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# SOME REMARKS ON THE STANLEY DEPTH FOR MULTIGRADED MODULES

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We show that Stanley's conjecture holds for any multigraded module $M$ over $S$, with $\operatorname{sdepth}(M) = 0$, where $S = K[x_1,\dots,x_n]$. Also, we give some bounds for the Stanley depth of the powers of the maximal irrelevant ideal in $S$.

**Keywords:** Stanley depth, monomial ideal.

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## Introduction

Let $K$ be a field and $S = K[x_1,\dots,x_n]$ the polynomial ring over $K$. Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module. A *Stanley decomposition* of $M$ is a direct sum $\mathscr{D} : M = \bigoplus_{i=1}^r m_iK[Z_i]$ as $K$-vector space, where $m_i \in M$, $Z_i \subset \{x_1,\dots,x_n\}$ such that $m_iK[Z_i]$ is a free $K[Z_i]$-module. The latter condition is needed, since the module $M$ can have torsion. We define $\operatorname{sdepth}(\mathscr{D}) = min_{i=1}^r |Z_i|$ and $\operatorname{sdepth}(M) = max\{\operatorname{sdepth}(M) | \ \mathscr{D}$ is a Stanley decomposition of $M\}$. The

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number sdepth$(M)$ is called the *Stanley depth* of $M$. Herzog, Vladoiu and Zheng show in [9] that this invariant can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. A computer implementation of this algorithm, with some improvements, is given by Rinaldo in [14].

Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module. Stanley's conjecture says that sdepth$(M) \geq$ depth$(M)$. The Stanley conjecture for $S/I$ was proved for $n \leq 5$ and in other special cases, but it remains open in the general case. See for instance, [4], [8], [10], [3] and [12]. Another interesting problem is to explicitly compute the sdepth. This is difficult, even in the case of monomial ideals! Some small progresses were made in [13], [9], [6], [7] and [15].

In the first section, we prove that the Stanley conjecture holds for modules with sdepth$(M) = 0$, see Theorem 1.4. As a consequence, it follows that any torsion free module $M$ has sdepth$(M) \geq 1$. In the second section, we give an upper bound for the Stanley depth of the powers of the maximal ideal $\mathbf{m} = (x_1,\dots,x_n) \subset S$, see Theorem 2.2. We conjecture that sdepth$(\mathbf{m}^k) = \lceil \frac{n}{k+1} \rceil$, for any positive integer $k$.

## 1. Stanley's conjecture for modules with sdepth zero.

Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module. We use an idea of Herzog, in order to obtain a decomposition of $M$, similar to the Janet decomposition given in [2]. For any $j \geq 1$, we have a natural surjective map $\varphi_j : M \to x_n^j M$ given by the multiplication with $x_n^j$. Obviously, $\varphi_j(x_nM) \subset x_n^{j+1}M$ and therefore $\varphi_j$ induces a natural surjection $\bar{\varphi}_j : M/x_nM \to x_n^jM/x_n^{j+1}M$. We write $L_j = Ker(\bar{\varphi}_j)$.

Note that $L_j \subset L_{j+1}$ for any $j$, since we have a natural surjection

$$x_n^jM/x_n^{j+1}M \to x_n^{j+1}M/x_n^{j+2}M$$

given by multiplication with $x_n$. As $M/x_nM$ is finitely generated, it follows that there exists a nonnegative integer $q$ such that $L_q = L_{q+1} = \cdots$ and moreover $x_n^jM/x_n^{j+1}M \cong x_n^{j+1}M/x_n^{j+2}M$ for any $j \geq q$. Now, we can prove the following Lemma.

