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ON THE MAXIMUM PRINCIPLE FOR VISCOSITY SOLUTIONS OF FULLY NONLINEAR ELLIPTIC EQUATIONS IN GENERAL DOMAINS

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We analyze the validity of the Maximum Principle for viscosity solutions of fully nonlinear second order elliptic equations in general unbounded domains under suitable structure conditions on the equation allowing notably quadratic growth in the gradient terms.

1. Introduction

Consider the fully nonlinear partial differential equation

$$F\left(x,u(x),Du(x),D^2u(x)\right) = 0\ ,\ x \in \Omega \qquad (\mathbf{PDE})$$

Here Du, D^2u denote, respectively, the gradient and the Hessian matrix of the function u. We will assume that the continuous function $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}^N \to \mathbb{R}$ is *non-decreasing* in the matrix variable with respect to the partial order on the space $\mathscr{S}^N$ of real $N \times N$ symmetric matrices induced by positive semi-definiteness.
Due to the fully nonlinear character of the *degenerate elliptic* equation **(PDE)**, it

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is appropriate to understand solutions in the generalized *viscosity sense*. A comprehensive well-posedness and regularity theory has been developed for viscosity solutions of equations of the above type, mainly in the case of a bounded domain Ω, see for example [9], [17], [21], [25] and the references therein.
Our purpose here is to present a few recent results, mostly taken from [13] and [15], concerning some qualitative properties, related to the validity of the Maximum Principle, of viscosity solutions of **(PDE)** in a *general*, possibly *unbounded domain* $\Omega \subset \mathbb{R}^N$. More precisely, we will analyze the validity of the Alexandrov-Bakelman-Pucci Maximum Principle and of the Phragmen-Lindelöf Principle for viscosity solutions of **(PDE)**. In this context, a relevant role will be played by some measure-geometric properties of the unbounded domain Ω, but not on its smoothness. Let us point that our results can be seen as extensions in several directions (namely, nonlinearity of the equation, generality of the domain and non smoothness of solutions) of the corresponding results for strong solutions (i.e., twice Sobolev-differentiable functions) of linear uniformly elliptic equations in non-divergence form

$$\mathrm{Tr}(A(x)D^2u) + b(x)\cdot Du + c(x)u = 0\ ,\ x \in \Omega \qquad (\mathbf{LPDE})$$

see for example [1], [3], [5], [28], [31].
Indeed, our results in Section 6 and 7 extend the validity of the Maximum Principle to *bounded above* viscosity subsolutions of **(PDE)** for degenerate elliptic nonlinear differential operators satisfying

$$F(x,t,p,X) \leq \mathscr{P}^+_{\lambda,\Lambda}(X) + b(x)|p| + b_2|p|^2 + c(x)t\,,$$

where $\mathscr{P}^+_{\lambda,\Lambda}$ is the Pucci maximal operator, see Section 2, in unbounded domains Ω such as spirals, complements of infinite hypersurfaces, periodic lattices of balls which can be seen as wide generalizations of cones and cylinders, see Section 5 and Subsection 7.2 for more examples.
Let us point out explicitly that we only suppose $c(x)$ to be non-positive and even, in the special case of narrow domains, we allow $c(x)$ to be bounded above by a small positive constant. It is well-known that if c is bounded above by a negative constant, then the Maximum Principle holds in any domain, see [23], [22]. On the other hand, as explained in Section 4, the Maximum Principle fails in exterior domains in the case $c(x) \leq 0$.
In Section 8 we show the boundedness of subsolutions from above can be relaxed in order to obtain some qualitative Phragmèn-Lindelöf theorems for subsolutions with polynomial (respectively, exponential) growth at infinity in domains of conical (respectively, cylindrical) type. Analogous results can be shown for supersolutions.
The somewhat related issue of Liouville type theorems for viscosity solutions of

(PDE) will not be touched in the present paper. Recent results in this direction can be found in [16], [12], [35].

After two short sections on basic assumptions, viscosity solutions and a quick review of results in the linear case, the paper addresses the following topics:

- the Alexandrov-Bakelman-Pucci Maximum Principle for **(LPDE)**,
- **G**- and **wG**- domains,
- the Alexandrov-Bakelman-Pucci **ABP** Maximum Principle for **(PDE)**,
- some extensions of **ABP** Maximum Principle for **(PDE)** to more general domains, solutions with exponential growth in narrow domains, equations with quadratic growth in Du,
- Phragmèn-Lindelöf theorems for **(PDE)**,
- Minimum Principles for **(PDE)**.

2. Basic assumptions

Let $\mathscr{S}^N$ be the set of $N \times N$ symmetric matrices endowed with partial ordering:

$$X \geq Y \quad \text{if and only if} \quad X - Y \quad \text{positive semidefinite.}$$

Following [9], we consider the class of continuous functions $F : \Omega \times \mathrm{I\!R} \times \mathrm{I\!R}^N \times \mathscr{S}^N \to \mathrm{I\!R}$ such that

$$\lambda \operatorname{Tr}(Q) \leq F(x,t,p,X+Q) - F(x,t,p,X) \leq \Lambda \operatorname{Tr}(Q) \quad \text{for any} \quad Q \geq O \quad (1)$$

for given positive constants λ and Λ (in the above, Tr denotes the trace of a matrix).

The left-hand side inequality is a *uniform ellipticity* condition. Note that conditions above imply that F is monotonically increasing and Lipschitz continuous in the matrix variable.

Our basic assumptions on F will be however a bit less restrictive; we will assume indeed just *non-decreasing* monotonicity (that is, *degenerate ellipticity*) in the matrix variable of function F and some extra structure condition that will be specified next.

