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POTENTIAL ANALYSIS FOR A CLASS OF DIFFUSION EQUATIONS: A GAUSSIAN BOUNDS APPROACH

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Let $\mathscr{H}$ be a linear second order partial differential operator with nonnegative characteristic form in a strip $S \subset \mathbb{R}^N \times \mathbb{R}$. We assume that $\mathscr{H}$ as a fundamental solution, smooth out of its poles and bounded from above and from below by Gaussian kernels modeled on subriemannian doubling distances in $\mathbb{R}^N$. Under these assumptions we show that $\mathscr{H}$ endows S with a structure of β-harmonic space. This allows us to study boundary value problems for L with a Perron-Wiener-Brelot-Bauer method, and to obtain pointwise regularity estimates at the boundary in terms of Wiener series modeled on the Gaussian kernels. Our analysis includes the proof of a scale invariant Harnack inequality for nonnegative solutions. We also show an application to the real hypersurphaces of $\mathbb{C}^{n+1}$ with given Levi-curvature.

1. Introduction and main results

In this note we present a series of results, obtained in collaboration with M. Bramanti, L. Brandolini and F. Uguzzoni (see [5],[6],[18]), related to a class of diffusion second order PDE's of the following type

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$$\mathscr{H} = \sum_{i,j=1}^{N} q_{i,j}(z)\partial^2_{x_i,x_j} + \sum_{j=1}^{N} q_j(z)\partial_{x_j} - \partial_t.$$

The coefficients $q_{i,j} = q_{j,i}, q_j$ are of class C^∞ in the strip

$$\begin{aligned} S &:= \{z = (x,t) \, : \, x \in \mathbb{R}^N, \, T_1 < t < T_2\} \\ &= \mathbb{R}^N \times]T_1, T_2[, \end{aligned}$$

where $-\infty \leq T_1 < T_2 \leq \infty$. We assume the characteristic form

$$q_{\mathscr{H}}(z,\xi) = \sum_{i,j=1}^{N} q_{i,j}(z)\xi_i\xi_j$$

is non-negative definite, and not identically zero, at any point $z \in S$. Moreover, the operator $\mathscr{H}$ is supposed to be hypoelliptic in S.

Together with the previous qualitative properties, we assume that $\mathscr{H}$ has a *fundamental solution* Γ satisfying the *Gaussian estimates*

$$\frac{1}{\Lambda} G_{b_0}(z,\zeta) \leq \Gamma(z,\zeta) \leq \Lambda G_{a_0}(z,\zeta),$$

where, $G_a(z,\zeta) = G_a(x,t;\xi,\tau) = 0$ if $t \leq \tau$, and

$$G_a(x,t;\xi,\tau) = \frac{1}{|B(x,\sqrt{t-\tau})|} \exp\left(-a\frac{d^2(x,\xi)}{t}\right)$$

if $t > \tau$. Λ, a_0, b_0 are positive constants.

Hereafter d is a *metric* in $\mathbb{R}^N$ and $|B(x,r)|$ denotes the Lebesgue measure of the d-ball $B(x,r)$. We also assume that the metric space $(\mathbb{R}^N,d)$ satisfies the following conditions:

- the d-topology is the Euclidean topology
- $\mathrm{diam}_d(\mathbb{R}^N) = \infty$
- $(\mathbb{R}^N,d)$ is a doubling space w.r to the Lebesgue measure, i.e. $0 < |B(0,2r)| \leq c_d|B(0,2r)|$, for every $x \in \mathbb{R}^N$, and $r > 0$
- $(\mathbb{R}^N,d)$ has the segment property, i.e. for every $x,y \in \mathbb{R}^N$ there exists $\gamma : [0,1] \to \mathbb{R}^N$, continuous and such that $d(x,y) = d(x,\gamma(t)) + d(\gamma(t),y)$ for every $t \in [0,1]$

In what follows, we shall denote by $|\mathscr{H}|$ the constant

$$|\mathscr{H}| := \Lambda + a_0 + b_0 + c_d$$

Under the previous assumptions, we proved several results, which can be summarized as follows.

