A NOTE ON THE SERRIN PROBLEM IN THE PLANE

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We investigate the stability of the radial symmetry for the overdetermined Serrin problem in a planar convex set. More precisely, we prove that, whenever we properly perturb both the boundary conditions and the data, then a convex solution is "close" to a suitable paraboloid and the domain is "close" to a ball with respect to the Hausdorff metric.

1. Introduction

In [14] Serrin proves that, if Ω is a smooth, bounded, connected, open subset of $\mathbb{R}^n$ ($n \geq 2$) and $u \in C^2(\overline{\Omega})$ satisfies the following problem

$$\begin{cases} \Delta u = n & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 1 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathbf{n}$ is the unit outer normal to $\partial\Omega$, then Ω must be the unit ball and $u = \frac{|x|^2 - 1}{2}$ up to a translation.

Serrin's proof is based on the so–called moving plane method and on a smart refinement of the maximum principle. After Serrin, this technique became very popular and it has been deeply exploited to prove symmetry results in many

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different situations (see for istance [9], [2], [12], [8], [10]). In [1] it is shown that the domain is close to the unit ball even if the boundary condition $\frac{\partial u}{\partial \mathbf{n}} = 1$ is perturbed in a suitable way. More recently in [5] and [6] the authors proved that a different approach to Serrin's result (see [4]), based on elementary integral inequalities and standard tools of convex analysis, is well suited to investigate stability issues. In this short note we focus our attention on the planar case, which deserves some special interests since, despite the restriction, it allows us to consider a general class of perturbations of the data in (1). Indeed we prove the following

Theorem 1.1. *Let Ω be a smooth, bounded, convex, open subset of $\mathbb{R}^2$ and $u \in C^2(\overline{\Omega})$ be a convex function satisfying*

$$\begin{cases} 2 - \varepsilon \leq \Delta u \leq 2 + \varepsilon & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

and

$$1 - \delta \leq \frac{\partial u}{\partial \mathbf{n}} \leq 1 + \delta \quad \text{on } \partial\Omega, \tag{3}$$

with ε, δ positive real numbers. Then Ω is close to a unit ball and u is almost radially symmetric in the following sense:

(i) there exist positive constants C_1 and C_2 such that

$$R(\Omega) - r(\Omega) \leq C_1 (\delta^2 + \varepsilon^2)^{1/4}, \tag{4}$$

$$r(\Omega) - C_2 (\delta^2 + \varepsilon^2)^{1/2} \leq 1 \leq R(\Omega) + C_2 (\delta^2 + \varepsilon^2)^{1/2}, \tag{5}$$

where $R(\Omega)$ and $r(\Omega)$ are the radii of the annulus containing the boundary of Ω and having the minimum width;

(ii) there exists a positive constant C_3 such that

$$\left\| u(x) - \frac{|x|^2 - R(\Omega)^2}{2} \right\|_{L^\infty(\Omega)} \leq C_3 (\delta^2 + \varepsilon^2)^{1/4} \quad (up \ to \ a \ translation). \tag{6}$$

We observe that, since the solution to (1) is a strictly convex function, the restriction to convex solutions in Theorem 1.1 seems not a hard one.

2. Notation and preliminaries

Let Ω be a convex subset of $\mathbb{R}^2$. The support function of Ω, $h_\Omega : \mathbb{R}^2 \to \mathbb{R}$, is defined by

$$h_\Omega(y) = \sup_{x \in \Omega} \langle y, x \rangle, \quad y \in \mathbb{R}^2;$$

hence it is a convex and positive homogeneous function of degree 1.

We say that a bounded, convex, open set Ω is a C_+^2 set if $\partial\Omega \in C^2$ and $\partial\Omega$ has non vanishing Gauss curvature. If Ω is of class C_+^2, then its Gauss map $\mathbf{n}_\Omega : x \to \mathbf{n}$ is a diffeomorphism between $\partial\Omega$ and the unit sphere S^1 and it is easily seen that

$$h_\Omega(y) = \langle y, \mathbf{n}_\Omega^{-1}(y/|y|)\rangle, \quad y \in \mathbb{R}^2 \setminus \{0\}.$$

Moreover, using $\mathbf{n}_\Omega$ as an admissible change of variables, whose Jacobian determinant is the Gauss curvature k_Ω, we can write any integral over $\partial\Omega$ as an integral over S^1. In particular it holds

$$|\partial\Omega| = \int_{\partial\Omega} k_\Omega \langle x, \mathbf{n}\rangle = \int_{S^1} h_\Omega(\omega)d\omega. \tag{7}$$

