Three algorithms to construct semistandard Young tableaux, and a generalization of Knuth's formula for the number of skew tableaux

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Dissertation

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Contents

Chapter 1

Introduction

Kostka numbers give the number of semistandard tableaux of given shape and weight, and they play a fundamental role in representation theory of symmetric groups (see [11]). Much work has been done on the problem of computing Kostka numbers, which is known to be #P complete (see [10]).

Throughout this thesis, n will denote a positive integer. We write $\mu \vDash n$ if μ is a composition of n, that is, a sequence $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ of nonnegative integers such that $|\mu| = \sum_{i=1}^{k} \mu_i = n$. In particular, if the sequence μ is non-increasing and $\mu_i > 0$ for all $1 \le i \le k$, then we write $\mu \vdash n$ and say that μ is a partition of n. We say that k is the height of μ and denote it by $h(\mu)$. We denote by D_μ the Young diagram of μ. More precisely,

$$D_\mu = \{(i, j) \in \mathbb{Z}^2 \mid 1 \le i \le k,\ 1 \le j \le \mu_i\}.$$

If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h) \vdash m \le n$ and $D_\lambda \subset D_\mu$, then the skew shape μ/λ is obtained by removing from D_μ all the boxes belonging to D_λ.

Let $\mu \vdash n$ and $\lambda \vDash n$. A semistandard Young tableau (SSYT) of shape μ and weight λ is a filling of the Young diagram D_μ with the numbers $1, 2, \dots, h(\lambda)$ in such a way that

(i) i occupies λ_i boxes, for $i = 1, 2, \dots, h(\lambda)$,

(ii) the numbers are strictly increasing down the columns and weakly increasing along the rows.

We denote by $\mathrm{STab}(\mu, \lambda)$ the set of all semistandard tableaux of shape μ and weight λ.

In Chapter 2, we study the *set*, rather than the number, of semistandard tableaux of given shape and weight. We do not assume the weight is a partition, rather, it is an arbitrary composition.

For compositions $a = (a_1, a_2, \dots, a_h)$ and $b = (b_1, b_2, \dots, b_k)$ of n, we say a dominates b, denoted $a \trianglerighteq b$, if $k \ge h$ and

$$\sum_{i=1}^{j} a_i \ge \sum_{i=1}^{j} b_i$$

for $j = 1, 2, \ldots, h$. The following is well known.

Theorem 1 ([5, p. 26, Exercise 2]). *Let μ and λ be partitions of n. Then* $\mathrm{STab}(\mu, \lambda)$ *is nonempty if and only if $\mu \trianglerighteq \lambda$.*

Let $\lambda(a)$ denote the partition of n associated with a composition a of n, that is, the partition obtained from a by rearranging the parts of a in non-increasing order. Then one can strengthen Theorem 1 using [2, Lemma 3.7.1], as follows:

Theorem 2 ([5, p. 50, Proposition 2]). *Let μ and a be a partition and composition of n, respectively. Then* $\mathrm{STab}(\mu, a)$ *is nonempty if and only if $\mu \trianglerighteq \lambda(a)$.*

This theorem is incorrectly stated in [2, Lemma 3.7.3], where $\mu \trianglerighteq \lambda(a)$ is replaced by $\mu \trianglerighteq a$. For example, let $\mu = (5, 3) \vdash 8$ and $a = (2, 6) \vDash 8$. Then $\mu \trianglerighteq a$, but $\mathrm{STab}(\mu, a) = \emptyset$.

In Chapter 2, we give explicit algorithms to produce an element of $\mathrm{STab}(\mu, a)$, thereby giving a direct proofs of Theorem 2. We also introduce a natural partial order on $\mathrm{STab}(\mu, a)$ and show that it has unique greatest and least elements, by showing that the elements produced by two of the three algorithms have the respective property. Although the proof of Theorem 2 using [2, Lemma 3.7.1] or [5, p. 50, Proposition 2] gives, in principle, a bijection between $\mathrm{STab}(\mu, a)$ and $\mathrm{STab}(\mu, \lambda(a))$, it does not give an efficient algorithm to describe an element of $\mathrm{STab}(\mu, a)$ unless the permutation required to transform a to $\lambda(a)$ is a transposition. We show that, in our Proposition 20 below, a direct approach for proving Theorem 2 along the line of [2, Lemma 3.7.3] can be justified, and we describe an algorithm to produce an element of $\mathrm{STab}(\mu, a)$ in this way.

In Chapter 3, we take an elementary approach to derive a generalization of Knuth's formula using Lassalle's explicit formula. In particular, we give a formula for the Kostka numbers of a shape $\mu \vdash n$ and weight $(m, 1^{n-m})$ for $m = 3,\ 4$.

The Kostka number $K(\mu, \lambda)$ is the number of semistandard Young tableaux (SSYT) of shape μ and weight λ. In particular, if $\lambda = (1^n)$ then such a tableau is called a standard Young tableau (SYT) of shape μ, and for a skew shape μ/ν and weight (1^{n-m}) such a tableau is called a skew SYT of skew shape μ/ν, where $\nu \vdash m \leq n$. We denote by $f^{\mu/\nu}$ the number of skew SYTs of skew shape μ/ν. Obviously, if $\lambda = (m, 1^{n-m}) \vdash n$ and $m \leq \mu_1$, then for all SSYTs of shape μ and weight λ, a box $(1, j) \in D_\mu$ is filled by 1 for $1 \leq j \leq m$, so $K(\mu, (m, 1^{n-m})) = f^{\mu/(m)}$. Naturally, if $\nu = \emptyset$ then f^μ is the number of SYTs of shape μ. We can easily compute f^μ using the hook formula (see [4]). There is a recurrence formula for Kostka numbers (see [7] and [9]), but we have no explicit formula for Kostka numbers.

For $z \in \mathbb{C}$, the falling factorial is defined by $[z]_n = z(z-1)\cdots(z-n+1) = n!\binom{z}{n}$, and $[z]_0 = 1$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$ and μ' be the conjugate of μ. Knuth [6, p.67, Exercise 19] shows:

$$f^{\mu/(2)} = \frac{f^\mu}{[n]_2}\left(\sum_{i=1}^{k}\binom{\mu_i}{2} - \sum_{j\geq 1}\binom{\mu'_j}{2} + \binom{n}{2}\right). \tag{1}$$

In fact, we can also compute $f^{\mu/\lambda}$ using [1, p.310], [3, Theorem] and [12, Corollary 7.16.3], but this requires evaluation of determinants and knowledge of Schur functions. If we compute $\lambda = (2)$ using [12, Corollary 7.16.3], then we get the following:

$$f^{\mu/(2)} = \frac{f^\mu}{[n]_2}\left(\sum_{i=1}^{k}\left(\binom{\mu_i}{2} - \mu_i(i-1)\right) + \binom{n}{2}\right). \tag{2}$$

Since the following equation is well known (see [9, (1.6)], also see Proposition 34 for a generalization):

$$\sum_{i=1}^{k} \mu_i(i-1) = \sum_{j\geq 1}\binom{\mu'_j}{2}, \tag{3}$$

we have (1). As previously stated, since $K(\mu,(m,1^{n-m})) = f^{\mu/(m)}$, we know the value of $K(\mu,(2,1^{n-2}))$ from (1), so we are interested in the extent to which (1) can be generalized to an arbitrary positive integer m. In fact, if $\lambda = (3)$ then we get the following using [12, Corollary 7.16.3]:

$$\begin{aligned} f^{\mu/(3)} = &\frac{f^\mu}{[n]_3}\left(\sum_{i=1}^{k}\left(\mu_i(i-1) + \binom{\mu_i}{2}\right) + (n-2)\sum_{i=1}^{k}\left(\binom{\mu_i}{2} - \mu_i(i-1)\right)\right) \\ &+ \frac{f^\mu}{[n]_3}\left(2\sum_{i=1}^{k}\left(\mu_i\binom{i-1}{2} + \binom{\mu_i}{3}\right) - 2\sum_{i=1}^{k}\binom{\mu_i}{2}(i-1) + \binom{n}{3} - \binom{n}{2}\right). \end{aligned} \tag{4}$$

The proof of (4) using Lassalle's explicit formula for characters will be given in Section 3.3.

In Chapter 4, we will prove Vershik's relations for the Kostka numbers. Given a partition $\mu = (\mu_1,\ldots,\mu_k) \vdash n$ and a composition $a = (a_1, a_2 \ldots, a_h) \vDash n$, we denote by $a^{(i)}$ the composition of $n-1$ defined by $a_i^{(i)} = a_i - 1$, and $a_j^{(i)} = a_j$ otherwise.

For $\mu = (\mu_1,\ldots,\mu_k) \vdash n$ and $\gamma \vdash n-1$, we write $\gamma \preceq \mu$ if $\gamma_i \leq \mu_i$ for all i with $1 \leq i \leq k$, and define

$$C(\mu,\gamma) = |\{i \mid 1 \leq i \leq k,\ \lambda(\mu^{(i)}) = \gamma\}|.$$

Vershik's relations for the Kostka numbers is as follows:

Theorem 3 ([2, p.143, Theorem 3.6.13] and [14, Theorem 4])**.** *For any $\lambda \vdash n$ and $\rho \vdash n-1$, we have*

$$\sum_{\substack{\mu\vdash n\\ \mu\succeq\rho}} K(\mu,\lambda) = \sum_{\substack{\gamma\vdash n-1\\ \gamma\preceq\lambda}} C(\lambda,\gamma)K(\rho,\gamma).$$

Theorem 3 can be proved using representation theory. As previously stated, since $K(\mu,\lambda) = |\operatorname{STab}(\mu,\lambda)|$, it is natural to expect a bijective proof of Theorem 3. In fact, Vershik [14, Theorem 4] claims to give a bijection from

$$\mathcal{L} = \bigcup_{\substack{\mu\vdash n\\ \mu\succeq\rho}} \operatorname{STab}(\mu,\lambda)$$

to

$$\mathcal{R} = \bigcup_{1 \leq x \leq h} \mathrm{STab}(\rho, \lambda^{(x)}),$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h) \vdash n$ and $\rho \vdash n-1$. In order to explain his proof, we call a tableau in $\mathrm{STab}(\mu, \lambda)$ a μ-tableau, and a tableau in $\mathrm{STab}(\rho, \lambda^{(x)})$ a ρ-tableau. Since μ-tableaux have one more box than ρ-tableaux, Vershik [14, Theorem 4] claims that removable of one box from μ-tableaux gives a bijection from $\mathcal{L}$ to $\mathcal{R}$. Vershik [14, Section 4] gives examples, each of which comes with a bijection. However, if $\lambda = (3,3,2) \vdash 8$ and $\rho = (4,3) \vdash 7$ then there is no bijection from $\mathcal{L}$ to $\mathcal{R}$ arising from removable of one box. More precisely, we consider two tableaux in $\mathcal{L}$ as follows:

$$A = \begin{matrix} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & & \end{matrix}, \quad E = \begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & \\ 3 & & & \end{matrix}.$$

The only ρ-tableau obtainable from A by removing one box is

$$Q = \begin{matrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & \end{matrix}.$$

Similarly, the only ρ-tableau obtainable from E by removing one box is Q.

In Chapter 4, we describe a bijection between $\mathcal{R}$ and $\mathcal{L}$ using tableau insertion and reverse insertion algorithms (see [5] and [8]). We note that, in our bijection, λ is allowed to be a composition which is not necessarily a partition.

This thesis is organized as follows. In Chapter 2, we introduce basic concepts and define a partial order on $\mathrm{STab}(\mu, a)$ in Section 2.1. We describe the procedure of constructing the greatest and least elements of $\mathrm{STab}(\mu, a)$ in Section 2.2 and 2.4, respectively. In Section 2.3, we justify the proof of [2, Lemma 3.7.3] in Proposition 20, using the ideas from Section 2.2.

In Chapter 3, we introduce basic concepts about binomial coefficients in Section 3.1, and we prove that $p_l[C(\mu)]$ can be written as a linear combination of $q^{\pm}_{r,t}$ in Section 3.2. We give an expression for $f^{\mu/(m)}$ in terms of $q^{\pm}_{r,t}$ for $m \leq 4$ in Section 3.3. We prove a generalization of (3) in Section 3.4.

In Chapter 4, we define a bumping route using the tableau insertion algorithm in Section 4.1. Similarly, we define a reverse bumping route using the reverse insertion algorithm in Section 4.2. Finally, in Section 4.3, we prove Theorem 3 by showing that the tableau insertion algorithm gives a bijection.

Chapter 2

Three algorithms to construct semistandard Young tableaux

2.1 A partial order on $\mathrm{STab}(\mu, a)$

For a composition $a = (a_1, a_2, \ldots, a_h) \vDash n$, we define

$$a^{(i)} = (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_h)$$

for each $1 \le i \le h$. Set

$$a' = (a_1, a_2, \ldots, a_{h-1}) \vDash n - a_h,$$
$$\tilde{a} = \begin{cases} (a_1, a_2, \ldots, a_h - 1) & \text{if } a_h \ge 2, \\ (a_1, a_2, \ldots, a_{h-1}) & \text{if } a_h = 1. \end{cases}$$

Then $\tilde{a} \vDash n - 1$. Let

$$q(a) = \max\{i \mid 1 \le i \le h,\ \lambda(a)_i = a_h\}.$$

Then

$$\lambda(a)_{q(a)} = a_h > \lambda(a)_{q(a)+1}. \tag{1}$$

Set

$$\tilde{\lambda}(a) = \lambda(a)^{(q(a))} \vdash n - 1.$$

For a partition $\mu = (\mu_1, \ldots, \mu_k) \vdash n$, we define

$$s(\mu, a) = \max\{i \mid 1 \le i \le k,\ \mu_i \ge a_h\}.$$

Clearly,

$$\mu_{s(\mu,a)} \ge a_h > \mu_{s(\mu,a)+1}. \tag{2}$$

For $\rho = (\rho_1, \rho_2, \ldots, \rho_h) \vdash m$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$, we write

$$\rho \preceq \mu$$

if $m \leq n$, $h \leq k$ and $\rho_i \leq \mu_i$ for all i with $1 \leq i \leq h$. For such ρ and μ, we say that μ/ρ is a *skew shape*, and we denote $D_\mu \setminus D_\rho$ by $D_{\mu/\rho}$. We say that the skew shape μ/ρ is totally disconnected if $\rho_i \geq \mu_{i+1}$ for all i with $1 \leq i \leq h$. Set

$$\mathcal{B}(\mu, a) = \{\rho \vdash n - a_h \mid \rho \trianglerighteq \lambda(a'),\ \rho \preceq \mu,\ \mu/\rho : \text{totally disconnected}\}.$$

Lemma 4. *For a composition $a \vDash n$, $\lambda(\tilde{a}) = \tilde{\lambda}(a)$.*

Proof. Immediate from the definition. □

Lemma 5. *Let p and q be positive integers. Let $\mu \vdash n$ and $\lambda \vdash n$ satisfy $\mu_p > \mu_{p+1}$, $\lambda_q > \lambda_{q+1}$ and $\mu \trianglerighteq \lambda$. Then the following are equivalent.*

(i) $\mu^{(p)} \trianglerighteq \lambda^{(q)}$,

(ii) *either $p \geq q$, or $p < q$ and $\sum_{i=1}^{j} \mu_i > \sum_{i=1}^{j} \lambda_i$ for all j with $p \leq j < q$.*

Proof. Observe

$$\sum_{i=1}^{j} \mu_i^{(p)} = \begin{cases} \sum_{i=1}^{j} \mu_i & \text{if } 1 \leq j < p, \\ \sum_{i=1}^{j} \mu_i - 1 & \text{otherwise,} \end{cases}$$
$$\sum_{i=1}^{j} \lambda_i^{(q)} = \begin{cases} \sum_{i=1}^{j} \lambda_i & \text{if } 1 \leq j < q, \\ \sum_{i=1}^{j} \lambda_i - 1 & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{i=1}^{j} \mu_i^{(p)} \geq \sum_{i=1}^{j} \lambda_i^{(q)} \iff \begin{cases} \sum_{i=1}^{j} \mu_i \geq \sum_{i=1}^{j} \lambda_i & \text{if } 1 \leq j < \min\{p, q\}, \\ \sum_{i=1}^{j} \mu_i \geq \sum_{i=1}^{j} \lambda_i - 1 & \text{if } q \leq j < p, \\ \sum_{i=1}^{j} \mu_i - 1 \geq \sum_{i=1}^{j} \lambda_i & \text{if } p \leq j < q, \\ \sum_{i=1}^{j} \mu_i - 1 \geq \sum_{i=1}^{j} \lambda_i - 1 & \text{if } \max\{p, q\} \leq j. \end{cases}$$

Since $\mu \trianglerighteq \lambda$, we have

$$\begin{aligned} \text{(i)} &\iff \sum_{i=1}^{j} \mu_i - 1 \geq \sum_{i=1}^{j} \lambda_i \quad \text{if } p \leq j < q, \\ &\iff \text{(ii)}. \end{aligned}$$

□

Definition 6. Let $\mu \vdash n$ and $a \vDash n$ with $\mu \trianglerighteq \lambda(a)$. A box of coordinate (i, μ_i) is *removable for the pair* (μ, a) if $\mu_i > \mu_{i+1}$ and $\mu^{(i)} \trianglerighteq \lambda(\tilde{a})$. We denote by $R(\mu, a)$ the set of all i such that (i, μ_i) is removable for the pair (μ, a).