(I) $\mathscr{H}$ endows S with a structure of β harmonic space (in the sense of Constantinescu&Cornea [8]) As a consequence, for every bounded open set $\Omega \subset \bar{\Omega} \subset S$, and for every $\varphi \in C(\partial\Omega)$, the Dirichlet Problem:

$$\begin{cases} \mathscr{H}u = 0 \quad \text{in} \quad \Omega \\ u|_{\partial\Omega} = \varphi \end{cases}$$

has a generalized solution $u = H^{\Omega}_{\varphi}$, in the sense of Perron-Wiener-Brelot-Bauer (PWBB solution).

(II) H^{Ω}_{φ} satisfies the boundary estimate

$$|H^{\Omega}_{\varphi}(z) - \varphi(z_0)| \le h(\varphi; z_0, z), \quad z_0 \in \partial\Omega,\ z \in \Omega$$

where $h(\varphi; z_0, z) \to 0$ as $z \to z_o$, for every φ, *if and only if* z_0 is a $\mathscr{H}$*- regular boundary point* for Ω.

We constructed $h(\varphi; .)$ by using a *Wiener-type series* of Γ *potentials*. As a consequence, $h(\varphi; .)$ can be estimated in terms of the Gaussian functions G_a's and some *geometric features* of Ω

(III) (Scale invariant Harnack inequality) If $\mathscr{H}u = 0$ and $u \ge 0$ in an open set containing $B(x_0, R) \times [t_0 - R^2, t_0]$, where $(x_0, t_0) =: z_0 \in S$, then

$$\max_{C_R(z_0)} u \le M\, u(z_0).$$

Here $C_R(z_0) := \overline{B_d(\xi_0, \gamma R)} \times [\tau_0 - \gamma R^2, \tau_0 - \frac{\gamma}{2} R^2]$

The constants $M > 0$ and $0 < \gamma < 1$ are independent of R and z_0. They *depend on the operator* $\mathscr{H}$ *only through the constant* $|\mathscr{H}|$.

We would like to close this Introduction by comparing our result in (III) with a remarkable well known Theorem in the Riemannian setting. Let (M, d) be a complete Riemannian manifold and let $\mathscr{H}$ its Heat operator (that is, a classical divergence form parabolic operator). Saloff-Coste and Grigoryan, starting from some deep ideas coming back to Nash and Moser, as implemented by Fabes and Strook, proved that

d – scaling invariant Harnack inequality for nonnegative solutions to $\mathscr{H}u = 0$

$$\Updownarrow$$

(M, d) is a doubling space + Gaussian bounds for the Heat kernel.

See [23] and he wide bibliograpy therein. Our result in (III) shows that, also in a *subriemannian setting* and for *nondivergence degenerate* parabolic operators

$$\text{Doubling property} + \text{Gaussian bounds} \implies \text{scale invariant Harnack inequality.}$$

2. Our main motivation

Many problems in geometric theory of several complex variables lead to fully nonlinear second order equations, whose linearizations are *nonvariational* operators modeled on vector fields satisfying rank conditions of Hörmander type. Here we would like to present one of these problems arising in the study of some curvature notions related to Levi form.

Let M be a real hypersurface, embedded in the Euclidean complex space $\mathbb{C}^{n+1}$. The Levi form of M at a point $p \in M$ is a Hermitian form on the complex tangent space whose eigenvalues $\lambda_1(p), ..., \lambda_n(p)$ determine a kind of *principal curvature* in the directions of each corresponding eigenvector. Then, given a generalized symmetric function s, in the sense of Caffarelli-Nirenberg-Spruck [7], one can define the s - Levi curvature of M at p, as follows:

$$S_p(M) = s(\lambda_1(p), ..., \lambda_n(p)),$$

see [1], [25], [26],[17]. When M is the graph of a function u and one imposes that its s-Levi curvature is equal to a given function, one obtains a second order fully nonlinear partial differential equation, which can be seen as the pseudoconvex counterpart of the usual fully nonlinear elliptic equations of Hessian type, as studied e.g. in [7]. In linearized form, the equations of this new class can be written as follows

$$\mathscr{L}u \equiv \sum_{i,j=1}^{2n} a_{ij} X_i X_j u = K(x, u, Du) \text{ in } \mathbb{R}^{2n+1} \tag{1}$$

where:

- the X_j's are first order differential operators, with coefficients depending on the gradient of u, which form a real basis for the complex tangent space to the graph of u;
- the matrix $\{a_{ij}\}$ depends on the function s and its entries depends on the first and second derivatives of u;
- K is a prescribed function.

It has to be noticed that $\mathscr{L}$ only involves $2n$ derivatives, while it lives in a space of dimension $2n+1$. Then, $\mathscr{L}$ is never elliptic, on any reasonable class of functions. However, the operator $\mathscr{L}$, when restricted to the set of strictly s-pseudoconvex functions, becomes *elliptic* along the $2n$ linearly independent directions given by the X_i's. The missing ellipticity direction can be recovered by a commutation. Precisely,

$$\dim\left(\operatorname{span}\left\{X_j, [X_i, X_j], i, j = 1, ..., 2n\right\}\right) = 2n+1$$

at any point, a kind of Hörmander rank condition of step 2 (see [17]).

The parabolic counterpart of $\mathscr{L}$, i.e. the operator

$$\mathscr{L} - \partial_t, \quad t \in \mathbb{R},\ x \in \mathbb{R}^{2n+1} \tag{2}$$

arises in studying the evolution by s - Levi curvature of a real hypersurface of $\mathbb{C}^{n+1}$, see[12], [19].

One of the main motivations of the present work is to provide the linear framework for the s-Levi equations and their parabolic counterpart.
For instance, a consequence of our Harnack inequality, applied to time-invariant solutions to $\mathscr{H}u = 0$, is the following: let u be a positive smooth strictly s-pseudoconvex solution to the Levi equation, $\mathscr{L}u = K$ with K of class C^∞. Then u satisfies a Harnack inequality of type:

$$\sup_{B_r} u \leq C \inf_{B_r} u$$

where B_r is the Carnot-Carathéodory ball of radius r, related to the vector fields $X_1, X_2, ..., X_{2n}$.
The unpleasant fact of this inequality is that the constant C depends on the solution u in an unspecified way. Understanding how C depends on u is an interesting and seemingly difficult open problem.

3. Main examples: nondivergence operators of Hörmander-type

Our basic examples of *diffusion operators*, satisfying the assumption stated in the Introduction, are suggested by the motivations presented in the previous section. These operators take the form

$$\mathscr{H} = \mathscr{L} - \partial_t = \sum_{i,j=1}^{q} a_{ij}(z) X_i X_j + \sum_{k=1}^{q} a_k(z) X_k - \partial_t$$

where:

- $X_1, X_2, \ldots, X_q$ are smooth vector fields in the open set $\Omega \subset \mathbb{R}^N$ satisfying the Hörmander condition

$$\text{rank Lie}\{X_i, i = 1,2,...,q\} = n \text{ at any point of } \Omega.$$

  Then $\mathscr{H}$ is hypoelliptic in Ω [11].

