &= 2 \dim X - 1 - \text{codim} B \end{aligned}$$

which proves the lemma. \square

Analogously we can prove the following.

Lemma 3.3. *Let Y be a general complete intersection in a product $X_1 \times X_2$ in a projective space. If $2 \dim Y \leq \dim X_1 + 1$, the projection $Y \rightarrow X_1$ is finite.*

Proof. We consider in the ambient space \mathbb{P}^n the closure of the locus

$$\begin{aligned} \{(x_1, x_2, x'_2, Y_1, \dots, Y_c) \in X_1 \times X_2 \times X_2 \times V_{\mathbf{d}} \mid x_2 \neq \\ \neq x'_2, (x_1, x_2), (x_1, x'_2) \in Y_1 \cap \dots \cap Y_c\} \end{aligned}$$

This variety has a natural projection to $X_1 \times X_2 \times X_2$ whose fibers have codimension $2c$. As in the previous lemma the projection to $V_{\mathbf{d}}$ has fibers of dimension at most 1 if the condition $2c \geq \dim(X_1 \times X_2 \times X_2) - 1$ is satisfied. When $c = \text{codim} Y$, this amounts to saying that the projection $Y \rightarrow X_1$ is finite and that

$$2 \dim(X_1 \times X_2) - 2 \dim Y \geq \dim(X_1 \times X_2 \times X_2) - 1$$

which is equivalent to

$$\dim Y \leq \dim X_1 + 1$$

as wanted. \square

We now obtain the desired statement.

Theorem 3.4. *Let X_1, \dots, X_m be smooth projective varieties with big cotangent bundle, all of dimension at least $d > 0$. Let Y be a general complete intersection in $X_1 \times \dots \times X_m$. If $\dim Y \leq \frac{d(m+1)+1}{2(d+1)}$, the cotangent bundle of Y is ample.*

Proof. The proof goes as in Bogomolov's theorem (see [2]). \square

Remark. Of course, in case the embedding of $X_1 \times \dots \times X_m$ in a projective space comes from embeddings of each X_i in a projective space followed by a Segre embedding, this theorem is just a consequence of Bogomolov's theorem 3.1. \square

4. Computation of Chern numbers.

We compute the Chern numbers of a complete intersection surface in a product.

Proposition 4.1. *Let X_1, \dots, X_m be smooth projective varieties with big cotangent bundle, all of dimension at least $d > 0$, such that $2 \leq \frac{d(m+1)+1}{2(d+1)}$. Let N be the dimension of $X = X_1 \times \dots \times X_m$. Embed X in a projective space \mathbb{P}^n . Let S be a surface, general complete intersection of degree $\mathbf{d} = (d_1, \dots, d_{N-2})$ in X . We have*

$$c_1^2(S) = c_1^2(X)|_S - 2|\mathbf{d}|c_1(X)|_S \cdot H|_S + |\mathbf{d}|^2 H^2|_S$$

$$c_2(S) = c_2(X)|_S - |\mathbf{d}|c_1(X)|_S \cdot H|_S + \left(\sum_{i \leq j} d_i d_j \right) H^2|_S$$

where H is a hyperplane in \mathbb{P}^n .

Proof. According to Theorem 3.4, the cotangent bundle of S is ample. The surface S is the intersection of X with hypersurfaces L_1, \dots, L_{N-2} of degrees d_1, \dots, d_{N-2} in \mathbb{P}^n . We want to compute the ratio c_1^2/c_2 for S . Since S is smooth, we have a short exact sequence of sheaves (see for example [5, p.182])

$$0 \longrightarrow \mathcal{T}_S \longrightarrow (\mathcal{T}_X)|_S \longrightarrow \mathcal{N}_{S/X} \longrightarrow 0$$

and we also know that, for a complete intersection,

$$\mathcal{N}_{S/X} \simeq (\mathcal{I}_S/\mathcal{I}_S^2)^* \simeq \bigoplus_{i=1}^{N-2} \mathcal{O}_S(L_i)$$