Now, let $u \in C(\overline{\Omega})$ be a convex function. The *conjugate* of u is the convex function v defined by

$$v(y) = \max\{\langle y, x\rangle - u(x) : x \in \overline{\Omega}\}, \quad y \in \mathbb{R}^2.$$

If u is strictly convex and C^1 in Ω, then v is of class C^1 in the set

$$Du(\Omega) = \{Du(x) : x \in \Omega\}.$$

Furthermore, $Dv(y)$ is equal to the point $x \in \overline{\Omega}$ such that $y = Du(x)$, that is $Dv = (Du)^{-1}$ in $Du(\Omega)$ and

$$v(y) + u(x) = \langle y, x\rangle \tag{8}$$

where $x = Dv(y)$ and $y = Du(x)$. Notice that Dv vanishes at $y = Du(0)$ and this, by the convexity of v and (8), implies that

$$\min_{\overline{Du(\Omega)}} v = v(Du(0)) = -u(0). \tag{9}$$

Moreover one can easily check that $u \in C_+^2(\Omega)$ (i.e. u is of class C^2 and D^2u is positive definite in Ω) if and only if $v \in C_+^2(Du(\Omega))$; in such a case, we have

$$D^2v(y) = D^2u(x)^{-1} \tag{10}$$

where $x = Dv(y)$ and $y = Du(x)$.

We refer, for example, to [11], [13] for more details about support functions, convex sets and conjugates of convex functions.

We end this section by recalling that the classical isoperimetric inequality in the plane states

$$|\Omega| \leq \frac{|\partial\Omega|^2}{4\pi}, \tag{11}$$

equality holding if and only if Ω is a ball. Moreover, in the class of convex sets, the well-known Bonnesen inequality holds (see [3])

$$(R(\Omega) - r(\Omega))^2 \leq \frac{|\partial\Omega|^2}{4\pi} - |\Omega|, \tag{12}$$

where $R(\Omega)$ and $r(\Omega)$ are the radii of the annulus containing $\partial\Omega$ and having the minimum width.

3. Proof of Theorem 1.1

Proof. First we summarize the main steps of the proof. We begin by proving that the measure of Ω is nearly π and its perimeter is nearly 2π. Then, using the stability inequality (12), we get the estimate (4). From this one we deduce (6) simply applying the maximum principle for the Poisson equation. For the seek of simplicity we shall assume that u achieves its minimum at 0.

Setting $E = Du(\Omega)$, (3) is equivalent to

$$(1 - \delta)B \subseteq E \subseteq (1 + \delta)B, \tag{13}$$

where B is the unit disk, while from the boundary condition '$u = 0$ on $\partial\Omega$' we obtain

$$v(y) = \langle y, Dv(y)\rangle = h_\Omega(y) \quad \text{for every } y \in \partial E. \tag{14}$$

Notice that, due to the monotonicity of the gradient of convex functions, the set E is starshaped with respect to the origin. By the arithmetic–geometric mean inequality and $\Delta u \leq 2 + \varepsilon$ we have

$$\det D^2 u \leq \left(\frac{2+\varepsilon}{2}\right)^2 \quad \text{and, equivalently,} \quad \det D^2 v \geq \left(\frac{2}{2+\varepsilon}\right)^2. \tag{15}$$

Integrating the first one over Ω we get

$$|E| \leq \left(\frac{2+\varepsilon}{2}\right)^2 |\Omega|$$

that implies, by (13),

$$|\Omega| \geq 4\left(\frac{1-\delta}{2+\varepsilon}\right)^2 \pi,$$

whence

$$|\Omega| \geq \pi - C\sqrt{\delta^2 + \varepsilon^2}, \tag{16}$$

where C is a positive constant.

Next we look for an estimate from above of $|\partial\Omega|$. To this aim let us integrate (2) over Ω; taking into account (3) we obtain

$$\frac{1-\delta}{2+\varepsilon}|\partial\Omega| \leq |\Omega| \leq \frac{1+\delta}{2-\varepsilon}|\partial\Omega|. \tag{17}$$

A straightforward computation shows

$$\langle x, Du\rangle \Delta u = \operatorname{div}(\langle x, Du\rangle Du) - \frac{1}{2}\operatorname{div}(x|Du|^2),$$

then it follows

$$\frac{1}{2+\varepsilon}\left[\operatorname{div}(\langle x, Du\rangle Du) - \frac{1}{2}\operatorname{div}(x|Du|^2)\right] \leq \langle x, Du\rangle \tag{18}$$

$$\leq \frac{1}{2-\varepsilon}\left[\operatorname{div}(\langle x, Du\rangle Du) - \frac{1}{2}\operatorname{div}(x|Du|^2)\right].$$