Lemma 7. *Let $\mu \vdash n$ and $a \vDash n$ satisfy $\mu \trianglerighteq \lambda(a)$. Then $s(\mu, a) \in R(\mu, a)$.*

Proof. Write $p = s(\mu, a)$, $q = q(a)$, $\lambda = \lambda(a)$. Then $\lambda_q > \lambda_{q+1}$ by (1) and $\mu_p > \mu_{p+1}$ by (2). Moreover, by Lemma 4, we have $\lambda(\tilde{a}) = \tilde{\lambda}(a) = \lambda^{(q)}$. Thus, in view of Lemma 5, it suffices to show

$$\text{either } p \geq q, \text{ or } p < q \text{ and } \sum_{i=1}^{j} \mu_i > \sum_{i=1}^{j} \lambda_i \text{ for all } j \text{ with } p \leq j < q.$$

Suppose $p < q$ and let $p \leq j < q$. Let $a = (a_1, \dots, a_h)$. If $j + 1 \leq i \leq q$, then $p < i \leq q$, so $\mu_i < a_h \leq \lambda_i$. Thus

$$\begin{aligned}
\sum_{i=1}^{j} \mu_i &= \sum_{i=1}^{q} \mu_i - \sum_{i=j+1}^{q} \mu_i \\
&\geq \sum_{i=1}^{q} \lambda_i - \sum_{i=j+1}^{q} \mu_i \\
&> \sum_{i=1}^{q} \lambda_i - \sum_{i=j+1}^{q} \lambda_i \\
&= \sum_{i=1}^{j} \lambda_i.
\end{aligned}$$

□

From Lemma 7, we find $R(\mu, a) \neq \emptyset$. Set

$$l(\mu, a) = \min R(\mu, a).$$

Lemma 8. *Let $\mu \vdash n$ and $a \vDash n$ satisfy $\mu \trianglerighteq \lambda(a)$. Then $l(\mu, a) \leq s(\mu, a)$.*

Proof. Immediate from Lemma 7. □

Lemma 9. *Let $\mu = (\mu_1, \mu_2, \dots, \mu_k) \vdash n$ and $a = (a_1, a_2, \dots, a_h) \vDash n$. Let i be an integer with $1 \leq i \leq k$. If there exists a tableau $T \in \mathrm{STab}(\mu, a)$ such that $T(i, \mu_i) = h$, then $i \in R(\mu, a)$.*

Proof. Since $T \in \mathrm{STab}(\mu, a)$ and $T(i, \mu_i) = h$, we have $(i + 1, \mu_i) \notin D_\mu$, so $\mu_i > \mu_{i+1}$. Also, $\tilde{T} = T|_{D_\mu \setminus \{(i,\mu_i)\}} \in \mathrm{STab}(\mu^{(i)}, \tilde{a})$. By [2, Lemma 3.7.1], we obtain $\mathrm{STab}(\mu^{(i)}, \lambda(\tilde{a})) \neq \emptyset$. Thus $\mu^{(i)} \trianglerighteq \lambda(\tilde{a})$ and $i \in R(\mu, a)$. □

In fact, the converse of Lemma 9 is also true. We will prove it in Section 2.3.

Lemma 10. *Let $\mu \vdash n$ and $a = (a_1, a_2, \dots, a_h) \vDash n$ satisfy $\mu \trianglerighteq \lambda(a)$. For $T \in \mathrm{STab}(\mu, a)$, we have*

$$l(\mu, a) \leq \min\{i \mid T(i, \mu_i) = h\} \leq s(\mu, a).$$

Proof. Write $q = \min\{i \mid T(i, \mu_i) = h\}$. By Lemma 9, we have $l(\mu, a) \leq q$. Since $a_h = |\{(i, j) \in D_\mu \mid T(i, j) = h\}| \leq \mu_q$, we have $q \leq s(\mu, a)$. □

We will show in Theorem 14, Lemma 18 and Theorem 26 that equality can be achieved in both of the inequalities above. In Section 2.2, we give an algorithm to construct $T \in \mathrm{STab}(\mu, a)$ such that $\min\{i \mid T(i, \mu_i) = h\} = s(\mu, a)$. In Sections 2.3 and 2.4, we give an algorithm to construct $T \in \mathrm{STab}(\mu, a)$ such that $\min\{i \mid T(i, \mu_i) = h\} = l(\mu, a)$.

Finally, we define a partial order on $\mathrm{STab}(\mu, a)$ and a partition $\rho(\mu, a) \vdash n - a_h$ as follows. We write $\mu \rhd \lambda$ to mean $\mu \trianglerighteq \lambda$ and $\mu \neq \lambda$.

Definition 11. Let $\mu = (\mu_1, \mu_2, \dots, \mu_k) \vdash n$ and $a = (a_1, a_2, \dots, a_h) \vDash n$ satisfy $\mu \trianglerighteq \lambda(a)$. For T, $S \in \mathrm{STab}(\mu, a)$, let

$$\tau^{(p)} = (|\{j \mid T(i, j) \leq p\}|)_{i=1}^{k}, \tag{3}$$

$$\sigma^{(p)} = (|\{j \mid S(i, j) \leq p\}|)_{i=1}^{k} \tag{4}$$

for all p with $1 \leq p \leq h$. We define that $S \leq T$ if, either $T = S$ or, $\tau^{(h)} = \sigma^{(h)}$, $\tau^{(h-1)} = \sigma^{(h-1)}, \dots, \tau^{(p+1)} = \sigma^{(p+1)}$, $\tau^{(p)} \rhd \sigma^{(p)}$ for some $1 \leq p \leq h$.

Since the relation $\trianglerighteq$ is a partial order, we see that $(\mathrm{STab}(\mu, a), \leq)$ is a partially ordered set.

Alternatively the partial order $\leq$ on $\mathrm{STab}(\mu, a)$ can be defined recursively as follows: for $T, S \in \mathrm{STab}(\mu, a)$, define τ and σ by

$$T^{-1}(\{1, \dots, h-1\}) = D_\tau, \tag{5}$$

$$S^{-1}(\{1, \dots, h-1\}) = D_\sigma, \tag{6}$$

respectively. We define $S \leq T$ if, either $\tau \rhd \sigma$, or $\tau = \sigma$ and $S|_{D_\sigma} \leq T|_{D_\sigma}$.

Definition 12. Let $\mu = (\mu_1, \dots, \mu_k) \vdash n$ and $a = (a_1, \dots, a_h) \vDash n$ satisfy $\mu \trianglerighteq \lambda(a)$. Define $\rho(\mu, a) = (\rho_1, \dots, \rho_{k-1}) \vdash n - a_h$ by setting

$$\rho_i = \begin{cases} \mu_i & \text{if } 1 \leq i < s, \\ \mu_s - (a_h - \mu_{s+1}) & \text{if } i = s, \\ \mu_{i+1} & \text{if } s < i \leq k-1, \end{cases}$$

where $s = s(\mu, a)$.

2.2 The greatest element of $\mathrm{STab}(\mu, a)$

Lemma 13. *Let $\mu = (\mu_1, \dots, \mu_k) \vdash n$ and $a = (a_1, \dots, a_h) \vDash n$ satisfy $\mu \trianglerighteq \lambda(a)$. Then $\rho(\mu, a)$ is the greatest element of $\mathcal{B}(\mu, a)$.*

Proof. Write $\rho = \rho(\mu, a)$ and $s = s(\mu, a)$. By (2), we have $\mu_s \geq a_h > \mu_{s+1}$. Thus $\rho \vdash n - a_h$ and $\mu_s > \rho_s \geq \mu_{s+1}$. So $\mu_i \geq \rho_i \geq \mu_{i+1}$ for all i with $1 \leq i \leq k$. This implies that $\rho \preceq \mu$ and μ/ρ is totally disconnected.

Next, we show that $\rho \trianglerighteq \lambda(a')$. Write $\lambda(a) = (\lambda_1, \dots, \lambda_h)$, $\lambda' = (\lambda'_1, \dots, \lambda'_{h-1}) = \lambda(a')$ and $q = q(a)$. Then $\lambda_q = a_h$ and

$$\lambda' = (\lambda_1, \lambda_2, \dots, \lambda_{q-1}, \lambda_{q+1}, \lambda_{q+2}, \dots, \lambda_h) \vdash n - \lambda_q.$$

Observe

$$\sum_{i=1}^{j} \rho_i = \begin{cases} \sum_{i=1}^{j} \mu_i & \text{if } 1 \le j < s, \\ \sum_{i=1}^{j+1} \mu_i - \lambda_q & \text{if } s \le j \le k-1 \end{cases}$$
$$\ge \begin{cases} \sum_{i=1}^{j} \lambda_i & \text{if } 1 \le j < s, \\ \sum_{i=1}^{j+1} \lambda_i - \lambda_q & \text{if } s \le j \le k-1 \end{cases} \tag{7}$$

since $\mu \trianglerighteq \lambda(a)$.

Case 1. $q < s$. By (7), we have

$$\sum_{i=1}^{j} \rho_i \ge \begin{cases} \sum_{i=1}^{j} \lambda_i & \text{if } 1 \le j < q, \\ \sum_{i=1}^{q-1} \lambda_i + \sum_{i=q+1}^{j+1} \lambda_i + \lambda_q - \lambda_{j+1} & \text{if } q \le j < s, \\ \sum_{i=1}^{j+1} \lambda_i - \lambda_q & \text{if } s \le j \le k-1 \end{cases}$$
$$\ge \begin{cases} \sum_{i=1}^{j} \lambda_i & \text{if } 1 \le j < q, \\ \sum_{i=1}^{q-1} \lambda_i + \sum_{i=q+1}^{j+1} \lambda_i & \text{if } q \le j < s, \\ \sum_{i=1}^{j+1} \lambda_i - \lambda_q & \text{if } s \le j \le k-1 \end{cases}$$
$$= \sum_{i=1}^{j} \lambda'_i.$$

Case 2. $s \le q$. By (7), we have

$$\sum_{i=1}^{j} \rho_i \ge \begin{cases} \sum_{i=1}^{j} \lambda_i & \text{if } 1 \le j < s, \\ \sum_{i=1}^{j} \lambda_i + \lambda_{j+1} - \lambda_q & \text{if } s \le j < \min\{q, k\}, \\ \sum_{i=1}^{j+1} \lambda_i - \lambda_q & \text{if } \min\{q, k\} \le j \le k \end{cases}$$
$$\ge \begin{cases} \sum_{i=1}^{j} \lambda_i & \text{if } 1 \le j < \min\{q, k\}, \\ \sum_{i=1}^{j+1} \lambda_i - \lambda_q & \text{if } q \le j \le k \end{cases}$$
$$\ge \sum_{i=1}^{j} \lambda'_i.$$

Thus we have $\rho \in \mathcal{B}(\mu, a)$.

It remains to show that $\rho \trianglerighteq \tau$ for all $\tau \in \mathcal{B}(\mu, a)$. Let $\tau \in \mathcal{B}(\mu, a)$ with $\tau \ne \rho$ and

$$r = \min\{i \mid 1 \le i \le k,\ \tau_i < \mu_i\}. \tag{8}$$

Then

$$1 \le r \le s \tag{9}$$

and

$$\sum_{i=r}^{k} (\mu_i - \tau_i) = a_h. \tag{10}$$

By the definition of $\mathcal{B}(\mu, a)$, we have

$$\mu_i \geq \tau_i \geq \mu_{i+1} \tag{11}$$

for all i with $1 \leq i \leq k$.

If $1 \leq j < s$, then

$$\sum_{i=1}^{j} \rho_i = \sum_{i=1}^{j} \mu_i \geq \sum_{i=1}^{j} \tau_i.$$

If $s \leq j \leq k$, then

$$\begin{aligned}
\sum_{i=1}^{j} \rho_i &= \sum_{i=1}^{j+1} \mu_i - a_h \\
&= \sum_{i=1}^{j+1} \mu_i - \sum_{i=r}^{k} (\mu_i - \tau_i) && \text{(by (10))} \\
&= \sum_{i=1}^{r-1} \mu_i + \sum_{i=r}^{j+1} \mu_i - \sum_{i=r}^{k} \mu_i + \sum_{i=r}^{k} \tau_i && \text{(by (9))} \\
&= \sum_{i=1}^{j} \tau_i + \sum_{i=j+1}^{k} \tau_i - \sum_{i=j+2}^{k} \mu_i && \text{(by (8))} \\
&= \sum_{i=1}^{j} \tau_i + \sum_{i=j+1}^{k} (\tau_i - \mu_{i+1}) \\
&\geq \sum_{i=1}^{j} \tau_i && \text{(by (11))}.
\end{aligned}$$

Therefore, $\rho \trianglerighteq \tau$. $\square$

Theorem 14. *Given $\mu \vdash n$ and $a = (a_1, a_2, \ldots, a_h) \vDash n$ such that $\mu \trianglerighteq \lambda(a)$, define ρ^i and a^i inductively by setting $\rho^0 = \mu$, $a^0 = a$, and for $1 \leq i \leq h$,*

$$\rho^i = \rho(\rho^{i-1}, a^{i-1}) \vdash \sum_{j=1}^{h-i} a_j,$$

$$a^i = (a^{i-1})' \vDash \sum_{j=1}^{h-i} a_j.$$

Define a tableau T of shape μ and weight a by

$$T(p,q) = h - i \text{ if } (p,q) \in D_{\rho^i/\rho^{i+1}}. \tag{12}$$

Then T is the greatest element of $\mathrm{STab}(\mu, a)$.