The class of functions defined by condition (1) contains two extremal operators which will play a crucial role in our analysis. For

$$\mathscr{A}_{\lambda,\Lambda} = \left\{ A \in \mathscr{S}^N : \lambda I \leq A \leq \Lambda I \right\}$$

a compact convex subset of $\mathscr{S}^N$, the convex function

$$X \to \sup_{A \in \mathscr{A}_{\lambda,\Lambda}} \mathrm{Tr}\,(AX) := \mathscr{P}^+_{\lambda,\Lambda}(X)$$

is the Pucci maximal operator, which is positively homogeneous

$$\mathscr{P}^+_{\lambda,\Lambda}(tX) = t\,\mathscr{P}^+_{\lambda,\Lambda}(X),\ t \geq 0$$

and subadditive

$$\mathscr{P}^+_{\lambda,\Lambda}(X+Y) \leq \mathscr{P}^+_{\lambda,\Lambda}(X) + \mathscr{P}^+_{\lambda,\Lambda}(Y).$$

Since any $X \in \mathscr{S}^N$ can be decomposed as $X = X^+ - X^-$ with $X^+ \geq 0$, $X^- \geq 0$ and $X^+X^- = 0$, it follows that

$$\mathscr{P}^+_{\lambda,\Lambda}(X) = \Lambda\,\mathrm{Tr}(X^+) - \lambda\,\mathrm{Tr}\,(X^-)$$

Respectively, the concave function

$$X \to \inf_{A \in \mathscr{A}_{\lambda,\Lambda}} \mathrm{Tr}\,(AX) := \mathscr{P}^-_{\lambda,\Lambda}(X)$$

is the Pucci minimal operator which is positively homogeneous and superadditive. Note that

$$\mathscr{P}^-_{\lambda,\Lambda}(X) = -\mathscr{P}^+_{\lambda,\Lambda}(-X) = \lambda\,\mathrm{Tr}(X^+) - \Lambda\,\mathrm{Tr}\,(X^-),$$

and, moreover,

$$\mathscr{P}^-_{\lambda,\Lambda}(X+Y) \leq \mathscr{P}^-_{\lambda,\Lambda}(X) + \mathscr{P}^+_{\lambda,\Lambda}(Y).$$

Note also that for any F satisfying (1) the following holds

$$\mathscr{P}^-_{\lambda,\Lambda}(X) + F(x,t,p,O) \leq F(x,t,p,X) \leq \mathscr{P}^+_{\lambda,\Lambda}(X) + F(x,t,p,O). \qquad (2)$$

Assuming F to be non-increasing with respect to $t \in [0,+\infty)$ and that

$$F(x,0,p,O) \leq b(x)|p|$$

for some non-negative function $b(x) \in C(\Omega) \cap L^\infty(\Omega)$, from the right-hand side of (2) we derive the *above structure condition*:

$$F(x,t,p,X) \leq \mathscr{P}^+_{\lambda,\Lambda}(X) + b(x)|p|, \qquad (\mathbf{ASC})$$

which will be sufficient for most results concerning viscosity subsolutions of **(PDE)**. The symmetric *below structure condition*

$$F(x,t,p,X) \geq \mathscr{P}^-_{\lambda,\Lambda}(X) - b(x)|p| \qquad (\mathbf{BSC})$$

will be used for supersolutions.

We point out that while condition (1) implies **(ASC)** and **(BSC)**, the converse is not true except in special cases. For instance, if F is linear in the matrix variable, then **(ASC)** is enough to imply uniform ellipticity of the operator $F(x,t,0,X)$. Indeed,

$$-F(x,t,0,-Y) = F(x,t,0,X+Y) - F(x,t,0,X) = F(x,t,0,Y)$$

so, using the relationship between the extremal Pucci operators, for $Y \geq 0$ we get

$$\lambda \operatorname{Tr}(Y) \leq \mathscr{P}^-_{\lambda,\Lambda}(Y) = -\mathscr{P}^+_{\lambda,\Lambda}(-Y) \leq$$
$$\leq F(x,t,0,X+Y) - F(x,t,0,X) \leq \mathscr{P}^+_{\lambda,\Lambda}(Y) \leq \Lambda \operatorname{Tr}(Y).$$

Some nonlinear degenerate elliptic operators fulfill our assumptions but not (1), for example

$$F(X) = \Lambda \left(\sum_{i=1}^{N} \varphi(\mu_i^+) \right) - \lambda \left(\sum_{i=1}^{N} \psi(\mu_i^-) \right)$$

Here, $\mu_i^\pm$, $i = 1, \dots N$, are the positive and negative eigenvalues of the matrix $X(x) \in \mathscr{S}^N$ and φ, ψ : $[0,+\infty) \to [0,+\infty)$ are continuous and nondecreasing functions such that

$$\varphi(s) \leq s \leq \psi(s)\,.$$

We will also consider the case of quadratic growth in the gradient. In this case we will employ the structure conditions

$$F(x,t,p,X) \leq \mathscr{P}^+_{\lambda,\Lambda} + b(x)|p| + b_2|p|^2 \qquad (\textbf{QASC})$$

and

$$F(x,t,p,X) \geq \mathscr{P}^-_{\lambda,\Lambda} - b(x)|p| - b_2|p|^2, \qquad (\textbf{QBSC})$$

where b_2 is a positive constant.

3. Viscosity solutions

A function $u \in USC(\Omega)$ is a viscosity subsolution of **(PDE)** if the inequality

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0$$

holds at any point $x_0 \in \Omega$ and for all quadratic polynomials φ touching from above the graph of u at x_0, i.e. $\varphi(x_0) = u(x_0)$ and $u(x) \geq \varphi(x)$ in a neighborhood of x_0.

Observe that u is a viscosity solution of $\Delta u \geq 0$ if and only if u is subharmonic function in the sense of potential theory: for any ball $B \subset \Omega$ and for all h such that $\Delta h = 0$ in B, the inequality $u \leq h$ on ∂B implies $u \leq h$ in B, see [10]

Viscosity supersolutions are defined in a symmetric way: a function $u \in LSC(\Omega)$ is a viscosity supersolution of **(PDE)** if the inequality

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0$$

holds at any point $x_0 \in \Omega$ and for all quadratic polynomials φ touching from below the graph of u at x_0.
A viscosity solution of **(PDE)** is a function $u \in C(\Omega)$ which is both a sub- and a supersolution. Observe also that any $u \in USC(\Omega) \cap C^2(\Omega)$ satisfying **(PDE)** in the viscosity sense is a classical solution of the equation and, conversely, that any classical solution $u \in C^2(\Omega)$ is a viscosity solution. Main general references on viscosity solutions are [17], [9], [14].