Since $\langle x, Du\rangle = \operatorname{div}(xu) - 2u$, we obtain

$$\int_\Omega \langle x, Du\rangle = 2\int_\Omega(-u). \tag{19}$$

Integrating (18) and recalling the boundary conditions of our problem, we obtain

$$\frac{(1-\delta)^2}{2(2+\varepsilon)}\int_{\partial\Omega}\langle x, \mathbf{n}\rangle \leq 2\int_\Omega(-u) \leq \frac{(1+\delta)^2}{2(2-\varepsilon)}\int_{\partial\Omega}\langle x, \mathbf{n}\rangle.$$

Since

$$\int_{\partial\Omega}\langle x, \mathbf{n}\rangle = \int_\Omega \operatorname{div}(x) = 2|\Omega|,$$

we get

$$\frac{(1-\delta)^2}{2(2+\varepsilon)}|\Omega| \leq \int_\Omega(-u) \leq \frac{(1+\delta)^2}{2(2-\varepsilon)}|\Omega|. \tag{20}$$

By (17), we finally have

$$\frac{(1-\delta)^3}{2(2+\varepsilon)^2}|\partial\Omega| \leq \int_\Omega(-u) \leq \frac{(1+\delta)^3}{2(2-\varepsilon)^2}|\partial\Omega|. \tag{21}$$

By (15), (8) and (21), it holds

$$\begin{aligned}
\int_E v &\leq \left(\frac{2+\varepsilon}{2}\right)^2\int_E v\det D^2 v = \left(\frac{2+\varepsilon}{2}\right)^2\int_\Omega v(Du(x)) \tag{22}\\
&= \left(\frac{2+\varepsilon}{2}\right)^2\int_\Omega(\langle x, Du\rangle - u) = \left(\frac{2+\varepsilon}{2}\right)^2\int_\Omega(\operatorname{div}(xu) - 3u)\\
&= 3\left(\frac{2+\varepsilon}{2}\right)^2\int_\Omega(-u) \leq c_1(\delta, \varepsilon)|\partial\Omega|,
\end{aligned}$$

where

$$c_1(\delta, \varepsilon) = \frac{3}{8}(1+\delta)^3 \left(\frac{2+\varepsilon}{2-\varepsilon}\right)^2 .$$

Since $\langle y, Dv \rangle = \mathrm{div}(yv) - 2v$, we obtain

$$\int_E v = \frac{1}{2}\left(\int_{\partial E} v \langle y, \mathbf{n}_E \rangle - \int_E \langle y, Dv \rangle\right) . \tag{23}$$

Introducing radial and angular coordinates $\rho = |y|$, $\omega = \dfrac{y}{\rho}$, and writing $h_\Omega(y) = h_\Omega(\rho\omega)$, from (13) and (7) we can deduce

$$
\begin{aligned}
\int_{\partial E} v \langle y, \mathbf{n}_E \rangle &= \int_{\partial E} h_\Omega(y) \langle y, \mathbf{n}_E \rangle \\
&= \int_{S^1} h_\Omega(\rho\omega)\rho^2 \, d\omega = \int_{S^1} h_\Omega(\omega)\rho^3 \, d\omega \\
&\geq (1-\delta)^3 \int_{S^1} h_\Omega(\omega)\, d\omega = (1-\delta)^3 |\partial\Omega| ,
\end{aligned}
\tag{24}
$$

then, putting together (23) and (24), we get

$$\int_E v \geq \frac{1}{2}\left((1-\delta)^3|\partial\Omega| - \int_E \langle y, Dv \rangle\right) . \tag{25}$$

On the other hand,

$$\int_E \langle y, Dv \rangle = \int_\Omega \langle Du, x \rangle \det D^2 u ,$$

hence, by (15), (19) and (21), we have

$$
\begin{aligned}
\int_E \langle y, Dv \rangle &\leq \left(\frac{2+\varepsilon}{2}\right)^2 \int_\Omega \langle Du, x \rangle \\
&= 2\left(\frac{2+\varepsilon}{2}\right)^2 \int_\Omega (-u) \leq \frac{(1+\delta)^3}{4}\left(\frac{2+\varepsilon}{2-\varepsilon}\right)^2 |\partial\Omega| .
\end{aligned}
$$