Proof. We prove the assertion by induction on h. Suppose first $h = 1$. Then $\mathrm{STab}(\mu, a)$ consists of a single element T, so that the assertion trivially holds.

Next suppose $h > 1$. Assume that the assertion holds for $h - 1$. Set $\nu = \rho^1$ and $b = a^1$. Since $\rho^1 \in \mathcal{B}(\mu, a)$ by Lemma 13, we have $\nu \trianglerighteq \lambda(b)$. Define ν^i and b^i inductively by setting $\nu^0 = \nu$, $b^0 = b$, and for $1 \le i < h$,

$$\nu^i = \rho(\nu^{i-1}, b^{i-1}) \vdash \sum_{j=1}^{h-1-i} b_j,$$

$$b^i = (b^{i-1})' \vDash \sum_{j=1}^{h-1-i} b_j.$$

Define a tableau T' of shape ν and weight b by

$$T'(p, q) = h - (i + 1) \text{ if } (p, q) \in D_{\nu^i/\nu^{i+1}}.$$

By the inductive hypothesis,

$$T' \in \mathrm{STab}(\nu, b), \tag{13}$$

$$T' \ge S' \text{ for all } S' \in \mathrm{STab}(\nu, b). \tag{14}$$

It is easy to show that $b^i = a^{i+1}$ and $\nu^i = \rho^{i+1}$ by induction on i, and the latter implies $T|_{D_\nu} = T'$. Then by (13) and the fact that μ/ν is totally disconnected, we obtain $T \in \mathrm{STab}(\mu, a)$.

It remains to show that $T \ge S$ for all $S \in \mathrm{STab}(\mu, a)$. Let $S \in \mathrm{STab}(\mu, a)$. Define a partition σ by (6). By Lemma 13, we have $\nu \trianglerighteq \sigma$. If $\nu \triangleright \sigma$, then $T \ge S$. If $\nu = \sigma$, then $T|_{D_\nu} = T' \ge S|_{D_\nu}$ by (14), hence $T \ge S$. □

From Theorem 14, we obtain a tableau $T \in \mathrm{STab}(\mu, a)$.

Algorithm 1. Input: $\mu \vdash n$ and $a \vDash n$ such that $\mu \trianglerighteq \lambda(a)$.
Output: $T \in \mathrm{STab}(\mu, a)$.
Initialization: $\nu := \mu$, $b := a$.
while $h(b) > 1$ **do**
  $T(i, j) := h(b)$ **where** $(i, j) \in D_{\nu/\rho(\nu,b)}$.
  $\nu \leftarrow \rho(\nu, b)$, $b \leftarrow b'$.
end
$T(1, j) := 1$ **where** $1 \le j \le \nu_1$.
Output T.

Example 15. Let $\mu = (4, 4, 1, 1) \vdash 10$ and $a = (1, 3, 2, 2, 2) \vDash 10$. Then a tableau $T \in \mathrm{STab}(\mu, a)$ is obtained via Algorithm 1.

ν	b	$\rho(\nu, b)$	T
$(4, 4, 1, 1)$	$(1, 3, 2, 2, 2)$	$(4, 3, 1)$	$T(2, 4) = T(4, 1) = 5$
$(4, 3, 1)$	$(1, 3, 2, 2)$	$(4, 2)$	$T(2, 3) = T(3, 1) = 4$
$(4, 2)$	$(1, 3, 2)$	(4)	$T(2, 1) = T(2, 2) = 3$
(4)	$(1, 3)$	(1)	$T(1, 2) = T(1, 3) = T(1, 4) = 2$
(1)	(1)		$T(1, 1) = 1$

Thus

$$T = \begin{matrix} 1 & 2 & 2 & 2 \\ 3 & 3 & 4 & 5 \\ 4 & & & \\ 5 & & & \end{matrix}$$

is the greatest element of $\mathrm{STab}(\mu, a)$.

2.3 Removable boxes

Throughout this section, let $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$ and $a = (a_1, a_2, \ldots, a_h) \vDash n$. Let $\lambda = \lambda(a) = (\lambda_1, \lambda_2, \ldots, \lambda_h)$, $q = q(a)$ and $l = l(\mu, a)$. We assume $\mu \trianglerighteq \lambda$.

Lemma 16. *Assume $a_h \geq 2$. For $i \in R(\mu, a)$ with $i \leq s(\mu^{(i)}, \tilde{a})$, we have $\rho(\mu^{(i)}, \tilde{a}) \in \mathcal{B}(\mu, a)$.*

Proof. Write $\rho = \rho(\mu^{(i)}, \tilde{a})$. Since $i \in R(\mu, a)$, we have $\mu^{(i)} \trianglerighteq \lambda(\tilde{a})$. Then by Lemma 13, $\rho \in \mathcal{B}(\mu^{(i)}, \tilde{a})$. Since $a_h \geq 2$, this implies $\rho \trianglerighteq \lambda(\tilde{a}') = \lambda(a')$. To prove $\rho \in \mathcal{B}(\mu, a)$, it remains to show that μ/ρ is totally disconnected. Since $\mu^{(i)}/\rho$ is totally disconnected, it is enough to show $\rho_{i-1} \geq \mu_i$. Since $i - 1 < s(\mu^{(i)}, \tilde{a})$, we obtain $\rho_{i-1} = \mu^{(i)}_{i-1} = \mu_{i-1} \geq \mu_i$. □

Lemma 17. *Assume $a_h \geq 2$. Then $r \leq s(\mu, a)$ if and only if $r \leq s(\mu^{(r)}, \tilde{a})$.*

Proof. Immediate from the definition. □

Lemma 18. *Define $\rho \vdash n - a_h$ by*

$$\rho = \begin{cases} \mu^{(l)} & \text{if } a_h = 1, \\ \rho(\mu^{(l)}, \tilde{a}) & \text{if } a_h \geq 2. \end{cases}$$

Then μ/ρ is totally disconnected, $\rho_l < \mu_l$ and $\rho \trianglerighteq \lambda(a')$.

Proof. If $a_h = 1$, then $\rho = \mu^{(l)} \trianglerighteq \lambda(\tilde{a}) = \lambda(a')$, since $l \in R(\mu, a)$. Thus the assertion holds.

Suppose $a_h \geq 2$. Then $l \leq s(\mu^{(l)}, \tilde{a})$ by Lemma 8, Lemma 17, and hence $\rho \in \mathcal{B}(\mu, a)$ by Lemma 16. Thus it remains to show that $\rho_l < \mu_l$. This can be shown as follows:

$$\begin{aligned} \rho_l &= \begin{cases} \mu^{(l)}_l - (\tilde{a}_h - \mu^{(l)}_{l+1}) & \text{if } l = s(\mu^{(l)}, \tilde{a}), \\ \mu^{(l)}_l & \text{if } l < s(\mu^{(l)}, \tilde{a}) \end{cases} \\ &\leq \mu^{(l)}_l \\ &< \mu_l, \end{aligned}$$

where the second inequality follows from the definition of $s(\mu^{(l)}, \tilde{a})$. □

From Lemma 18, we obtain a tableau $U \in \mathrm{STab}(\mu, a)$.

Algorithm 2. **Input**: $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$ and $a = (a_1, a_2, \ldots, a_h) \vDash n$ such that $\mu \trianglerighteq \lambda(a)$.
Output: $U \in \mathrm{STab}(\mu, a)$.
Initialization: $\nu := \mu$, $b := a$.
while $h(b) > 1$ **do**
  $l := l(\nu, b)$.
  if $b_{h(b)} = 1$ **then** $\rho := \nu^{(l)}$.
  else
  $\rho := \rho(\nu^{(l)}, \tilde{b})$.
  end if
  $U(i, j) := h(b)$ **where** $(i, j) \in D_{\nu/\rho}$.
  $\nu \leftarrow \rho$, $b \leftarrow b'$.
end
$U(1, j) := 1$ **where** $1 \leq j \leq \nu_1$.
Output U.

Example 19. Let $\mu = (4, 4, 1, 1) \vdash 10$ and $a = (1, 3, 2, 2, 2) \vDash 10$. Then a tableau $U \in \mathrm{STab}(\mu, a)$ is obtained via Algorithm 2.

ν	b	ρ	U
$(4,4,1,1)$	$(1,3,2,2,2)$	$(4,3,1)$	$U(2,4) = U(4,1) = 5$
$(4,3,1)$	$(1,3,2,2)$	$(3,3)$	$U(1,4) = U(3,1) = 4$
$(3,3)$	$(1,3,2)$	$(3,1)$	$U(2,2) = U(2,3) = 3$
$(3,1)$	$(1,3)$	(1)	$U(1,2) = U(1,3) = U(2,1) = 2$
(1)	(1)		$U(1,1) = 1$

Thus

$$U = \begin{array}{cccc} 1 & 2 & 2 & 4 \\ 2 & 3 & 3 & 5 \\ 4 & & & \\ 5 & & & \end{array}.$$

We note that the least element of $\mathrm{STab}(\mu, a)$ is

$$S = \begin{array}{cccc} 1 & 2 & 2 & 4 \\ 2 & 3 & 5 & 5 \\ 3 & & & \\ 4 & & & \end{array}.$$

In Section 2.4, we will show that there exists a unique least element of $\mathrm{STab}(\mu, a)$ whenever $\mu \trianglerighteq \lambda(a)$, and give an algorithm to construct it.

Proposition 20. *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$ and $a = (a_1, a_2, \ldots, a_h) \vDash n$, and assume $\mu \trianglerighteq \lambda(a)$. Let r be an integer with $1 \leq r \leq k$. Then there exists a tableau $T \in \mathrm{STab}(\mu, a)$ such that $T(r, \mu_r) = h$ if and only if $r \in R(\mu, a)$.*

Proof. The "only if" part has been proved in Lemma 9. We prove the "if" part by induction on n. If $n = 1$ then it is obvious. Let $r \in R(\mu, a)$, $s = s(\mu, a)$ and $s' = s(\mu^{(r)}, \tilde{a})$.

If $a_h \geq 2$ then define $\rho \vdash n - a_h$ by

$$\rho = \begin{cases} \rho(\mu, a) & \text{if } r > s', \\ \rho(\mu^{(r)}, \tilde{a}) & \text{otherwise.} \end{cases} \tag{15}$$

From Lemma 13 and 16, we have $\rho \in \mathcal{B}(\mu, a)$, so

$$\rho \trianglerighteq \lambda(a'). \tag{16}$$

If $a_h = 1$, then define by $\rho = \mu^{(r)}$. From the definition of $R(\mu, a)$, (16) holds in this case also.

Since (16) implies $R(\rho, \lambda(a')) \neq \emptyset$, the inductive hypothesis implies that there exists a tableau $T' \in \mathrm{STab}(\rho, a')$. Define a tableau T of shape μ and weight a by

$$T(i,j) = \begin{cases} T'(i,j) & \text{if } (i,j) \in D_\rho, \\ h & \text{if } (i,j) \in D_{\mu/\rho}. \end{cases}$$

It remains to show that $T(r, \mu_r) = h$. This will follow if we can show $\rho_r < \mu_r$. If $a_h = 1$, then $\rho_r = \mu_r - 1 < \mu_r$. Suppose $a_h \geq 2$. If $r > s'$ then we have $r > s' \geq s$ by the definition of s' and s. Since $r \in R(\mu, a)$, we have $\rho_r = \mu_{r+1} < \mu_r$. If $r \leq s'$, then

$$\begin{aligned} \rho_r &= \begin{cases} \mu_r^{(r)} - (\tilde{a} - \mu_{r+1}^{(r)}) & \text{if } r = s', \\ \mu_r^{(r)} & \text{if } r < s' \end{cases} \\ &\leq \mu_r^{(r)} \\ &= \mu_r - 1 \\ &< \mu_r, \end{aligned}$$

where the second inequality follows from the definition of s'. □

Proposition 20 justifies the proof of [2, Lemma 3.7.3]. It also gives an alternative proof of the "if" part of Theorem 2.

2.4 The least element of $\mathrm{STab}(\mu, a)$

Throughout this section, we let $\mu = (\mu_1, \mu_2, \dots, \mu_k) \vdash n$ and $a = (a_1, a_2, \dots, a_h) \vDash n$. We assume $\mu \trianglerighteq \lambda(a)$. For a sequence $(i_1, i_2, \dots, i_j)$ of positive integers, we abbreviate the partition

$$(\cdots((\mu^{(i_1)})^{(i_2)})\cdots)^{(i_j)}$$

of $n - j$, as $\mu^{(i_1, i_2, \dots, i_j)}$.

Lemma 21. *We have*

$$R(\mu, a) = \{i \mid l(\mu, a) \leq i \leq k,\ \mu_i > \mu_{i+1}\}.$$

Proof. Write $q = q(a)$, $\lambda = \lambda(a)$ and $l = l(\mu, a)$. Then $\lambda_q > \lambda_{q+1}$ by (1). Let $l \le i \le k$ and $\mu_i > \mu_{i+1}$. From Lemma 5 and the definition of l, we have $\mu^{(i)} \trianglerighteq \mu^{(l)} \trianglerighteq \lambda^{(q)}$. Thus $i \in R(\mu, a)$. □

For each i with $1 \le i \le k$, set

$$R(\mu, a, i) = \{r \in R(\mu, a) \mid r \ge i\}.$$

From Lemma 21, we have $k \in R(\mu, a, i)$ for each i with $1 \le i \le k$. Set

$$l(\mu, a, i) = \min R(\mu, a, i).$$

Clearly, $l(\mu, a, 1) = l(\mu, a)$.

