Studies on non-amorphous association schemes and spin models

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Contents

Chapter 1

Introduction

In this dissertation, we will mainly study the following two topics: non-amorphous association schemes and spin models. These contents are joint work with the supervisor Akihiro Munemasa. The first half is a study about association schemes themselves, and the second half is a study which applies association schemes to other mathematical objects. In both cases, an association scheme is the starting point.

In 1973, P. Delsarte [16] suggested the concept of association schemes to treat coding theory and combinatorial designs integrally. Roughly speaking, an association scheme on a finite set X is a partition of $X \times X$ into relations which exhibits nice regularity properties. The regularity properties of the relations are conveniently described in terms of their adjacency matrices: they span an algebra of matrices with unit I closed under transposition. This algebra is called the Bose-Mesner algebra of the association scheme. It is also an algebra with unit J (the all-one matrix) for the Hadamard (i.e., entrywise) product of matrices. We always assume here that the ordinary matrix product in the Bose-Mesner algebra is commutative. In this case, Bose-Mesner algebras and association schemes are equivalent concepts.

First we consider a problem about non-amorphous association scheme. If each nontrivial relation in a symmetric association scheme is strongly regular, then an arbitrary partition of the set of nontrivial relations gives rise to an association scheme. Association schemes satisfying this conclusion is called *amorphous*. A.V. Ivanov conjectured in [23] that if each nontrivial relation in an association scheme is strongly regular, then the association scheme must be amorphous. This conjecture turned out to be false. A counterexample was given by van Dam [13] in the case where the association scheme is imprimitive. Later, van Dam [14] gave an example of primitive non-amorphous association schemes in which every nontrivial relation is a strongly regular graph, as a fusion scheme of the cyclotomic scheme of class

45 on $GF(2^{12})$. We are interested in the construction of counterexamples to Ivanov's conjecture in the primitive case.

In chapter 2, we are to present a new example of primitive non-amorphous association schemes in which every nontrivial relation is a strongly regular graph, as a fusion scheme of the cyclotomic scheme of class 75 on $GF(2^{20})$. We also propose an infinite family of parameters of association schemes containing both of these two examples.

In chapter 3, we are to present the following: Let $\mathcal{X}$ be a pseudocyclic association scheme in which all the nontrivial relations are strongly regular graphs with the same eigenvalues. We prove that the principal part of the first eigenmatrix of $\mathcal{X}$ is a linear combination of an incidence matrix of a symmetric design and the all-ones matrix. Amorphous pseudocyclic association schemes are examples of such association schemes whose associated symmetric design is trivial. We present several non-amorphous examples, which are either cyclotomic association schemes, or their fusion schemes. Special properties of symmetric designs guarantee the existence of further fusions, and the two known non-amorphous association schemes of class 4 discovered by van Dam and in chapter 3, are recovered in this way. We also give another pseudocyclic non-amorphous association scheme of class 7 on $GF(2^{21})$, and a new pseudocyclic amorphous association scheme of class 5 on $GF(2^{12})$.

Lastly, we consider the construction of spin models. Spin models were introduced by Jones [27] to produce invariants of links. A spin model is defined as a square matrix W with nonzero complex entries satisfying certain conditions. Kawagoe, Munemasa and Watatani [28] generalized it by dropping the symmetric condition. The fact that association schemes and their Bose-Mesner algebras provide a convenient and natural framework for the study of spin models was first pointed out by Jaeger [24]. Jaeger, Matsumoto, and Nomura [25] showed that a spin model belongs to the Bose-Mesner algebra of a self-dual association scheme.

For a spin model W, it is known that W^TW^{-1} is a permutation matrix, and its order is called the index of W. F. Jaeger and K. Nomura found spin models of index 2, by modifying the construction of symmetric spin models from Hadamard matrices. We are interested in the construction of spin models of even index. The reason is that in [26], the author wrote that the link invariant of spin models of odd index is gauge equivalent to the link invariant of symmetric spin models, but one could still expect to obtain new non-symmetric spin models in the case where m is a power of 2. In chapter 4, we give a construction of spin models of an arbitrary even index from any Hadamard matrix. In particular, we show that our spin models of indices a power of 2 are new.

Chapter 2

A New Example of Non-Amorphous Association Schemes

2.1 Introduction

Let X be a finite set with cardinality n. Let $(X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme of class d on X. Let $P = (p_{i,j})_{\substack{0 \le i \le d \\ 0 \le j \le d}}$ and $Q = (q_{i,j})_{\substack{0 \le i \le d \\ 0 \le j \le d}}$ be the first and the second eigenmatices of $(X, \{R_i\}_{i=0}^d)$ respectively. We refer [5] for notation and general theory of association schemes.

Let $\{\Lambda_j\}_{j=0}^{d'}$ be a partition of $\{0, 1, \ldots, d\}$ with $\Lambda_0 = \{0\}$. We define $R_{\Lambda_j} = \bigcup_{\ell \in \Lambda_j} R_\ell$. If $(X, \{R_{\Lambda_j}\}_{j=0}^{d'})$ forms an association scheme, then we call $(X, \{R_{\Lambda_j}\}_{j=0}^{d'})$ a *fusion* scheme of $(X, \{R_i\}_{i=0}^d)$. If $(X, \{R_{\Lambda_j}\}_{j=0}^{d'})$ is an association scheme for any partition $\{\Lambda_j\}_{j=0}^{d'}$ of $\{0, 1, \ldots, d\}$ with $\Lambda_0 = \{0\}$, then $(X, \{R_i\}_{i=0}^d)$ is called *amorphous*.

There is a simple criterion in terms of P for a given partition $\{\Lambda_j\}_{j=0}^{d'}$ to give rise to a fusion scheme (due to Bannai [1], Muzychuk [34]): There exists a partition $\{\Delta_i\}_{i=0}^{d'}$ of $\{0, 1, \ldots, d\}$ with $\Delta_0 = \{0\}$ such that each (Δ_i, Λ_j)-block of the first eigenmatrix P has a constant row sum. The constant row sum turns out to be the (i, j) entry of the 1st eigenmatrix of the fusion scheme.

Let q be a prime power and e be a divisor of $q - 1$. Fix a primitive element α of the multiplicative group of the finite field $\mathrm{GF}(q)$. Then $\langle \alpha^e \rangle$ is a subgroup of index e and its cosets are $\alpha^i \langle \alpha^e \rangle$ $(0 \le i \le e - 1)$. We define $R_0 = \{(x, x) | x \in \mathrm{GF}(q)\}$ and $R_i = \{(x, y) | x - y \in \alpha^i \langle \alpha^e \rangle, x, y \in \mathrm{GF}(q)\}$ $(1 \le i \le e)$. Then $(\mathrm{GF}(q), \{R_i\}_{i=0}^e)$ forms an association scheme and is called the *cyclotomic* scheme of class e on $\mathrm{GF}(q)$.

Baumert, Mills and Ward [7] gave a necessary and sufficient condition for a cyclotomic scheme to be amorphous. See also [12]. Ito, Munemasa and Yamada constructed amorphous association schemes over Galois rings. Clearly, in an amorphous association scheme, every nontrivial relation is a strongly regular graph. A. V. Ivanov [23] conjectured the converse also holds, but later it was disproved by van Dam [13]. Since the counterexample given in [13] was an imprimitive association scheme, it remained as an unsolved problem to find a primitive non-amorphous association scheme in which every nontrivial relation is a strongly regular graph. In [14], van Dam constructed a non-amorphous 4-class fusion scheme of the cyclotomic scheme of class 45 on $\mathrm{GF}(2^{12})$ with the following first eigenmatrix:

$$\begin{pmatrix} 1 & 3276 & 273 & 273 & 273 \\ 1 & -52 & 17 & 17 & 17 \\ 1 & 12 & -15 & -15 & 17 \\ 1 & 12 & -15 & 17 & -15 \\ 1 & 12 & 17 & -15 & -15 \end{pmatrix}. \tag{2.1}$$

This was the first and the only known primitive non-amorphous association scheme in which every nontrivial relation is a strongly regular graph.

