LE MATEMATICHE
Vol. LXII (2007) – Fasc. II, pp. 345–353

SINGULAR DIMENSION OF SPACES OF REAL FUNCTIONS

DARKO ŽUBRINIĆ

Let X be a space of measurable real functions defined on a fixed open set $\Omega \subseteq \mathbb{R}^N$. It is natural to define the singular dimension of X as the supremum of Hausdorff dimension of singular sets of all functions in X. We say that $f \in X$ is a maximally singular function in X if the Hausdorff dimension of its singular set is the largest possible. The paper discusses recent results about singular dimension of Banach spaces of functions, existence and density of maximally singular functions, and provides some open problems.

1. Introduction

Many spaces of real functions contain nice elements. For example, the Lebesgue spaces $L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, contain C_0^∞-functions as a dense subspace. But what are the wildest representatives of $L^p(\mathbb{R}^N)$? In general, given a space of real functions X it is natural to ask how do its wildest representatives look like. Here by wild functions we mean those having the corresponding singular set (to be defined below) with as large as possible Hausdorff dimension. It turns out that many classical spaces of functions indeed possess the wildest representatives, that we call maximally singular functions. Moreover, they usually form a dense

Entrato in redazione 1 gennaio 2007
AMS 2000 Subject Classification: 46E30
Keywords: Singular set, fractal set, singular dimension, maximally singular function, function spaces

subset in these spaces. See the first table below. Hence, the basic questions concerning a given space of functions X that we would like to answer are the following:

1. What is the supremum of Hausdorff dimension of singular sets of functions in X? We call it singular dimension of X.
2. Are there maximally singular functions in X? If so, are they dense in X?
3. How do singular sets o maximally singular functions look like?

Since the singular sets are negligible and of fractal nature, we need some basic notions from the analysis of fractal sets, in particular the notions of the upper box dimension and Hausdorff dimension, see Falconner [1]. The history of the study of singularities of functions in various spaces of functions is very rich, and goes back to 1950s, see [6] for a short survey.

2. Definition of singular dimension and maximally singular functions

Let a Lebesgue measurable function $f : \Omega \to \mathbb{R}$ be given, where $\Omega \subseteq \mathbb{R}^N$ is a fixed open set. We say that $a \in \Omega$ is a singular point of f if

$$f(x) \geq C|x-a|^{-\gamma}$$

a.e. in a neighbourhood of a for some positive constants C and γ depending on f. The set of all singular points of f is denoted by $\operatorname{Sing} f$, and we call it the singular set of f. If $\operatorname{Sing} f \neq \emptyset$ we say that the function f is singular. For any nonempty set X of measurable functions $f : \Omega \to \mathbb{R}$ we define the *singular dimension* of X by

$$\text{s-dim}\, X = \sup\{\dim_H(\operatorname{Sing} f) : f \in X\}. \tag{1}$$

If the supremum is achieved for some $f \in X$ and $\operatorname{Sing} f \neq \emptyset$, we say that the function f is *maximally singular* in X, or MS-function for short. The notions of singular dimension and MS-function have been introduced in [6] and [7] respectively.

There are many classical spaces of functions for which $\operatorname{Sing} f$ is empty for all $f \in X$, since the notion of singularity is too strong. For this reason we also consider weaker types of singularities of $f \in X$, say logarithmic and iterated logarithmic singularities. Given a measurable function $f : \Omega \to \mathbb{R}$ we therefore consider the extended singular set

$$\text{e-Sing}\, f = \{a \in \Omega : \limsup_{r\to 0} \frac{1}{r^N} \int_{B_r(a)} f(x)\, dx = +\infty\}.$$

It is easy to see that it contains $\mathrm{Sing}\, f$. Here by $B_r(a)$ we denote the open ball of radius r around a. If $\text{e-Sing}\, f \neq \emptyset$ we say that f is extended singular function. Analogously as in (1) we can then define the *upper singular dimension* of a space (or just a nonempty set) of functions X by

$$\text{s-}\overline{\dim}\, X = \sup\{\dim_H(\text{e-Sing}\, f) : f \in X\}, \tag{2}$$

see [6], and also [9]. It is clear that $\text{s-dim}\, X \leq \text{s-}\overline{\dim}\, X$, and the inequality may be strict. For example, for

$$X = \bigcap_{1 \leq p < \infty} L^p(\mathbb{R}^N)$$

we have $\text{s-dim}\, X = 0$ while $\text{s-}\overline{\dim}\, X = N$, see [6].

