LE MATEMATICHE
Vol. LXII (2007) - Fasc. I, pp. 47-54

CLEAN RINGS OF SKEW HURWITZ SERIES

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In this paper we study the transfer of some algebraic properties from the ring R to the ring of skew Hurwitz series $T = (HR, \sigma)$ where σ is an automorphism of R and vice versa. We show that $T = (HR, \sigma)$ is a clean (strongly clean) ring if and only if R is clean (strongly clean). Different properties of skew Hurwitz series are studied such as simplicity, primeness and semiprime.

1. Introduction.

Throughout this paper R denotes an associative ring with identity. An element $a \in R$ is called clean if it can be expressed as the sum of an idempotent and a unit elements. An element $a \in R$ is called *strongly clean* if its clean and the idempotent and the unit commute. A ring R is called *clean* (*strongly clean*) ring if every element of R is. Clearly units and idempotents are clean (strongly clean) elements and a is clean (strongly clean) if and only if $1 - a$ is clean (strongly clean). It follows that every local ring is clean (strongly clean).

The ring $T = (HR, \sigma)$ of *skew Hurwitz series* over a ring R and $\sigma \in Aut(R)$ is defined as follows: the elements of $T = (HR, \sigma)$ are

Entrato in redazione il 28 Novembre 2006.

functions $f : \mathbb{N} \to R$, where $\mathbb{N}$ is the set of all natural numbers, the operation of addition in $T = (HR, \sigma)$ is component wise and the operation of multiplication for each $f, g \in T$ is defined by

$$(fg)(n) = \sum_{k=0}^{n} \binom{n}{k} f(k)\sigma^k(g(n-k))$$

for all $n \in \mathbb{N}$, where $\binom{n}{k}$ is the binomial coefficient.

If one identifies a skew formal power series $\sum_{n=0}^{\infty} a_n x^n \in R[[x, \sigma]]$ with the function f such that $f(n) = a_n$, then multiplication in T is similar to the usual product of skew formal power series, except that binomial coefficients appear in each term in the product introduced above.

It can be easily shown that T is a ring with identity h_1, defined by $h_1(0) = 1$ and $h_1(n) = 0$ for all $n \geq 1$. For a function $f \in T$, we defined $\text{supp}(f) = \{n \in \mathbb{N} \mid f(n) \neq 0\}$ and $\pi(f)$ denotes the minimal element in $\text{supp}(f)$. It is clear that R is canonically embedded as a subring of T via $0 \neq r \in R \mapsto h_r \in T$, where $h_r(0) = r, h_r(n) = 0$ for every $n \geq 1$ (then $\text{supp}(h_r) = \{0\}$) . For more details we refer to [3].

Following [5, p. 129] $\text{char}(R) = 0$ if and only if $nx = 0$, where n is a positive integer implies $x = 0$, for $x \in R$. This is a stronger condition than the usual definition that no positive multiple of the identity vanishes.

Han and Nicholson [1] studied the clean property in some ring extensions. They showed that the ring $R[[x]]$ of formal power series is clean ring if and only if R is clean which isn't the case of the ring of polynomials.

In a series of papers [7, 8, 9] Keigher demonstrated that the ring HR of *Hurwitz series* over a commutative ring R with identity has many interesting applications in differential algebra.

Recently, Hassanein et al [4] showed that the ring HR of Hurwitz series is clean (strongly clean) if and only if R is clean (strongly clean). Also necessary and sufficient conditions for HR to be simple or prime were studied. It is natural to ask if any or all of these properties can be extended to the skew Hurwitz series rings.

The motivation of this paper is to extend the results in [4] to the ring (HR, σ) of the skew Hurwitz series over the ring R.

2. Transfer of Some Properties Between R and $T = (HR, \sigma)$.

At the beginning of this section we study the properties of units in the ring $T = (HR, \sigma)$ of skew Hurwitz series, where σ is an automorphism of R.

We need the following result.