- $A(z) = (a_{ij}(z))_{i,j=1}^q$ is a symmetric matrix such that

$$\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}(z)\xi_i\xi_j \leq \lambda|\xi|^2$$

  for every $z = (x,t) \in S$, $\xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q$

A natural distance for the operator $\mathscr{H}$ is the *Carnot-Carathéodory* distance d generated by the vector fields $X_1, X_2, \ldots, X_q$. We would like to stress that d is well defined since the system $X = \{X_1, X_2, \ldots, X_q\}$ satisfies the Hörmander rank condition

In [6]we proved that X can be extended to a system of Hörmander vector fields, defined all over $\mathbb{R}^N$, in such a way that the associated Carnot-Carathéodory distance satisfies all the assumptions stated in the Introduction. Precisely:

- the d-topology is the Euclidean topology
- $\text{diam}_d(\mathbb{R}^N) = \infty$
- $(\mathbb{R}^N, d)$ is a doubling space w.r to the Lebesgue measure, i.e.
  $0 < |B(0,2r)| \leq c_d|B(0,2r)|$, for every $x \in \mathbb{R}^N$, and$r > 0$
- $(\mathbb{R}^N, d)$ has the segment property, i.e. for every $x, y \in \mathbb{R}^N$
  there exists$\gamma : [0,1] \to \mathbb{R}^N$, continuous, such that
  $d(x,y) = d(x,\gamma(t)) + d(\gamma(t),y)$ for every $t \in [0,1]$

We also extended the operator $\mathscr{H}$ to the whole $\mathbb{R}^{N+1}$ in such a way that outside of a compact set in the spatial variable, it becomes the classical Heat operator.

Still denoting by $\mathscr{H}$ the extended operator, under the qualitative assumption $a_{i,j}, a_j$ C^∞, we proved that $\mathscr{H}$ as a global fundamental solution Γ such that

$$\frac{1}{\Lambda} G_{b_0}(z,\zeta) \leq \Gamma(z,\zeta) \leq \Lambda G_{a_0}(z,\zeta)$$

and, for every $\alpha = (\alpha_i, \alpha_j.\alpha_k)$ with $|\alpha| \leq 2$,

$$|D^\alpha \Gamma(z,\zeta)| \leq \Lambda|t-\tau|^{-\frac{|\alpha|}{2}} G_{a_0}(z,\zeta).$$

Here we have used the notation $D^\alpha := X_i^{\alpha_i} X_j^{\alpha_j} \partial_t^{\alpha_k}$ and $|\alpha| := \alpha_i + \alpha_j + 2\alpha_k$.

In these inequalities, Λ, a_0, b_0 are positive structural constants: they only depend on

- the doubling constant of d
- the constant λ in

$$\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}(z)\xi_i\xi_j \leq \lambda|\xi|^2$$

- the d-Hölder norms of the coefficients $a_{i,j}$ and a_j.

As a consequence: *all our results extend to the operators*

$$\mathscr{H} := \mathscr{L} - \partial_t := \sum_{i,j=1}^{q} a_{ij}(z)X_iX_j + \sum_{k=1}^{q} a_k(z)X_k - \partial_t$$

with d-Hölder continuous coefficients $a_{i,j}$ and a_j.

To close this section, we have to say that the method we followed is based on an adaptation to our sub-riemannian setting of the classical Levi parametrix method, as in [2]. In doing that we used a large amount of ideas, techniques and results due to Rotschild&Stein [22], Jerison&Sanchez-Calle [13], Fefferman&Sanchez-Calle [10] and to Kusuoka&Stroock [14], [15], [16].

4. General diffusion operators

Let us consider the operator

$$\mathscr{H} = \sum_{i,j=1}^{N} q_{i,j}(z)\partial^2_{x_i,x_j} + \sum_{j=1}^{N} q_j(z)\partial_{x_j} - \partial_t$$

in the strip $S = \mathbb{R}^N \times]T_1, T_2[, \ -\infty \leq T_1 < T_2 \leq \infty$. We assume, together with the smoothness of the coefficients and the hypoellipticity of $\mathscr{H}$, the d-Gaussian bounds of Γ-the fundamental solution of $\mathscr{H}$-, the equivalence between the d-topology and the Euclidean one, the doubling and the segment property of the distance d, as stated in the Introduction.

All the results we present in this section have been proved in [18].