It is natural to define extended maximally singular functions $f \in X$ (EMS-functions for short) as those for which $\dim_H(\text{e-Sing}\, f) = \text{s-}\overline{\dim}\, X$.

The set of all MS-functions in X is denoted by $\mathscr{M}\mathscr{S}(X)$, while the set of all EMS-functions is denoted by $\mathscr{E}\mathscr{M}\mathscr{S}(X)$, that is,

$$\begin{aligned} \mathscr{M}\mathscr{S}(X) &= \{f \in X : \dim_H(\mathrm{Sing}\, f) = \text{s-dim}\, X\}, \\ \mathscr{E}\mathscr{M}\mathscr{S}(X) &= \{f \in X : \dim_H(\mathrm{Sing}\, f) = \text{s-}\overline{\dim}\, X\}. \end{aligned}$$

These sets have been introduced and studied in [13]. It is clear that either $\mathscr{M}\mathscr{S}(X) = \mathscr{E}\mathscr{M}\mathscr{S}(X)$ (when $\text{s-dim}\, X = \text{s-}\overline{\dim}\, X$), or $\mathscr{M}\mathscr{S}(X) \cap \mathscr{E}\mathscr{M}\mathscr{S}(X) = \emptyset$ (when $\text{s-dim}\, X < \text{s-}\overline{\dim}\, X$).

3. Results

3.1. Spaces of real functions

The following table provides the results about singular dimension of Lebesgue spaces $L^p(\mathbb{R}^N)$, $1 \leq p < \infty$, Sobolev spaces $W^{k,p}(\mathbb{R}^N)$, Bessel potential spaces $L^{\alpha,p}(\mathbb{R}^N)$, Besov spaces $B_\alpha^{p,q}(\mathbb{R}^N)$, Lizorkin-Triebel spaces $F_\alpha^{p,q}(\mathbb{R}^N)$, and finally Hardy spaces $H^1(\mathbb{R}^N)$, collected from [6], [7], [11], and [13]. In the table we assume that $1 < p < \infty$, $1 < q < \infty$, except for L^p-spaces. We do not know if the claims hold for $p = 1$ as well.

X	s-dim$X$	s-$\overline{\dim}X$	$\mathscr{M}\mathscr{S}(X)$
$L^p(\mathbb{R}^N)$, $1 \le p < \infty$	N	N	dense in X
$W^{k,p}(\mathbb{R}^N)$, $kp < N$	$N - kp$	$N - kp$	dense in X
$L^{\alpha,p}(\mathbb{R}^N)$, $\alpha p < N$	$N - \alpha p$	$N - \alpha p$	dense in X
$B_\alpha^{p,q}(\mathbb{R}^N)$, $\alpha p < N$	$N - \alpha p$	$N - \alpha p$	dense in X
$F_\alpha^{p,q}(\mathbb{R}^N)$, $\alpha p < N$	$N - \alpha p$	$N - \alpha p$	dense in X
$H^1(\mathbb{R}^N)$	N	N	dense in X?

The values of singular dimension are in accordance with intuition. For example, from s-dim$W^{k,p}(\mathbb{R}^N) = N - kp$ when $kp < N$, we see that the larger k or p, the smaller the dimensional size of singular sets of functions. Recall also that by the Sobolev imbedding theorem for $kp \ge N$ we have $\operatorname{Sing} f = \emptyset$ for all $f \in W^{k,p}(\mathbb{R}^N)$, and hence s-dim$W^{k,p}(\mathbb{R}^N) = 0$.

For $kp = N$ or $\alpha p = N$ the table also holds except for the density in the last column, since in these cases $\mathscr{M}S(X) = \emptyset$. But in these cases the set $\mathscr{E}MS(X)$ is dense in X and s-$\overline{\dim}W^{k,p}(\mathbb{R}^N) = 0$.