Lemma 22. *Let μ and μ' be partitions of n. Suppose that $i \in R(\mu, a)$ and $i' \in R(\mu', a)$ satisfy $i \le i'$, $\mu_{i'} \ge \mu'_{i'}$ and $\mu_j = \mu'_j$ for all j with $j > i'$. Then*

$$R(\mu'^{(i')}, \tilde{a}, i') \subseteq R(\mu^{(i)}, \tilde{a}, i).$$

Proof. Let $r \in R(\mu'^{(i')}, \tilde{a}, i')$. Since $i \le i' \le r$, we have

$$\delta_{i,r} \le \delta_{i',r}, \tag{17}$$

and

$$\mu'^{(i')}_{r+1} = \mu'_{r+1} = \mu_{r+1} = \mu^{(i)}_{r+1}. \tag{18}$$

Thus

$$\begin{aligned}
\mu^{(i)}_r &= \mu_r - \delta_{i,r} \\
&\ge \mu'_r - \delta_{i,r} \\
&\ge \mu'_r - \delta_{i',r} && \text{(by (17))} \\
&= \mu'^{(i')}_r \\
&> \mu'^{(i')}_{r+1} \\
&= \mu^{(i)}_{r+1} && \text{(by (18))}.
\end{aligned}$$

It remains show that $\mu^{(i,r)} \trianglerighteq \lambda(\tilde{\tilde{a}})$. If $1 \le q < r$, then

$$\sum_{j=1}^{q} \mu^{(i,r)}_j = \sum_{j=1}^{q} \mu^{(i)}_j \ge \sum_{j=1}^{q} \lambda(\tilde{a})_j \ge \sum_{j=1}^{q} \lambda(\tilde{\tilde{a}})_j,$$

since $i \in R(\mu, a)$. If $r \le q$, then $i' \le q$, and hence $\sum_{j>q} \mu_j = \sum_{j>q} \mu'_j$. Thus

$$\begin{aligned}
\sum_{j=1}^{q} \mu^{(i,r)}_j &= \sum_{j=1}^{q} \mu_j - 2 \\
&= \sum_{j=1}^{q} \mu'_j - 2
\end{aligned}$$

$$= \sum_{j=1}^{q} \mu_j'^{(i',r)}$$
$$\geq \sum_{j=1}^{q} \lambda(\tilde{\tilde{a}})_j,$$

by $r \in R(\mu'^{(i')}, \tilde{a})$. □

Lemma 23. *Assume $a_h \geq 2$, $r \in R(\mu, a)$, and $r \leq s(\mu, a)$. Then $s(\mu, a) \leq s(\mu^{(r)}, \tilde{a})$ In particular, $R(\mu^{(r)}, \tilde{a}, r) \neq \emptyset$ and $l(\mu^{(r)}, \tilde{a}, r) \leq s(\mu^{(r)}, \tilde{a})$.*

Proof. Since $a_h \geq 2$, we have

$$\begin{aligned}\tilde{a}_h &= a_h - 1 \\ &\leq a_h - \delta_{r,s(\mu,a)} \\ &\leq \mu_{s(\mu,a)} - \delta_{r,s(\mu,a)} \\ &= \mu^{(r)}_{s(\mu,a)}.\end{aligned}$$

Thus $s(\mu^{(r)}, \tilde{a}) \geq s(\mu, a) \geq r$, and hence $s(\mu^{(r)}, \tilde{a}) \in R(\mu^{(r)}, \tilde{a}, r)$ by Lemma 7. □

Notation 24. Let $r \in R(\mu, a)$ and suppose $r \leq s(\mu, a)$. Define a^i, l_i and μ^i inductively by setting $a^0 = a$, $l_0 = r$, $\mu^0 = \mu$ and for $0 \leq i < n$,

$$\begin{aligned}a^{i+1} &= \widetilde{a^i} \vDash n - i - 1, \\ l_{i+1} &= \begin{cases} l(\mu^i, a^i, 1) & \text{if } i \in A, \\ l(\mu^i, a^i, l_i) & \text{otherwise}, \end{cases} \\ \mu^{i+1} &= (\mu^i)^{(l_{i+1})} \vdash n - i - 1,\end{aligned}$$

where $A = \{a_h, a_h + a_{h-1}, \ldots, a_h + \cdots + a_2\}$.

In order to check l_{i+1} and μ^{i+1} are well-defined, we show

$$\mu^i \trianglerighteq \lambda(a^i) \qquad (0 \leq i < n), \tag{19}$$
$$R(\mu^i, a^i, l_i) \neq \emptyset \qquad (0 \leq i < n,\ i \notin A), \tag{20}$$
$$l_{i+1} \leq s(\mu^i, a^i) \qquad (0 \leq i < n). \tag{21}$$

Indeed, (19)–(20) guarantee that l_{i+1} is defined as an element of $R(\mu^i, a^i)$, even when $i \notin A$, so μ^{i+1} is also defined.

We prove (19)–(21) by induction on i. If $i = 0$ then, as $\mu \trianglerighteq \lambda(a)$, (19) holds. Also, (20) holds since $r \in R(\mu, a, r)$. Since $0 \notin A$, we have $l_1 = l(\mu, a, l_0) = r \leq s(\mu, a)$. Thus (21) holds for $i = 0$ as well.

Assume (19)–(21) hold for some $i \in \{0, 1, \ldots, n-2\}$. Since $l_{i+1} \in R(\mu^i, a^i)$, (19) holds for $i + 1$.

If $i + 1 \notin A$, then $a^i_{h(a^i)} \geq 2$. Also $l_{i+1} \leq s(\mu^i, a^i)$ by induction. Lemma 23 then implies $R(\mu^{i+1}, a^{i+1}, l_{i+1}) \neq \emptyset$ and $l_{i+2} \leq s(\mu^{i+1}, a^{i+1})$, so (20)–(21) hold for $i + 1$ as well.

If $i+1 \in A$, then $l_{i+2} = l(\mu^{i+1}, a^{i+1}, 1) = \min R(\mu^{i+1}, a^{i+1}) \leq s(\mu^{i+1}, a^{i+1})$ by Lemma 7. Thus (21) holds for $i+1$ as well.

Clearly, $\mu^i = \mu^{(l_1,\dots,l_i)}$.

Lemma 25. *Let $T \in \mathrm{STab}(\mu, a)$. With reference to Notation 24, suppose $T^{-1}(h) = \{(t_1, t'_1), (t_2, t'_2), \dots, (t_{a_h}, t'_{a_h})\}$ and $r \leq t_1 \leq t_2 \leq \cdots \leq t_{a_h}$. Then $l_i \leq t_i$ for $1 \leq i \leq a_h$. In particular, $\mu^{(t_1,\dots,t_{a_h})} \trianglerighteq \mu^{a_h}$.*

Proof. We prove the assertion by induction on i. If $i = 1$, then $l_1 = r \leq t_1$.

Assume $l_1 \leq t_1, \dots, l_i \leq t_i$ hold for some i with $1 \leq i < a_h$. We aim to show $l_{i+1} \leq t_{i+1}$ by deriving

$$R(\mu^{(t_1,\dots,t_i)}, a^i, t_i) \subseteq R(\mu^i, a^i, l_i) \tag{22}$$

from Lemma 22. In order to do so, we need to verify the hypotheses of Lemma 22. By the definition of l_i, we have $l_i \in R(\mu^{i-1}, a^{i-1})$. Since the restriction of T to $D_{\mu^{(t_1,\dots,t_{i-1})}}$ is an element of $\mathrm{STab}(\mu^{(t_1,\dots,t_{i-1})}, a^{i-1})$, Lemma 9 implies $t_i \in R(\mu^{(t_1,\dots,t_{i-1})}, a^{i-1})$, and our inductive hypothesis shows $l_i \leq t_i$. Similarly, we have

$$t_{i+1} \in R(\mu^{(t_1,\dots,t_i)}, a^i, t_i). \tag{23}$$

Since $l_p \leq t_p \leq t_i$ for $1 \leq p \leq i-1$ by our inductive hypothesis,

$$\begin{aligned} \mu^{i-1}_{t_i} &= \mu^{(l_1,\dots,l_{i-1})}_{t_i} \\ &= \mu_{t_i} - |\{p \mid 1 \leq p \leq i-1,\ l_p = t_i\}| \\ &\geq \mu_{t_i} - |\{p \mid 1 \leq p \leq i-1,\ t_p = t_i\}| \\ &= \mu^{(t_1,\dots,t_{i-1})}_{t_i}. \end{aligned}$$

Finally, for $j > t_i$, we have $\mu^{i-1}_j = \mu^{(l_1,\dots,l_{i-1})}_j = \mu_j = \mu^{(t_1,\dots,t_{i-1})}_j$, since $l_p \leq t_p \leq t_i$ for $1 \leq p \leq i-1$. Therefore, we have verified all the hypotheses of Lemma 22, and we obtain (22).

Now

$$\begin{aligned} l_{i+1} &= l(\mu^i, a^i, l_i) \\ &= \min R(\mu^i, a^i, l_i) \\ &\leq \min R(\mu^{(t_1,\dots,t_i)}, a^i, t_i) && \text{(by (22))} \\ &\leq t_{i+1} && \text{(by (23))}. \end{aligned}$$

□

Theorem 26. *Let $\mu = (\mu_1, \mu_2, \dots, \mu_k) \vdash n$ and $a = (a_1, a_2, \dots, a_h) \vDash n$, and suppose $\mu \trianglerighteq \lambda(a)$. Let $r \in R(\mu, a)$ and suppose $r \leq s(\mu, a)$. Define a^i, l_i and μ^i as in Notation 24. Define a tableau S of shape μ and weight a by*

$$S(l_{i+1}, \mu^i_{l_{i+1}}) = t,$$

where $0 \leq i < n$ and $\sum_{j=1}^{t-1} a_j < n - i \leq \sum_{j=1}^{t} a_j$. Then S is the least element of the subposet

$$\{T \in \mathrm{STab}(\mu, a) \mid \min\{i \mid T(i, \mu_i) = h\} \geq r\} \tag{24}$$

In particular, if $r = l(\mu, a)$, then S is the least element of $\mathrm{STab}(\mu, a)$.

Proof. Note that the tableau S is well-defined. Indeed, by the definition of μ^i, we have

$$D_{\mu^i} = D_{\mu^{i+1}} \cup \{(l_{i+1}, \mu^i_{l_{i+1}})\}. \tag{25}$$

So $D_\mu = \{(l_{i+1}, \mu^i_{l_{i+1}}) \mid 0 \leq i < n\}$.

Next, we prove the statement by induction on n. If $n = 1$ then $\mu = a = (1)$, so it is obvious.

Assume that the statement holds for $n-1$. We apply Notation 24 with r, μ, a replaced by $l_2, \nu = \mu^1, b = a^1$, respectively. This is admissible since $l_2 \in R(\mu^1, a^1) = R(\nu, b)$ and $l_2 \leq s(\mu^1, a^1) = s(\nu, b)$ by (21). Define b^i, l'_i and ν^i inductively by setting $b^0 = b$, $l'_0 = l_2$, $\nu^0 = \nu$ and for $0 \leq i < n-1$,

$$\begin{aligned} b^{i+1} &= \widetilde{b^i} \models n-i-2, \\ l'_{i+1} &= \begin{cases} l(\nu^i, b^i, 1) & \text{if } i \in B, \\ l(\nu^i, b^i, l'_i) & \text{otherwise,} \end{cases} \\ \nu^{i+1} &= (\nu^i)^{(l'_{i+1})} \vdash n-i-2, \end{aligned}$$

where

$$B = \begin{cases} \{b_{h-1}, b_{h-1}+b_{h-2}, \dots, b_{h-1}+\cdots+b_2\} & \text{if } a_h = 1, \\ \{b_h, b_h+b_{h-1}, \dots, b_h+\cdots+b_2\} & \text{otherwise,} \end{cases}$$

$$b = \begin{cases} (b_1, \dots, b_{h-1}) & \text{if } a_h = 1, \\ (b_1, \dots, b_h) & \text{otherwise.} \end{cases}$$

Define a tableau $\tilde{S}$ of shape ν and weight b by

$$\tilde{S}(l'_{i+1}, \nu^i_{l'_{i+1}}) = t,$$

where $\sum_{j=1}^{t-1} b_j < n-1-i \leq \sum_{j=1}^{t} b_j$. By the inductive hypothesis, $\tilde{S}$ is the least element of the set

$$\{\tilde{T} \in \mathrm{STab}(\nu, b) \mid \min\{i \mid \tilde{T}(i, \nu_i) = h(b)\} \geq l_2\}. \tag{26}$$

It is easy to see that $b^i = a^{i+1}$ for $0 \leq i < n$. We show that

$$l'_i = l_{i+1} \text{ and } \nu^i = \mu^{i+1} \quad (1 \leq i < n) \tag{27}$$

by induction on i. Since $0 \notin B$, we have

$$\begin{aligned} l'_1 &= l(\nu^0, b^0, l'_0) \\ &= l(\mu^1, a^1, l_2) \\ &= \begin{cases} l(\mu^1, a^1, l(\mu^1, a^1, 1)) & \text{if } 1 \in A, \\ l(\mu^1, a^1, l(\mu^1, a^1, l_1)) & \text{otherwise} \end{cases} \end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} l(\mu^1, a^1, 1) & \text{if } 1 \in A, \\ l(\mu^1, a^1, l_1) & \text{otherwise} \end{cases} \\
&= l_2. \qquad (28)
\end{aligned}
$$

Then $\nu^1 = (\nu^0)^{(l'_1)} = (\mu^1)^{(l_2)} = \mu^2$.

Assume $i \geq 2$ and $l'_{i-1} = l_i$ and $\nu^{i-1} = \mu^i$. Since $i-1 \in B$ if and only if $i \in A$, we have

$$
\begin{aligned}
l'_i &= \begin{cases} l(\nu^{i-1}, b^{i-1}, 1) & \text{if } i-1 \in B, \\ l(\nu^{i-1}, b^{i-1}, l'_{i-1}) & \text{otherwise} \end{cases} \\
&= \begin{cases} l(\mu^i, a^i, 1) & \text{if } i \in A, \\ l(\mu^i, a^i, l_i) & \text{otherwise} \end{cases} \\
&= l_{i+1}.
\end{aligned}
$$

Then $\nu^i = (\nu^{i-1})^{(l'_i)} = (\mu^i)^{(l_{i+1})} = \mu^{i+1}$.

Next we show

$$S|_{D_\nu} = \tilde{S}. \qquad (29)$$

First, since $b = a^1$, we obtain

$$\sum_{i=1}^{j} b_i = \begin{cases} \sum_{i=1}^{j} a_i & \text{if } j < h, \\ \sum_{i=1}^{h} a_i - 1 & \text{if } j = h. \end{cases}$$

Suppose that $\sum_{j=1}^{t-1} b_j < n-1-i \leq \sum_{j=1}^{t} b_j$. Then

$$\sum_{j=1}^{t-1} a_j < n-(i+1) \leq \sum_{j=1}^{t} a_j,$$

so $\tilde{S}(l'_{i+1}, \nu^i_{l'_{i+1}}) = t = S(l_{i+2}, \mu^{i+1}_{l_{i+2}})$. Thus, we have proved (29).

Next we show $S \in \mathrm{STab}(\mu, a)$. If $l_1 = 1$ then this is clear, since $\tilde{S} \in \mathrm{STab}(\nu, b)$. Suppose $l_1 \geq 2$. Since $(l_1 - 1, \mu_{l_1}) \in D_\nu$, there exists an $i \in \{1, 2, \ldots, n-1\}$ such that $(l_1 - 1, \mu_{l_1}) = (l_{i+1}, \mu^i_{l_{i+1}})$. Since $l_1 \leq l_2 \leq \cdots \leq l_{a_h}$, we have

$$i + 1 > a_h = n - \sum_{j=1}^{h-1} a_j = n - \sum_{j=1}^{h-1} b_j,$$

and hence

$$n - 1 - (i-1) \leq \sum_{j=1}^{h-1} b_j.$$

This implies

$$\tilde{S}(l'_i, \nu^{i-1}_{l'_i}) \leq h - 1. \qquad (30)$$

Now

$$\begin{aligned} S(l_1-1,\mu_{l_1}) &= S(l_{i+1},\mu^i_{l_{i+1}}) \\ &= \tilde{S}(l'_i,\nu^{i-1}_{l'_i}) && \text{(by (27), (29))} \\ &< h && \text{(by (30))} \\ &= S(l_1,\mu_{l_1}). \end{aligned}$$

Since $\tilde{S} \in \mathrm{STab}(\nu,b)$, this implies $S \in \mathrm{STab}(\mu,a)$.

It remains to show that $S \leq T$ for all T in the set (24). Define partitions τ and σ by (5) and (6), respectively.