In this paper, we present another such example.

Theorem 2.1.1. *The cyclotomic scheme of class* 75 *on* $\mathrm{GF}(2^{20})$ *has a non-amorphous fusion scheme of class* 4 *with the following first eigenmatrix :*

$$\begin{pmatrix} 1 & 838860 & 69905 & 69905 & 69905 \\ 1 & -820 & 273 & 273 & 273 \\ 1 & 204 & -239 & -239 & 273 \\ 1 & 204 & -239 & 273 & -239 \\ 1 & 204 & 273 & -239 & -239 \end{pmatrix}. \tag{2.2}$$

2.2 Restrictions on the first eigenmatrix

In general, if an association scheme $(X, \{R_i\}_{i=0}^d)$ has the following first eigenmatrix (2.3), then for each relation R_i $(i = 1, 2, 3, 4)$, (X, R_i) is a strongly regular graph, and $(X, \{R_i\}_{i=0}^d)$ is not amorphous.

$$P = \left(\begin{array}{cc|ccc} 1 & k_1 & k_2 & k_2 & k_2 \\ 1 & s_1 & r_2 & r_2 & r_2 \\ \hline 1 & r_1 & s_2 & s_2 & r_2 \\ 1 & r_1 & s_2 & r_2 & s_2 \\ 1 & r_1 & r_2 & s_2 & s_2 \end{array}\right). \tag{2.3}$$

Indeed, clearly $r_2 \neq s_2$, so $(X, \{R_0, R_1 \cup R_2, R_3 \cup R_4\})$ is not an association scheme.

Lemma 2.2.1. *Let* $(X, \{R_i\}_{i=0}^4)$ *be an association scheme with the first eigenmatrix* (2.3). *Then* r_1 *and* s_1 *are integers, and*

$$|X| = \frac{(r_1 - s_1)^2(s_1 + 4)}{4(s_1 + 3r_1 + 4)}, \tag{2.4}$$

$$k_1 = \frac{r_1(r_1 s_1 + 4r_1 - s_1^2 + 4)}{s_1 + 3r_1 + 4}, \tag{2.5}$$

$$k_2 = -\frac{r_1 s_1 + 4r_1 - s_1^2 + 4}{12} \tag{2.6}$$

$$r_2 = -\frac{1}{3}(s_1 + 1), \tag{2.7}$$

$$s_2 = \frac{-3r_1 + s_1 - 2}{6}, \tag{2.8}$$

$$m_1 = \frac{1}{4}k_1, \tag{2.9}$$

$$m_2 = m_3 = m_4 = -\frac{s_1 + 4}{12 r_1} k_1. \tag{2.10}$$

Proof. By [5, Chap.2, Theoerem 4.1], we have $m_2 = m_3 = m_4$. By [5, Chap.2, Theoerem 3.5], the second eigenmatrix Q is given by

$$Q = \begin{pmatrix} 1 & m_1 & m_2 & m_2 & m_2 \\ 1 & \frac{s_1 m_1}{k_1} & \frac{r_1 m_2}{k_1} & \frac{r_1 m_2}{k_1} & \frac{r_1 m_2}{k_1} \\ 1 & \frac{r_2 m_1}{k_2} & \frac{s_2 m_2}{k_2} & \frac{s_2 m_2}{k_2} & \frac{r_2 m_2}{k_2} \\ 1 & \frac{r_2 m_1}{k_2} & \frac{s_2 m_2}{k_2} & \frac{r_2 m_2}{k_2} & \frac{s_2 m_2}{k_2} \\ 1 & \frac{r_2 m_1}{k_2} & \frac{r_2 m_2}{k_2} & \frac{s_2 m_2}{k_2} & \frac{s_2 m_2}{k_2} \end{pmatrix}. \tag{2.11}$$

Since $PQ = |X|I$, we have

$$0 = (PQ)_{1,0} = 1 + s_1 + 3r_2,$$
$$0 = (PQ)_{2,0} = 1 + r_1 + 2s_2 + r_2.$$

These give (2.7) and (2.8). Also, we have

$$0 = (PQ)_{1,2} = m_1 \left(1 + \frac{r_1 s_1}{k_1} + \frac{2r_2 s_2 + r_2^2}{k_2}\right),$$
$$0 = (PQ)_{2,3} = m_2 \left(1 + \frac{r_1^2}{k_1} + \frac{s_2^2 + 2r_2 s_2}{k_2}\right).$$

Thus

$$\begin{pmatrix} \frac{1}{k_1} & \frac{1}{k_2} \end{pmatrix} \begin{pmatrix} r_1 s_1 & r_1^2 \\ 2r_2 s_2 + r_2^2 & s_2^2 + 2r_2 s_2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \end{pmatrix}.$$

Solving this and substituting (2.7) and (2.8), we obtain (2.5) and (2.6). Finally, (2.4), (2.9) and (2.10) follow from

$$\begin{aligned} |X| &= (PQ)_{0,0} = 1 + k_1 + 3k_2, \\ |X| &= (PQ)_{1,1} = m_1 \left(1 + \frac{s_1^2}{k_1} + 3\frac{r_2^2}{k_2}\right), \\ |X| &= (PQ)_{2,2} = m_2 \left(1 + \frac{r_1^2}{k_1} + 2\frac{s_2^2}{k_2} + \frac{r_2^2}{k_2}\right). \end{aligned}$$

Finally, we show that r_1 and s_1 are integers. Since they are algebraic integers, it suffices to show that they are rational. Suppose contrary. Then by (2.7), r_2 is irrational, so r_2 and s_2 are algebraic conjugate. Thus $m_1 + m_4 = m_2 + m_3$, as the multiplicities of r_2 and s_2 in (X, R_2) are equal. By (2.10), this implies $m_1 = m_2$. On the other hand, the same argument applied to (X, R_1) implies $m_1 = 3m_2$, which is a contradiction. □

Recall that a symmetric association scheme $(X, \{R_i\}_{i=0}^d)$ is formally self-dual if its first eigenmatrix P coincides with its second eigenmatrix, after permuting the rows and the columns of P. We say that a strongly regular graph (X, R) is formally self-dual if the associated association scheme of class 2 is formally self-dual. Note that, we can see easily from (2.9) and (2.11) that any association scheme with the first eigenmatrix (2.3) is not formally self-dual. In van Dam's example with the first eigenmatrix (2.1), however, the strongly regular graph (X, R_1) is formally self-dual. If we adopt this as an assumption, then we have a one-parameter family of possible first eigenmatrices:

Lemma 2.2.2. *Let $(X, \{R_i\}_{i=0}^4)$ be an association scheme with the first eigenmatrix* (2.3)*, and assume that the strongly regular graph (X, R_1) is formally self-dual. Then $s_1 = -4r_1 - 4$, $r_1 \equiv 0 \pmod 6$ and $|X| = (5r_1 + 4)^2$.*

Proof. By the assumption, $k_1 \in \{m_1, |X| - m_1 - 1\}$. By (2.9) and (2.10), we obtain $s_1 = -4r_1 - 4$. Then by (2.4), we obtain $|X| = (5r_1 + 4)^2$. Also by (2.8), $s_2 = \frac{-7r_1 - 6}{6}$, and hence $r_1 \equiv 0 \pmod 6$. □

Setting $r = \frac{r_1}{6}$, the first eigenmatrix of an association scheme satisfying the hypotheses of Lemma 2.2.2 has the following form:

$$P = \begin{pmatrix} 1 & k_1 & k_2 & k_2 & k_2 \\ 1 & -4(6r+1) & 8r+1 & 8r+1 & 8r+1 \\ 1 & 6r & -7r-1 & -7r-1 & 8r+1 \\ 1 & 6r & -7r-1 & 8r+1 & -7r-1 \\ 1 & 6r & 8r+1 & -7r-1 & -7r-1 \end{pmatrix}, \tag{2.12}$$

where $k_1 = 12(6r+1)(10r+1)$ and $k_2 = (6r+1)(10r+1)$.