**Lemma 1.1.** *Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module and $q$ such that $L_q = L_{q+1} = \cdots$. Then we have the following decomposition of $M$, as $K$-vector space:*

$$M \cong M/x_nM \oplus \cdots \oplus x_n^{q-1}M/x_n^qM \oplus x_n^qM/x_n^{q+1}M[x_n].$$

*Proof.* Note that, since $M$ is graded, $\bigcap x_n^jM = 0$. Therefore, we have

$$M = M/x_nM \oplus x_nM = M/x_nM \oplus x_nM/x_n^2M \oplus x_n^2M = \cdots = \bigoplus_{j\geq 0} x_n^jM/x_n^{j+1}M.$$

Since $x_n^jM/x_n^{j+1}M \cong x_n^{j+1}M/x_n^{j+2}M$ for any $j \geq q$, the proof of Lemma is complete. □

Note that each factor $x_n^jM/x_n^{j+1}M$ naturally carries the structure of a multigraded $S'$-module, where $S' = K[x_1,\dots,x_{n-1}]$. Also, if $M = S/I$, where $I \subset S$ is a monomial ideal, the above decomposition is exactly the Janet decomposition of $S/I$, with respect to the variable $x_n$.

**Lemma 1.2.** *Let $M$ be a multigraded $S$-module. Then* $\operatorname{sdepth}(M) = n$ *if and only if $M$ is free.*

*Proof.* If $M$ is free, it follows that $M \cong \bigoplus_{i=1}^r S(-a_i)$, where $a_i \in \mathbb{Z}^n$ are some multidegrees. Therefore, $M$ has a basis $\{e_1,\dots,e_n\}$ where $e_i$ correspond to $1 \in S(-a_i)$. Therefore $M = \bigoplus e_iS$ is a Stanley decomposition of $M$ and thus $\operatorname{sdepth}(M) = n$. Conversely, given a Stanley decomposition $M = \bigoplus e_iS$, it follows that $M \cong \bigoplus_{i=1}^r S(-a_i)$, where $deg(e_i) = a_i$. □

**Lemma 1.3.** *Let $M$ be a graded $K[x]$-module. Then, the following are equivalent:*

*(1) $M$ is free.*
*(2) $M$ is torsion free.*
*(3)* $\operatorname{depth}(M) = 1$.
*(4)* $\operatorname{sdepth}(M) = 1$.

*Proof.* The equivalences $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ are well known. $(4) \Leftrightarrow (1)$ is the case $n = 1$ of the previous Lemma. □

Let $\mathbf{m} = (x_1,\dots,x_n) \subset S$ be the maximal irrelevant ideal. Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module. We denote $sat(M) = (0 :_M \mathbf{m}^\infty) = \bigcup_{k\geq 1}(0 :_M \mathbf{m}^k)$ the *saturation* of $M$. It is well known, that $\operatorname{depth}(M) = 0$ if and only if $\mathbf{m} \in Ass(M)$ if and only if $sat(M) \neq 0$. On the other hand, $sat(M/sat(M)) = 0$. Note that if $I \subset S$ is a monomial ideal, then $sat(S/I) = I^{sat}/I$, where $I^{sat} = (I : \mathbf{m}^\infty)$ is the saturation of the ideal $I$. We prove the following generalization of [7, Theorem 1.5].

**Theorem 1.4.** *Let $M$ be a multigraded $S$-modules. If* $\operatorname{sdepth}(M) = 0$ *then* $\operatorname{depth}(M) = 0$. *Conversely, if* $\operatorname{depth}(M) = 0$ *and $dim_K(M_a) \leq 1$ for any $a \in \mathbb{Z}^n$, then* $\operatorname{sdepth}(M) = 0$.

*Proof.* We use induction on $n$. If $n = 1$, then we are done by Lemma 1.3. Suppose $n > 1$. We consider the decomposition

$$(*)\ \ M \cong M/x_nM \oplus \cdots \oplus x_n^{q-1}M/x_n^qM \oplus x_n^qM/x_n^{q+1}M[x_n],$$

given by Lemma 1.2. We define $M_j := x_n^j M / x_n^{j+1} M$ for $j \in [q]$. As sdepth$(M) = 0$, it follows that sdepth$(M_j) = 0$ for some $j < q$. We have $M_j = sat(M_j) \oplus M/sat(M_j)$, where $sat(M_j)$ is the saturation of $M_j$ as a $S'$-module. If there exists some nonzero element $m \in sat(M_j)$ such that $x_n^j m = 0$, it follows that $m \in sat(M)$ and thus $sat(M) \neq 0$.