It is possible to construct a subspace X of $L^1(\mathbb{R}^N)$ such that s-dim$X = \alpha$ and s-$\overline{\dim}X = \beta$ for any prescribed $\alpha < \beta$ in $[0,N]$, see [13].

3.2. Elliptic boundary value problems

It is also of interest to know how large (in the sense of Hausdorff dimension) can be singular sets of solutions of the class of boundary value problems

$$-\Delta u = F(x), \quad u \in H_0^1(\Omega), \tag{3}$$

for various right-hand sides $F \in L^2(\Omega)$. This question can be regarded as the problem of nonregularity of solutions in a given class of boundary value problems. We believe that this is a legitimite and natural question in regularity theory of boundary value problems. Therefore we define the solution set of the class of problems (3), see [12],

$$X(\Omega,2) = \{u \in H_0^1(\Omega) : -\Delta u \in L^2(\Omega)\},$$

where the inclusion $-\Delta u \in L^2(\Omega)$ has to be understood in the weak sense.

More generally, the solution set of the class of boundary value problems

$$-\Delta_p u = F(x), \quad u \in W_0^{1,p}(\Omega), \tag{4}$$

for $F \in L^{p'}(\Omega)$, is

$$X(\Omega, p) = \{u \in W_0^{1,p}(\Omega) : -\Delta_p u \in L^{p'}(\Omega)\}.$$

Here p is fixed, $1 < p < \infty$, $p' = p/(p-1)$, and Δ_p is the p-Laplacian operator. Our aim is study the corresponding singular dimension of solution sets, i.e. to answer the question how large can be the set of singularities of weak solutions in a given class of boundary value problems.

In the following table Ω is an arbitrary open domain in $\mathbb{R}^N$ for $p = 2$, that is, for $X(\Omega, 2)$, while for $p \neq 2$ we assume that Ω is a bounded domain with sufficiently smooth boudnary, see [13] for details. Note that in the latter case the set $X(\Omega, p)$ is not a vector space. The table also indicates the corresponding open problems. The density in the table refers to the space $H_0^1(\Omega)$ for $p = 2$ (this can be streghtened to H^2-topology if Ω is sufficiently regular, see [13]), and to the $W_0^{1,p}$-topology for $p \neq 2$.

X	s-dim$X$	s-$\overline{\text{dim}}X$	$\mathscr{M}\mathscr{S}(X)$
$X(\Omega,2)$, $N \geq 4$	$N-4$	$N-4$	dense in X for $N \geq 5$
$X(\Omega,p)$, $N \geq pp'$, $p > 2$	$N-pp'$	$N-pp'$	$\neq \emptyset$ for $N > pp'$?
$X(\Omega,p)$, $N \geq pp'$, $p < 2$	?	?	$\neq \emptyset$ for $N > pp'$?

Maximally singular functions in $X(\Omega, 2)$, $N \geq 5$, are generated as solutions of (3) by maximally singular right-hand sides $F \in L^2(\Omega)$. For $N = 4$ we have $\text{Sing}\, u = \emptyset$ for all $u \in X(\Omega, 2)$, although u may possess weaker singularities. For $N \leq 3$ all solutions in $X(\Omega, 2)$ are continuous. We do not know if $X(\Omega, p)$ possesses maximally singular functions for $p \neq 2$, $N > pp'$. For $p > 2$ and $N < pp'$, using the Simon regularity result [4] for p-Laplace equations, combined with an imbedding result for Besov spaces, it is possible to show that $\text{Sing}\, u = \emptyset$ for each $u \in X(\Omega, p)$, hence, s-dim$X(\Omega, p) = \emptyset$, see [13] for details.

3.3. Open problems

Besides the open problems indicated in the above tables, there are many classical Banach spaces of real functions for which their singular dimensions are not known. This is the case for example with the following spaces (for their definitions see e.g. Kufner, John, Fučik [3]):

X	s-dim$X$	s-$\overline{\dim}X$	$\mathscr{M}\mathscr{S}(X)$
BMO spaces	?	?	dense in X?
Morrey spaces	?	?	dense in X?
Campanato spaces	?	?	dense in X?
Orlicz spaces	?	?	dense in X?
Lorentz spaces	?	?	dense in X?