A first trivial remark is that

$$\Omega \longrightarrow \mathscr{H}(\Omega) := \{u \in C^\infty(\Omega) \ : \ \mathscr{H}u = 0\}$$

is a linear sheaf of functions on S. We agree to call $\mathscr{H}(\Omega)$ the space of the $\mathscr{H}$-harmonic functions in Ω.

We first proved that $(S, \mathscr{H})$ is a β-harmonic space. Precisely:

- There exists a strictly positive $\mathscr{H}$-harmonic functions on every shrinked strip $\mathbb{R}^N \times]T, T_2[$, $T_1 < T < T_2$. This function is given by

$$u(x,t) = \int_{\mathbb{R}}^{N} \Gamma(x,t;\xi,T)d\xi,$$

- There exists a family of bounded open set $\{V\}$, that form a basis of the Euclidean topology of S, such that the Dichlet problem

$$\begin{cases} \mathscr{H}u = 0 \quad \text{in} \quad V \\ u|_{\partial V} = \varphi \end{cases}$$

  has a classical solution for every $\varphi \in C(\partial V)$. This follows from a Bony's argument that uses the non-total degeneracy of $q_{\mathscr{H}}$, the characteristic form of $\mathscr{H}$ [4].

- $\{\Gamma(.,\zeta) : \zeta \in S\}$ is a family of $\mathscr{H}$-superharmonic functions separating the points of S.

- (Doob convergence property)

$$\mathscr{H}(\Omega) \ni u_n \nearrow u, \quad u < \infty \text{ in a dense subset of } \Omega$$

$$\Downarrow$$

$$u \in \mathscr{H}(\Omega)$$

  This follows from the hypoellipticity of $\mathscr{H}$, by using an argument again due to Bony [4].

General results from Abstract Potential Theory [8], now imply that

$$H_{\varphi}^{\Omega} := \inf\{u \ : u \text{ is } \mathscr{H} - \text{superharmonic in } \Omega : \liminf u \geq \varphi \text{ on } \partial\Omega\}$$

is $\mathscr{H}$-harmonic in Ω, for every bounded open set $\Omega \subset \bar{\Omega} \subset S$, and for every $\varphi \in C(\partial\Omega)$. H_{φ}^{Ω} is the PWBB generalized solution to

$$\begin{cases} \mathscr{H}u = 0 \quad \text{in} \quad \Omega \\ u|_{\partial\Omega} = \varphi \end{cases}$$

By using several devices from Balayage Theory in Abstract Harmonic Spaces, we also have estimated the behavior of H_{φ}^{Ω} at any point of $\partial\Omega$. Indeed, we proved that

$$|H_{\varphi}^{\Omega}(z) - \varphi(z_0)| \leq h(\varphi; z_0, z), \quad z_0 \in \partial\Omega, \, z \in \Omega.$$

We constructed the function $h(\varphi;.)$ by using a *Wiener-type series* of Γ *potentials*. As a consequence, $h(\varphi;.)$ can be estimated in terms of the Gaussian functions G_a's and of some *geometric properties* of the open set Ω. In particular, it turns out that $h(\varphi;z_0,z)\to 0$ as $z\to z_o$, for every $\varphi\in C(\partial\Omega)$, *if and only if* z_0 is a $\mathscr{H}$*- regular boundary point* for Ω.

As one can expect, our estimates of the function $h(\varphi;.)$ take simpler form when Ω is a cylindrical domain. As an example: let us consider an open set $\Omega = D\times]a,b[$, where $D\subset\mathbb{R}^N$ and $T_1<a<b<T_2$. Assume that, for a suitable $\theta\in]0,1[$ and for every r sufficiently small,

$$|B(x_0,r)\setminus D|\geq \theta|B(x_0,r)|,\quad x_0\in\partial D.$$

Then, for every $t_0\in]a,b[$, the point $z_0=(x_0,t_0)$ is $\mathscr{H}$-regular for the Dirichlet problem. Moreover, if the function φ is d-Hölder continuous at $z_0=(x_0,t_0)$, then there exists $\gamma\in]0,1[$ such that

$$|H^{\Omega}_{\varphi}(z)-\varphi(z_0)|\leq C_{\varphi}((d(x_0,x))^4+(t-t_0)^2)^{\frac{\gamma}{4}}.$$