2.3 Construction of a new example

We consider the problem of realizing (2.12) as the first eigenmatrix of a cyclotomic association scheme. By Lemma 2.2.2, $|X|$ is even, so we assume $|X|$ is a power of 2. Put $30r + 4 = \sqrt{|X|} = 2^g$. Then $2^g \equiv 4 \pmod 5$, and hence $g = 4h + 2$ for some nonnegative integer h. In this case, $|X| = 2^{8h+4}$ and $r = \frac{2}{15}(16^h - 1)$.

When $h = 0$, we have

$$P = \begin{pmatrix} 1 & 12 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 & -1 \end{pmatrix}.$$

This is realized as the first eigenmatrix of an association scheme belonging to an infinite family of imprimitive non-amorphous association schemes. appeared in [13], and it is also mentioned in [14].

The case $h = 1$ gives the matrix (2.1) which was realized in [14].

When $h = 2$, we obtain the following matrix:

$$P = \begin{pmatrix} 1 & 838860 & 69905 & 69905 & 69905 \\ 1 & -820 & 273 & 273 & 273 \\ 1 & 204 & -239 & -239 & 273 \\ 1 & 204 & -239 & 273 & -239 \\ 1 & 204 & 273 & -239 & -239 \end{pmatrix}. \tag{2.13}$$

This is realized as a fusion scheme of the cyclotomic scheme of class 75 on $\mathrm{GF}(2^{20})$. Let α be a primitive element satisfying

$$\alpha^{20} + \alpha^{10} + \alpha^9 + \alpha^7 + \alpha^6 + \alpha^5 + \alpha^4 + \alpha + 1 = 0.$$

Let

$$H_j = \{(x, y) \mid x - y \in \alpha^j \langle \alpha^{75} \rangle\} \quad (j = 0, 1, \dots, 74)$$

By computer, we have verified that the graph Γ on $\mathrm{GF}(2^{20})$ with edge set

$$R_2 = H_0 \cup H_3 \cup H_6 \cup H_9 \cup H_{12}$$

is a strongly regular graph with eigenvalues $69905, 273, -239$. Clearly, each of the graphs with edge sets

$$R_3 = H_{25} \cup H_{28} \cup H_{31} \cup H_{34} \cup H_{37},$$
$$R_4 = H_{50} \cup H_{53} \cup H_{56} \cup H_{59} \cup H_{62}$$

are isomorphic to Γ. Moreover, since $H_0 \cup H_{25} \cup H_{50}$ is one of the relations in the 25-class cyclotomic amorphous association scheme on $\mathrm{GF}(2^{20})$, the union $R_2 \cup R_3 \cup R_4$ is a strongly regular graph with eigenvalues $209715, 819, -205$, by [12, Theorem 2]. Hence the complement Γ_1 of this union is strongly regular with eigenvalues $838860, 204, -820$. Let R_0 denote the diagonal relation on $\mathrm{GF}(2^{20})$, and let R_1 denote the edge set of Γ_1. Then the association scheme $(\mathrm{GF}(2^{20}), \{R_i\}_{i=0}^4)$ has the character table as described in (2.13). This completes the proof of Theorem 2.1.1.

Chapter 3

Pseudocyclic Association Schemes and Strongly Regular Graphs

3.1 Introduction

A.V. Ivanov's conjecture [23], though disproved by E.R. van Dam, asserted that, if each nontrivial relation in a symmetric association scheme is strongly regular, then an arbitrary partition of the set of nontrivial relations gives rise to an association scheme. Association schemes satisfying this conclusion is called *amorphous* (or *amorphic*). A counterexample to A.V. Ivanov's conjecture was given by van Dam in [13] for the imprimitive case, and in [14] for the primitive case. Presently there are only a few primitive counterexamples known. An example due to van Dam [14] has the first eigenmatrix given by

$$\begin{pmatrix} 1 & 3276 & 273 & 273 & 273 \\ 1 & -52 & 17 & 17 & 17 \\ 1 & 12 & -15 & -15 & 17 \\ 1 & 12 & -15 & 17 & -15 \\ 1 & 12 & 17 & -15 & -15 \end{pmatrix}, \tag{3.1}$$

and another one is due to the authors [19] with the first eigenmatrix given by

$$\begin{pmatrix} 1 & 838860 & 69905 & 69905 & 69905 \\ 1 & -820 & 273 & 273 & 273 \\ 1 & 204 & -239 & -239 & 273 \\ 1 & 204 & -239 & 273 & -239 \\ 1 & 204 & 273 & -239 & -239 \end{pmatrix}. \tag{3.2}$$

A symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ is said to be pseudocyclic if the nontrivial multiplicities $m_1, \dots, m_d$ of $\mathcal{X}$ coincide. The first eigenmatrix of a pseudocyclic association scheme is of the form

$$P = \begin{pmatrix} 1 & f & \dots & f \\ 1 & & & \\ \vdots & & P_0 & \\ 1 & & & \end{pmatrix},$$

where f denotes the common nontrivial multiplicities, as well as the common nontrivial valencies. The submatrix P_0 is called the principal part of P. If we restrict A.V. Ivanov's conjecture to the pseudocyclic case, it asserts that for pseudocyclic association scheme in which each nontrivial relation is strongly regular, the principal part of its first eigenmatrix is a linear combination of I and J, after a suitable permutation of rows.

Cyclotomic association schemes in which each nontrivial relation is strongly regular, have been investigated in its own right. It follows from McEliece's theorem ([33], see also [39, Lemma 2.8]) that the number of nontrivial eigenvalues of the cyclotomic association scheme of class e over $\mathrm{GF}(p^m)$ is the same as that of weights in the irreducible cyclic code $c(p, m, e)$ (see [39, Definition 2.2]). In this sense, such cyclotomic association schemes correspond to two-weight irreducible cyclic codes. Moreover, under this correspondence, subfield codes, semiprimitive codes correspond to amorphous cyclotomic association scheme which are imprimitive, primitive, respectively. Primitive amorphous cyclotomic association schemes were investigated by Baumert, Mills and Ward [7], and Brouwer, Wilson and Xiang [12]. Non-amorphous cyclotomic association schemes in which every nontrivial relation is strongly regular, are thus equivalent to exceptional two weight irreducible cyclic codes in the sense of Schmidt and White [39]. Therefore, cyclotomic association schemes corresponding to exceptional two weight irreducible cyclic codes are pseudocyclic counterexamples to A.V. Ivanov's conjecture, and there are eleven such codes in [39].

One of the purpose of this paper is to show that both of the counterexamples with first eigenmatrices (3.1), (3.2) are derived from some pseudocyclic association schemes $\mathcal{X}_1, \mathcal{X}_2$, respectively, of class 15 which are also counterexamples themselves. It turns out that, the principal part of the first eigenmatrix of $\mathcal{X}_1$ or $\mathcal{X}_2$ is expressed by an incidence matrix of $\mathrm{PG}(3, 2)$. In a more general setting, we prove in Theorem 2.1.1 that the principal part of the first eigenmatrix is a linear combination of an incidence matrix of a symmetric design and the all-ones matrix. For an amorphous pseudocyclic association scheme of class d, the associated symmetric design is the complete 2-$(d, d-1, d-2)$ design. When the associated symmetric design is a projective space, we show in Theorem 3.4.1 that the existence of certain fusion schemes follows from special properties of projective spaces. This gives an explanation for the existence of the fusion schemes of class 4 in $\mathcal{X}_1, \mathcal{X}_2$. Moreover, the two pseudocyclic association schemes $\mathcal{X}_1, \mathcal{X}_2$ of class 15 give rise to two pseudocyclic amorphous fusion schemes of class 5. We also give a pseudocyclic class 7 fusion scheme of the cyclotomic association scheme of class 49 on $\mathrm{GF}(2^{21})$. Its associated design is $\mathrm{PG}(2, 2)$.