4. The ABP Maximum Principle for linear inequalities

The classical **(ABP)** estimate in a *bounded* domain Ω of $\mathbb{R}^N$, see [20], is

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \operatorname{diam}(\Omega) \, ||f||_{L^N(\Omega)} \qquad (\mathbf{ABP})$$

where u is any $C(\overline{\Omega}) \bigcap W^{2,N}(\Omega)$ solution of the uniformly elliptic inequality

$$\operatorname{Tr}\left(A(x) D^2 u\right) + b(x) \cdot Du \geq f(x) \ \text{ in } \Omega\,.$$

As an immediate consequence of **(ABP)**, the Maximum Principle holds:
if $c \leq 0$ and $u \in C(\overline{\Omega}) \bigcap W^{2,N}(\Omega)$ satisfies

$$\operatorname{Tr}\left(A(x) D^2 u\right) + b(x) \cdot Du + c(x) u \geq 0 \ \text{ in } \Omega\,, \tag{3}$$

then

$$u \leq 0 \ \text{ on } \partial\Omega \quad \text{implies} \quad u \leq 0 \ \text{ in } \ \Omega. \qquad (\mathbf{MP})$$

Extensions of **(ABP)** and **(MP)** to *unbounded domains of finite measure* have been established in [4] for *bounded above* solutions of (3), substituting **(ABP)** the dependence on $\operatorname{diam}(\Omega)$ with the Lebesgue measure $|\Omega|$ of the domain:

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \, |\Omega|^{\frac{2}{N}} \, ||f||_{L^\infty(\Omega)},$$

On the other hand, it is well-known that **(MP)** does not hold in an arbitrary *unbounded domain of infinite measure*:

$$u(x) = 1 - 1/|x|^{N-2}, N \geq 3\,,$$

is harmonic on the exterior domain $\Omega = \mathbb{R}^N \setminus \overline{B}_1(0)$, $u \equiv 0$ on $\partial\Omega$ but $u > 0$ in Ω.
Note also that **(MP)** may not hold for unbounded above solutions:

$$u(x) = u(x_1, x_2) = e^{x_1} \sin x_2$$

is harmonic and unbounded on the unbounded plane strip

$$\Omega = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : |x_2| < \pi \right\}$$

and $u \equiv 0$ on $\partial\Omega$, but u changes sign in Ω. To enforce the validity of **(MP)** in unbounded domain of infinite measure some condition on the domain needs therefore to be imposed. For the results in Sections 6 and 7 we will consider only bounded above solutions, while this restriction will be relaxed in Section 8

5. G- and wG- domains

An improved form of the **(ABP)** estimate for bounded above strong solutions of linear inequalities as above has been established in [5]:

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C R(\Omega) \, ||f||_{L^n(\Omega)} \qquad (\mathbf{ABP})_{\mathbf{G}}$$

depending on a new geometric constant $R(\Omega)$ for domains satisfying the following condition **(G)** requiring, roughly speaking, that there is *enough boundary near every point* in Ω:

(G) for fixed numbers $\sigma, \tau \in (0,1)$, there exists a positive real number $R(\Omega)$ such that for any $y \in \Omega$ there exists an N-dimensional ball B_{R_y} of radius $R_y \leq R(\Omega)$ satisfying

$$y \in B_{R_y}\,, \quad |B_{R_y} \setminus \Omega_{y,\tau}| \geq \sigma |B_{R_y}|$$

where $\Omega_{y,\tau}$ is the connected component of $\Omega \cap B_{R_y/\tau}$ containing y.

Note that condition **(G)** implies in particular

$$\sup_{y \in \Omega} \operatorname{dist}(y, \partial\Omega) < \infty .$$

Condition (**G**) is satisfied by domains with finite measure with

$$R(\Omega) = C(N) |\Omega|^{\frac{1}{N}}$$

and also for a large class of unbounded domains with infinite measure. For instance, cylinders and slabs:

$$\Omega = \mathbb{R}^k \times \omega,\ k \geq 1,$$

where ω is a bounded domain in $\mathbb{R}^{N-k}$ of diameter d satisfy (**G**) with $R(\Omega) = d$. For this the reason we will sometimes in the sequel refer to **G** - domains as to *cylindrical domains*. As another example, the complement of a *periodic lattice* of balls of period l:

$$\Omega = \mathbb{R}^N \setminus \sum_{\mathbf{k} \in \mathbb{Z}^\mathbf{N}} (l\mathbf{k} + \mathbf{B_1}(\mathbf{0})),$$

satisfy (**G**) with $R(\Omega) = l$. A further example is provided by the complement of a *plane spiral* with constant step σ, in polar cohordinates

$$\Omega = \mathbb{R}^2 \setminus \{\rho = \frac{\sigma}{2\pi}\theta\}$$

In this case, $R(\Omega) = s$.

More general domains, which satisfy a weaker form (**wG**) of condition (**G**), have been considered in [7], [37]. Condition (**wG**) is as follows:

(**wG**) there exist constants $\sigma, \tau \in (0,1)$ such that for all $y \in \Omega$ there is a ball B_{R_y} of radius R_y containing y such that

$$|B_{R_y} \setminus \Omega_{y,\tau}| \geq \sigma |B_{R_y}|$$

where $\Omega_{y,\tau}$ is the connected component of $\Omega \cap B_{R_y/\tau}$ containing y.

Observe that condition (**wG**) with $R_y = O(1)$ as $|y| \to \infty$ implies condition (**G**)
.