where \mathcal{I}_S is the ideal sheaf of S in X . So, for the Chern classes,

$$c(\mathcal{T}_X)|_S = c(\mathcal{T}_S) \cdot c(\mathcal{N}_{S/X})$$

$$c(\mathcal{N}_{S/X}) = c\left(\bigoplus_{i=1}^{N-2} \mathcal{O}_S(L_i)\right) = \prod_{i=1}^{N-2} (1 + c_1(\mathcal{O}_S(L_i)))$$

Let us compute $c_1^2(S)$:

$$c_1(\mathcal{T}_X)|_S = c_1(\mathcal{T}_S) + \sum_{i=1}^{N-2} c_1(\mathcal{O}_S(L_i))$$

which implies, with $c_1(S) = c_1(\mathcal{T}_S)$ and $c_1(X) = c_1(\mathcal{T}_X)$,

$$c_1(S) = c_1(X)|_S - \sum_{i=1}^{N-2} c_1(\mathcal{O}_S(L_i))$$

Now we remember that, if H is a hyperplane in \mathbb{P}^n , we have for all $i \in \{1, \dots, N-2\}$, $L_i \sim d_i H$, so that

$$c_1(S) = c_1(X)|_S - |\mathbf{d}|H|_S$$

and

$$\begin{aligned} c_1^2(S) &= (c_1(X) - |\mathbf{d}|H|_X)|_S^2 \\ &= c_1^2(X)|_S - 2|\mathbf{d}|c_1(X)|_S \cdot H|_S + |\mathbf{d}|^2 H^2|_S \end{aligned}$$

Let us compute $c_2(S)$:

$$c_2(\mathcal{T}_X)|_S = c_2(\mathcal{T}_S) + c_1(\mathcal{T}_S) \cdot \sum_{i=1}^{N-2} c_1(\mathcal{O}_S(L_i)) + \sum_{i<j} c_1(\mathcal{O}_S(L_i)) \cdot c_1(\mathcal{O}_S(L_j))$$

so that

$$\begin{aligned} c_2(S) &= c_2(\mathcal{T}_X)|_S - c_1(\mathcal{T}_S) \cdot \sum_{i=1}^{N-2} c_1(\mathcal{O}_S(L_i)) - \sum_{i<j} c_1(\mathcal{O}_S(L_i)) \cdot c_1(\mathcal{O}_S(L_j)) \\ &= c_2(X)|_S - c_1(S) \cdot \sum_{i=1}^{N-2} L_i|_S - \sum_{i<j} (L_i \cdot L_j)|_S \\ &= \left(c_2(X) - (c_1(X) - |\mathbf{d}|H|_X) \cdot \sum_{i=1}^{N-2} L_i|_X - \sum_{i<j} (L_i \cdot L_j)|_X \right)|_S \\ &= \left(c_2(X) - |\mathbf{d}|(c_1(X) - |\mathbf{d}|H|_X) \cdot H|_X - \sum_{i<j} d_i d_j H^2|_X \right)|_S \end{aligned}$$

So, if we put

$$(1) \quad a := c_1(X) \cdot H^{N-1}|_X \quad \text{and} \quad b := H^N|_X$$

we can write

$$c_1^2(S) = \left(\prod_{i=1}^{N-2} d_i \right) (c_1^2(X) \cdot H^{N-2}|_X - 2a|\mathbf{d}| + b|\mathbf{d}|^2)$$

and

$$\begin{aligned} c_2(S) &= \left(\prod_{i=1}^{N-2} d_i \right) \left(c_2(X) \cdot H^{N-2}|_X - a|\mathbf{d}| + b \left(|\mathbf{d}|^2 - \sum_{i < j} d_i d_j \right) \right) \\ &= \left(\prod_{i=1}^{N-2} d_i \right) \left(c_2(X) \cdot H^{N-2}|_X - a|\mathbf{d}| + b \sum_{i \leq j} d_i d_j \right). \quad \square \end{aligned}$$