Suppose first that $\min\{i \mid T(i,\mu_i)=h\} > l_1$. Write $T^{-1}(h) = \{(t_1,t'_1),\ldots,(t_{a_h},t'_{a_h})\}$ with $l_1 < t_1 \leq t_2 \leq \cdots \leq t_{a_h}$. Then Lemma 25 implies $\tau = \mu^{(t_1,\ldots,t_{a_h})} \trianglerighteq \mu^{a_h} = \sigma$. Since $l_1 < t_1$, we have $\tau \triangleright \sigma$. Thus $S \leq T$.

Next suppose that $\min\{i \mid T(i,\mu_i)=h\} = l_1$. Set $\tilde{T} = T|_{D_\nu}$ and observe $\tilde{T} \in \mathrm{STab}(\nu,b)$. Set $m = \min\{i \mid \tilde{T}(i,\nu_i) = h(b)\}$. By Lemma 9, we have $m \in R(\nu,b)$, so $m \geq l(\nu,b)$. If $a_h = 1$, then $l_2 = l(\nu,b)$, so $m \geq l_2$. If $a_h \geq 2$, then $h(b) = h$, so $m \geq l_1$. Thus $m \geq l(\nu,b,l_1) = l_2$. Therefore, $\tilde{T}$ belong to the set (26). This implies $\tilde{S} \leq \tilde{T}$, and hence either $\tau \triangleright \sigma$, or $\tau = \sigma$ and $\tilde{S}|_{D_\sigma} \leq \tilde{T}|_{D_\sigma}$. Since $\tilde{S}|_{D_\sigma} = S|_{D_\sigma}$ and $\tilde{T}|_{D_\tau} = T|_{D_\tau}$, the recursive definition of the partial order implies $S \leq T$. □

Algorithm 3. Input: $\mu = (\mu_1,\mu_2,\ldots,\mu_k) \vdash n$ and $a = (a_1,a_2,\ldots,a_h) \vDash n$ such that $\mu \trianglerighteq \lambda(a)$.
Output: $S \in \mathrm{STab}(\mu,a)$.
Initialization: $\nu := \mu$, $b := a$, $m := n$ and $l' := 1$.
while $m > 1$ **do**
  $h := h(b)$ and $l := l(\nu,b,l')$.
  $S(l,\nu_l) := h$.
  if $b_h = 1$, **then** $l' \leftarrow 1$.
  else $l' \leftarrow l$.
  $\nu \leftarrow \nu^{(l)}$, $b \leftarrow \tilde{b}$ and $m \leftarrow m-1$.
end
$S(1,1) := 1$.
Output S.

Example 27. Let $\mu = (4,4,1,1) \vdash 10$ and $a = (1,3,2,2,2) \vDash 10$. Then a tableau

$S \in \mathrm{STab}(\mu, a)$ is obtained via Algorithm 3.

ν	b	m	l'	h	l	S
$(4,4,1,1)$	$(1,3,2,2,2)$	10	1	5	2	$S(2,4) = 5$
$(4,3,1,1)$	$(1,3,2,2,1)$	9	2	5	2	$S(2,3) = 5$
$(4,2,1,1)$	$(1,3,2,2)$	8	1	4	1	$S(1,4) = 4$
$(3,2,1,1)$	$(1,3,2,1)$	7	1	4	4	$S(4,1) = 4$
$(3,2,1)$	$(1,3,2)$	6	1	3	2	$S(2,2) = 3$
$(3,1,1)$	$(1,3,1)$	5	2	3	3	$S(3,1) = 3$
$(3,1)$	$(1,3)$	4	1	2	1	$S(1,3) = 2$
$(2,1)$	$(1,2)$	3	1	2	1	$S(1,2) = 2$
$(1,1)$	$(1,1)$	2	1	2	2	$S(2,1) = 2$
(1)	(1)	1	1			$S(1,1) = 1$

Thus

$$S = \begin{matrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 5 & 5 \\ 3 & & & \\ 4 & & & \end{matrix}$$

and S is the least element of $\mathrm{STab}(\mu, a)$.

Remark 28. Let a and b be compositions of n with $\lambda(a) = \lambda(b)$. Then there exists a bijection from $\mathrm{STab}(\mu, a)$ to $\mathrm{STab}(\mu, b)$ using [2, Lemma 3.7.1], but they are not isomorphic as partially ordered sets. For example, let $\mu = (4,4,1,1) \vdash 10$, $a = (1,3,2,2,2) \vDash 10$ and $b = (1,2,2,2,3) \vDash 10$. Then $(\mathrm{STab}(\mu, b), \leq)$ is a totally ordered set, while $(\mathrm{STab}(\mu, a), \leq)$ contains two incomparable tableaux:

$$T = \begin{matrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 4 & 5 \\ 3 & & & \\ 5 & & & \end{matrix},$$

$$S = \begin{matrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 3 & 5 \\ 4 & & & \\ 5 & & & \end{matrix}.$$

Indeed, define $\tau^{(p)}$ and $\sigma^{(p)}$ by (3) and (4), respectively. Then $\tau^{(3)} = (4,1,1)$ and $\sigma^{(3)} = (3,3)$ are incomparable. Thus $(\mathrm{STab}(\mu, a), \leq)$ is not totally ordered, hence it is not isomorphic to $(\mathrm{STab}(\mu, b), \leq)$.

Chapter 3

Generalization of Knuth's formula for the number of skew tableaux

3.1 Binomial coefficients

Throughout this section, $h,\ l,\ r$ and t be nonnegative integers. We denote by $S(n,k)$ the Stirling numbers of the second kind. First of all, we define

$$\mathcal{C}(r,t) = t!S(r+1,t+1).$$

Then

$$\begin{aligned}\mathcal{C}(r,t) &= t!S(r+1,t+1)\\ &= t!(S(r,t)+(t+1)S(r,t+1))\\ &= t\mathcal{C}(r-1,t-1)+(t+1)\mathcal{C}(r-1,t), \end{aligned} \tag{1}$$

since $S(r+1,t+1) = S(r,t)+(t+1)S(r,t+1)$.

Set

$$\varphi_l(h,r,t) = \binom{l}{h}\mathcal{C}(h,r)\mathcal{C}(l-h,t). \tag{2}$$

Clearly,

$$\begin{aligned}\varphi_l(h,r,t) &= \binom{l}{l-h}\mathcal{C}(l-h,t)\mathcal{C}(h,r)\\ &= \varphi_l(l-h,t,r). \end{aligned} \tag{3}$$

We define

$$R_l(t) = \sum_{i=1}^{t} i^l.$$

Lemma 29. *We have*

$$R_{l+1}(t) = (t+1)R_l(t) - \sum_{i=1}^{t} R_l(i).$$

Proof. We have

$$\begin{aligned}(t+1)R_l(t) &= (t+1)\sum_{i=1}^{t} i^l \\ &= \sum_{i=1}^{t} i^{l+1} + \sum_{i=1}^{t}\sum_{j=1}^{i} j^l \\ &= R_{l+1}(t) + \sum_{i=1}^{t} R_l(i).\end{aligned}$$

□

Lemma 30. *We have*

$$R_l(t) = \sum_{i=0}^{l} \mathcal{C}(l,i)\binom{t}{i+1}.$$

Proof. Setting $n = q = 0$ in [2, Proposition 5.1.2]. We have

$$\sum_{k=0}^{l}\binom{k}{m} = \binom{l+1}{m+1}. \tag{4}$$

We prove the statement by induction on l. If $l = 0$, then the statement holds since $\mathcal{C}(0,0) = 1$. Assume that the statement holds for $l-1$. Then

$$\begin{aligned}R_l(t) &= (t+1)R_{l-1}(t) - \sum_{j=1}^{t} R_{l-1}(j) && \text{(by Lemma 29)} \\ &= (t+1)\sum_{i=0}^{l-1}\mathcal{C}(l-1,i)\binom{t}{i+1} - \sum_{j=1}^{t}\sum_{i=0}^{l-1}\mathcal{C}(l-1,i)\binom{j}{i+1} \\ &= \sum_{i=0}^{l-1}(i+2)\mathcal{C}(l-1,i)\binom{t+1}{i+2} - \sum_{i=0}^{l-1}\mathcal{C}(l-1,i)\binom{t+1}{i+2} && \text{(by (4))} \\ &= \sum_{i=0}^{l-1}(i+1)\mathcal{C}(l-1,i)\binom{t+1}{i+2} \\ &= \sum_{i=0}^{l-1}(i+1)\mathcal{C}(l-1,i)\binom{t}{i+2} + \sum_{i=0}^{l-1}(i+1)\mathcal{C}(l-1,i)\binom{t}{i+1} \\ &= \sum_{i=1}^{l} i\mathcal{C}(l-1,i-1)\binom{t}{i+1} + \sum_{i=0}^{l-1}(i+1)\mathcal{C}(l-1,i)\binom{t}{i+1} \\ &= \sum_{i=0}^{l}\left(i\mathcal{C}(l-1,i-1) + (i+1)\mathcal{C}(l-1,i)\right)\binom{t}{i+1} \\ &= \sum_{i=0}^{l}\mathcal{C}(l,i)\binom{t}{i+1} && \text{(by (1))}.\end{aligned}$$

□

Lemma 31. *For $z \in \mathbb{C}$, we have*

$$z^l = \sum_{i=0}^{l} \mathcal{C}(l,i) \binom{z-1}{i}.$$

Proof. From [2, p.211, (4.65)], we have

$$z^l = \sum_{i=0}^{l} S(l,i)[z]_i,$$

so

$$\begin{aligned}
z^l &= \sum_{i=0}^{l} S(l,i)[z]_i \\
&= \sum_{i=0}^{l} S(l,i) z [z-1]_{i-1} \\
&= \sum_{i=0}^{l} S(l,i)[z-1]_{i-1}(z-i+i) \\
&= \sum_{i=0}^{l} S(l,i)[z-1]_i + \sum_{i=1}^{l} i S(l,i)[z-1]_{i-1} \\
&= \sum_{i=0}^{l} S(l,i)[z-1]_i + \sum_{i=0}^{l-1} (i+1) S(l,i+1)[z-1]_i \\
&= \sum_{i=0}^{l} \left(S(l,i) + (i+1)S(l,i+1)\right)[z-1]_i \\
&= \sum_{i=0}^{l} S(l+1,i+1)[z-1]_i \\
&= \sum_{i=0}^{l} i! S(l+1,i+1) \binom{z-1}{i} \\
&= \sum_{i=0}^{l} \mathcal{C}(l,i) \binom{z-1}{i}.
\end{aligned}$$

□

3.2 $p_l[C(\mu)]$ and $q^{\pm}_{r,t}$

Let l be a nonnegative integer. Let $C(\mu) = \{j - i \mid (i,j) \in D_\mu\}$ be the multiset of contents of the partition μ, and

$$p_l[C(\mu)] = \sum_{(i,j) \in D_\mu} (j-i)^l$$

be the lth power sum symmetric function evaluated at the contents of μ.

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$, and let $r,\ t$ be nonnegative integers. We define

$$q^{\pm}_{r,t} = \sum_{i=1}^{k} \left(\binom{\mu_i}{r+1}\binom{i-1}{t} \pm \binom{\mu_i}{t+1}\binom{i-1}{r} \right). \tag{5}$$

Observe that if $r = t$ then

$$q^{-}_{r,r} = 0, \tag{6}$$

and

$$q^{+}_{r,t} = q^{+}_{t,r}, \tag{7}$$
$$q^{-}_{r,t} = -q^{-}_{t,r}. \tag{8}$$

Proposition 32. *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$ and l be a nonnegative integer. Then*

$$p_{2l+1}[C(\mu)] = \sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h} (-1)^h \varphi_{2l+1}(h,r,t) q^{-}_{t,r},$$
$$p_{2l}[C(\mu)] = \sum_{h=0}^{l-1}\sum_{r=0}^{h}\sum_{t=0}^{2l-h} (-1)^h \varphi_{2l}(h,r,t) q^{+}_{r,t} + \frac{1}{2}(-1)^l \sum_{r=0}^{l}\sum_{t=0}^{l} \varphi_{2l}(l,r,t) q^{+}_{r,t}.$$

Proof. By the definition of $p_l[C(\mu)]$, we get the following:

$$\begin{aligned}
p_l[C(\mu)] &= \sum_{i=1}^{k}\sum_{j=1}^{\mu_i} (j-i)^l \\
&= \sum_{i=1}^{k}\sum_{j=1}^{\mu_i}\sum_{h=0}^{l} (-1)^{l-h}\binom{l}{h} j^h i^{l-h} \\
&= \sum_{i=1}^{k}\sum_{h=0}^{l} (-1)^{l-h}\binom{l}{h} i^{l-h} R_h(\mu_i) \\
&= \sum_{i=1}^{k}\sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{l-h} (-1)^{l-h}\binom{l}{h} \mathcal{C}(h,r)\mathcal{C}(l-h,t)\binom{\mu_i}{r+1}\binom{i-1}{t} \\
&= \sum_{i=1}^{k}\sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{l-h} (-1)^{l-h}\varphi_l(h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t} \qquad \text{(by (2))},
\end{aligned}$$

where the fourth equality follows from Lemma 30 and Lemma 31. Thus

$$\begin{aligned}
p_{2l+1}[C(\mu)] &= \sum_{i=1}^{k}\sum_{h=0}^{2l+1}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h} (-1)^{2l+1-h}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t} \\
&= \sum_{i=1}^{k}\sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h} (-1)^{2l+1-h}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k}\sum_{h=l+1}^{2l+1}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{2l+1-h}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t}\\
&=\sum_{i=1}^{k}\sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{h-1}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t}\\
&\quad+\sum_{i=1}^{k}\sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{h}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{t+1}\binom{i-1}{r}\\
&=\sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{h}\varphi_{2l+1}(h,r,t)\\
&\qquad\cdot\left\{\sum_{i=1}^{k}\binom{\mu_i}{t+1}\binom{i-1}{r}-\sum_{i=1}^{k}\binom{\mu_i}{r+1}\binom{i-1}{t}\right\}\\
&=\sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{h}\varphi_{2l+1}(h,r,t)q^{-}_{t,r},
\end{aligned}
$$

where the third equality can be shown as follows:

$$
\begin{aligned}
&\sum_{h=l+1}^{2l+1}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{2l+1-h}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t}\\
&\quad=\sum_{h=0}^{l}\sum_{r=0}^{2l+1-h}\sum_{t=0}^{h}(-1)^{h}\varphi_{2l+1}(2l+1-h,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t}\\
&\quad=\sum_{h=0}^{l}\sum_{r=0}^{2l+1-h}\sum_{t=0}^{h}(-1)^{h}\varphi_{2l+1}(h,t,r)\binom{\mu_i}{r+1}\binom{i-1}{t} \qquad \text{(by (3))}\\
&\quad=\sum_{h=0}^{l}\sum_{r=0}^{h}\sum_{t=0}^{2l+1-h}(-1)^{h}\varphi_{2l+1}(h,r,t)\binom{\mu_i}{t+1}\binom{i-1}{r}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
p_{2l}[C(\mu)]&=\sum_{h=0}^{l-1}\sum_{r=0}^{h}\sum_{t=0}^{2l-h}(-1)^{h}\varphi_{2l}(h,r,t)q^{+}_{r,t}\\
&\quad+\sum_{i=1}^{k}\sum_{r=0}^{l}\sum_{t=0}^{l}(-1)^{l}\varphi_{2l}(l,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t}\\
&=\sum_{h=0}^{l-1}\sum_{r=0}^{h}\sum_{t=0}^{2l-h}(-1)^{h}\varphi_{2l}(h,r,t)q^{+}_{r,t}+\frac{1}{2}(-1)^{l}\sum_{r=0}^{l}\sum_{t=0}^{l}\varphi_{2l}(l,r,t)q^{+}_{r,t},
\end{aligned}
$$

where the second equality can be shown as follows:

$$
\sum_{i=1}^{k}\sum_{r=0}^{l}\sum_{t=0}^{l}(-1)^{l}\varphi_{2l}(l,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t}
$$

$$
\begin{aligned}
&= \frac{1}{2}(-1)^l \sum_{i=1}^{k}\sum_{r=0}^{l}\sum_{t=0}^{l} \varphi_{2l}(l,r,t)\binom{\mu_i}{r+1}\binom{i-1}{t} \\
&\quad + \frac{1}{2}(-1)^l \sum_{i=1}^{k}\sum_{r=0}^{l}\sum_{t=0}^{l} \varphi_{2l}(l,r,t)\binom{\mu_i}{t+1}\binom{i-1}{r} \\
&= \frac{1}{2}(-1)^l \sum_{r=0}^{l}\sum_{t=0}^{l} \varphi_{2l}(l,r,t) q^{+}_{r,t}.
\end{aligned}
$$