3.2 Preliminaries

We refer the reader to [5] for notation and general theory of association schemes. Let $\mathcal{X} = (X, \{R_i\}_{i=0}^{d})$ be a symmetric association scheme of class d on X. Let P be the first eigenmatrix of $\mathcal{X}$. Let $\{\Lambda_j\}_{j=0}^{d'}$ be a partition of $\{0, 1, \dots, d\}$ with $\Lambda_0 = \{0\}$, and we set $R_{\Lambda_j} = \bigcup_{\ell \in \Lambda_j} R_\ell$. If $(X, \{R_{\Lambda_j}\}_{j=0}^{d'})$ forms an association scheme, then we call $(X, \{R_{\Lambda_j}\}_{j=0}^{d'})$ a *fusion* scheme of $\mathcal{X}$. If $(X, \{R_{\Lambda_j}\}_{j=0}^{d'})$ is an association scheme for any partition $\{\Lambda_j\}_{j=0}^{d'}$ of $\{0, 1, \dots, d\}$ with $\Lambda_0 = \{0\}$, then $\mathcal{X}$ is called *amorphous*. We refer the reader to a recent article [15] for details on amorphous association schemes.

There is a simple criterion in terms of P for a given partition $\{\Lambda_j\}_{j=0}^{d'}$ to give rise to a fusion scheme (due to Bannai [1], Muzychuk [34]): There exists a partition $\{\Delta_i\}_{i=0}^{d'}$ of $\{0, 1, \dots, d\}$ with $\Delta_0 = \{0\}$ such that each (Δ_i, Λ_j)-block of the first eigenmatrix P has a constant row sum. The constant row sum turns out to be the (i, j) entry of the first eigenmatrix of the fusion scheme.

An association scheme $\mathcal{X}$ of class d having the nontrivial multiplicities $m_1 = \ldots = m_d$ is called *pseudocyclic*. It is known that, in a pseudocyclic association scheme $\mathcal{X}$, all the nontrivial valencies coincide (see [5, p.76],or [11, Proposition 2.2.7]). By the *principal part* of the first eigenmatrix, we mean the lower-right $d \times d$ submatrix of the first eigenmatrix.

Let q be a prime power and let e be a divisor of $q-1$. Fix a primitive element α of the multiplicative group of the finite field $\mathrm{GF}(q)$. Then $\langle \alpha^e \rangle$ is a subgroup of index e and its cosets are $\alpha^i \langle \alpha^e \rangle$ $(0 \le i \le e-1)$. We define $R_0 = \{(x, x) \mid x \in \mathrm{GF}(q)\}$ and $R_i = \{(x, y) \mid x - y \in \alpha^i \langle \alpha^e \rangle,\ x, y \in \mathrm{GF}(q)\}$ $(1 \le i \le e)$. Then $(\mathrm{GF}(q), \{R_i\}_{i=0}^{e})$ forms an association scheme and is called the *cyclotomic association scheme*, or *cyclotomic scheme*, for short, of class e on $\mathrm{GF}(q)$. A cyclotomic scheme is a pseudocyclic association scheme.

Suppose $q = p^m$, where p is a prime. The cyclotomic scheme of class e on $\mathrm{GF}(q)$ is amorphous if and only if m is even and e divides $p^{m'} + 1$ for some divisor m' of $m/2$. This is essentially due to Baumert, Mills and Ward [7], but see also [12].

3.3 A symmetric design in the first eigenmatrix

Theorem 3.3.1. *Let $(X, \{R_i\}_{i=0}^d)$ be a pseudocyclic association scheme of class d. Assume that the graphs (X, R_i) $(i = 1, \ldots, d)$ are all strongly regular with the same eigenvalues. Then there exists a symmetric 2-(d, k, λ) design $\mathcal{D}$ such that the principal part of the first eigenmatrix of $\mathcal{X}$ is given by $rM + s(J - M)$, where M is an incidence matrix of $\mathcal{D}$, r and s are the nontrivial eigenvalues of the graphs (X, R_i).*

Proof. By the assumption, the principal part P_0 can be expressed as $P_0 = rM+s(J-M)$ for some $(0,1)$-matrix M. Then by the orthogonality relations (see [5, Chapter II, (3.10)]), we find

$$P_0 J = -J, \quad fJ + P_0 P_0^T = |X|I,$$

where f denotes the common nontrivial multiplicities. The former implies

$$MJ = -\frac{sd+1}{r-s} J,$$

hence $k = -(sd+1)/(r-s)$ is a positive integer. The latter implies

$$MM^T = \frac{1}{(r-s)^2}(|X|I + (s^2 d + 2s - f)J).$$

This implies that M is an incidence matrix of a symmetric design on d points with block size k. □

The assumption that the eigenvalues of the strongly regular graphs appearing as the nontrivial relations are the same, seems redundant. We have verified that the conclusion of Theorem 3.3.1 holds without this assumption for $d \le 4$.

Next we show the existence of further fusions. We denote by $\mathbf{1}_n$ the column vector of length n whose entries are all 1.

Corollary 3.3.2. *Under the same assumptions as in Theorem 3.3.1, $\mathcal{X}$ has a fusion scheme of class 3 with the first eigenmatrix*

$$\begin{pmatrix} 1 & f & (k-1)f & (d-k)f \\ 1 & r & (k-1)r & (d-k)s \\ 1 & r & (\lambda-1)r + (k-\lambda)s & (k-\lambda)r + (d-2k+\lambda)s \\ 1 & s & \lambda r + (k-1-\lambda)s & (k-\lambda)r + (d-2k+\lambda)s \end{pmatrix}. \tag{3.3}$$

In particular, there exists a fusion scheme of class 2 *with the first eigenmatrix*

$$\begin{pmatrix} 1 & kf & (d-k)f \\ 1 & kr & (d-k)s \\ 1 & \lambda r + (k-\lambda)s & (k-\lambda)r + (d-2k+\lambda)s \end{pmatrix}. \tag{3.4}$$

Proof. Let M be an incidence matrix of the design $\mathcal{D}$, so that $P_0 = rM + s(J - M)$ holds. Without loss of generality, we may assume that the first k columns of M correspond to the set of points on a block B of $\mathcal{D}$, and that B is represented by the first row of M. Let F denote the $d \times 3$ matrix defined by

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{1}_{k-1} & 0 \\ 0 & 0 & \mathbf{1}_{d-k} \end{pmatrix}.$$

Then we have

$$MF = \begin{pmatrix} 1 & k-1 & 0 \\ \mathbf{1}_{k-1} & (\lambda-1)\mathbf{1}_{k-1} & (k-\lambda)\mathbf{1}_{k-1} \\ 0 & \lambda\mathbf{1}_{d-k} & (k-\lambda)\mathbf{1}_{d-k} \end{pmatrix}.$$

It follows that the matrix P_0F has 3 distinct rows, which are precisely those of the 3×3 lower-right submatrix of (3.3). By the Bannai–Muzychuk criterion, we obtain a fusion scheme of class 3 with the first eigenmatrix given by (3.3). Fusing the first two relations of this class 3 association scheme, we obtain a class 2 association scheme with the first eigenmatrix given by (3.4). □

Amorphous pseudocyclic association schemes satisfy the conditions of Theorem 3.3.1. However, the symmetric design appearing in the principal part of the first eigenmatrix is the complete 2-$(d, d-1, d-2)$ design. The conclusion of Corollary 3.3.2 is trivially true for amorphous association schemes. The nontrivial part of Corollary 3.3.2 is that it holds also for non-amorphous association schemes.