This particular boundary regularity results played a crucial role in our proof of the invariant Harnack inequality for nonnegative solutions to $\mathscr{H}u=0$. Here, as in [3], we adapted to our setting the Fabes&Strook's implementation [9] of the "'hold" idea by Nash [21]. The main difficulty we encountered was the construction, and the estimates from above and from below, of the Green function for cylindrical domains like $C(z_0,r):=B(x_0,r)\times]t_0-r^2,t_0+r^2[$. The difficulty arises from the fact that, in general, the *parabolic boundary* of these domains are not $\mathscr{H}$-regular for the Dirichlet problem.

To overcome this difficulty, for every fixed δ, and $\theta\in]0,1[$, we constructed domains $A(x_0,r)$ such that

- $B(x_0,\delta r)\subset A(x_0,r)\subset B(x_0,r)$,
- $|B(y,r)\setminus A(y,r)|\geq\theta|B(x_0,r)|,\quad$ for every $y\in\partial A(x_0,r)$

Then: from the previous boundary regularity results for cylindrical domains, the parabolic boundary of the cylinder

$$\mathbf{C}(z_0,r)=A(x_0,r)\times]t_0-r^2,t_0+r^2[$$

is $\mathscr{H}$-regular for the Dirichlet problem. As a conseguence: the Green function for such a domain exists and satisfies "right" Gaussian bounds from above and from below. We would like to explicitly remark that in our construction of the domains $A(x_0,r)$'s, as well as in the proof of the Gaussian bounds for their Green functions, the *segment property* of the metric d played a crucial role.

We close this section by mentioning that, if the coefficient of $\mathscr{H}$ are t independent, i.e. if

$$\mathscr{H} = \sum_{i,j=1}^{N} q_{i,j}(x)\partial^2_{x_i,x_j} + \sum_{j=1}^{N} q_j(x)\partial_{x_j} - \partial_t := \mathscr{L} - \partial_t$$

then every solution to $\mathscr{L}u = 0$ also solves $\mathscr{H}u = 0$. As a consequence: our Harnack inequality trivially extends to $\mathscr{L}$. Precisely, let

$$\mathscr{L}u = 0, u \geq 0 \text{ in an open set } D \subset \mathbb{R}^N, \text{ such that } B(x_0, 2r) \subset D$$

Then

$$\max_{B(x_0,r)} u \leq M \inf_{B(x_0,r)} u(x_0).$$

where M is a positive structural constant.

5. An application: graphs having the same Levi-curvature

Let $u,\ v : D \to \mathbb{R}$ be smooth functions, where D is a subset of $\mathbb{R}^{2n+1}$.

Assume

- $u \leq v$ in D
- u and v are s-strictly pseudoconvex
- the graphs of u and v have the same s-Levi curvature.

Then $w := v - u \geq 0$ and satisfies

$$\mathscr{L}w \equiv \sum_{i,j=1}^{2n} b_{ij} X_i X_j w = 0 \text{ in } D$$

The vector fields X_j's take the form $X_j\partial_{x_j} + a_j(\nabla u)\partial_{x_{2n+1}}$, and satisfie the rank conditions

$$\dim(\mathrm{span}(\{X_j, [X_i, X_j] : i, j = 1, \dots, 2n+1\})) = 2n+1$$

at any point of D [17]. Then, if $B(x_0, 2r) \subset D$,

$$\max_{B(x_0,r)} w \leq M \inf_{B(x_0,r)} w(x_0).$$

where $B(x_0, r)$ is the Carnot-Carathéodory distance related to the vector fields X_j's.

In particular if $u(x_0) = v(x_0)$ at a point $x_0 \in D$ and D is connected, then

$$u \equiv v \text{ in } \Omega$$

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