Typical examples of unbounded domains satisfying (**wG**) but not (**G**) are *nondegenerate cones* of $\mathbb{R}^N$ (and their unbounded subsets). For those sets, (**wG**) holds with $R_y = O(|y|)$ as $|y| \to \infty$. Such domains will be referred to as *conical domains*.
A less standard example of **wG**-domain is the complement of the *logarithmic spiral*; in polar cohordinates,

$$\Omega = \mathbb{R}^2 \setminus \left\{\rho = \mathrm{e}^\theta,\ \theta \geq 0\right\}$$

Condition (**wG**) is satisfied with $R_y = O(|y|)$ as $|y| \to \infty$. Observe in this respect that in the case

$$\Omega = \mathbb{R}^2 \setminus \{\rho = h(\theta),\ \theta \geq 0\},$$

with $h(\theta)$ growing faster than exponentially, condition (**wG**) might not be verified with the above bound $R_y = O(|y|)$.

In a conical domain, the following variant of the improved **(ABP)** estimate of [5] was proved in [37] for bounded above strong solutions of (3), provided that $b(x) = O(1/|x|)$ as $|x| \to \infty$,

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \sup_{y\in\Omega} ||(|x|+1)f(x)||_{L^n(\Omega_{y,\tau})} \qquad (\mathbf{ABP})_{\mathbf{con}}$$

This yields **(MP)** under the above assumptions. However, no bound on the growth of R_y at infinity is required to have **(MP)** in the case $b \equiv 0$, see [7].
The above mentioned results are based on a suitable version of the Krylov-Safonov Growth Lemma, which can be obtained using a boundary weak Harnack inequality.
Recently, interior estimates as Harnack inequalities have been extended to viscosity setting, where it is natural to carry out up to the boundary [36], [8], [9], [27], [25].

6. The ABP Maximum Principle for (PDE)

The next result from [13] is a generalization of $(\mathbf{ABP})_{\mathbf{G}}$ and $(\mathbf{ABP})_{\mathbf{con}}$ estimates and, consequently, of the Maximum Principle to viscosity solutions of fully nonlinear equations.

Theorem 1. Let $u \in USC(\overline{\Omega})$ with $\sup_{\Omega} u < +\infty$ be a viscosity solution of

$$F(x,u,Du,D^2u) \geq f(x) \ , \ x \in \Omega,$$

where $f \in C(\Omega) \cap L^\infty(\Omega)$ and Ω is a domain of $\mathbb{R}^N$ satisfying condition (**wG**) . Assume that F is continuous and elliptic and that **(ASC)** holds for given constants $0 < \lambda \leq \Lambda$ and some $0 < b \in C(\Omega) \bigcap L^\infty(\Omega)$. Assume moreover that (**wG**) is satisfied for some R_y such that

$$Rb := \sup_{y\in\Omega} R_y \, ||b||_{L^\infty(\Omega_{y,\tau})} < \infty. \qquad (\blacksquare\mathbf{b})$$

Then,

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \sup_{y\in\Omega} R_y \, ||f^-||_{L^N(\Omega_{y,\tau})} \qquad (\mathbf{ABP})_{\mathbf{wG}}$$

for some positive constant C depending on $N,\ \lambda,\ \Lambda,\ \sigma,\ \tau$ and Rb.

We refer for the proof to [13]. As an immediate consequence of $(\mathbf{ABP})_{\mathbf{wG}}$, the Maximum Principle holds:
if $u \in USC(\overline{\Omega})$ is bounded above and

$$F(x,u,Du,D^2u) \geq 0 \ , \ x \in \Omega \, ,$$

then

$$u \leq 0 \text{ on } \partial\Omega \quad \text{implies} \quad u \leq 0 \text{ in } \Omega. \qquad (\mathbf{MP})$$

Indeed, if $b \equiv 0$, i.e. when F does not depend on first-order derivatives, condition ($\blacksquare$**b**) is trivially satisfied in any (**wG**)-domain. In general, however, some condition relating the *size of the domain* with the *size of first order terms* is crucial for the validity of **(MP)** in unbounded domains. Indeed,

$$u(x) = u(x_1,x_2) = \left(1 - \mathrm{e}^{1-x_1^\alpha}\right)\left(1 - \mathrm{e}^{1-x_2^\alpha}\right)$$

with $0 < \alpha < 1$, is bounded and strictly positive in the cone

$$\Omega = \left\{ x = (x_1,x_2) \in \mathbb{R}^2 \, : \, x_1 > 1, \, x_2 > 1 \right\}$$

while satisfying

$$u_{|_{\partial\Omega}} = 0 \quad , \quad \Delta u + B(x) \cdot Du = 0 \quad \text{in } \Omega$$

with B given by

$$B(x) = B(x_1,x_2) = \left(\frac{\alpha}{x_1^{1-\alpha}} + \frac{1-\alpha}{x_1} \ , \ \frac{\alpha}{x_2^{1-\alpha}} + \frac{1-\alpha}{x_2} \right)$$

Since Ω satisfies (**wG**) with $R_y = O(|y|)$ as $|y| \to \infty$ and that the structure condition **(ASC)** holds with $b(x) = |B(x)|$, condition ($\blacksquare$**b**) fails in this example. If one only suppose b to be bounded, then in order to enforce ($\blacksquare$**b**) the requirement is that $\sup_{y\in\Omega} R_y \leq R_0 < +\infty$ in (**wG**), i.e. the stronger condition (**G**), so that the $(\mathbf{ABP})_{\mathbf{wG}}$ estimate reduces in this case to the $(\mathbf{ABP})_{\mathbf{G}}$ estimate in [5]. For a convex conical domain, one can always choose balls B_{R_y} in condition (**wG**) in such a way that $\frac{dist(B_{R_y},0)}{|y|} \geq \varepsilon > 0$ for all $y \in \Omega$. Hence, choosing $|b(x)| \leq \frac{b_0}{(1+|x|^2)^{\frac{1}{2}}}$, then ($\blacksquare$**b**) is fulfilled and this leads to the $(\mathbf{ABP})_{\mathbf{con}}$ estimate for convex conical domains, see [13].
In the intermediate case of parabolically shaped domains, defined for $k > 1$ by the inequalities

$$|x'| \equiv \sqrt{\sum_{i=1}^{N-1} x_i} < x_N^{1/k}, x_N > 0,$$

for which (**wG**) holds with $R = O(x_N^{1/k})$, a similar argument can be used to show that ($\blacksquare$**b**) holds provided $b(x) = O(1/x_N^{1/k})$ as $|x| \to \infty$. Note that cylindrical and conical domains can be seen as limiting cases of above situation when, respectively, $k \to +\infty$ and $k \to 1$.