□

By Proposition 32, we have

$$
\begin{aligned}
p_0[C(\mu)] &= \frac{1}{2} q^{+}_{0,0} = n, \\
p_1[C(\mu)] &= q^{-}_{0,0} + q^{-}_{1,0} \\
&= q^{-}_{1,0}, && \text{(by (6))} \\
p_2[C(\mu)] &= 2q^{+}_{0,1} + 2q^{+}_{0,2} - q^{+}_{1,0} - q^{+}_{1,1} \\
&= q^{+}_{0,1} + 2q^{+}_{0,2} - q^{+}_{1,1}, && \text{(by (7))} \\
p_3[C(\mu)] &= -2q^{-}_{1,0} + 6q^{-}_{2,0} + 6q^{-}_{3,0} - 3q^{-}_{0,1} - 9q^{-}_{1,1} - 6q^{-}_{2,1} \\
&= q^{-}_{1,0} + 6q^{-}_{2,0} + 6q^{-}_{3,0} - 6q^{-}_{2,1} && \text{(by (6) and (8)).} \qquad (9)
\end{aligned}
$$

3.3 Main results

Let $\mu,\ \lambda \vdash n$. We denote by $\chi^{\mu}(\lambda)$ the value of the character of the Specht module S^{μ} evaluated at a permutation π belonging to the conjugacy class of type λ. From [2, Example 5.3.3], we have

$$
\begin{aligned}
\chi^{\mu}(2,1^{n-2}) &= \frac{f^{\mu}}{[n]_2} 2p_1[C(\mu)], \\
\chi^{\mu}(3,1^{n-3}) &= \frac{f^{\mu}}{[n]_3} 3\left(p_2[C(\mu)] - \binom{n}{2}\right), \\
\chi^{\mu}(4,1^{n-4}) &= \frac{f^{\mu}}{[n]_4} 4\left(p_3[C(\mu)] - (2n-3)p_1[C(\mu)]\right), \\
\chi^{\mu}(5,1^{n-5}) &= \frac{f^{\mu}}{[n]_5} 5\left(p_4[C(\mu)] - (3n-10)p_2[C(\mu)] - 2p_1[C(\mu)]^2 + 5\binom{n}{3} - 3\binom{n}{2}\right), \\
\chi^{\mu}(6,1^{n-6}) &= \frac{f^{\mu}}{[n]_6} 6\left(p_5[C(\mu)] + (25-4n)p_3[C(\mu)] + 2(3n-4)(n-5)p_1[C(\mu)]\right) \\
&\quad - \frac{f^{\mu}}{[n]_6} 36 p_1[C(\mu)]p_2[C(\mu)]. \qquad (10)
\end{aligned}
$$

Remark 33. In [2, Example 5.3.3], the coefficient of $d_3(\lambda)$ (in this paper, we denote by $p_3[C(\mu)]$) in the character value $\hat{\chi}^{\lambda}_{6,1^{n-6}}$ is $24(7-n)$. Since c^{λ}_6 and c^{λ}_7 are incorrect in [2, p.251], the value of the character $\hat{\chi}^{\lambda}_{6,1^{n-6}}$ is also incorrect. In fact, the coefficient of $d_3(\lambda)$ in the character value $\hat{\chi}^{\lambda}_{6,1^{n-6}}$ is $6(25-4n)$, as given in (10).

We obtain [2, Example 5.3.8]:

$$\chi^{\mu}(2,2,1^{n-4}) = \frac{f^{\mu}}{[n]_4} 4\left(p_1[C(\mu)]^2 - 3p_2[C(\mu)] + 2\binom{n}{2}\right). \tag{11}$$

In general, for $\mu \vdash n$ and $\lambda \vdash m \leq n$, the character $\chi^{\mu}(\lambda, 1^{n-m})$ can be expressed as a polynomial of $c_r^{\mu}(t)$ using Lassalle's explicit formula [2, Theorem 5.3.11].

For any $i \geq 1$, $m_i(\mu) = |\{j \mid \mu_j = i\}|$ is the multiplicity of i in μ. Set

$$z_{\mu} = \prod_{i \geq 1} i^{m_i(\mu)} m_i(\mu)!.$$

Let $\mu \vdash n$ and $\lambda \vdash m \leq n$. From [13, Theorem 3.1], we have

$$f^{\mu/\lambda} = \sum_{\nu \vdash m} z_{\nu}^{-1} \chi^{\mu}(\nu, 1^{n-m}) \chi^{\lambda}(\nu).$$

If $\lambda = (m)$, then

$$\begin{aligned} f^{\mu/(m)} &= \sum_{\nu \vdash m} z_{\nu}^{-1} \chi^{\mu}(\nu, 1^{n-m}) \chi^{(m)}(\nu) \\ &= \sum_{\nu \vdash m} z_{\nu}^{-1} \chi^{\mu}(\nu, 1^{n-m}). \end{aligned} \tag{12}$$

We already proved that $p_l[C(\mu)]$ can be expressed as a linear combination of $q^{\pm}_{r,t}$ (Proposition 32), so the character value $\chi^{\mu}(\lambda, 1^{n-m})$ can be written as a polynomial in $q^{\pm}_{r,t}$ using Lassalle's explicit formula [2, Theorem 5.3.11]. We compute $\chi^{\mu}(m, 1^{n-m})$ for $2 \leq m \leq 4$ and $\chi^{\mu}(2,2,1^{n-4})$ using (9), (10) and (11).

$$\begin{aligned} \chi^{\mu}(2,1^{n-2}) &= \frac{f^{\mu}}{[n]_2} 2p_1[C(\mu)] \\ &= \frac{f^{\mu}}{[n]_2} 2q^{-}_{1,0}, \\ \chi^{\mu}(3,1^{n-3}) &= \frac{f^{\mu}}{[n]_3} 3\left(p_2[C(\mu)] - \binom{n}{2}\right) \\ &= \frac{f^{\mu}}{[n]_3} 3\left(q^{+}_{0,1} + 2q^{+}_{0,2} - q^{+}_{1,1} - \binom{n}{2}\right), \\ \chi^{\mu}(4,1^{n-4}) &= \frac{f^{\mu}}{[n]_4} 4\left(p_3[C(\mu)] - (2n-3)p_1[C(\mu)]\right) \\ &= \frac{f^{\mu}}{[n]_4} 4\left((4-2n)q^{-}_{1,0} + 6q^{-}_{2,0} + 6q^{-}_{3,0} - 6q^{-}_{2,1}\right), \\ \chi^{\mu}(2,2,1^{n-4}) &= \frac{f^{\mu}}{[n]_4} 4\left(p_1[C(\mu)]^2 - 3p_2[C(\mu)] + 2\binom{n}{2}\right) \end{aligned}$$

$$= \frac{f^\mu}{[n]_4} 4 \left((q^-_{1,0})^2 - 3q^+_{0,1} - 6q^+_{0,2} + 3q^+_{1,1} + 2\binom{n}{2} \right). \tag{13}$$

Substituting (13) into (12), we find

$$\begin{aligned}
f^{\mu/(2)} &= \frac{1}{z_{(2)}}\chi^\mu(2, 1^{n-2}) + \frac{1}{z_{(1,1)}}\chi^\mu(1^n) \\
&= \frac{1}{2}\frac{f^\mu}{[n]_2} \cdot 2q^-_{1,0} + \frac{1}{2} f^\mu \\
&= \frac{f^\mu}{[n]_2} \left(q^-_{1,0} + \binom{n}{2} \right), \qquad (14) \\
f^{\mu/(3)} &= \frac{1}{z_{(3)}}\chi^\mu(3, 1^{n-3}) + \frac{1}{z_{(2,1)}}\chi^\mu(2, 1^{n-2}) + \frac{1}{z_{(1,1,1)}}\chi^\mu(1^n) \\
&= \frac{1}{3}\frac{f^\mu}{[n]_3} \cdot 3\left(q^+_{0,1} + 2q^+_{0,2} - q^+_{1,1} - \binom{n}{2} \right) + \frac{1}{2}\frac{f^\mu}{[n]_2} \cdot 2q^-_{1,0} + \frac{1}{6} f^\mu \\
&= \frac{f^\mu}{[n]_3} \left(q^+_{0,1} + 2q^+_{0,2} - q^+_{1,1} + (n-2)q^-_{1,0} + \binom{n}{3} - \binom{n}{2} \right), \qquad (15)
\end{aligned}$$

and

$$\begin{aligned}
f^{\mu/(4)} &= \frac{1}{z_{(4)}}\chi^\mu(4, 1^{n-4}) + \frac{1}{z_{(3,1)}}\chi^\mu(3, 1^{n-3}) + \frac{1}{z_{(2,2)}}\chi^\mu(2, 2, 1^{n-4}) \\
&\quad + \frac{1}{z_{(2,1,1)}}\chi^\mu(2, 1^{n-2}) + \frac{1}{z_{(1,1,1,1)}}\chi^\mu(1^n) \\
&= \frac{1}{4}\frac{f^\mu}{[n]_4} \cdot 4\left((4-2n)q^-_{1,0} + 6q^-_{2,0} + 6q^-_{3,0} - 6q^-_{2,1} \right) \\
&\quad + \frac{1}{3}\frac{f^\mu}{[n]_3} \cdot 3\left(q^+_{0,1} + 2q^+_{0,2} - q^+_{1,1} - \binom{n}{2} \right) \\
&\quad + \frac{1}{8}\frac{f^\mu}{[n]_4} \cdot 4\left((q^-_{1,0})^2 - 3q^+_{0,1} - 6q^+_{0,2} + 3q^+_{1,1} + 2\binom{n}{2} \right) \\
&\quad + \frac{1}{4}\frac{f^\mu}{[n]_2} \cdot 2q^-_{1,0} + \frac{1}{24} f^\mu \\
&= \frac{f^\mu}{[n]_4} \left(\frac{1}{2}(n-2)(n-7)q^-_{1,0} + 6q^-_{2,0} + 6q^-_{3,0} - 6q^-_{2,1} + \frac{1}{2}(q^-_{1,0})^2 \right) \\
&\quad + \frac{f^\mu}{[n]_4} \left((n - \frac{9}{2})q^+_{0,1} + (2n-9)q^+_{0,2} - (n - \frac{9}{2})q^+_{1,1} \right) \\
&\quad + \frac{f^\mu}{[n]_4} \left(\binom{n}{4} - 3\binom{n}{3} + 2\binom{n}{2} \right).
\end{aligned}$$

We get (2) and (4) by substituting (5) into (14) and (15), respectively.

3.4 A generalization of a polynomial identity for a partition and its conjugate

Proposition 34. *Let μ be a partition of an integer. Then μ' is the conjugate of μ if and only if*

$$\sum_{i=1}^{k}\binom{\mu_i}{t+1}\binom{i-1}{r}=\sum_{j\geq 1}\binom{\mu'_j}{r+1}\binom{j-1}{t}.$$

for all nonnegative integers r and t.

Proof. First, we show the "only if" part. Then

$$\begin{aligned}\sum_{j\geq 1}\binom{\mu'_j}{r+1}\binom{j-1}{t} &= \sum_{j\geq t+1}\sum_{\substack{J\subseteq\{1,2,\ldots,\mu_1\},\\ |J|=t+1,\\ \max J=j}}|\{I \mid I\times J\subseteq D_\mu,\ |I|=r+1\}| \\ &= \sum_{i=r+1}^{k}\sum_{\substack{I\subseteq\{1,2,\ldots,k\},\\ |I|=r+1,\\ \max I=i}}|\{J \mid I\times J\subseteq D_\mu,\ |J|=t+1\}| \\ &= \sum_{i=r+1}^{k}\sum_{\substack{I\subseteq\{1,2,\ldots,k\},\\ |I|=r+1,\\ \max I=i}}|\{J \mid \max J\leq \mu_i,\ |J|=t+1\}| \\ &= \sum_{i=r+1}^{k}\sum_{\substack{I\subseteq\{1,2,\ldots,k\},\\ |I|=r+1,\\ \max I=i}}|\{J \mid J\subseteq\{1,2,\ldots,\mu_i\},\ |J|=t+1\}| \\ &= \sum_{i=r+1}^{k}\sum_{\substack{I\subseteq\{1,2,\ldots,k\},\\ |I|=r+1,\\ \max I=i}}\binom{\mu_i}{t+1} \\ &= \sum_{i=r+1}^{k}\binom{\mu_i}{t+1}\binom{i-1}{r}.\end{aligned}$$

Next, let λ be the conjugate of μ. Set $h(\lambda)=h$. Then

$$\begin{aligned}\sum_{j=1}^{h}\binom{\lambda_j}{r+1}\binom{j-1}{t} &= \sum_{i=1}^{k}\binom{\mu_i}{t+1}\binom{i-1}{r} \\ &= \sum_{j\geq 1}\binom{\mu'_j}{r+1}\binom{j-1}{t}.\end{aligned} \tag{16}$$

Setting $h(\mu')=l$ and $r=0$ in (16), we have

$$\sum_{j=1}^{h}\lambda_j\binom{j-1}{t}=\sum_{i=1}^{l}\mu'_i\binom{i-1}{t}. \tag{17}$$

Suppose $h > l$ and set $t = h-1$ in (17), then $\lambda_h = 0$. Similarly, suppose $h < l$ and set $t = l-1$ in (17). Then $\mu'_l = 0$, and both cases are contradictions. Thus $h = l$.

We show that $\lambda_{h-i} = \mu'_{h-i}$ for all i with $0 \le i \le h-1$ by induction on i. If $i = 0$, setting $t = h-1$ in (17), then $\lambda_h = \mu'_h$.