Examples of cyclotomic schemes satisfying the conditions of Theorem 3.3.1 have been investigated thoroughly by Schmidt and White [39], and some of the exceptional examples were already found by Langevin [30]. The smallest example in [39, Table 1] is the cyclotomic scheme of class 11 on $\mathrm{GF}(3^5)$, which gives a unique symmetric 2-$(11, 5, 2)$ design by Theorem 3.3.1. Its associated strongly regular graph is the coset graph of the ternary Golay code (see [8]), which was later recognized as the cyclotomic graph by van Lint and Schrijver [31] in 1981. In this sense, a counterexample to A.V. Ivanov's conjecture [23] could be considered known before the conjecture was announced in 1991. We note that the fusion schemes of this cyclotomic association scheme obtained by Corollary 3.3.2 were already pointed out by Delsarte [16, Example 2 on p.93], in 1973.

There are three more pseudocyclic association schemes satisfying the conditions of Theorem 3.3.1, which are not cyclotomic schemes, but fusions of cyclotomic schemes. They will be given in the next section.

3.4 Projective spaces and fusion schemes

Let q be a prime power, m an integer greater than 1. By $\mathrm{PG}(m,q)$ we mean the symmetric 2-(d,k,λ) design consisting of the points and hyperplanes of the projective space $\mathrm{PG}(m,q)$ of dimension m over $\mathrm{GF}(q)$, where $d = (q^{m+1}-1)/(q-1)$, $k=(q^m-1)/(q-1)$, $\lambda=(q^{m-1}-1)/(q-1)$. Let M be the hyperplane-point incidence matrix of $\mathrm{PG}(m,q)$, and suppose that the columns of M are indexed by the points of $\mathrm{PG}(m,q)$ in such a way that the last $q+1$ columns correspond to the set of points on a line $L=\{\beta_1,\dots,\beta_{q+1}\}$. Consider the following $d\times(q+2)$ matrix

$$F_1 = \begin{pmatrix} \mathbf{1}_{d-q-1} & 0 \\ 0 & I_{q+1} \end{pmatrix},$$

If the rows of M are indexed by λ hyperplanes containing L, $k-\lambda$ hyperplanes which meet L at β_1, $k-\lambda$ hyperplanes which meet L at β_2, and so on, then we have

$$MF_1 = \begin{pmatrix} (k-q-1)\mathbf{1}_\lambda & & J_{\lambda\times(q+1)} & \\ (k-1)\mathbf{1}_{k-\lambda} & \mathbf{1}_{k-\lambda} & & 0 \\ \vdots & & \ddots & \\ (k-1)\mathbf{1}_{k-\lambda} & 0 & & \mathbf{1}_{k-\lambda} \end{pmatrix}, \tag{3.5}$$

A spread of $\mathrm{PG}(3,q)$ is a set of lines which partition the set of points. A spread in $\mathrm{PG}(3,q)$ exists for any prime power q. Let $\mathcal{S}=\{L_1,\dots,L_{q^2+1}\}$ be a spread in $\mathrm{PG}(3,q)$. Let M be the plane-point incidence matrix of $\mathrm{PG}(3,q)$, and suppose that the columns of M are indexed in accordance with the partition $\mathcal{S}$ of the points of $\mathrm{PG}(3,q)$. Consider the following $(q^2+1)(q+1)\times(q^2+1)$ matrix

$$F_2 = \begin{pmatrix} \mathbf{1}_{q+1} & & 0 \\ & \ddots & \\ 0 & & \mathbf{1}_{q+1} \end{pmatrix}.$$

If the rows of M are indexed by $q+1$ planes containing L_1, $q+1$ planes containing L_2, and so on, then we have

$$MF_2 = \begin{pmatrix} (q+1)\mathbf{1}_{q+1} & & \mathbf{1}_{q+1} \\ & \ddots & \\ \mathbf{1}_{q+1} & & (q+1)\mathbf{1}_{q+1} \end{pmatrix}. \tag{3.6}$$

Theorem 3.4.1. *Let $\mathcal{X}$ be an association scheme of class $d = (q^{m+1}-1)/(q-1)$ with the first eigenmatrix*

$$P = \begin{pmatrix} 1 & f\mathbf{1}_d^T \\ \mathbf{1}_d & rM + s(J-M) \end{pmatrix},$$

where M is an incidence matrix of $\mathrm{PG}(m,q)$. Let $k = (q^m-1)/(q-1)$. Then the following statements hold.

(i) *There exists a fusion scheme of class $q+2$ with the first eigenmatrix*

$$\begin{pmatrix} 1 & (d-q-1)f & f\mathbf{1}_{q+1}^T \\ 1 & (k-q-1)r + (d-k)s & r\mathbf{1}_{q+1}^T \\ \mathbf{1}_{q+1} & ((k-1)r + (d-k-q)s)\mathbf{1}_{q+1} & (r-s)I + sJ \end{pmatrix}. \tag{3.7}$$

(ii) *If $m = 3$, then there exists an amorphous fusion scheme of class q^2+1 with the first eigenmatrix*

$$P = \begin{pmatrix} 1 & (q+1)f\mathbf{1}_{q^2+1}^T \\ \mathbf{1}_{q^2+1} & q(r-s)I + (r+sq)J \end{pmatrix}.$$

Proof. (i) We can see easily from (3.5) that the matrix $(rM+s(J-M))F_1$ has $q+2$ distinct rows, which are precisely those of the lower-right $(q+2)\times(q+2)$ submatrix of (3.7). Then the result follows from the Bannai–Muzychuk criterion.

(ii) The proof is similar to (i), noting that the matrix $(rM+s(J-M))F_2$ has q^2+1 distinct rows. □

Example 1. Let α be an arbitrary primitive element of $\mathrm{GF}(2^{12})$, and let

$$H_j = \{(x,y) \mid x-y \in \alpha^j\langle\alpha^{45}\rangle\} \quad (j \in \mathbb{Z}).$$

For a fixed integer a which is relatively prime to 15, we put

$$R_k = \bigcup_{i=0}^{2} H_{a(3(k-1)+5i)}.$$

By computer, we have verified that the graph Γ_k on $\mathrm{GF}(2^{12})$ with edge set R_k is a strongly regular graph with eigenvalues $273, 17, -15$, for each $k \in \{1, \dots, 15\}$. In fact, these graphs are one of the strongly regular graphs discovered by de Lange [29]. Together with the diagonal relation R_0, we obtain a 15-class pseudocyclic association scheme $(\mathrm{GF}(2^{12}), \{R_i\}_{i=0}^{15})$ satisfying the hypothesis of Theorem 3.3.1. The principal part of the first eigenmatrix is a linear combination of the all-ones matrix and an incidence matrix of a symmetric 2-$(15, 7, 3)$ design. Since this matrix is circulant by the definition of R_k, the design is isomorphic to $\mathrm{PG}(3, 2)$ by [18, p. 984].

By Theorem 3.4.1(i), we obtain a 4-class fusion scheme with the first eigenmatrix given by (3.1). By Theorem 3.4.1(ii), we obtain a 5-class pseudocyclic amorphous association scheme. We have verified by computer that this amorphous association scheme is not isomorphic to the amorphous cyclotomic association scheme of class 5 on $\mathrm{GF}(2^{12})$.

Example 2. Let α be an arbitrary primitive element of $\mathrm{GF}(2^{20})$, and let

$$H_j = \{(x, y) \mid x - y \in \alpha^j \langle \alpha^{75} \rangle\} \quad (j \in \mathbb{Z}).$$

For a fixed integer a which is relatively prime to 15, we put

$$R_k = \bigcup_{i=0}^{4} H_{a(5(k-1)+3i)}.$$

By computer, we have verified that the graph Γ_k on $\mathrm{GF}(2^{20})$ with edge set R_k is a strongly regular graph with eigenvalues $69905, 273, -239$, for each $k \in \{1, \dots, 15\}$. Together with the diagonal relation R_0, we obtain a 15-class pseudocyclic association scheme $(\mathrm{GF}(2^{20}), \{R_i\}_{i=0}^{15})$ satisfying the hypothesis of Theorem 3.3.1. The principal part of the first eigenmatrix is a linear combination of the all-ones matrix and an incidence matrix of a 2-$(15, 7, 3)$ design. Since this matrix is circulant by the definition of R_k, the design is isomorphic to $\mathrm{PG}(3, 2)$ by [18, p. 984].

