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COMPONENTWISE LINEARITY OF IDEALS ARISING FROM GRAPHS

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Let G be a simple undirected graph on n vertices. Francisco and Van Tuyl have shown that if G is chordal, then $\bigcap_{\{x_i,x_j\}\in E_G}\langle x_i,x_j\rangle$ is componentwise linear. A natural question that arises is for which $t_{ij}>1$ the intersection ideal $\bigcap_{\{x_i,x_j\}\in E_G}\langle x_i,x_j\rangle^{t_{ij}}$ is componentwise linear, if G is chordal. In this report we show that $\bigcap_{\{x_i,x_j\}\in E_G}\langle x_i,x_j\rangle^{n-1}$ is componentwise linear for all $n\geq 3$, if G is a complete graph. We give also an example where G is chordal, but the intersection ideal is not componentwise linear for any $t>1$.

1. Introduction

Let G be a simple graph on n vertices, E_G the edge set of G and V_G the vertex set of G. Let $R=k[x_1,\ldots,x_n]$ be the polynomial ring over a field k. The *edge ideal* of G is the quadratic squarefree monomial ideal $\mathscr{I}(G)=\langle\{x_ix_j\}\mid\{x_i,x_j\}\in E_G\rangle\subset R$. Then we define the *squarefree Alexander dual* of $\mathscr{I}(G)$ as $\mathscr{I}(G)^\vee=\bigcap_{\{x_i,x_j\}\in E_G}\langle x_i,x_j\rangle$. To call $\mathscr{I}(G)^\vee$ the squarefree Alexander dual of $\mathscr{I}(G)$ is natural since it is the Stanley–Reisner ideal of the simplicial complex $\Delta^\vee$ that is the Alexander dual simplicial complex of Δ, where Δ in turn is the simplicial complex whose Stanley–Reisner ideal is $\mathscr{I}(G)$.

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In [4] Herzog and Hibi give the following definition. Given a graded ideal $I \subset R$, we denote by $I_{\langle d \rangle}$ the ideal generated by the elements of degree d that belong to I. Then we say that a (graded) ideal $I \subset R$ is *componentwise linear* if $I_{\langle d \rangle}$ has a linear resolution for all d.

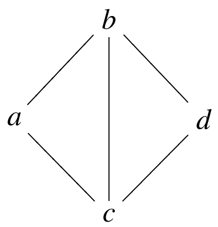
If the graph G is chordal, that is, every cycle of length $m \geq 3$ in G has a chord, then it is proved by Francisco and Van Tuyl in [2] that $\mathscr{I}(G)^V$ is componentwise. (The authors then use the result to show that all chordal graphs are sequentially Cohen-Macaulay.)

In this report we examine componentwise linearity of ideals arising from complete graphs and of the form $\bigcap_{\{x_i,x_j\} \in E_G} \langle x_i, x_j \rangle^{n-1}$.

2. A counterexample

There exists a chordal graph G such that $\bigcap_{\{x_i,x_j\} \in E_G} \langle x_i, x_j \rangle^t$ is not componentwise linear for any $t > 1$.

Let G be the chordal graph



Denote the intersection $\bigcap_{\{i,j\} \in E_G} \langle i, j \rangle^t$ by $I_4^{(t)}$. We have that

$$I_4^{(1)} = \bigcap_{\{i,j\} \in E_G} \langle i, j \rangle = \langle bc, abd, acd \rangle$$

and

$$I_4^{(2)} = \bigcap_{\{i,j\} \in E_G} \langle i, j \rangle = \langle b^2c^2, abcd, a^2b^2d^2, a^2c^2d^2 \rangle.$$

We claim that for $t > 1$ the ideal has the form

$$I_4^{(t)} = \langle b^t c^t, b^{t-1} c^{t-1} ad \rangle + J_t,$$

where J_t is an ideal generated of elements of degree at least $2t+1$. This is evidently true for $t = 1$. Now, for $t+1$ we may write the ideal as

$$I_4^{(t+1)} = \langle a,b \rangle \langle a,b \rangle^t \cap \langle a,c \rangle \langle a,c \rangle^t \cap \langle b,c \rangle \langle b,c \rangle^t \cap \langle b,d \rangle \langle b,d \rangle^t \cap \langle c,d \rangle \langle c,d \rangle^t.$$

Assuming our claim holds for $I_4^{(t)}$, it is clear that no generator of $I_4^{(t+1)}$ has degree strictly less than $2t+2$. Furthermore one sees that the only generators of degree equal to $2t+2$ are $b^{t+1}c^{t+1}$ and b^tc^tad. This proves our claim.

Consider the minimal free resolution of $I_4^{(t)}$. Its degree $2t$-part is

$$0 \to R(-(2t+2)) \to R^2(-2t) \to (b^tc^t, b^{t-1}c^{t-1}ad) \to 0,$$

which clearly is non-linear.

3. Intersections for complete graphs

Let K_n be a complete graph on n vertices, that is, $\{x_i,x_j\} \in E_{K_n}$ for all $1 \leq i \neq j \leq n$. We write $K_n^{(n-1)} = \bigcap_{\{x_i,x_j\}\in E_{K_n}} \langle x_i, x_j\rangle^{n-1}$. We show that the ideal $K_n^{(n-1)}$ is componentwise linear for all $n \geq 3$. Recall that a *vertex cover* of a graph G is a subset $A \subset V_G$ such that every edge of G is incident to at least one vertex of A. One can show that $\mathscr{I}(G)^V = \langle x_{i_1}\cdots x_{i_k} \mid \{x_{i_1},\dots,x_{i_k}\} \text{ a vertex cover of } G\rangle$. A *t-vertex cover* (or a *vertex cover of order t) of* G is a vector $\mathbf{a} = (a_1,\dots,a_n)$ with $a_i \in \mathbb{N}$ such that $a_i + a_j \geq t$ for all $\{x_i,x_j\} \in E_G$.