Clearly, the answer will depend on various parameters entering the definition of a space.

3.4. Structure of singular sets

As for the structure of possible singular sets of MS-functions, we illustrate this in the case of Lebesgue and Sobolev spaces. Let a Lebesgue space $L^p(\mathbb{R}^N)$ be given, $1 \le p < \infty$. Let A_n be a sequence of subsets in $\mathbb{R}^N$ such that

$$\overline{\dim}_B A_n < N, \quad \dim_H A_n \to N \quad \text{as } n \to \infty. \tag{5}$$

Note that the condition $\overline{\dim}_B A_n < N$ implies that A_n's are negligible, and the second condition in (5) implies that $\dim_H(\cup_n A_n) = N$, due to countable stability of the Hausdorff dimension. The sets A_n with properties (5) can be constructed for example as Cartesian products of generalized Cantor sets in $[0,1]$ (see Falconner [1]) with $[0,1]^{N-1}$, that we call Cantor grills. Then there exists a MS-function $f \in L^p(\mathbb{R}^N)$, that is, $\dim_H(\operatorname{Sing} f) = N$, such that $\cup_n A_n \subseteq \operatorname{Sing} f$, and it can be described by an explicit formula. See [7] or [10] for details.

In the case of Sobolev spaces $W^{k,p}(\mathbb{R}^N)$, where $kp < N$, $1 < p < \infty$, we take a sequence of subsets A_n such that

$$\overline{\dim}_B A_n < N - kp, \quad \dim_H A_n \to N - kp \quad \text{as } n \to \infty. \tag{6}$$

Then there exists an explicit maximally singular Sobolev function $f \in W^{k,p}(\mathbb{R}^N)$ such that $\cup_n A_n \subseteq \operatorname{Sing} f$. See [7] or [10].

It would be of interest to know in general the structure of singular sets of MS-functions in a given space X. For example, if X is a Sobolev space as above, is it true that for *any* MS-function $f \in X$ there exists a sequence of subsets A_n in $\mathbb{R}^N$ satisfying (6), such that $\cup_n A_n \subseteq \operatorname{Sing} f$?

4. Comments and perspectives

4.1. MS-functions as an educational tool

We believe that the question of finding the optimal upper bound of the (Hausdorff dimensional) size of singular sets of functions in a given space X is legitimate. It is of interest to know functions in X possessing singular sets as large as possible in the sense of Hausdorff dimension. The construction of such functions is not obvious even in the case of simplest spaces of functions, like the Lebesgue spaces.

Nevertheless, the construction of maximally singular Lebesgue functions can be carried out using elementary means. We hope that it might serve as an educational tool already at the level of beginning undergraduate students of mathematics. It enables to better understand the complexity of Lebesgue spaces. See [7] and [10] for explicit construction of a Lebesgue integrable function $f \in L^1(a,b)$ such that its singular set has the Hausdorff dimension equal to 1. Even graduate students of mathematics specializing in Analysis most often have experience with integrable functions permitting no other singularity than a single point, defined by $f : B_r(a) \to \mathbb{R}$, $B_r(a) \subset \mathbb{R}^N$, $f(x) = |x-a|^{-\gamma}$, $\gamma < N$.

4.2. Other definitions of singular dimension?

In [10] we have shown that the upper box dimension is not a suitable mean for measuring the size of singular sets of functions from a given space X. Namely, if we replace $\dim_H$ by $\overline{\dim}_B$ in (1), then the corresponding singular dimension of X would be either zero or N, and therefore useless.

We do not know if it has sense to use any other fractal dimension instead of Hausdorff dimension in the definition of singular dimension (1), for example packing dimension.