For the converse, we assume $depth(M) > 0$. It follows that $x_n sat(M_j) \subset sat(M_{j+1})$ for any $j < q$. Since $sat(M_j/sat(M_j)) = 0$, by induction hypothesis, it follows that sdepth$(M_j/sat(M_j)) \geq 1$. Therefore, $(*)$ implies

$$(**) M \cong \bigoplus_{j=0}^{q-1} M_j/sat(M_j) \oplus M_q/sat(M_q)[x_n] \oplus \bigoplus_{j=0}^{q-1} sat(M_j) \oplus sat(M_q)[x_n].$$

Also, $\bigoplus_{j=0}^{q-1} sat(M_j) \oplus sat(M_q)[x_n] = \bigoplus_{j=0}^{q} \bigoplus_{\bar{m} \in sat(M_j)/sat(M_{j-1})} mK[x_n]$ since $dim_K(M_a) \leq 1$, and therefore, by $(**)$, we obtain a Stanley decomposition of $M$ with it's sdepth $\geq 1$! □

**Corollary 1.5.** *If $M$ is torsion free, then* sdepth$(M) \geq 1$.

*Proof.* Obviously, since $M$ is torsion free, we have depth$(M) \geq 1$. □

**Example 1.6.** (Dorin Popescu, [12]) The condition $dim_K(M_a) \leq 1$ is essential in the second part of Theorem 1.4. Let $S = K[x_1, x_2]$ and consider the module $M := (Se_1 \oplus Se_2)/(x_1 z, x_2 z$, where $z = x_1 e_2 - x_2 e_1$. $M$ is multigraded with $deg(e_1) = deg(x_1) = (1,0)$ and $deg(e_2) = deg(x_2) = (0,1)$. Note that $dim_K(M_a) = 1$ for any $a \in \mathbb{Z}^2 \setminus \{(1,1)\}$ and $dim_K(M_{(1,1)}) = 2$. Since $z \in Soc(M)$, it follows that depth$(M) = 0$. We have a Stanley decomposition of $M$,

$$M = \bar{e}_1 K[x_2] \oplus \bar{e}_1 x_1 K[x_1] \oplus \bar{e}_2 K[x_1] \oplus \bar{e}_2 x_2 K[x_2] \oplus \bar{e}_1 x_1 x_2 K[x_1, x_2],$$

where $\bar{e_1}, \bar{e_2}$ are the images of $e_1$ and $e_2$ in $M$. It follows that sdepth$(M) \geq 1$ and thus sdepth$(M) = 1$, since $M$ is not free.

**Remark 1.7.** Let $M$ be a torsion free finitely generated $\mathbb{Z}^n$-graded $S$-module. Then we have an inclusion $0 \to M \to F$, where $F$ is a free module with the same rank as $M$. Let $Q := F/M$. Is it true that sdepth$(M) \geq$ sdepth$(Q) + 1$? In particular, if $I \subset S$ is a monomial ideal, is it true that sdepth$(I) \geq$ sdepth$(S/I) + 1$?

If this result were true, then by depth$(M) =$ depth$(Q) + 1$, if $Q$ satisfy Stanley's conjecture, then $M$ also satisfy Stanley's conjecture. Note that, in general we cannot expect that sdepth$(M) =$ sdepth$(Q) + 1$. Take for instance $M = \mathbf{m} = (x_1, \ldots, x_n) \subset S$ and $Q = k = S/\mathbf{m}$. It is known from [9] and [5] that sdepth$(\mathbf{m}) = \lceil \frac{n}{2} \rceil$, but sdepth$(k) = 0$. It would be interesting to characterize those modules $M$ with sdepth$(M) =$ sdepth$(Q) + 1$. Or, at least, the monomials ideals $I \subset S$ with sdepth$(I) =$ sdepth$(S/I) + 1$.

We end this section with the following example.