7. Some extensions

In this section we propose a few results establishing the validity of the Maximum Principle in some situations not directly covered by Theorem 1: more general domains, solutions with *exponential growth in narrow domains* and, finally, equations with *quadratic growth* in Du.

7.1. More general domains

The next result, whose proof consists in a globalization argument based on iterated applications of Theorem 1, see [7], [37], shows the validity of **(MP)** in domains which are, roughly speaking, piecewise -(**wG**) .

Theorem 2. Let $u \in USC(\overline{\Omega})$ with $\sup_\Omega u < +\infty$ be a viscosity solution of

$$F(x,u,Du,D^2u) \geq 0 \ , \ x \in \Omega, \qquad (\mathbf{PDE})$$

where $f \in C(\Omega) \cap L^\infty(\Omega)$. Assume that F is continuous and elliptic and that **(ASC)** holds for given constants $0 < \lambda \leq \Lambda$ and some $0 < b \in C(\Omega) \cap L^\infty(\Omega)$. Assume moreover that there exists a closed set H with the following properties:

(i) **(MP)** holds for bounded from above viscosity solutions in each connected component of $\Omega \setminus H$,

(ii) there exist constants $\sigma, \tau \in (0,1)$ such that for all $y \in H$ there is a ball B_{R_y} of radius R_y containing y such that

$$|B_{R_y} \setminus \Omega_{y,\tau}| \geq \sigma |B_{R_y}|$$

where $\Omega_{y,\tau}$ is the connected component of $\Omega \cap B_{R_y/\tau}$ containing y,

(iii)
$$\sup_{y \in H \cap \Omega} R_y \, \|b\|_{L^\infty(\Omega_{y,\tau})} < \infty \qquad (\blacksquare\mathbf{b})_\mathbf{H}$$

Then the Maximum Principle holds:

$$u \leq 0 \ \text{ on } \partial\Omega \quad \text{implies} \quad u \leq 0 \ \text{ in } \Omega. \qquad (\mathbf{MP})$$

By Theorem 1, condition (i) holds in particular if each connected component Ω' of $\Omega\backslash H$ is a **wG**-domain satisfying condition (■**b**).
Theorem 2 applies, for example, to non-convex cones such as

$$\Omega = \{(x_1,x_2) \in \mathbb{R}^2 \ : \ x_2 > -|x_1|\}$$

(take $H = \{(x_1,x_1) \in \mathbb{R}^2 \ : \ x_1 > 0\} \cup \{(x_1,-x_1) \in \mathbb{R}^2 \ : \ x_1 < 0\}$). Using again Theorem 2, **(MP)** can therefore be extended to the *cut plane*

$$\Omega = \mathbb{R}^2 \setminus \left\{(x_1,0) \in \mathbb{R}^2 \ : \ x_1 \leq 0\right\}$$

by taking $H = \left\{(x_1,-x_1) \in \mathbb{R}^2 \ : \ x_1 < 0\right\}$.
A similar argument works for further general domains, even in higher dimensions, for instance in the complement $\Omega = \mathbb{R}^N\backslash H$ of a hypersurface $H = \varphi(\Gamma)$, where Γ is a $(N-1)$ - dimensional cone, e.g.

$$\Gamma = \{x = (x_1,\ldots,x_N) \in \mathbb{R}^N : x_i \leq 0, \ i = 1,\ldots,N-1, \ x_N = 0\}$$

and φ a continuous function with sub-linear growth, i.e.

$$|\varphi(x)| \leq h + k|x|.$$

Let $K \subset \mathbb{R}^N$ be the set of all closed balls with some fixed radius ρ centered at the points of a $(N-1)$ - dimensional cone Γ with integer coordinates; one can check that $\Omega = \mathbb{R}^N\backslash K$ turns out to be an admissible domain for Corollary 2.

7.2. Narrow domains

The next result shows that the Maximum Principle may hold for *unbounded* solutions of **(PDE)**, provided the unbounded domain satisfies an appropriate *narrowness* condition, related to the rate of growth at infinity of the solution. More precisely, consider the unbounded cylinder

$$\Omega = \mathbb{R}^k \times \omega \quad \text{with} \quad k + h = N, \quad h,k \geq 1\,,$$

where ω is a bounded domain of $\mathbb{R}^h$ with diameter $\operatorname{diam}(\omega)$. As pointed out before above this is typical example of **G**-domain.

Theorem 3. For F as in Theorem 1 and Ω as above, suppose $\|b\|_{L^\infty(\Omega)} \leq b_1$ and let

$$u \in USC(\overline{\Omega})\ , \ F(x,u,Du,D^2u) \geq 0 \ \ ,x \in \Omega\ ,$$

with

$$u \leq 0 \text{ on } \partial\Omega\ , \ u^+(x) = o(e^{\beta|x|}) \text{ as } |x| \to +\infty\ .$$

Then, for any $\beta > 0$ there exists $d = d(N,\lambda,\Lambda,b_1,\beta) > 0$ such that, if $\operatorname{diam}(\omega)$ $< d$, then $u \leq 0$ in Ω. Conversely, for any fixed $d > 0$ there exists $\beta = \beta(N,\lambda,\Lambda,$ $b_1,d)$ such that if $\operatorname{diam}(\omega) < d$, then $u \leq 0$ in Ω.

Qualitative results of this type for general linear uniformly elliptic operators can be found in [1], see also [6] for semilinear operators. Note that when $F(D^2) = \Delta$ and Ω is the 2 - dimensional strip $\mathbb{R} \times (0,d)$, there is a precise quantitative relationship between the diameter d and the growth exponent β, namely $\beta = \pi/d$, see [18].