One can also try to define i-maximally singular functions $f \in X$ as those for which the corresponding singular set $\operatorname{Sing} f$ is maximal with respect to inclusion. But it is easy to see that i-maximally singular functions do not exist. Indeed, if $f \in X$ were i-maximally singular, then clearly $A := \operatorname{Sing} f \neq \Omega$, since otherwise we would have $f \equiv \infty$. Take any $b \in \Omega \setminus A$, and let $f \in X$ be a function such that $\operatorname{Sing} f = \{b\}$ and $0 \le f(x) \le C|x-b|^{-\gamma}$ a.e. in Ω (we assume that X has this natural property). Then $f + g \in X$, and $\operatorname{Sing}(f+g) = A \cup \{b\}$, contrary to the assumption that f is i-maximally singular.

4.3. Integrable singular functions and Minkowski content

Many classical spaces of functions are defined as subspaces of a space of Lebesgue integrable functions. In [8], the repertoire of Lebesgue integrable functions of the form $d(x,A)^{-\gamma}$, defined on an r-neighbourhood A_r of a bounded set $A \subset \mathbb{R}^N$ for some fixed positive r (A_r is also called the Minkowski sausage around A, a term coined by B. Mandelbrot), has been extended to those with large singular sets, understood in the sense of the upper box dimension. In the study of integrability of such functions the Minkowski content of fractal sets A plays an important role, see [8] and the references therein. There we have introduced the notion of swarming functions in order to easily construct various fractal sets with desired box dimension and with partial control over the corresponding Minkowski contents.

In the construction of MS-functions one usually exploits fractal sets for which their Hausdorff and box dimensions are equal (like in the case of generalized Cantor sets), or are at least sufficiently close to each other (see for example (5) or (6)), together with suitable integral representations of functions in X. An illustration of this idea can be seen in [2], where maximally singular Sobolev and Bessel potential functions have been constructed, and in [11], where MS-functions in Besov and Lizorkin-Triebel spaces have been constructed. The construction of MS-functions has been extended to the case of general Banach spaces of real functions, see [13].

Acknowledgement. I express my gratitude to Professor Patrizia Pucci for information about regularity results due to J. Simon [4]. Also, many thanks to the organizers of the 2006 Third International Symposium dedicated to the memory of Gaetano Fichera in Taormina, Italy, and in particular to Professor Francesco Nicolosi.

REFERENCES

[1] K. Falconer, *Fractal Geometry*, Wiley, 1990.

[2] L. Horvat - D. Žubrinić, *Maximally singular Sobolev functions*, J. Math. Anal. Appl. **304** (2005), 531–541.

[3] A. Kufner - Õ. John - S. Fučik, *Function Spaces*, Academia, Prague, 1977.

[4] J. Simon, *Sur des équations aux dérivées partielles non linéaires*, Thése. Paris, 1977.

[5] C. Tricot, *Curves and Fractal Dimension*, Springer–Verlag, 1995.

[6] D. Žubrinić, *Singular sets of Sobolev functions*, C. R. Acad. Sci., Analyse mathématique, Série I **334** (2002), 539–544.

[7] D. Žubrinić, *Singular sets of Lebesgue integrable functions*, Chaos, Solitons and Fractals **21** (2004), 1281–1287.

[8] D. Žubrinić, *Analysis of Minkowski contents of fractal sets and applications*, Real Anal. Exchange **31** (2005), 315–354.

[9] D. Žubrinić, *Extended singular set of potentials*, Math. Inequal. Appl. **8** (2005), 173–177.

[10] D. Žubrinić, *Hausdorff dimension of singular sets of Sobolev functions and applications*, Proceedings of the 5th ISAAC Conference – 2005, University of Catania (2005), to appear.

[11] D. Žubrinić, *Maximally singular functions in Besov spaces*, Archiv der Math. (Basel) **87** (2006), 154–162.

[12] D. Žubrinić, *Generating singularities of solutions of p-Laplace equations on fractal sets*, Rocky Mountain J. Math. (2007), to appear.

[13] D. Žubrinić, *Singular dimension and maximally singular functions*, preprint (2007).

DARKO ŽUBRINIĆ
Department of Applied Mathematics
University of Zagreb,
Unska 3, 10000 Zagreb, Croatia
e-mail: `darko.zubrinic@fer.hr`