**Example 1.8.** Let $M_i := syz_i(K)$ the $i$-th syzygy module of $K$. It is known that $\operatorname{depth}(M_i) = i$ for all $0 \leq i \leq n$. The problem of computing $\operatorname{sdepth}(M_i)$ is a chellenging problem. Obviously, $\operatorname{sdepth}(M_0) = \operatorname{sdepth}(K) = 0$. On the other hand, $\operatorname{sdepth}(M_1) = \operatorname{sdepth}(\mathbf{m}) = \lceil \frac{n}{2} \rceil$. Also, $\operatorname{sdepth}(M_n) = \operatorname{sdepth}(S) = n$. We claim that $\operatorname{sdepth}(M_{n-1}) = n-1$.

Indeed, $M_{n-1} = Coker(S \xrightarrow{\psi} S^n)$, where we define $S^n = \bigoplus_{i=1}^n Se_i$ and we set $\psi(1) := x_1e_1 + \cdots + x_ne_n$. Therefore, $M_{n-1} := S\bar{e}_1 + \cdots + S\bar{e}_n$, where $\bar{e}_i$ are the class of $e_i$ in $M_{n-1}$ for all $i \in [n]$. Note that $\bar{e}_1, \ldots, \bar{e}_{n-1}$ are linearly independent in $M_{n-1}$, since the only relation in $M_{n-1}$ is $x_1\bar{e}_1 + \cdots + x_{n-1}\bar{e}_n = -x_n\bar{e}_n$. It follows that, $M_{n-1} = S\bar{e}_1 \oplus \cdots \oplus S\bar{e}_{n-1} \oplus K[x_1, \ldots, x_{n-1}]\bar{e}_n$, and therefore $\operatorname{sdepth}(M_{n-1}) \geq n-1$. On the other hand, $\operatorname{sdepth}(M_{n-1}) \leq n-1$, since $M$ is not free. Thus $\operatorname{sdepth}(M_{n-1}) = n-1$.

## 2. Bounds for the sdepth of powers of the maximal irrelevant ideal

Let $\mathbf{m} = (x_1, \ldots, x_n)$ be the maximal irrelevant ideal of $S$. Let $k \geq 1$ be an integer. In this section, we will give some upper bounds for $\operatorname{sdepth}(\mathbf{m}^k)$. In order to do so, we consider the following poset, associated to $\mathbf{m}^k$,

$$P := \{u \in \mathbf{m}^k \; monomial \; : \; u | x_1^k x_2^k \cdots x_n^k\},$$

where $u \leq v$ if and only if $u|v$. For any $u \in P$, we denote $\rho(u) = |\{j : x_j^k | u\}|$. Note that, by [9, Theorem 2.4], there exists a partition of $P = \bigoplus_{i=1}^r [u_i, v_i]$, i.e. a disjoint sum of intervals $[u_i, v_i] = \{u \in P : \; u_i|u \text{ and } u|v_i\}$, such that $\min_{i=1}^r \{\rho(v_i)\} = \operatorname{sdepth}(\mathbf{m}^k)$.

We write $P_d = \{u \in P : deg(u) = d\}$, where $k \leq d \leq kn$, and $\alpha_d := |P_d|$. First, we want to compute the numbers $\alpha_d$.

**Lemma 2.1.** *We the above notations, we have:*

$$\alpha_d = \sum_{i \geq 0} (-1)^i \binom{n}{i} \binom{n+d-i(k+1)-1}{n-1}.$$

*Proof.* We fix $d \geq k$. For any $j \in [n]$, we write $A_j := \{u \in S : \; deg(u) = d, x_j^{k+1}|u\}$. Obviously, $P_d := S_d \setminus (A_1 \cup A_2 \cup \cdots \cup A_n)$, where $S_d$ is the set of all monomials of degree $d$ in $S$. For any nonempty subset $I \subset [n]$, we write $A_I := \bigcap_{i \in I} A_i$. By inclusion-exclusion principle,

$$|A_1 \cup \cdots \cup A_n| = \sum_{\emptyset \neq I \subset [n]} (-1)^{|I|-1} |A_I|.$$

Note that a monomial $u \in A_I$ can be written as $u = w \cdot \prod_{i \in I} x_i^{k+1}$. Therefore, $|A_I| = \binom{n+d-i(k+1)-1}{n-1}$. Now, one can easily get the required conclusion. □

**Theorem 2.2.** *Let $a \leq \lceil \frac{n}{2} \rceil$ be a positive integer. Then* $\operatorname{sdepth}(\mathbf{m}^k) \leq \lceil \frac{n}{k+1} \rceil$. *In particular, if $k \geq n-1$, then* $\operatorname{sdepth}(\mathbf{m}^k) = 1$.