In the proof of the theorem below, we use the following definition and proposition.

Definition 3.1. A monomial ideal I is said to have *linear quotients*, if for some degree ordering of the minimal generators $f_1,\dots,f_r$ and all $k > 1$, the colon ideals $\langle f_1,\dots,f_{k-1}\rangle : f_k$ are generated by a subset of $\{x_1,\dots,x_n\}$.

Proposition 3.2 (Proposition 2.6 in [3] and Lemma 4.1 in [1])**.** *If I is a homogeneous ideal with linear quotients, then I is componentwise linear.*

Theorem 3.3. *The ideal $K_n^{(n-1)}$ is componentwise linear for all $n \geq 3$.*

Proof. For calculating an explicit generating system of $K_n^{(n-1)}$ we will use t-vertex covers. Pick any monomial m in $K_n^{(n-1)}$ and, for some k and l, consider the maximal t_k, t_l such that $x_k^{t_k}x_l^{t_l}$ is a factor in m. As m is contained in $\langle x_k, x_l\rangle^{n-1}$ we must have $t_k + t_k \geq n-1$. Hence, $K_n^{(n-1)}$ is generated by the monomials of the form $\mathbf{x}^{\mathbf{a}}$, where $\mathbf{a}$ is an $(n-1)$-cover of K_n. That is, the sum of the two lowest exponents in every (monomial) generator of $K_n^{(n-1)}$ is at least $n-1$.

Now assume that $n-1 = 2m+1$ is odd. Using the degree lexicographic ordering $x_1 \prec x_2 \prec \cdots \prec x_n$ on the the minimal generators we get

$$\begin{aligned} K_n^{(n-1)} = K_n^{(2m+1)} = \quad & \Big\langle x_1^m \textstyle\prod_{i\neq 1} x_i^{m+1}, \quad \dots \quad , x_n^m \prod_{i\neq n} x_i^{m+1}, \\ & x_1^{m-1} \textstyle\prod_{i\neq 1} x_i^{m+2}, \quad \dots \quad , x_n^{m-1} \prod_{i\neq 1} x_i^{m+2}, \\ & \qquad \vdots \\ & \textstyle\prod_{i\neq 1} x_i^{2m+1}, \quad \dots \quad , \prod_{i\neq n} x_i^{2m+1}\Big\rangle. \end{aligned}$$

This order on the minimal generators satisfies the condition in Definition 3.1. Hence, $K_n^{(n-1)}$ has linear quotients and is componentwise linear by Proposition 3.2.

If $n-1=2m$ is even, then the degree lexicographic ordering yields the sequence

$$\begin{array}{rlll} K_n^{(n-1)} = K_n^{(2m)} = & \big\langle \prod_{i=1}^{2m} x_i^m, & x_1^{m-1}\prod_{i\neq 1} x_i^{m+1}, & \dots \quad , x_n^{m-1}\prod_{i\neq n} x_i^{m+1}, \\ & & x_1^{m-2}\prod_{i\neq 1} x_i^{m+2}, & \dots \quad , x_n^{m-2}\prod_{i\neq 1} x_i^{m+2}, \\ & & & \vdots \\ & & \prod_{i\neq 1} x_i^{2m}, & \dots \qquad , \prod_{i\neq n} x_i^{2m}\big\rangle, \end{array}$$

which also satisfies the condition in Definition 3.1, and the same result follows. $\square$

Example 3.4.

$$K_6^{(5)} = \Big\langle \{x_j^2 \prod_{i\neq j} x_i^3\}_{1\leq j\leq 6},\ \{x_j \prod_{i\neq j} x_i^4\}_{1\leq j\leq 6},\ \{\prod_{i\neq j} x_i^5\}_{1\leq j\leq 6}\Big\rangle$$

and

$$K_7^{(6)} = \big\langle \prod_{i=1}^{7} x_i^3,\ \{x_j^2 \prod_{i\neq j} x_i^4\}_{1\leq j\leq 7},\ \{x_j \prod_{i\neq j} x_i^5\}_{1\leq j\leq 7},\ \{\prod_{i\neq j} x_i^6\}_{1\leq j\leq 7}\big\rangle.$$

4. Problems and generalizations

We want to check whether the result in Section 3 is valid for complete hypergraphs. We would also like to investigate the relation between sequentially Cohen-Macaulayness and componentwise linearity for non-squarefree ideals.

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REFERENCES

[1] A. Conca - J. Herzog, *Castelnuovo-Mumford regularity of products of ideals*, Collect. Math. **54** 2 (2003), 137–152.

[2] C. Francisco - A. van Tuyl, *Sequentially Cohen-Macaulay edge ideals*, Proc. Amer. Math. Soc. **135** (2007), 2327–2337.

[3] C. Francisco - A. van Tuyl, *Some families of componentwise linear monomial ideals*, Nagoya Math. J. **187** (2007), 115–156.

[4] J. Herzog - T. Hibi, *Componentwise linear ideals*, Nagoya Math. J. **153** (1999), 141–153.

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