Proposition 2.1. *Let R be a ring. Then f is a unit of HR if and only if $f(0)$ is a unit of R.*

Proposition 2.2. *Let R be a ring. Then f is a unit of $T = (HR, \sigma)$ if and only if $f(0)$ is a unit of R.*

Proof. Clearly, if $f \in T = (HR, \sigma)$ is invertible, then $f(0)$ is invertible in R. Conversely, suppose that $f \in T$ such that $f(0) = a$ is invertible in R. Let $b \in R$ such that $ab = 1 = ba$ and define an element $g \in T$ by $g(0) = b$ and inductively by $g(n) = -b\left(\sum_{k=1}^{n} \binom{n}{k} f(k)\sigma^k(g(n-k))\right)$, for all $n \geq 1$. It is not difficult to check that $fg = h_1 = gf$. □

Following the same procedure in Theorem 2.3 [4], we can extend that Theorem to the skew Hurwitz series ring as follows

Theorem 2.3. *Let R be a ring and σ be an automorphism of R. Then $T = (HR, \sigma)$ is a clean ring if and only if R is.*

For strongly clean rings we have the following

Proposition 2.4. *Suppose that R is a ring and σ an automorphism of R. If R is a clean ring such that every idempotent e of R is central and $\sigma(e) = e$, then T is a strongly clean ring.*

Proof. Suppose that R is a clean ring. Then, by Theorem 2.3, T is a clean ring and every element $f \in T$ can be represented as $f = (f - h_e) + h_e$ where e is an idempotent of R. So, it is sufficient to show that $fh_e = h_e f$.

Since e is central in R, then $(fh_e)(s) = f(s)\sigma^s(h_e(0)) = f(s)\sigma^s(e) = f(s)e = ef(s) = h_e(0)f(s) = (h_e f)(s)$ for each $s \in$ supp $fh_e)$ and the Proposition is proved. □

The following example shows that Proposition 2.4. is not true without the assumption that $\sigma(e) = e$ for all idempotents $e \in R$.

Example 2.5. Take $R = \mathbb{Z}_4 \oplus \mathbb{Z}_4$, with the usual operations of componentwise addition and multiplication; R is clearly a commutative clean ring. Therefore, R is strongly clean. Now let $\sigma : R \to R$ be defined by $\sigma(a, b) = (b, a)$, then σ is an automorphism of R and there are idempotents in R with $\sigma(e) \neq e$ (e.g., if $e = (0, 1) \in R$). By direct computations we can easily show that $T = (HR, \sigma)$ is not strongly clean.

Corollary 2.6. *Let R be a ring and σ an automorphism of R. If $T = (HR, \sigma)$ is a strongly clean ring, then R is strongly clean.*

Proof. Using Theorem 2.3, R is a clean ring and every element $r \in R$ can be represented as $r = f(0) + g(0)$, where f is an idempotent element of T and $g \in U(T)$. So it is sufficient to show that $f(0)g(0) = g(0)f(0)$. Since f, g commute in T, then $f(0)g(0) = f(0)\sigma^0(g(0)) = (fg)(0) = (gf)(0) = g(0)\sigma^0(f(0)) = g(0)f(0)$ and the corollary is proved. □

Definition 2.7. Let σ be an endomorphism of R. σ is called a *rigid endomorphism* if $r\sigma(r) = 0$ implies $r = 0$ for $r \in R$. A ring R is said to be σ*-rigid* if there exists a rigid endomorphism σ of R.

Remark 2.8. Recall that a ring R is reduced if R has no nonzero nilpotent elements. Observe that reduced rings are abelian, i.e., all idempotents are central.

Corollary 2.9. *Let R be a reduced ring, σ an automorphism of R and $\sigma(e) = e$. If R is a clean ring, then T is a strongly clean ring.*

Proof. Since R is a reduced ring, then every idempotent e of R is central and by using Proposition 2.4. T is strongly clean. □

The following example shows that it is possible to find automorphisms of a reduced ring which are not rigid.

Example 2.10. Consider the ring $R = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} | a \equiv b(mod 2)\}$, with the usual operations of componentwise addition and multiplication; R is clearly a commutative reduced ring. Now let $\sigma : R \to R$ be defined by $\sigma(a, b) = (b, a)$, then σ is an automorphism of R. Note that R is not σ-rigid.