*Proof.* Let $a = \lceil \frac{n}{k+1} \rceil$ and assume, by contradiction, that $\operatorname{sdepth}(\mathbf{m}^k) \geq a+1$. Obviously, by Lemma 2.1, $\alpha_k = \binom{n+k-1}{n-1}$ and $\alpha_{k+1} = \binom{n+k}{n-1} - n$. We consider a partition of $\mathscr{P} : P_{n,k} = \bigcup_{i=1}^{r} [x^{c_i}, x^{d_i}]$ with $\operatorname{sdepth}(\mathscr{D}(\mathscr{P})) = a+1$. Note that $\mathbf{m}^k$ is minimally generated by all the monomials of degree $k$ in $S$. We can assume that $S_k = \{x^{c_i} | i = 1, \dots, N\}$, where $N = \binom{n+k-1}{n-1}$. We consider an interval $[x^{c_i}, x^{d_i}]$. If $c_i = x_j^k$, then by $\rho(x^{d_i}) \geq a+1$, it follows that in $[x^{c_i}, x^{d_i}]$ are at least $a$ distinct monomials of degree $k+1$. If $c_i(j) < k$ for all $j \in [n]$, then, in $[x^{c_i}, x^{d_i}]$ are at least $a+1$ distinct monomials of degree $k+1$.

We assume that $k \geq \lceil \frac{n-a}{a} \rceil$. Since $\mathscr{P} : P_{n,k} = \bigcup_{i=1}^{r} [x^{c_i}, x^{d_i}]$ is a partition of $P_{n,k}$, by above considerations, it follows that $\alpha_{k+1} \geq na + (\alpha_k - n)(a+1)$. Therefore, $\binom{n+k}{k-1} \geq (a+1)\binom{n+k-1}{n-1}$. This implies $n+k \geq (k+1)(a+1) \geq (k+1)(\frac{n}{k+1}+1) = n+k+1$, a contradiction. □

We conjecture that $\operatorname{sdepth}(\mathbf{m}^k) \leq \lceil \frac{n}{k+1} \rceil$. Using the computer, see [14], one can prove that this conjecture is true for small $n$. Also, the conjecture is true for $k = 1$, from [9], [5]. We end this section with the following proposition.

**Proposition 2.3.** *Let $I \subset S$ be a monomial ideal. Then* $\operatorname{sdepth}(\mathbf{m}^k I) = 1$ *for $k \gg 0$.*

*Proof.* We consider the $K$-algebra $A := \bigoplus_{i \geq 0} \mathbf{m}^i I / \mathbf{m}^{i+1} I$ and denote $A_i$ the $i^{th}$ graded component of $A$. Note that $H(A,i) := dim_K(A_i) = |G(\mathbf{m}^i I)|$, where $G(\mathbf{m}^i I)$ is the set of minimal monomial generators of $\mathbf{m}^i I$. Since $A$ is a finitely generated $K$-algebra, it follows that the Hilbert function $H(A,i)$ is polynomial for $i \gg 0$.

Therefore, $lim_{i\to\infty} H(A,i)/H(A,i+1) = 1$. Note that there are exactly $H(A,i+1)$ monomials of degree $i+1$ in $\mathbf{m}^i I$. Suppose $\operatorname{sdepth}(\mathbf{m}^i I) \geq 2$. As in the proof of Theorem 2.2, it follows that $H(A,i+1) \geq 2(H(A,i)-n)+n$, which is false for $i \gg 0$, since it contradicts that $\lim_{i\to\infty} H(A,i)/H(A,i+1) = 1$. □

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