Assume moreover that the d_i are multiple of the same integer d . So $d_i = e_i d$ for all $i \in \{1, \dots, N - 2\}$. The ratio is

$$\begin{aligned} (2) \quad \frac{c_1^2(S)}{c_2(S)} &= \frac{c_1^2(X) \cdot H^{N-2}|_X - 2a|\mathbf{d}| + b|\mathbf{d}|^2}{c_2(X) \cdot H^{N-2}|_X - a|\mathbf{d}| + b \sum_{i \leq j} d_i d_j} \\ &= \frac{c_1^2(X) \cdot H^{N-2}|_X - 2ad|\mathbf{e}| + bd^2|\mathbf{e}|^2}{c_2(X) \cdot H^{N-2}|_X - ad|\mathbf{e}| + bd^2 \sum_{i \leq j} e_i e_j} \end{aligned}$$

Letting d go to infinity, we obtain

$$\lim_{d \rightarrow +\infty} \frac{c_1^2(S)}{c_2(S)} = \frac{|\mathbf{e}|^2}{\sum_{1 \leq i \leq j \leq N-2} e_i e_j}$$

which is a rational number between 1 and 2.

Theorem 4.2. *The set of values of the fraction above is dense in the interval (1, 2).*

We want to rephrase Theorem 4.2 as follows. Set $T := |\mathbf{e}|$ and $U := e_1^2 + \dots + e_{N-2}^2$. Then

$$\frac{|\mathbf{e}|^2}{\sum_{1 \leq i \leq j \leq N-2} e_i e_j} = \frac{T^2}{(T^2 + U)/2} = \frac{2T^2}{T^2 + U}$$

The statement of the theorem is equivalent to ask for the values of

$$\frac{T^2 + U}{2T^2} = \frac{1}{2} \left(1 + \frac{U}{T^2} \right)$$

to be dense in $(1/2, 1)$ or for the values of

$$\frac{T^2}{U}$$

to be dense in $(1, +\infty)$. The new formulation is the following.

Theorem 4.3. *The set of rational numbers*

$$\left\{ \frac{\left(\sum_{i=1}^M e_i\right)^2}{\sum_{i=1}^M e_i^2} \mid M \geq 4, e_1, \dots, e_M \in \mathbb{N}^* \right\}$$

is dense in the interval $(1, +\infty)$.

Proof. Fix $M \geq 4$ and define a function

$$\begin{aligned} f_M : (\mathbb{R}^{+*})^M &\longrightarrow \mathbb{R}^+ \\ (e_1, \dots, e_M) &\longmapsto \frac{\left(\sum_{i=1}^M e_i\right)^2}{\sum_{i=1}^M e_i^2} \end{aligned}$$

Observe that for all real numbers $\lambda > 0$,

$$f_M(\lambda e_1, \dots, \lambda e_M) = f_M(e_1, \dots, e_M)$$

This implies $f_M((\mathbb{N}^*)^M) = f_M((\mathbb{Q}^{+*})^M)$. Since \mathbb{Q}^+ is dense in \mathbb{R}^+ , $f_M((\mathbb{Q}^{+*})^M)$ is dense in $f_M((\mathbb{R}^{+*})^M)$. To prove the theorem, we have only to show that

$$\bigcup_{M \geq 4} \text{Im}(f_M) = (1, +\infty)$$

It is easy to see that $\text{Im}(f_M) \subseteq (1, +\infty)$. On the other hand, since we have