The proof of Theorem 3 is based on the construction of a suitable sequence of smooth barrier functions Φ_k on finite cylinders $\bar{C}_k = \bar{B}_k(0) \times \bar{\omega}$, $k \in \mathbb{N}$, such that

$$\mathscr{P}^+ \left(D^2\Phi_k(x)\right) + b_1 |D\Phi_k(x)| \leq 0 \ \text{ in } \ C_k,$$
$$\Phi_k \geq 0 \ \text{ in } \ \bar{C}_k, \ \ \Phi_k \geq u^+ \ \text{ on } \ \partial\bar{C}_k \backslash \partial\Omega$$

and for each fixed $x \in \Omega$

$$\lim_{k\to\infty} \Phi_k(x) = 0$$

It is a familiar technique in the case of a linear operator to use **(MP)** in bounded domains C_k, considering differences $u - \Phi_k$, and then passing to the limit as $k \to \infty$. The difficulty in implementing this procedure in the present nonlinear setting is overcome by the use of the structure condition **(ASC)**, together with the superadditivity of the maximal Pucci operator, since standard calculus rules apply since Φ_k is twice continuously differentiable, see [13] for details.

A similar result holds for viscosity subsolutions with polynomial growth $u(x) = O(|x|^\alpha)$ in angular sectors $\Omega = \mathbb{R}^k \times \omega$, where ω is a cone in $\mathbb{R}^h$ and $h+k = N$, provided **(ASC)** holds true with $b(x) = O(1/|x|)$ as $|x| \to \infty$. In this case, in order to get **(MP),** the opening of the cone has to be sufficiently small depending on the exponent α and the various structure parameters.

7.3. Quadratic dependence on Du

We have considered up to now second-order fully nonlinear operators with linear growth in the gradient variable. Let us briefly describe how the previous results can be extended to the case of a *quadratic growth* in the gradient variable. A fundamental tool in the proof of Theorem 1, see [13], is the *weak Harnack inequality* for functions $w \in LSC(A)$ satisfying in the viscosity sense in a domain A of $\mathbb{R}^N$ the partial differential inequality

$$w \geq 0, \qquad \mathscr{P}^-_{\lambda,\Lambda}(D^2 w) - b(x)|Dw| \leq g(x) \tag{4}$$

with $b, g \in C(A) \cap L^\infty(A)$, namely:

there exist positive numbers C, p depending on N, λ, Λ, b_1 such that

$$\left(\frac{1}{|B_1|}\int_{B_1} w^p\right)^{1/p} \leq C\left(\inf_{B_2} w + \|g\|_{L^n(B_4)}\right) \tag{5}$$

where $B_1 \subset B_2 \subset B_4 \subset A$ are concentric balls of radii $1,2$ and 4, respectively and

$$b_1 = \|b\|_{L^\infty(B_4)}$$

see [20] for linear operators and [9] for the case $b \equiv 0$.
Consider now an additive quadratic gradient term to the operator in the above inequality,

$$\mathscr{P}^-_{\lambda,\Lambda}(D^2v) - b(x)|Dv| - b_2|Dv|^2 \leq g(x), \tag{6}$$

where b_2 is a positive constant, and suppose that

$$0 \leq v \leq M \ \text{ in } \ A\,. \tag{7}$$

If v is a viscosity solution of (6), the function $w = h^{-1}(v)$, where h is smooth non-negative increasing and convex, satisfies

$$\mathscr{P}^-_{\lambda,\Lambda}(D^2w) + \lambda\frac{h''(w)}{h'(w)}|Dw|^2 - b(x)|Dw| - b_2h'(w)|Dw|^2 \leq \frac{g(x)}{h'(w)}$$

in the viscosity sense. The proof of this fact requires some viscosity calculus together with the superadditivity and the ellipticity of $\mathscr{P}^-_{\lambda,\Lambda}$. Solving the ordinary differential equation

$$\lambda h''(t) - b_2(h')^2(t) = 0$$

one finds

$$h(t) = \frac{\lambda}{b_2}\log\left(1 - \frac{b_2t}{\lambda}\right)^{-1},$$

which satisfies the required properties for $t \in [0, \frac{\lambda}{b_2})$. Correspondingly, the function

$$w = \frac{\lambda}{b_2}\left(1 - e^{-\frac{b_2v}{\lambda}}\right)$$

is a solution of (4) with right-hand side $\hat{g} = g\,(1 - b_2w/\lambda)$ instead of g.
Applying the weak Harnack inequality (5) to w and observing that

$$\frac{1 - e^{-\frac{b_2M}{\lambda}}}{\frac{b_2M}{\lambda}}\,v \leq w \leq v\,,$$

we derive a weak Harnack inequality for positive and bounded above solutions of (6) , namely

$$\left(\frac{1}{|B_1|}\int_{B_1} v^p\right)^{1/p} \leq C\left(\inf_{B_2} v + \|g\|_{L^n(B_4)}\right) \tag{8}$$

Observe that the constant C will also depend in this case on b_2M. The dependence on the upper bound M in the estimate seems to be unavoidable, see [36] and [27]. Note that this dependence occurs also in the Alexandrov-Bakelman-Pucci estimate for second-order elliptic operators, see [26]. Since we will only need to deal with bounded above non-negative supersolutions and to keep finite the constants appearing in the estimates, this fact it is not relevant for the applications, see below in this Section, of estimate (8) to the Maximum Principle.

A *boundary version* of inequality (8) can be easily obtained in a natural way in the viscosity framework. Indeed, let A be a bounded domain in $\mathbb{R}^N$ and $B_R, B_{R/\tau}$, $\tau \in (0,1)$, be concentric balls such that

$$A \cap B_R \neq \emptyset \ , \ B_{R/\tau} \backslash A \neq \emptyset \ .$$

For $v \in LSC(\bar{A})$, $v \geq 0$, consider the following lower semicontinuous extension of v:

$$v_m^-(x) = \begin{cases} \min(v(x);m) & \text{if } x \in A \\ m & \text{if } x \notin A \end{cases}$$

where

$$m = \inf_{x \in \partial A \cap B_{R/\tau}} v(x)$$

Rescaling and using a covering argument as in [5], from (8) we deduce the *boundary weak Harnack inequality*

$$\left(\frac{1}{|B_R|} \int_{B_R} (v_m^-)^p \right)^{1/p} \leq C^* \left(\inf_{A \cap B_R} v + R \, \|g^+\|_{L^n(A \cap B_{R/\tau})} \right) \tag{9}$$

where p and C^* are positive constants depending on $N, \lambda, \Lambda, \tau, b_2M$ and on $R\|b\|_{L^\infty(A \cap B_{R/\tau})}$.