By Theorem 3.4.1(i), we obtain a 4-class fusion scheme with the first eigenmatrix given by (3.2). By Theorem 3.4.1(ii), we obtain a 5-class pseudocyclic amorphous association scheme. We conjecture that the strongly regular graphs in this association scheme are not isomorphic to a cyclotomic strongly regular graph, and hence our amorphous association scheme is new.

Example 3. Let α be an arbitrary primitive element of $\mathrm{GF}(2^{21})$, and let

$$H_j = \{(x, y) \mid x - y \in \alpha^j \langle \alpha^{49} \rangle\} \quad (j \in \mathbb{Z}).$$

For a fixed integer a which is relatively prime to 7, we put

$$R_k = \bigcup_{i=0}^{6} H_{a(7(k-1)+i)}.$$

By computer, we have verified that the graph Γ_k on $\mathrm{GF}(2^{21})$ with edge set R_k is a strongly regular graph with eigenvalues $299593, 585, -439$, for each $k \in \{1, \dots, 7\}$. Together with the diagonal relation R_0, we obtain a 7-class pseudocyclic association scheme $(\mathrm{GF}(2^{21}), \{R_i\}_{i=0}^{7})$ satisfying the hypothesis of Theorem 3.3.1. The principal part of the first eigenmatrix is a linear combination of an incidence matrix of $\mathrm{PG}(2, 2)$ and the all-ones matrix.

By Theorem 3.4.1(i), we obtain a non-amorphous 4-class fusion scheme of the cyclotomic scheme of class 49 on $\mathrm{GF}(2^{21})$ with the following first eigenmatrix:

$$\begin{pmatrix} 1 & 1198372 & 299593 & 299593 & 299593 \\ 1 & -1756 & 585 & 585 & 585 \\ 1 & 292 & -439 & -439 & 585 \\ 1 & 292 & -439 & 585 & -439 \\ 1 & 292 & 585 & -439 & -439 \end{pmatrix}. \tag{3.8}$$

This gives the third counterexample to A.V. Ivanov's conjecture having class 4.

Chapter 4

Spin Models Constructed from Hadamard Matrices

4.1 Introduction

The notion of spin model was introduced by V.F.R. Jones [27] to construct invariants of knots and links. The original definition due to Jones requires that a spin model be a symmetric matrix, but later by K. Kawagoe, A. Munemasa, and Y. Watatani [28], a general definition allowing non-symmetric matrices is given. In this paper, we consider spin models which are not necessarily symmetric.

Let X be a non-empty finite set. We denote by $Mat_X(\mathbb{C}^*)$ the set of square matrices with non-zero complex entries whose rows and columns are indexed by X. For $W \in Mat_X(\mathbb{C}^*)$ and x, $y \in X$, the (x, y)-entry of W is denoted by $W(x, y)$. A spin model $W \in Mat_X(\mathbb{C}^*)$ is defined to be a matrix which satisfies two conditions (type II and type III; see Section 4.2).

One of the examples of spin models is a Potts model, defined as follows. Let X be a finite set with r elements, and let $I, J \in Mat_X(\mathbb{C}^*)$ be the identity matrix and the all 1's matrix, respectively. Let u be a complex number satisfying

$$\begin{aligned} (u^2 + u^{-2})^2 = r \text{ if } r \geq 2, \\ u^4 = 1 \text{ if } r = 1. \end{aligned} \tag{4.1}$$

Then a Potts model A_u is defined as

$$A_u = u^3 I - u^{-1}(J - I).$$

As examples of spin models, we know only Potts models [27, 24], spin models on finite abelian groups [4, 6], Jaeger's Higman-Sims model [24], Hadamard models [36, 26], non-symmetric Hadamard models [26], and tensor products of these. Apart from spin models on finite abelian groups, non-symmetric Hadamard models are essentially the only known family of non-symmetric spin models.

If W is a spin model, then by [26, Proposition 2], $R = W^T W^{-1}$ is a permutation matrix. The order of R as a permutation is called the *index* of the spin model W.

A *Hadamard matrix* of order r is a square matrix H of size r with entries ± 1 satisfying $HH^T = I$. In [26], F. Jaeger and K. Nomura constructed *non-symmetric Hadamard models,* which are spin models of index 2:

$$W = \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes A_u & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \xi H \\ \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \xi H^T & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes A_u \end{pmatrix}, \tag{4.2}$$

where ξ is a primitive 8-th root of unity, $A_u \in Mat_X(\mathbb{C}^*)$ is a Potts model, and $H \in Mat_X(\mathbb{C}^*)$ is a Hadamard matrix.

Note that non-symmetric Hadamard models are a modification of the earlier Hadamard models ([26], see also [26, Section 5]), defined by

$$W' = \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes A_u & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \omega H \\ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \omega H^T & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes A_u \end{pmatrix}, \tag{4.3}$$

where ω is a 4-th root of unity.

To construct spin models of index $m > 2$, it seems natural to consider an $m \times m$ block matrix $W = (W_{i,j})_{i,j \in \mathbb{Z}_m}$ such that each block W_{ij} is the tensor product of two matrices like those in (4.2) and (4.3):

$$W_{ij} = S_{ij} \otimes T_{ij} \qquad (i, j \in \mathbb{Z}_m). \tag{4.4}$$

Such matrices appeared in [21, Proposition 6.2], with the matrices $S_{ij} \in Mat_{\mathbb{Z}_m}(\mathbb{C}^*)$ given by

$$S_{ij}(\ell, \ell') = \eta^{(\ell-\ell')(i-j)} \qquad (\ell,\, \ell' \in \mathbb{Z}_m), \tag{4.5}$$

where η is a primitive m-th root of unity.

In this paper, we construct an infinite class of spin models of even index containing non-symmetric Hadamard models. Also, we construct an infinite class of symmetric spin models containing Hadamard models. Our main result is as follows:

Theorem 4.1.1. *Let r be a positive integer, and let m be an even positive integer. Define $Y = \{1, \ldots, r\}$, $X_i = \{(i, \ell, x) \mid \ell \in \mathbb{Z}_m,\ x \in Y\}$ for $i \in \mathbb{Z}_m$, and $X = X_0 \cup \cdots \cup X_{m-1}$. Let A_u, $H \in \mathrm{Mat}_Y(\mathbb{C}^*)$ be a Potts model and a Hadamard matrix, respectively. Define V_{ij} for $i, j \in \mathbb{Z}_m$ by*

$$V_{ij} = \begin{cases} A_u & \text{if } i - j \text{ is even}, \\ H & \text{if } (i,j) \equiv (0,1) \pmod 2, \\ H^T & \text{if } (i,j) \equiv (1,0) \pmod 2. \end{cases} \tag{4.6}$$

Then the following statements hold:

(i) *Let a be a primitive $2m^2$-th root of unity. Let $W \in \mathrm{Mat}_X(\mathbb{C}^*)$ be the matrix whose (α, β) entry is given by $a^{2m(\ell-\ell')(i-j)+\epsilon(i,j)}V_{ij}(x, y)$ for $\alpha = (i, \ell, x), \beta = (j, \ell', y) \in X$, where $\epsilon(i, j) = (i-j)^2 + m(i-j)$. Then W is a spin model of index m.*

(ii) *Let η be a primitive m-th root of unity, and let b be an m^2-th root of unity. Let $W' \in \mathrm{Mat}_X(\mathbb{C}^*)$ be the matrix whose (α, β) entry is given by $\eta^{(\ell-\ell')(i-j)}b^{\delta(i,j)}V_{ij}$ for $\alpha = (i, \ell, x), \beta = (j, \ell', y) \in X$, where $\delta(i, j) = (i - j)^2$. Then W' is a symmetric spin model.*

Note that, in order for $a^{\epsilon(i,j)}$ and $b^{\delta(i,j)}$ to be well-defined, we need to identify $\mathbb{Z}_m$ with the subset $\{0, 1, \ldots, m-1\}$ of integers.