Assume that the assertion holds for some $i \in \{0, 1, \ldots, h-2\}$. Let $t = h-(i+2)$ in (17). By the inductive hypothesis, we have

$$\sum_{j=h-i}^{h} \lambda_j \binom{j-1}{h-i-2} = \sum_{j=h-i}^{h} \mu'_j \binom{j-1}{h-i-2}.$$

Therefore, $\lambda_{h-i-1} = \mu'_{h-i-1}$ since $\binom{j-1}{h-i-2} = 0$ for all j with $1 \le j \le h-j-2$. Thus $\lambda = \mu'$ and μ' is the conjugate of μ. □

From Proposition 34, we have

$$q^{\pm}_{r,t} = \sum_{i=1}^{k} \binom{\mu_i}{r+1}\binom{i-1}{t} \pm \sum_{j\ge 1} \binom{\mu'_j}{r+1}\binom{j-1}{t}. \tag{18}$$

By substituting (18) into (14) and (15), we get (1) and

$$\begin{aligned}
f^{\mu/(3)} &= \frac{f^\mu}{[n]_3}\left(q^+_{0,1} + 2q^+_{0,2} - q^+_{1,1} + (n-2)q^-_{1,0} + \binom{n}{3} - \binom{n}{2}\right) \\
&= \frac{f^\mu}{[n]_3}\left(q^+_{1,0} + 2q^+_{2,0} - q^+_{1,1} + (n-2)q^-_{1,0} + \binom{n}{3} - \binom{n}{2}\right) \qquad \text{(by (7))} \\
&= \frac{f^\mu}{[n]_3}\left(\left(\sum_{i=1}^{k}\binom{\mu_i}{2} + \sum_{j\ge1}\binom{\mu'_j}{2}\right) + 2\left(\sum_{i=1}^{k}\binom{\mu_i}{3} + \sum_{j\ge1}\binom{\mu'_j}{3}\right)\right) \\
&\quad - \frac{f^\mu}{[n]_3}\left(\sum_{i=1}^{k}\binom{\mu_i}{2}(i-1) + \sum_{j\ge1}\binom{\mu'_j}{2}(j-1)\right) \\
&\quad + \frac{f^\mu}{[n]_3}\left((n-2)\left(\sum_{i=1}^{k}\binom{\mu_i}{2} - \sum_{j\ge1}\binom{\mu'_j}{2}\right) + \binom{n}{3} - \binom{n}{2}\right),
\end{aligned}$$

respectively.

Chapter 4

A bijective proof of Vershik's relations for the Kostka numbers

4.1 Insertion

Throughout this chapter, $\lambda \vDash n$. For a positive integer i, we define $\lambda^i \vDash n+1$ as follows:

$$\lambda^i_j = \begin{cases} \lambda_j + 1 & \text{if } j = i, \\ \lambda_j & \text{otherwise.} \end{cases}$$

In this section, we let $\mu \vdash n$ and $T \in \mathrm{STab}(\mu, \lambda)$. We also let x be a positive integer.

Definition 35. The *bumping route* of (T, x) is defined as the sequence $\overrightarrow{R}(T, x) = (j_1, j_2, \ldots)$ with integer entries defined as follows: first, $\overrightarrow{R}(T, x) = 0$ if $\{q \mid T(1, q) > x\} = \emptyset$. Otherwise, $j_1 = \min\{q \mid T(1, q) > x\}$ and for $p \geq 2$,

$$j_p = \begin{cases} 0 & \text{if } j_{p-1} = 0 \text{ or } T(p, \mu_p) \leq T(p-1, j_{p-1}), \\ \min\{q \mid T(p, q) > T(p-1, j_{p-1})\} & \text{otherwise.} \end{cases}$$

If $j_p = 0$ for $p > l$ then we write $\overrightarrow{R}(T, x) = (j_1, j_2, \ldots, j_l)$. We denote by $l(\overrightarrow{R}(T, x))$ the length of $\overrightarrow{R}(T, x)$, that is, $l(\overrightarrow{R}(T, x)) = \max\{l \mid j_l \neq 0\}$. Note that if $\overrightarrow{R}(T, x) = 0$, then we define $l(\overrightarrow{R}(T, x)) = 0$.

By previous definition, clearly, we have

$$T(1, j_1 - 1) \leq x < T(1, j_1), \tag{1}$$

$$T(p, j_p - 1) \leq T(p-1, j_{p-1}) < T(p, j_p) \ (2 \leq p \leq l), \tag{2}$$

$$T(l+1, \mu_{l+1}) \leq T(l, j_l), \tag{3}$$

whenever $T(\cdot, \cdot)$ is defined. Moreover, we get $j_p \geq j_{p+1}$ for all positive integers p. Indeed, we may assume $p < l$. Then we have $T(p+1, j_p) > T(p, j_p)$. Thus

$$j_p \geq \min\{q \mid T(p+1, q) > T(p, j_p)\}$$

$$= j_{p+1}.$$

For the remainder of this section, we let $\overrightarrow{R}(T,x) = (j_1, j_2, \ldots, j_l)$, where $l = l(\overrightarrow{R}(T,x))$.

Lemma 36. *We have $\mu_{l+1} < j_l$. In particular, $\mu^{l+1} \vdash n+1$.*

Proof. If $l = 0$, then it is obvious. Suppose $l \geq 1$. Then by (3) we have $(l+1, j_l) \notin D_\mu$. Since $(l, j_l) \in D_\mu$, we have $\mu_{l+1} < j_l$. In particular, $\mu_{l+1} < \mu_l$. □

Definition 37. We define a *insertion* or *bumping* tableau T_x of shape μ^{l+1} and weight λ^x as follows: if $l = 0$, then define T_x by

$$T_x(p,q) = \begin{cases} T(p,q) & \text{if } (p,q) \in D_\mu, \\ x & \text{if } (p,q) = (1, \mu_1 + 1). \end{cases} \tag{4}$$

Otherwise, define T_x by

$$T_x(p,q) = \begin{cases} x & \text{if } (p,q) = (1, j_1), \\ T(p-1, j_{p-1}) & \text{if } q = j_p,\ 2 \leq p \leq l, \\ T(l, j_l) & \text{if } (p,q) = (l+1, \mu_{l+1}+1), \\ T(p,q) & \text{otherwise.} \end{cases} \tag{5}$$

Lemma 38. *We have $T_x(p, j_p) < T(p, j_p)$ for all p with $1 \leq p \leq l$.*

Proof. We have

$$\begin{aligned} T_x(p, j_p) &= \begin{cases} x & \text{if } p = 1, \\ T(p-1, j_{p-1}) & \text{if } 2 \leq p \leq l \end{cases} && \text{(by (5))} \\ &< T(p, j_p) && \text{(by (1) and (2)).} \end{aligned}$$

□

Lemma 39. *We have $T_x \in \mathrm{STab}(\mu^{l+1}, \lambda^x)$.*

Proof. By Lemma 36, we have $\mu^{l+1} \vdash n+1$. To show that T_x is a semistandard tableau, it suffices to prove

$$\begin{aligned} &T_x(p, j_p - 1) \leq T_x(p, j_p) \leq T_x(p, j_p + 1), \\ &T_x(l+1, \mu_{l+1}) \leq T_x(l+1, \mu_{l+1}+1), \\ &T_x(p-1, j_p) < T_x(p, j_p) < T_x(p+1, j_p), \\ &T_x(l, \mu_{l+1}+1) < T_x(l+1, \mu_{l+1}+1), \end{aligned} \tag{6}$$

whenever $T_x(\cdot,\cdot)$ is defined.

For all p with $1 \leq p \leq l$, we have

$$T_x(p, j_p - 1) = T(p, j_p - 1)$$

$$
\begin{aligned}
&\leq \begin{cases} x & \text{if } p = 1, \\ T(p-1, j_{p-1}) & \text{if } 2 \leq p \leq l \end{cases} && \text{(by (1) and (2))} \\
&= T_x(p, j_p) \\
&< T(p, j_p) && \text{(by Lemma 38)} \\
&\leq T(p, j_p + 1) \\
&= T_x(p, j_p + 1).
\end{aligned}
$$

Also,

$$
\begin{aligned}
T_x(l+1, \mu_{l+1}) &= T(l+1, \mu_{l+1}) \\
&\leq \begin{cases} x & \text{if } \overrightarrow{R}(T, x) = 0, \\ T(l, j_l) & \text{otherwise} \end{cases} && \text{(by (3))} \\
&= T_x(l+1, \mu_{l+1} + 1) && \text{(by (4) and (5)).}
\end{aligned}
$$

For all p with $2 \leq p \leq l$, we have

$$
\begin{aligned}
T_x(p-1, j_p) &= \begin{cases} x & \text{if } p = 2,\ j_1 = j_2, \\ T(1, j_2) & \text{if } p = 2,\ j_1 > j_2, \\ T(p-2, j_{p-2}) & \text{if } 3 \leq p \leq l,\ j_{p-1} = j_p, \\ T(p-1, j_p) & \text{if } 3 \leq p \leq l,\ j_{p-1} > j_p \end{cases} \\
&< \begin{cases} T(1, j_1) & \text{if } p = 2, \\ T(p-1, j_{p-1}) & \text{if } 3 \leq p \leq l \end{cases} && \text{(by (1) and (2))} \\
&= T_x(p, j_p) && \text{(by (5)).}
\end{aligned}
$$

For all p with $1 \leq p \leq l$, we have

$$
\begin{aligned}
T_x(p, j_p) &< T(p, j_p) && \text{(by Lemma 38)} \\
&\leq \begin{cases} T(p, j_p) & \text{if } j_p = j_{p+1}, \\ T(p+1, j_p) & \text{if } j_p > j_{p+1} \end{cases} \\
&= T_x(p+1, j_p) && \text{(by (5)).}
\end{aligned}
$$

Finally, we have $\mu_{l+1} < j_l$ by Lemma 36, so

$$
\begin{aligned}
T_x(l, \mu_{l+1} + 1) &= \begin{cases} T(l-1, j_{l-1}) & \text{if } \mu_{l+1} + 1 = j_l, \\ T(l, \mu_{l+1} + 1) & \text{if } \mu_{l+1} + 1 < j_l \end{cases} \\
&< T(l, j_l) && \text{(by (2))} \\
&= T_x(l+1, \mu_{l+1} + 1) && \text{(by (5)).}
\end{aligned}
$$

Thus we proved (6). □

4.2 Reverse insertion

In this section, let $\rho \vdash n-1$ and let l be a nonnegative integer such that $\rho^{l+1} \vdash n$. We also let $S \in \mathrm{STab}(\rho^{l+1}, \lambda)$.

Definition 40. The *reverse bumping route* of (S, l) is defined as the sequence $\overleftarrow{R}(S,l) = (j'_1, j'_2, \ldots, j'_l)$ with integer entries defined as follows: first, $\overleftarrow{R}(S,l) = 0$ if $l = 0$. Otherwise, $j'_l = \max\{q \mid S(l,q) < S(l+1, \rho_{l+1}+1)\}$ and

$$j'_p = \max\{q \mid S(p,q) < S(p+1, j'_{p+1})\}$$

for all p with $1 \leq p < l$.

By previous definition, clearly, we have

$$S(p, j'_p) < S(p+1, j'_{p+1}) \leq S(p, j'_p + 1) \; (1 \leq p < l), \tag{7}$$
$$S(l, j'_l) < S(l+1, \rho_{l+1}+1) \leq S(l, j'_l + 1). \tag{8}$$

Moreover, we get $j'_p \geq j'_{p+1}$ for all p with $1 \leq p < l$. Indeed, since $S(p, j'_{p+1}) < S(p+1, j'_{p+1})$, we have

$$\begin{aligned} j'_p &= \max\{q \mid S(p,q) < S(p+1, j'_{p+1})\} \\ &\geq j'_{p+1}. \end{aligned}$$

For the remainder of this section, we let $\overleftarrow{R}(S,l) = (j'_1, j'_2, \ldots, j'_l)$.

Definition 41. We define

$$x(S,l) = \begin{cases} S(1, \rho_1 + 1) & \text{if } l = 0, \\ S(1, j'_1) & \text{otherwise.} \end{cases}$$

We define a *reverse insertion* or *reverse bumping* tableau S^l of shape ρ and weight $\lambda^{(x(S,l))}$ as follows: if $l = 0$, then define $S^l = S|_{D_\rho}$. Otherwise, define S^l by

$$S^l(p,q) = \begin{cases} S(p+1, j'_{p+1}) & \text{if } q = j'_p,\ 1 \leq p < l, \\ S(l+1, \rho_{l+1}+1) & \text{if } (p,q) = (l, j'_l), \\ S(p,q) & \text{otherwise.} \end{cases} \tag{9}$$

Lemma 42. *We have $S(p, j'_p) < S^l(p, j'_p)$ for all p with $1 \leq p \leq l$.*

Proof. We have

$$\begin{aligned} S(p, j'_p) &< \begin{cases} S(p+1, j'_{p+1}) & \text{if } 1 \leq p < l, \\ S(l+1, \rho_{l+1}+1) & \text{if } p = l \end{cases} && \text{(by (7) and (8))} \\ &= S^l(p, j'_p) && \text{(by (9))}. \end{aligned}$$

□

Lemma 43. *Let* $x = x(S, l)$. *Then*

(i) $S^l \in \mathrm{STab}(\rho, \lambda^{(x)})$,

(ii) $\overrightarrow{R}(S^l, x) = \overleftarrow{R}(S, l)$,

(iii) $(S^l)_x = S$.

Proof. (i) If $l = 0$ then $S^l = S|_{D_\rho}$ and $x = S(1, \rho_1 + 1)$, so $S^l \in \mathrm{STab}(\rho, \lambda^{(x)})$. Suppose $l \geq 1$. By Definition 41, S^l is a tableau of shape ρ and weight $\lambda^{(x)}$. It suffices to prove that

$$\begin{aligned} &S^l(p, j'_p - 1) \leq S^l(p, j'_p) \leq S^l(p, j'_p + 1), \\ &S^l(p-1, j'_p) < S^l(p, j'_p) < S^l(p+1, j'_p), \end{aligned} \tag{10}$$

whenever $S^l(\cdot, \cdot)$ is defined.

For all p with $1 \leq p \leq l$, we have

$$\begin{aligned} S^l(p, j'_p - 1) &= S(p, j'_p - 1) \\ &\leq S(p, j'_p) \\ &< S^l(p, j'_p) && \text{(by Lemma 42)} \\ &= \begin{cases} S(p+1, j'_{p+1}) & \text{if } 1 \leq p < l, \\ S(l+1, \rho_{l+1} + 1) & \text{if } p = l \end{cases} && \text{(by (9))} \\ &\leq \begin{cases} S(p, j'_p + 1) & \text{if } 1 \leq p < l, \\ S(l, j'_l + 1) & \text{if } p = l \end{cases} && \text{(by (7) and (8))} \\ &= S^l(p, j'_p + 1) && \text{(by (9)).} \end{aligned}$$

For $2 \leq p \leq l$, we have

$$\begin{aligned} S^l(p-1, j'_p) &= \begin{cases} S(p-1, j'_p) & \text{if } j'_{p-1} > j'_p, \\ S(p, j'_p) & \text{if } j'_{p-1} = j'_p \end{cases} && \text{(by (9))} \\ &\leq S(p, j'_p) \\ &< S^l(p, j'_p) && \text{(by Lemma 42).} \end{aligned}$$

If $l = 1$, then $\rho_2 + 1 \leq j'_1$ since $S(1, \rho_2 + 1) < S(2, \rho_2 + 1)$, so $(2, \rho_2 + 1) \notin D_\rho$. Suppose $l \geq 2$. For $1 \leq p < l$, we have

$$\begin{aligned} S^l(p, j'_p) &= S(p+1, j'_{p+1}) \\ &< \begin{cases} S(p+1, j'_p) & \text{if } j'_p > j'_{p+1},\ 1 \leq p \leq l-2, \\ S(p+2, j'_{p+2}) & \text{if } j'_p = j'_{p+1},\ 1 \leq p \leq l-2, \\ S(l, j'_{l-1}) & \text{if } j'_p > j'_{p+1},\ p = l-1, \\ S(l+1, \rho_{l+1} + 1) & \text{if } j'_p = j'_{p+1},\ p = l-1 \end{cases} && \text{(by (7) and (8))} \\ &= S^l(p+1, j'_p). \end{aligned}$$

Thus we proved (10).