$$f_M(1, \dots, 1) = \frac{M^2}{M} = M$$

and

$$\begin{aligned} f_M(1, \dots, 1, M^2) &= \frac{(M-1 + M^2)^2}{M-1 + M^4} \\ &= \frac{M^4 + M^2 + 1 - 2M + 2M^3 - 2M^2}{M^4 + M - 1} \\ &= 1 + \frac{2M^3 - M^2 - 3M + 2}{M^4 + M - 1} =: 1 + \varepsilon(M) \end{aligned}$$

the whole interval $[1 + \varepsilon(M), M]$ is contained in the image of f_M . So

$$\bigcup_{M \geq 4} \text{Im}(f_M) \supseteq \bigcup_{M \geq 4} [1 + \varepsilon(M), M] = (1, \infty)$$

and the theorem is proved. □

Note that the hypothesis $2 \leq \frac{d(m+1)+1}{2(d+1)}$ in Proposition 4.1 is equivalent to

$$m \geq 3 + \frac{3}{d}$$

Taking for the X_i curves of genus at least 2 and $m = N \geq 6$, we see that Theorem 1.3 is proved.

We also have, as a corollary, the following result.

Proposition 4.4. *For any fixed product of projective varieties $X = X_1 \times \dots \times X_m$ as above, there are only a finite number of ratios $\frac{c_1^2(S)}{c_2(S)}$ greater than or equal to 2 for a general complete intersection surface S in X .*

We recall the following general fact (see, for example, [1]):

Proposition 4.5. *If F is any surface of general type, $c_2(F) > 0$.*

Proof of Proposition 4.4. With the notation of proposition 4.1, we have

$$\frac{c_1^2(S)}{c_2(S)} = \frac{c_1^2(X) \cdot H^{N-2}|_X - 2a|\mathbf{d}| + b|\mathbf{d}|^2}{c_2(X) \cdot H^{N-2}|_X - a|\mathbf{d}| + b \sum_{i \leq j} d_i d_j}$$

so $c_1^2(S) \geq 2c_2(S)$ if and only if

$$c_1^2(X) \cdot H^{N-2}|_X - 2a|\mathbf{d}| + b|\mathbf{d}|^2 \geq 2 \left(c_2(X) \cdot H^{N-2}|_X - a|\mathbf{d}| + b \sum_{i \leq j} d_i d_j \right)$$

or, equivalently, if and only if

$$\left(c_1^2(X) - 2c_2(X) \right) \cdot H^{N-2}|_X \geq b \left(2 \sum_{i \leq j} d_i d_j - |\mathbf{d}|^2 \right) = b \sum_{i=1}^{N-2} d_i^2 = \left(\sum_{i=1}^{N-2} d_i^2 \right) H^N|_X$$

Since $H^N|_X > 0$, we have

$$\sum_{i=1}^{N-2} d_i^2 \leq \frac{\left(c_1^2(X) - 2c_2(X) \right) \cdot H^{N-2}|_X}{H^N|_X}$$

We have now to consider two cases:

- if $(c_1^2(X) - 2c_2(X)) \cdot H^{N-2}|_X \leq 0$, the set of possible vectors \mathbf{d} is empty;

- if $(c_1^2(X) - 2c_2(X)) \cdot H^{N-2}|_X = \alpha > 0$, every d_i is bounded by

$$\sqrt{\frac{\alpha}{H^N|_X}}$$

so the number of vectors \mathbf{d} is finite. \square

5. Explicit computations for linear sections of a product of curves.

Let us fix curves X_1, \dots, X_N of respective genera $g_i \geq 2$ and embed each X_i in a projective space via a multiple $l_i K_{X_i}$ of the canonical bundle, with $l_i \geq 1$ (≥ 2 if X_i is hyperelliptic) as a curve of degree $\alpha_i = l_i(2g_i - 2)$.