Observe now that if $u \in USC(\bar{\Omega})$ is a viscosity solution of

$$F(x, u(x), Du(x), D^2u(x)) \geq f(x) \ , \ x \in \Omega,$$

bounded above by some constant $M > 0$ and F satisfies the following growth condition

$$F(x,t,p,X) \leq \mathscr{P}^+_{\lambda,\Lambda} + b(x)|p| + b_2(x)|p|^2 \qquad (\mathbf{QASC})$$

then it is easy to check by viscosity calculus that the function $v = M - u^+ \in LSC(\bar{\Omega})$ is a bounded above, nonnegative viscosity solution of

$$\mathscr{P}^-_{\lambda,\Lambda}(D^2 v) - b(x)|Dv| - b_2|Dv|^2 \leq f^-(x) \ , \ x \in \Omega$$

Now, if Ω is a (**wG**) -domain, then applying the boundary weak Harnack inequality (9) to v in $A = \Omega_{y,\tau}$ with $g = f^-$, then after some computations we obtain a *localized* form of the **(ABP)** estimate:

$$u^+(y) \leq (1-\theta_y) \sup_{\Omega} u^+ + \theta_y \sup_{\partial\Omega} u^+ + R_y \|f^-\|_{L^N(\Omega_{y,\tau})}, \tag{10}$$

for every $y \in \Omega$ and some costant $\theta_y \in (0,1)$ depending on N, λ, Λ, σ, τ, b_2M and on y through the quantity $R_y \|b\|_{L^\infty(\Omega_{y,\tau})}$.
Note that the above estimate (10) actually holds for all values $u(z)$, $z \in \Omega \cap B_{R_y}$, and therefore it may be considered as an extension of the Krylov-Safonov Growth Lemma, see [28], Section 1.4 to the present setting.
Furthermore, under condition (■**b**), the constant θ_y in (10) can be bounded above by a positive constant $\theta < 1$ independent of y. Hence, taking the supremum on both sides of (10) we find again the estimate

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \sup_{y\in\Omega} R_y \|f^-\|_{L^N(\Omega_{y,\tau})} \tag{$\mathbf{ABP}_{\mathbf{wG}}$}$$

for some positive constant C, which in this case depends on N, λ, Λ, σ, τ, b_2M and Rb.
Therefore, if Ω is a (**wG**) -domain, F is elliptic, condition **(QASC)** holds and $u \in USC(\overline{\Omega})$ is bounded above and satisfies

$$F\left(x, u(x), Du(x), D^2u(x)\right) \geq 0\,,\ x \in \Omega, \tag{\textbf{PDE}}$$

in the viscosity sense, then the Maximum Principle holds:

$$u \leq 0 \text{ on } \partial\Omega \quad \text{implies} \quad u \leq 0 \text{ in } \Omega\,.$$

8. Phragmen-Lindelöf type theorems

One form of the classical Phragmèn-Lindelöf Maximum Principle is as follows: if

$$\Delta u \geq 0$$

in an unbounded angular sector $\Omega \subset \mathbb{R}^2$ of opening $\frac{\pi}{\alpha}$ and $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ on Ω provided

$$u(x) = O(|x|^\alpha) \quad \text{as} \quad |x| \to +\infty,$$

see for example [34]. Several variants and extensions of this result to smooth solutions of linear and nonlinear elliptic inequalities in more general unbounded domains of $\mathbb{R}^N$ can be found in the literature, see for example [19], [24], [33],

[32], [34], [1], [31], [30], [38].

Subsection 7.2 contains some qualitative Phragmèn-Lindelöf type results for viscosity solutions in cylinders and cones. Thus, one may expect that **(MP)** should hold in more general (**G**) - and (**wG**) - domains of cylindrical or conical type under a suitable exponential, respectively, polynomial growth of subsolutions at infinity. This can be indeed proved to be true by a suitable refinement of the techniques previously described. Let us briefly comment on this issue and refer to [15] for a detailed treatment of this topic.
The construction of the barrier functions used in the proofs of the above mentioned results relies heavily on the simple geometry of cylinders and cones and cannot easily carried over to more general (**G**) - or (**wG**) - domains. The extension to general (**G**) - and (**wG**) - domains relies instead on the validity of the Maximum Principle for operators satisfying the structure condition

$$F(x,t,p,X) \leq \mathscr{P}^{+}_{\lambda,\Lambda} + b(x)|p| + c(x)t, \qquad (\mathbf{ASC}+)$$

where we allow the coefficient $c(x)$ to be positive.
A careful analysis show that if $c^{+}(x)$ is sufficiently small, namely $c^{+}(x) \leq c_1$, in the case of cylindrical domains, and $c^{+}(x) \leq c_1/|x|^2$ as $|x| \to \infty$ in the case of conical domains, for a small positive constant c_1 depending on the structure of F and on the geometric parameters occurring in the (**G**) or (**wG**) conditions, then **(MP)** still holds for viscosity subsolutions of **(PDE)** provided F satisfies **(ASC+)**. This remark, coupled with the method of barriers, lead to Phragmèn-Lindelöf type theorems as announced above. Two model statements in this direction are as follows, see [15]:

Theorem 4. Assume that Ω is a (**wG**)- domain of conical type and that F satisfies **(ASC)** with $|b(x)| \leq \frac{b_0}{(1+|x|^2)^{\frac{1}{2}}}$. Then, there exists $\alpha > 0$, depending on F and Ω, such that if $u \in USC(\Omega)$ is a viscosity solution of

$$F(x,u,Du,D^2u) \geq 0 \ \text{ in } \Omega$$

with $u \leq 0$ on $\partial\Omega$ and $u(x) = O(|x|^{\alpha})$ as $|x| \to +\infty$, then $u \leq 0$ in Ω.
The same conclusion holds for solutions of

$$F(x,u,Du,D^2u) + c(x)u \geq 0 \ \text{ in } \Omega$$

if $c^{+}(x) \leq \frac{c_0}{1+|x|^2}$ for small enough $c_0 > 0$.