(ii) Let $\overrightarrow{R}(S^l, x) = (j_1, \ldots, j_{l'})$ where $l' = l(\overrightarrow{R}(S^l, x))$. If $l = 0$ then $x = x(S, l) = S(1, \rho_1 + 1)$ and $S^l = S|_{D_\rho}$. Since $S^l \in \mathrm{STab}(\rho, \lambda^{(x)})$, we have $S^l(1, q) = S(1, q) \leq x$ for all q with $1 \leq q \leq \rho_1$, so $\{q \mid S^l(1, q) > x\} = \emptyset$. Thus $\overrightarrow{R}(S^l, x) = 0$.

Suppose $l \geq 1$. Note that, for all p with $1 \leq p \leq l$, we have

$$S^l(p, j'_p - 1) = S(p, j'_p - 1) \leq S(p, j'_p). \tag{11}$$

We prove $j_p = j'_p$ by induction on p. If $p = 1$ then

$$\begin{aligned} j_1 &= \min\{q \mid S^l(1, q) > x\} \\ &= \min\{q \mid S^l(1, q) > S(1, j'_1)\} \\ &= j'_1 && \text{(by Lemma 42 and (11))}. \end{aligned}$$

Assume $j_{p-1} = j'_{p-1}$ for some $2 \leq p \leq l$. Then

$$\begin{aligned} j_p &= \min\{q \mid S^l(p, q) > S^l(p-1, j_{p-1})\} \\ &= \min\{q \mid S^l(p, q) > S^l(p-1, j'_{p-1})\} \\ &= \min\{q \mid S^l(p, q) > S(p, j'_p)\} && \text{(by (9))} \\ &= j'_p && \text{(by Lemma 42 and (11))}. \end{aligned}$$

Since

$$\begin{aligned} S^l(l+1, \rho_{l+1}) &= S(l+1, \rho_{l+1}) && \text{(by (9))} \\ &\leq S(l+1, \rho_{l+1} + 1) \\ &= S^l(l, j'_l) && \text{(by (9))} \\ &= S^l(l, j_l), \end{aligned}$$

we have $j_p = 0$ for $p > l$, so $l' = l$.

(iii) By (ii), we have $\overrightarrow{R}(S^l, x) = \overleftarrow{R}(S, l)$. Then $(S^l)_x \in \mathrm{STab}(\rho^{l+1}, \lambda)$ by (i) and Lemma 39. Suppose first $l = 0$. For $(p, q) \in D_{\rho^1}$, we have

$$\begin{aligned} (S^l)_x(p, q) &= \begin{cases} S^l(p, q) & \text{if } (p, q) \in D_\rho, \\ x & \text{if } (p, q) = (1, \rho_1 + 1) \end{cases} \\ &= \begin{cases} S(p, q) & \text{if } (p, q) \in D_\rho, \\ S(1, \rho_1 + 1) & \text{if } (p, q) = (1, \rho_1 + 1) \end{cases} \\ &= S(p, q). \end{aligned}$$

Suppose $l \geq 1$. For $(p, q) \in D_{\rho^{l+1}}$, we have

$$(S^l)_x(p, q) = \begin{cases} x & \text{if } (p, q) = (1, j_1), \\ S^l(p-1, j_{p-1}) & \text{if } q = j_p,\ 2 \leq p \leq l, \\ S^l(l, j_l) & \text{if } (p, q) = (l+1, \rho_{l+1} + 1), \\ S^l(p, q) & \text{otherwise} \end{cases}$$

$$= \begin{cases} x & \text{if } (p,q) = (1, j'_1), \\ S^l(p-1, j'_{p-1}) & \text{if } q = j'_p,\ 2 \le p \le l, \\ S^l(l, j'_l) & \text{if } (p,q) = (l+1, \rho_{l+1}+1), \\ S^l(p,q) & \text{otherwise} \end{cases}$$
$$= S(p,q).$$

□

4.3 Vershik's relations for the Kostka numbers

Lemma 44. *Let $\mu \vdash n$, $\lambda \vDash n$ and $T \in \mathrm{STab}(\mu, \lambda)$. Let x be a positive integer and $l = l(\overrightarrow{R}(T,x))$. Then*

(i) $\overleftarrow{R}(T_x, l) = \overrightarrow{R}(T,x)$.

(ii) $(T_x)^l = T$.

Proof. If $l = 0$, then (i) clearly holds, and $(T_x)^l = T_x|_{D_\mu} = T$. Suppose $l \ge 1$ and let $\overleftarrow{R}(T_x, l) = (j'_1, \dots, j'_l)$ and $\overrightarrow{R}(T,x) = (j_1, \dots, j_l)$.

(i) Note that, for all p with $1 \le p \le l$, we have

$$T_x(p, j_p + 1) = T(p, j_p + 1) \ge T(p, j_p). \tag{12}$$

We prove $j'_p = j_p$ by induction on $l - p$. If $p = l$ then

$$\begin{aligned} j'_l &= \max\{q \mid T_x(l,q) < T_x(l+1, \mu_{l+1}+1)\} \\ &= \max\{q \mid T_x(l,q) < T(l, j_l)\} && \text{(by (5))} \\ &= j_l && \text{(by Lemma 38 and (12)).} \end{aligned}$$

Assume $j'_{p+1} = j_{p+1}$ for some $1 \le p < l$. Then

$$\begin{aligned} j'_p &= \max\{q \mid T_x(p,q) < T_x(p+1, j'_{p+1})\} \\ &= \max\{q \mid T_x(p,q) < T_x(p+1, j_{p+1})\} \\ &= \max\{q \mid T_x(p,q) < T(p, j_p)\} && \text{(by (5))} \\ &= j_p && \text{(by Lemma 38 and (12)).} \end{aligned}$$

(ii) By (i), we have $\overleftarrow{R}(T_x, l) = \overrightarrow{R}(T,x)$. Then $x(T_x, l) = T_x(1, j'_1) = T_x(1, j_1) = x$, so $(T_x)^l \in \mathrm{STab}(\mu, \lambda)$ by Lemma 43 (i). For $(p,q) \in D_\mu$, we have

$$(T_x)^l(p,q) = \begin{cases} T_x(p+1, j'_{p+1}) & \text{if } q = j'_p,\ 1 \le p < l, \\ T_x(l+1, \mu_{l+1}+1) & \text{if } (p,q) = (l, j'_l), \\ T_x(p,q) & \text{otherwise} \end{cases} \qquad \text{(by (9))}$$

$$
\begin{aligned}
&= \begin{cases} T_x(p+1, j_{p+1}) & \text{if } q = j_p,\ 1 \le p < l, \\ T_x(l+1, \mu_{l+1}+1) & \text{if } (p,q) = (l, j_l), \\ T_x(p,q) & \text{otherwise} \end{cases} \\
&= T(p,q).
\end{aligned}
$$

□

Before proving the main result, for $\lambda \vDash n$, we let

$$\operatorname{Supp}(\lambda) = \{i \mid \lambda_i > 0\}.$$

Theorem 45. *Let $\lambda \vDash n$ and $\rho \vdash n-1$. Then the map*

$$
\begin{array}{ccc}
\bigcup_{x \in \operatorname{Supp}(\lambda)} \operatorname{STab}(\rho, \lambda^{(x)}) & \to & \bigcup_{\substack{\mu \vdash n \\ \mu \succeq \rho}} \operatorname{STab}(\mu, \lambda) \\
T & \mapsto & T_x
\end{array}
$$

is a bijection.

Proof. The map is well-defined by Lemma 39. Suppose $S \in \operatorname{STab}(\mu, \lambda)$ for some $\mu \vdash n$ with $\mu \succeq \rho$. Then $\mu = \rho^{l+1}$ for some l. Set $x = x(S, l)$. By Lemma 43, $S^l \in \operatorname{STab}(\rho, \lambda^{(x)})$ and $(S^l)_x = S$, so the map is a surjection.

Let $T \in \operatorname{STab}(\rho, \lambda^{(x)})$ and $S \in \operatorname{STab}(\rho, \lambda^{(x')})$. Suppose $T_x = S_{x'} \in \operatorname{STab}(\mu, \lambda)$ for some $\mu \vdash n$ with $\mu \succeq \rho$. Then $\mu = \rho^{l+1}$ for some l, so $l = l(\overrightarrow{R}(T, x)) = l(\overrightarrow{R}(S, x'))$. By Lemma 44, we have $T = (T_x)^l = (S_{x'})^l = S$, so the map is an injection. □

Remark 46. Let $\mu \vdash n$ and let X be a set of positive integers. Define

$$
\begin{aligned}
\mathcal{W}(X, n) &= \{\lambda \vDash n \mid \lambda_i \ge 0,\ \operatorname{Supp}(\lambda) \subseteq X\}, \\
\operatorname{STab}_X(\mu) &= \bigcup_{\lambda \in \mathcal{W}(X,n)} \operatorname{STab}(\mu, \lambda).
\end{aligned}
$$

For $\rho \vdash n-1$, the map

$$
\begin{array}{ccc}
\operatorname{STab}_X(\rho) \times X & \to & \bigcup_{\substack{\mu \vdash n \\ \mu \succeq \rho}} \operatorname{STab}_X(\mu) \\
(T, x) & \mapsto & T_x
\end{array}
\tag{13}
$$

is a bijection (see [8, p.399, 10.60]). This follows from Theorem 45. Indeed, collecting the bijections of Theorem 45 for all $\lambda \in \mathcal{W}(X, n)$, we obtain a bijection

$$
\begin{array}{ccc}
\bigcup_{\lambda \in \mathcal{W}(X,n)} \bigcup_{x \in \operatorname{Supp}(\lambda)} \operatorname{STab}(\rho, \lambda^{(x)}) \times \{x\} & \to & \bigcup_{\lambda \in \mathcal{W}(X,n)} \bigcup_{\substack{\mu \vdash n \\ \mu \succeq \rho}} \operatorname{STab}(\mu, \lambda) \\
(T, x) & \mapsto & T_x
\end{array}.
\tag{14}
$$

Then the codomain of the bijection (14) equals that of (13), while

$$
\begin{aligned}
\bigcup_{\lambda \in \mathcal{W}(X,n)} \bigcup_{x \in \operatorname{Supp}(\lambda)} \operatorname{STab}(\rho, \lambda^{(x)}) \times \{x\} &= \bigcup_{\nu \in \mathcal{W}(X,n-1)} \operatorname{STab}(\rho, \nu) \times X \\
&= \operatorname{STab}_X(\rho) \times X.
\end{aligned}
$$

Corollary 47 (Vershik's relations for the Kostka numbers)**.** *For* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h) \vdash n$ *and* $\rho \vdash n-1$*, we have*

$$\sum_{\substack{\mu \vdash n \\ \mu \succeq \rho}} K(\mu, \lambda) = \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} C(\lambda, \gamma) K(\rho, \gamma).$$

Proof.

$$\begin{aligned}
\sum_{\substack{\mu \vdash n \\ \mu \succeq \rho}} K(\mu, \lambda) &= \sum_{\substack{\mu \vdash n \\ \mu \succeq \rho}} |\operatorname{STab}(\mu, \lambda)| \\
&= \sum_{1 \le x \le h} |\operatorname{STab}(\rho, \lambda^{(x)})| && \text{(by Theorem 45)} \\
&= \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} \sum_{\substack{1 \le x \le h \\ \lambda(\lambda^{(x)}) = \gamma}} |\operatorname{STab}(\rho, \lambda^{(x)})| \\
&= \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} \sum_{\substack{1 \le x \le h \\ \lambda(\lambda^{(x)}) = \gamma}} |\operatorname{STab}(\rho, \gamma)| && \text{(by [2, Lemma 3.7.1])} \\
&= \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} |\{x \mid 1 \le x \le h,\ \lambda(\lambda^{(x)}) = \gamma\}| |\operatorname{STab}(\rho, \gamma)| \\
&= \sum_{\substack{\gamma \vdash n-1 \\ \gamma \preceq \lambda}} C(\lambda, \gamma) K(\rho, \gamma).
\end{aligned}$$

□

Now, we compare Vershik's bijection with ours using [14, Example 1].

Example 48 ([14, Example 1])**.** Let $\lambda = (3, 2, 1) \vdash 6$ and $\rho = (4, 1) \vdash 5$. Then

$$\mu\text{-tableaux}: A = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 3 \end{array}, \ B = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ 2 \end{array}, \ C = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 3 \end{array},$$

$$D = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 \end{array}, \ E = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 \\ 3 \end{array};$$

$$\rho\text{-tableaux}: L = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 3 \end{array}, \ M = \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 \end{array}, \ N = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 3 \end{array},$$

$$P = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 \end{array}, \ Q = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 \end{array}.$$

We remove one box from the first row in A and B, one box from the second row in C and D, and one box $(3, 1)$ in E in order to obtain ρ-tableaux. Then we have a bijection as follows:

$$A \leftrightarrow L; \quad B \leftrightarrow M; \quad C \leftrightarrow N; \quad D \leftrightarrow P; \quad E \leftrightarrow Q.$$

The bijection given by Theorem 45 is:

$$\begin{array}{ll} L \leftrightarrow L_1 = E; & M \leftrightarrow M_1 = D; \\ N \leftrightarrow N_2 = A; & P \leftrightarrow P_2 = C; \\ Q \leftrightarrow Q_3 = B. & \end{array}$$

Finally, we give an example, for which there is no bijection arising from removable of one box.

Example 49. Let $\lambda = (3,3,2) \vdash 8$ and $\rho = (4,3) \vdash 7$. Then

$$\mu\text{-tableaux}: A = \begin{array}{ccccc} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & & \end{array}, \ B = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & & \end{array}, \ C = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & & \end{array},$$

$$D = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \end{array}, \ E = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & \\ 3 & & & \end{array}, \ F = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & \\ 3 & & & \end{array};$$

$$\rho\text{-tableaux}: L = \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & \end{array}, \ M = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & \end{array}, \ N = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & 3 & \end{array},$$

$$P = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 3 & 3 & \end{array}, \ Q = \begin{array}{cccc} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & \end{array}, \ R = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & \end{array}.$$

As mentioned in the Introduction, μ-tableaux A and E result in ρ-tableau Q, so there is no bijection between μ-tableaux and ρ-tableaux arising from removable of one box. The bijection given by Theorem 45 is:

$$\begin{array}{ll} L \leftrightarrow L_1 = E; & M \leftrightarrow M_1 = F; \\ N \leftrightarrow N_2 = D; & P \leftrightarrow P_2 = C; \\ Q \leftrightarrow Q_3 = A; & R \leftrightarrow R_3 = B. \end{array}$$

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