Remark 4.1.2. In Theorem 4.1.1 (i), if we define S_{ij} by (4.5) with $\eta = a^{2m}$, and T_{ij} by $T_{ij} = a^{\epsilon(i,j)}V_{ij}$, then the (X_i, X_j)-block of the matrix W is given by (4.4). Similarly, in Theorem 4.1.1 (ii), (4.4) holds with $T_{ij} = b^{\delta(i,j)}V_{ij}$.

The spin models W, W' given in Theorem 4.1.1 are determined by a Hadamard matrix H of order r, a complex number u satisfying (4.1), and a primitive $2m^2$-th root of unity a or an m^2-th root of unity b, respectively. Throughout this paper, we denote by $W_{H,u,a}$, $W'_{H,u,b}$ the spin models given by Theorem 4.1.1 (i), (ii), respectively.

Observe that, for any spin models W_i $(i = 1, 2)$ of indices m_i, their tensor product $W_1 \otimes W_2$ is also a spin model of index $\mathrm{LCM}(m_1, m_2)$. In Section 4.5, we show that the non-symmetric spin model $W_{H,u,a}$ whose index is a power of 2 is new in the following sense:

Theorem 4.1.3. *Let H be a Hadamard matrix of order r. Let $W_{H,u,a}$ be a spin model given in Theorem 4.1.1 (i), whose index m is a power of 2. If $r > 4$, then $W_{H,u,a}$ cannot be decomposed into a tensor product of known spin models.*

We note that the list of known spin models is given in Section 5. Jaeger and Nomura [26, p.278] expected that new non-symmetric spin models of index a power of 2 should be found, and our results confirm this expectation.

4.2 Type II and Type III conditions on block matrices of tensor products

First we define a spin model. A *type II matrix* on a finite set X is a matrix $W \in Mat_X(\mathbb{C}^*)$ which satisfies the *type II condition*:

$$\sum_{x \in X} \frac{W(\alpha, x)}{W(\beta, x)} = n\delta_{\alpha,\beta} \qquad \text{(for all } \alpha, \beta \in X). \tag{4.7}$$

Let $W^- \in Mat_X(\mathbb{C}^*)$ be defined by $W^-(x, y) = W(y, x)^{-1}$. Then the type II condition is written as $WW^- = nI$. Hence, if W is a type II matrix, then W is non-singular with $W^{-1} = n^{-1}W^-$.

A type II matrix $W \in Mat_X(\mathbb{C}^*)$ is called a *spin model* if W satisfies the *type III condition*:

$$\sum_{x \in X} \frac{W(\alpha, x)W(\beta, x)}{W(\gamma, x)} = D\frac{W(\alpha, \beta)}{W(\alpha, \gamma)W(\gamma, \beta)} \qquad \text{(for all } \alpha, \beta, \gamma \in X) \tag{4.8}$$

for some nonzero real number D with $D^2 = n$, which is independent of the choice of $\alpha, \beta, \gamma \in X$.

Let m be a positive integer. In this section, assuming that W is an $m \times m$ block matrix with blocks of the form (4.4), we will establish conditions on T_{ij} under which W satisfies the type II and type III conditions. Some parts of these conditions are already given in [21, Proposition 5.1, Proposition 6.2].

Let η be a primitive m-th root of unity, and let S_{ij} be the matrix of size m defined by (4.5) for $i, j \in \mathbb{Z}_m$. Let r be a positive integer, and define $Y = \{1, \ldots, r\}$, $X_i = \{(i, \ell, x) \mid \ell \in \mathbb{Z}_m,\ x \in Y\}$ for $i \in \mathbb{Z}_m$, and $X = X_0 \cup \cdots \cup X_{m-1}$. Let $T_{ij} \in Mat_Y(\mathbb{C}^*)$ be a matrix for $i, j \in \mathbb{Z}_m$, and let W_{ij} be the matrix defined by (4.4). Let $W \in Mat_X(\mathbb{C}^*)$ be the matrix whose (X_i, X_j)-block is W_{ij} for $i, j \in \mathbb{Z}_m$. Then

$$W((i, \ell, x), (j, \ell', y)) = S_{ij}(\ell, \ell')T_{ij}(x, y). \tag{4.9}$$

Lemma 4.2.1 ([21, Proposition 5.1])**.** *The matrix W is a type II matrix if and only if T_{ij} is a type II matrix for all $i, j \in \mathbb{Z}_m$.*

Lemma 4.2.2. *The matrix W satisfies the type III condition* (4.8) *if and only if the following equality holds for all $i_1, i_2, i_3 \in \mathbb{Z}_m$ and $x_1, x_2, x_3 \in Y$:*

$$\sum_{x \in Y} \frac{T_{i_1,i_0}(x_1, x)T_{i_2,i_0}(x_2, x)}{T_{i_3,i_0}(x_3, x)} = \frac{D}{m} \cdot \frac{T_{i_1,i_2}(x_1, x_2)}{T_{i_1,i_3}(x_1, x_3)T_{i_3,i_2}(x_3, x_2)}, \tag{4.10}$$

where $i_0 = i_1 + i_2 - i_3$ mod *m.*

Proof. The type III condition (4.8) for $\alpha = (i_1, \ell_1, x_1)$, $\beta = (i_2, \ell_2, x_2)$, $\gamma = (i_3, \ell_3, x_3)$ is equivalent to

$$\sum_{i,\ell \in \mathbb{Z}_m} \frac{\eta^{(\ell_1-\ell)(i_1-i)}\eta^{(\ell_2-\ell)(i_2-i)}}{\eta^{(\ell_3-\ell)(i_3-i)}} \sum_{x\in Y} \frac{T_{i_1,i}(x_1,x)T_{i_2,i}(x_2,x)}{T_{i_3,i}(x_3,x)}$$
$$= D\frac{\eta^{(\ell_1-\ell_2)(i_1-i_2)}}{\eta^{(\ell_1-\ell_3)(i_1-i_3)}\eta^{(\ell_3-\ell_2)(i_3-i_2)}} \cdot \frac{T_{i_1,i_2}(x_1,x_2)}{T_{i_1,i_3}(x_1,x_3)T_{i_3,i_2}(x_3,x_2)}.$$

By a direct computation, we obtain

$$\frac{\eta^{(\ell_1-\ell)(i_1-i)}\eta^{(\ell_2-\ell)(i_2-i)}}{\eta^{-(\ell_3-\ell)(i_3-i)}} \cdot \frac{\eta^{(\ell_1-\ell_3)(i_1-i_3)}\eta^{(\ell_3-\ell_2)(i_3-i_2)}}{\eta^{(\ell_1-\ell_2)(i_1-i_2)}}$$
$$= \eta^{(\ell_1+\ell_2-\ell_3-\ell)(i_1+i_2-i_3-i)}.$$

So (4.8) is equivalent to

$$\sum_{i\in\mathbb{Z}_m}\Big(\sum_{\ell\in\mathbb{Z}_m} \eta^{(\ell_1+\ell_2-\ell_3-\ell)(i_1+i_2-i_3-i)}\Big) \sum_{x\in Y} \frac{T_{i_1,i}(x_1,x)T_{i_2,i}(x_2,x)}{T_{i_3,i}(x_3,x)}$$
$$= D\frac{T_{i_1,i_2}(x_1,x_2)}{T_{i_1,i_3}(x_1,x_3)T_{i_3,i_2}(x_3,x_2)}. \tag{4.11}$$

Since η is a primitive m-th root of unity and $i_0 = i_1 + i_2 - i_3 \bmod m$, we have

$$\sum_{\ell\in\mathbb{Z}_m} \eta^{(\ell_1+\ell_2-\ell_3-\ell)(i_1+i_2-i_3-i)} = m\delta_{i,i_0}.$$

Thus (4.11) is equivalent to (4.10). □

We remark that in [21, Proposition 6.2] only the necessity of (4.10) for the type III condition is proved.