Theorem 5. Assume that Ω is a (**wG**) domain of $\mathbb{R}^N$ of cylindrical type and that F satisfies **(ASC)** with $|b(x)| \leq b_0$. Then, there exists $\alpha > 0$, depending on F and Ω, such that if $u \in USC(\Omega)$ is a viscosity solution of

$$F(x,u,Du,D^2u) \geq 0 \ \text{ in } \Omega$$

with $u \leq 0$ on $\partial\Omega$ and $u(x) = O(e^{\alpha|x|})$ as $|x| \to +\infty$, then $u \leq 0$ in Ω.
The same conclusion holds for solutions of

$$F(x,u,Du,D^2u) + c(x)u \geq 0 \ \text{ in } \Omega$$

if $c^+(x) \leq c_0$ for small enough $c_0 > 0$.

Theorems above extend in particular the results of [28] in the direction of more general unbounded domains as well as of viscosity solutions of non necessarily uniformly elliptic fully nonlinear differential inequalities containing lower order terms.
Let us point finally that in view of the discussion in Subsection 7.3, Phragmèn-Lindelöf type theorems as above could also be proved for second-order fully nonlinear operators with quadratic growth in the gradient-variable.

9. Minimum Principles

The previous Sections were concerned with Maximum Principles for subsolutions of **(LPDE)** under the above structure condition **(ASC)** or also **(QASC)**. In a similar manner, Minimum Principles can be stated for supersolutions under the below structure conditions **(BSC)** and **(QBSC)**. Here below we list briefly this kind of results.

Theorem 6. Let $u \in LSC(\overline{\Omega})$ such that $\sup_\Omega u^- \leq M$ for a positive constant M. Suppose that u is a viscosity solution of

$$F(x,u,Du,D^2u) \leq f(x) \ \ , \ \ x \in \Omega\,,$$

where $f \in C(\Omega) \cap L^\infty(\Omega)$ and Ω is a domain of $\mathbb{R}^N$ satisfying condition (**wG**) for some $\sigma, \tau \in (0,1)$.
Assume that F is continuous and elliptic and that **(BSC)** holds for given constants $0 < \lambda \leq \Lambda$, $b_2 > 0$ and some $0 < b \in C(\Omega) \bigcap L^\infty(\Omega)$.
Assume moreover that (**wG**) is satisfied with R_y such that

$$Rb := \sup_{y\in\Omega} R_y \,\|b\|_{L^\infty(\Omega_{y,\tau})} < \infty. \qquad (\blacksquare\mathbf{b})$$

holds, as in Theorem 1. Then,

$$\sup_{\Omega} u^- \leq \sup_{\partial\Omega} u^- + C \sup_{y\in\Omega} R_y \, \|f^+\|_{L^N(\Omega_{y,\tau})} \qquad (\mathbf{ABP})_{\mathbf{wG}}$$

for some positive constant C depending on $N,\ \lambda,\ \Lambda,\ \sigma,\ \tau, b_2M$ and Rb.

As an immediate consequence of $(\mathbf{ABP})_{\mathbf{wG}}$, the Maximum Principle holds: if $u \in LSC(\overline{\Omega})$ satisfies

$$F(x,u,Du,D^2u) \leq 0 \ , \ x \in \Omega,$$

then

$$u \geq 0 \ \text{ on } \partial\Omega \quad \text{implies} \quad u \geq 0 \ \text{ in } \ \Omega. \qquad (\mathbf{mP})$$

Theorem 7. Let $u \in LSC(\overline{\Omega})$ with $\sup_\Omega u^- < +\infty$ be a viscosity solution of

$$F(x,u,Du,D^2u) \leq 0 \ , \ x \in \Omega, \qquad (\mathbf{PDE})$$

where $f \in C(\Omega) \cap L^\infty(\Omega)$.
Assume that F is continuous and elliptic and that **(BSC)** holds for given constants $0 < \lambda \leq \Lambda$, $b_2 > 0$ and some $0 < b \in C(\Omega) \cap L^\infty(\Omega)$. Assume moreover that there exists a closed set H with the following properties:

(i) **(mP)** holds for bounded from above viscosity solutions in each connected component of $\Omega \setminus H$,

(ii) there exist constants $\sigma, \tau \in (0,1)$ such that for all $y \in H$ there is a ball B_{R_y} of radius R_y containing y such that

$$|B_{R_y} \setminus \Omega_{y,\tau}| \geq \sigma |B_{R_y}|$$

where $\Omega_{y,\tau}$ is the connected component of $\Omega \cap B_{R_y/\tau}$ containing y,

(iii)

$$\sup_{y\in H\cap\Omega} R_y \, \|b\|_{L^\infty(\Omega_{y,\tau})} < \infty \qquad (\blacksquare\mathbf{b})_{\mathbf{H}}$$

Then the Minimum Principle holds:

$$u \geq 0 \ \text{ on } \partial\Omega \quad \text{implies} \quad u \geq 0 \ \text{ in } \ \Omega. \qquad (\mathbf{mP})$$

The argument of Subsection 7.3 can be used to show that the above results continue to hold for operators with a quadratic growth in the gradient, i.e. using condition **(QBSC)** instead of **(BSC)**.
Similarly, the condition of boundedness from below for supersolutions u can be weakened to allow a polynomial growth of u^- at infinity in the case of conical domains, an exponential growth in the case of cylindrical domains. This leads to Phragmèn-Lindelöf Principles for fully nonlinear second-order operators satisfying **(BSC)** or also **(QBSC)**, analogous to those ones of Theorems 4 and 5.

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