Set $X = X_1 \times \dots \times X_N$ and embed X in a projective space \mathbb{P}^n via a Segre embedding. If $\pi_i : X \rightarrow X_i$ is the projection on the i -th factor and $K_{\widehat{X}_i} = \pi_i^* K_{X_i}$, and if H is a hyperplane in \mathbb{P}^n , we have

$$H|_X = \sum_{i=1}^N l_i K_{\widehat{X}_i}$$

For each $i \in \{1, \dots, N\}$, let p_i be a point in X_i . For all $j \in \{1, \dots, N\}$ and for all multi-indices

$$I_j = (i_1, \dots, i_j)$$

with $1 \leq i_1 < \dots < i_j \leq N$, set

$$X_{I_j} := X_{i_1} \times \dots \times X_{i_j}$$

with this notation, we have for each I_j a projection

$$\pi_{I_j} : X \longrightarrow X_{I_j}$$

Let $X_{\widehat{I}_j}$ be the fibre of π_{I_j} over the point $(p_{i_1}, \dots, p_{i_j})$. We have

$$H|_X = \sum_{i=1}^N \alpha_i X_{\widehat{(i)}}$$

We need to compute $H^j|_X$:

$$H^j|_X = j! \sum_{I_j} \left(\prod_{k \in I_j} \alpha_k \right) X_{\widehat{I}_j}$$

Writing $X_i \times X_j$ instead of $X_{\widehat{\{1, \dots, N\} \setminus \{i, j\}}}$, we obtain

$$\begin{aligned} H^N|_X &= N! \prod_i \alpha_i \\ H^{N-1}|_X &= (N-1)! \sum_i \left(\prod_{k \neq i} \alpha_k \right) X_i \\ H^{N-2}|_X &= (N-2)! \sum_{i < j} \left(\prod_{k \neq i, j} \alpha_k \right) (X_i \times X_j) \end{aligned}$$

Moreover,

$$c_1(X) = \sum_i \pi_i^* c_1(X_i) = - \sum_i K_{\widehat{i}} \sim - \sum_i (2g_i - 2) X_{\widehat{i}}$$

and

$$c_2(X) = \sum_{i < j} \pi_i^* c_1(X_i) \cdot \pi_j^* c_1(X_j) = \sum_{i < j} K_{\widehat{i}} \cdot K_{\widehat{j}} \sim \sum_{i < j} (2g_i - 2)(2g_j - 2) X_{\widehat{(i, j)}}$$

Since $c_1^2(X_i) = 0$ for all $i \in \{1, \dots, N\}$, we always have $c_1^2(X) = 2c_2(X)$ in $H^4(X, \mathbb{Z})$.

Proposition 5.1. *A general complete intersection surface S in a product of curves satisfies the inequality*

$$c_1^2(S) < 2c_2(S)$$

Proof. It is just a consequence of Proposition 4.4 and of $c_1^2(X) = 2c_2(X)$ for X a product of curves. \square

In this case we can compute Chern classes more explicitly and obtain a precise numerical result.

Proposition 5.2. *Let S be a complete intersection surface of degree $\mathbf{d} = (d_1, \dots, d_{N-2})$ in a product of curves $X = X_1 \times \dots \times X_N$ such that X_i is embedded in the projective space via the l_i -canonical bundle. Then*

$$\frac{c_1^2(S)}{c_2(S)} = 2 - \frac{N(N-1) \left(\sum_{i=1}^{N-2} d_i^2 \right)}{\sum_{1 \leq i < j \leq N} \frac{1}{l_i l_j} + (N-1) \left(|\mathbf{d}| \sum_{i=1}^N \frac{1}{l_i} + N \sum_{1 \leq i \leq j \leq N-2} d_i d_j \right)}$$

In particular, when S is a linear section, we have

$$\frac{c_1^2(S)}{c_2(S)} = 2 - \frac{N(N-1)(N-2)}{\sum_{1 \leq i < j \leq N} \frac{1}{l_i l_j} + (N-1)(N-2) \left(\frac{N(N-1)}{2} + \sum_{i=1}^N \frac{1}{l_i} \right)}$$

Recall that we need to take $N \geq 6$ in order to be able to apply Bogomolov's theorem to get a surface S with ample cotangent bundle.