Proposition 2.11. *Let R be a ring,* $\mathrm{char}(R) = 0$ *and σ be an automorphism of R. Then R is a σ-rigid if and only if $T = (HR, \sigma)$ is a reduced ring. In this case $\sigma(e) = e$, for every idempotent $e \in R$.*

Proposition 2.12. *Let R be a σ-rigid ring where σ is an automorphism of R and* $\mathrm{char}(R) = 0$. *If R is a clean ring, then T is a strongly clean ring.*

Proof. Since R is a σ-rigid ring, then R is a reduced ring and $\sigma(e) = e$ for all $e^2 = e \in R$ by using Proposition 2.11. Therefore T is a strongly clean ring by using Corollary 2.9. $\square$

3. Simplicity, Primeness and Semiprimeness of Skew Hurwitz Series Ring.

In this section we give some properties of a skew Hurwitz series ring. Let R be a ring with an automorphism σ. For any σ-ideal I of R, σ induces a ring endomorphism of R/I given by $r + I \mapsto \sigma(r) + I$. We denote this endomorphism by σ again and note that σ is an automorphism of R/I if and only if $\sigma(I) = I$.

Remark 3.1.

i) To every right ideal I in R it corresponds a right ideal (HI, σ) in $T = (HR, \sigma)$ where $(HI, \sigma) = \{f \in T \mid f(n) \in I \text{ for all } n \in \mathbb{N}\}$.

ii) In case I is a σ-ideal (resp. left ideal) of a ring R, then (HI, σ) is a σ-ideal (resp. left ideal) of T.

Proposition 3.2. *If I is a σ-ideal of R, then $(HR, \sigma)/(HI, \sigma) \cong (H(R/I), \sigma)$.*

Proof. We consider a mapping $\varphi : R \to R/I$ defined by $\varphi(r) = r + I = \overline{r}$ which is a canonical epimorphism of rings. Now we define a mapping $\varphi^* : (HR, \sigma) \to (H(R/I), \sigma)$ by the following

$$\varphi^*(f) = f', \text{ s.t., } f'(n) = \varphi(f(n)) = \overline{f(n)} \text{ for all } n \in \mathbb{N}$$

It is easy to verify that φ^* is an epimorphism. First, φ^* is a well defined since if $f = g$ where $f, g \in T = (HR, \sigma)$, then $n = \pi(f) = \pi(g)$ and $f(n) = g(n)$ for all $n \in \mathbb{N}$. Hence $f(n) + I = g(n) + I$. Therefore $f^{'}(n) = g^{'}(n)$ and $f^{'} = g^{'}$. Also, $Ker(\varphi^*) = \{f \in T \mid f(n) \in I$, i.e., $f \in (HI, \sigma)\} = (HI, \sigma)$. Therefore $T/(HI, \sigma) \cong (H(R/I), \sigma)$. □

Theorem 3.3. ([3], Theorem 2.6 (i)). *If R is a prime ring with* char$(R) = 0$. *Then for any automorphism σ of R, the ring $T = (HR, \sigma)$ is a prime ring.*

Proposition 3.4. *Let R be a ring with* char$(R) = 0$ *and σ be an automorphism of R. Then I is a σ-prime ideal of R if and only if (HI, σ) is a prime ideal of $T = (HR, \sigma)$.*

Proof. Since I is a σ-ideal of R, we have $T/(HI, \sigma) \cong (H(R/I), \sigma)$ by Proposition 3.2. Since I is a σ-ideal of R, then R/I is a prime ring and hence $(H(R/I), \sigma)$ is a prime ring by Theorem 3.3. Therefore $T/(HI, \sigma)$ is a prime ring and hence (HI, σ) is a prime ideal of T. □

Theorem 3.5. *R is a σ-simple ring if and only if $T = (HR, \sigma)$ is a simple ring where σ is an automorphism of R.*

Proof. Suppose that T isn't simple. Then there exists a proper two sided ideal $M \neq 0$ of T. Hence there exists a proper two sided σ-ideal of R generated by $\{f(s) \mid f \in M, s = \pi(f)\}$ which contradicts the σ-simplicity of R.