Substituting in (25), we finally obtain

$$\int_E v \geq c_2(\delta, \varepsilon) |\partial\Omega| , \tag{26}$$

where

$$c_2(\delta, \varepsilon) = \frac{1}{2}\left[(1-\delta)^3 - \frac{(1+\delta)^3}{4}\left(\frac{2+\varepsilon}{2-\varepsilon}\right)^2\right] .$$

Notice that $c_2(\delta, \varepsilon) \leq c_1(\delta, \varepsilon)$ and $\displaystyle\lim_{(\delta,\varepsilon)\to(0,0)} c_1(\delta, \varepsilon) = \lim_{(\delta,\varepsilon)\to(0,0)} c_2(\delta, \varepsilon) = 3/8.$

By (15), (22) and (26) we get

$$0 \leq \int_E \left[\left(\frac{2+\varepsilon}{2} \right)^2 \det D^2 v - 1 \right] v \leq [c_1(\delta, \varepsilon) - c_2(\delta, \varepsilon)] \, |\partial \Omega| \,,$$

that implies

$$\left(\frac{2+\varepsilon}{2} \right)^2 \int_E \det D^2 v - |E| \leq \frac{c_1(\delta, \varepsilon) - c_2(\delta, \varepsilon)}{m} \, |\partial \Omega| \,, \qquad (27)$$

where

$$m = \min v = -u(0) > 0 \,.$$

Thanks to (13), (27) yields

$$\left(\frac{2+\varepsilon}{2} \right)^2 |\Omega| \leq (1+\delta)^2 \pi + \frac{c_1(\delta, \varepsilon) - c_2(\delta, \varepsilon)}{m} \, |\partial \Omega| \,,$$

whence, by (17),

$$|\partial \Omega| \leq c_3(\delta, \varepsilon) \, \pi \,, \qquad (28)$$

where

$$c_3(\delta, \varepsilon) = (1+\delta)^2 \left[\frac{(2+\varepsilon)(1-\delta)}{4} + \frac{c_2(\delta, \varepsilon) - c_1(\delta, \varepsilon)}{m} \right]^{-1} .$$

Since (9) and (20) imply $m \geq \dfrac{(1-\delta)^2}{2(2+\varepsilon)}$, it is not hard to compute $|c_3(\delta, \varepsilon) - 2| \leq C\sqrt{\delta^2 + \varepsilon^2}$, where C is a positive constant. Then (28) yields

$$|\partial \Omega| \leq 2\pi + C\sqrt{\delta^2 + \varepsilon^2} \,. \qquad (29)$$

Coupling (16) and (29), estimates (4) and (5) easily follow from (12).

Finally, (6) follows directly from the comparison principle for Poisson equation. Indeed, up to a translation, we can assume

$$B(0, r(\Omega)) = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < r(\Omega)\}$$
$$\subseteq \Omega$$
$$\subseteq B(0, R(\Omega)) = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < R(\Omega)\} \,.$$

Let us consider the functions

$$u_1(x) = \left(\frac{2-\varepsilon}{2} \right) \frac{x^2 + y^2 - r(\Omega)^2}{2}$$

and

$$u_2(x) = \left(\frac{2+\varepsilon}{2}\right)\frac{x^2+y^2-R(\Omega)^2}{2};$$

they are, respectively, solutions to

$$\begin{cases} \Delta u_1 = 2-\varepsilon & \text{in } B(0, r(\Omega)) \\ \\ u_1(x) = 0 & \text{when } \sqrt{x^2+y^2} = r(\Omega) \end{cases}$$

and

$$\begin{cases} \Delta u_2 = 2+\varepsilon & \text{in } B(0, R(\Omega)) \\ \\ u_2(x) = 0 & \text{when } \sqrt{x^2+y^2} = R(\Omega). \end{cases}$$

Then, since $u = 0$ on $\partial\Omega$ and $u_2 < 0$ on $\partial\Omega$, by comparison principle $u_2 < u$ in Ω. Using this fact and again the comparison principle, we can easily infer that $u_2 \le u \le u_1$ in $B(0, r(\Omega))$. Thus

$$||u-u_2||_{L^\infty(\Omega)} \le ||u_1-u_2||_{L^\infty(\Omega)},$$

where, in order to give sense to the right hand side, we extend u_1 to 0 outside $B(0, r(\Omega))$. Estimate (6) immediately follows from (4).   □

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