Proof. We prove the first assertion. The second one follows when we set $\mathbf{d} = (1, \dots, 1)$. Assume S is a complete intersection in X , so that $S \sim \left(\prod_{i=1}^{N-2} d_i \right) H^{N-2}|_X$. We have, using notation (1),

$$\begin{aligned} a &= c_1(X) \cdot H^{N-1}|_X \\ &= -(N-1)! \left(\sum_{i=1}^N (2g_i - 2) X_{\widehat{(i)}} \right) \cdot \left(\sum_j \left(\prod_{k \neq j} \alpha_k \right) X_j \right) \\ &= -(N-1)! \sum_j \left((2g_j - 2) \prod_{k \neq j} \alpha_k \right) \\ &= -(N-1)! \left(\prod_i (2g_i - 2) \right) \sum_j \prod_{k \neq j} l_k \end{aligned}$$

and

$$b = H^N|_X = N! \prod_i \alpha_i = N! \left(\prod_{i=1}^N (2g_i - 2) \right) \left(\prod_{i=1}^N l_i \right)$$

Moreover,

$$\begin{aligned} c_2(X) \cdot H^{N-2}|_X &= \left(\sum_{i < j} (2g_i - 2)(2g_j - 2) X_{\widehat{(i,j)}} \right) \cdot H^{N-2}|_X \\ &= (N-2)! \sum_{i < j} (2g_i - 2)(2g_j - 2) \left(\prod_{k \neq i,j} \alpha_k \right) \\ &= (N-2)! \left(\prod_{i=1}^N (2g_i - 2) \right) \sum_{i < j} \left(\prod_{k \neq i,j} l_k \right) \end{aligned}$$

Now it follows from Proposition 4.1 and the fact that $c_1^2(X) = 2c_2(X)$

that we have

$$\begin{aligned} \frac{c_1^2(S)}{c_2(S)} &= 2 + \frac{b(|\mathbf{d}|^2 - 2 \sum_{i \leq j} d_i d_j)}{c_2(X) \cdot H^{N-2}|_X - a|\mathbf{d}| + b \sum_{i \leq j} d_i d_j} \\ &= 2 + \frac{-N(N-1) (\sum_i d_i^2) \prod_i l_i}{\sum_{i < j} \prod_{k \neq i, j} l_k + (N-1) (|\mathbf{d}| \sum_i \prod_{j \neq i} l_j + N \left(\sum_{i \leq j} d_i d_j \right) \prod_i l_i)} \\ &= 2 - \frac{N(N-1) (\sum_i d_i^2)}{\sum_{i < j} \frac{1}{l_i l_j} + (N-1) \left(|\mathbf{d}| \sum_i \frac{1}{l_i} + N \sum_{i \leq j} d_i d_j \right)} \end{aligned}$$

where in the second equality we simply observe that all terms are multiples of

$$(N-2)! \left(\prod_{i=1}^N (2g_i - 2) \right)$$

and in the third equality we divide both the numerator and the denominator by $\left(\prod_{i=1}^{N-2} l_i \right)$. □

Remarks and open problems.

It is interesting to note that, to the best of our knowledge, not many examples of surfaces with ample cotangent bundle are known. Some of them come from Hirzebruch’s construction, which uses arrangements of lines in \mathbb{P}^2 , and satisfy a criterion for ampleness which can be found in [8]. Another such criterion is proved by Spurr in [9] to construct double covers of some special Hirzebruch surfaces. For most of the above examples the Chern ratio is not greater than 2, and we know only a few sporadic examples with $c_1^2 \geq 2c_2$, while a result of Miyaoka (see [7]) shows the ampleness of the cotangent bundle for surfaces with $c_1^2 = 3c_2$, i.e., quotients of the unit ball.

So the question about density in [2, 3] is still open.

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