Conversely, suppose that R isn't a σ-simple ring. Hence there exists a nonzero σ-ideal I of R. Consequently (HI, σ) is a proper two sided ideal of T which is a contradiction. □

We recall that an ideal I of a ring R is a maximal ideal if R/I is a simple ring.

Proposition 3.6. *Let I be a σ-ideal of a ring R. Then I is a maximal ideal of R if and only if (HI, σ) is a maximal ideal of T.*

Proof. The proof is similar to that of Proposition 3.4. □

Theorem 3.7. *Suppose σ is an automorphism of R satisfying the condition that the right annihilators of any principal right ideal of R is σ-ideal. Then, if $T = (HR, \sigma)$ is a semiprime ring, so is R.*

Proof. Suppose R is not a semiprime ring. Then there exists a nonzero element a of R with $aRa = 0$. This means that $a \in r_R(aR)$ from our hypothesis $\sigma(r_R(aR)) \subseteq r_R(aR)$. It follows that $\sigma(a)$ (hence $\sigma^n(a)$ for all $n \in \mathbb{N}$) is in $r_R(aR)$ or $aR\sigma^n(a) = 0$ for all $n \in \mathbb{N}$. This implies that $h_a T h_a = 0$ showing that T is not a semiprime ring. □

Definition 3.8. A proper σ-ideal I of R is said to be a σ-semiprime ideal if whenever A is an ideal of R and m is an integer such that $A\sigma^t(A) \subseteq I$ for all $t \geq m$, then $A \subseteq I$.

It is not difficult to show that a proper σ-ideal I of R is a σ-prime ideal if and only if whenever A and B are ideals of R and m is an integer such that $A\sigma^t(B) \subseteq I$ for all $t \geq m$, then $A \subseteq I$ or $B \subseteq I$. This shows that any σ-prime ideal is a σ-semiprime ideal. The ring R is said to be σ-prime or σ-semiprime according as its zero ideal is a σ-prime or σ-semiprime ideal.

Theorem 3.9. *Let R be a ring with* $\mathrm{char}(R) = 0$ *and σ is any automorphism of R. If R is a σ-semiprime ring, then $T = (HR, \sigma)$ is a semiprime ring.*

Proof. Suppose that T is not a semiprime ring, then there exists a nonzero ideal D such that $D^2 = 0$. Let $I_n = \{g(n) \in R \mid g \in D,\ n = \pi(g)\} \subset R$, $I = \bigcup_{n\in\mathbb{N}} I_n$ and J be the ideal of R generated by I. Let $g, f \in D$ and $h \in T$ where $\pi(g) = n$, $\pi(f) = m$ and $\pi(h) = s$. Hence $fhg \in D^2 = 0$. Consequently,

$$\begin{aligned} 0 = (fhg)(m+s+n) &= \binom{m+s+n}{m} f(m)\sigma^m(hg)(s+n) \\ &= \binom{m+s+n}{m}\binom{s+n}{s} f(m)\sigma^m(h(s)\sigma^s(g(n))) \\ &= \binom{m+s+n}{m}\binom{s+n}{s} f(m)\sigma^m(h(s))\sigma^{m+s}(g(n)) \end{aligned}$$

Since $\mathrm{char}(R) = 0$, then $f(m)\sigma^m(h(s))\ \sigma^{m+s}(g(n)) = 0$. Let $K = Rf(m)R = \langle f(m)\rangle$ be the principal ideal generated by $f(m)$. Hence

$K\sigma^t(J) = 0$ for all $t \geq m$ and since $K \subseteq J$, then $K\sigma^t(K) \subseteq K\sigma^t(J) = 0$ which contradicts the fact that R is a σ-semiprime ring. $\square$

Acknowledgements.

The author expresses his deep sense of gratitude to the referee for her/his valuable suggestions. Also I like to express my deepest gratitude to Professor *M. H. Fahmy* for his sincere help during the preparation of this paper.

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