Let z_m be the permutation matrix of order m:

$$z_m = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}.$$

We define the permutation matrix R of size $n = m^2 r$ by $R = I_m \otimes z_m \otimes I_r$, where I_m and I_r are the identity matrices of size m and r, respectively. The order of R is m.

Lemma 4.2.3. *The matrix W satisfies $W^T W^{-1} = R$ if and only if $T_{ij} = \eta^{i-j} T_{ji}^T$ holds for all $i, j \in \mathbb{Z}_m$.*

Proof. For $\alpha = (i, \ell, x)$ and $\beta = (j, \ell', y) \in X$,

$$\begin{aligned} W^T(\alpha, \beta) &= W(\beta, \alpha) \\ &= \eta^{(\ell'-\ell)(j-i)} T_{j,i}(y, x), \\ (RW)(\alpha, \beta) &= ((I_m \otimes z_m \otimes I_r)W)((i, \ell, x), (j, \ell', y)) \\ &= W((i, \ell - 1, x), (j, \ell', y)) \\ &= \eta^{(\ell-1-\ell')(i-j)} T_{ij}(x, y) \\ &= \eta^{(\ell'-\ell)(j-i)} \eta^{-(i-j)} T_{ij}(x, y). \end{aligned}$$

Therefore $R = W^T W^{-1}$ if and only if $T_{ji}(y, x) = \eta^{-(i-j)} T_{ij}(x, y)$ holds for all $i, j \in \mathbb{Z}_m$ and $x, y \in Y$. □

4.3 Proof of Theorem 4.1.1

From Remark 4.1.2, the results in Section 2 can be used for the matrices W and W' given in Theorem 4.1.1, if we define T_{ij} according to Remark 4.1.2.

For a mapping g from $\mathbb{Z}^2$ to $\mathbb{Z}$, we denote by λ_g the mapping from $\mathbb{Z}^4$ to $\mathbb{Z}$ defined by

$$\lambda_g(i_1, i_2, i_3, i_4) = g(i_1, i_4) + g(i_2, i_4) - g(i_3, i_4) + g(i_1, i_3) + g(i_3, i_2) - g(i_1, i_2). \tag{4.12}$$

Recall that we regard $\mathbb{Z}_m$ as the subset $\{0, 1, \ldots, m-1\}$ of $\mathbb{Z}$, and $\delta, \epsilon : \mathbb{Z}^2 \to \mathbb{Z}$ are defined by $\delta(i, j) = (i - j)^2$, $\epsilon(i, j) = \delta(i, j) + m(i - j)$, respectively.

Lemma 4.3.1. *For all $i_1, i_2, i_3, i_4 \in \mathbb{Z}$, we have*

$$\begin{aligned} \lambda_\delta(i_1, i_2, i_3, i_4) &= (i_1 + i_2 - i_3 - i_4)^2, \\ \lambda_\epsilon(i_1, i_2, i_3, i_4) &= (i_1 + i_2 - i_3 - i_4)(i_1 + i_2 - i_3 - i_4 + m). \end{aligned}$$

In particular, if $i_0 = i_1 + i_2 - i_3 \pmod{m}$, then

$$\begin{aligned} \lambda_\delta(i_1, i_2, i_3, i_0) &\equiv 0 \pmod{m^2}, \\ \lambda_\epsilon(i_1, i_2, i_3, i_0) &\equiv 0 \pmod{2m^2}. \end{aligned}$$

Proof. Straightforward. □

In [26, §5.1], the following is used to construct non-symmetric or symmetric Hadamard models:

Lemma 4.3.2 ([26, §5.1]). *Let A_u, $H \in \mathrm{Mat}_Y(\mathbb{C}^*)$ be a Potts model and a Hadamard matrix, respectively. Then the following holds for all $x_1, x_2, x_3 \in Y$:*

$$\sum_{y\in Y} \frac{A_u(x_1,y)A_u(x_2,y)}{A_u(x_3,y)} = D_u \frac{A_u(x_1,x_2)}{A_u(x_1,x_3)A_u(x_3,x_2)}, \tag{4.13}$$

$$\sum_{y\in Y} A_u(x_1,y)H(y,x_2)H(y,x_3) = D_u \frac{H(x_1,x_2)H(x_1,x_3)}{A_u(x_2,x_3)}, \tag{4.14}$$

$$\sum_{y\in Y} A_u(x_1,y)H(x_2,y)H(x_3,y) = D_u \frac{H(x_2,x_1)H(x_3,x_1)}{A_u(x_2,x_3)}, \tag{4.15}$$

$$\sum_{y\in Y} \frac{H(y,x_1)H(y,x_2)}{A_u(x_3,y)} = D_u A_u(x_1,x_2)H(x_3,x_1)H(x_3,x_2), \tag{4.16}$$

$$\sum_{y\in Y} \frac{H(x_1,y)H(x_2,y)}{A_u(x_3,y)} = D_u A_u(x_1,x_2)H(x_1,x_3)H(x_2,x_3), \tag{4.17}$$

where

$$D_u = \begin{cases} -u^2 - u^{-2} & \text{if } |Y| \geq 2, \\ u^2 & \text{if } |Y| = 1. \end{cases}$$

We now prove Theorem 4.1.1. Since A_u and H are type II matrices, so are the matrices $T_{ij} = a^{\epsilon(i,j)}V_{ij}$ or $b^{\delta(i,j)}V_{ij}$. Thus, Lemma 4.2.1 implies that $W_{H,u,a}$ and $W'_{H,u,b}$ are type II matrices.

We claim

$$\sum_{y\in Y} \frac{V_{i_1,i_0}(x_1,y)V_{i_2,i_0}(x_2,y)}{V_{i_3,i_0}(x_3,y)} = D_u \frac{V_{i_1,i_2}(x_1,x_2)}{V_{i_1,i_3}(x_1,x_3)V_{i_3,i_2}(x_3,x_2)} \tag{4.18}$$

for all $i_1, i_2, i_3 \in \mathbb{Z}_m$ and $x_1, x_2, x_3 \in Y$, where $i_0 = i_1 + i_2 - i_3 \bmod m$. Indeed, let $i_1, i_2, i_3 \in \mathbb{Z}_m$. Then

$$(4.18) \iff \begin{cases} (4.13) & \text{if} \quad (i_1,i_2,i_3) \equiv (0,0,0),(1,1,1) \pmod 2, \\ (4.14) & \text{if} \quad (i_1,i_2,i_3) \equiv (0,1,1) \pmod 2, \\ (4.15) & \text{if} \quad (i_1,i_2,i_3) \equiv (1,0,0) \pmod 2, \\ (4.16) & \text{if} \quad (i_1,i_2,i_3) \equiv (1,1,0) \pmod 2, \\ (4.17) & \text{if} \quad (i_1,i_2,i_3) \equiv (0,0,1) \pmod 2. \end{cases}$$

Moreover, when $(i_1,i_2,i_3) \equiv (1,0,1),(0,1,0) \pmod 2$, (4.18) is equivalent to (4.14), (4.15), respectively, with x_1 and x_2 switched. Therefore, (4.18) holds in all cases by Lemma 4.3.2.

First, we show that $W_{H,u,a}$ and $W'_{H,u,b}$ satisfy the condition (4.10). From Lemma 4.3.1 we have

$$a^{\lambda_\epsilon(i_1,i_2,i_3,i_0)} = 1 \quad \text{and} \quad b^{\lambda_\delta(i_1,i_2,i_3,i_0)} = 1.$$

In view of (4.12), these imply

$$c^{g(i_1,i_0)+g(i_2,i_0)-g(i_3,i_0)} = c^{g(i_1,i_2)-g(i_1,i_3)-g(i_3,i_2)}, \tag{4.19}$$

where $(c, g) = (a, \epsilon), (b, \delta)$. Combining (4.18) and (4.19), we obtain

$$\sum_{y\in Y} \frac{c^{g(i_1,i_0)}V_{i_1,i_0}(x_1, y)c^{g(i_2,i_0)}V_{i_2,i_0}(x_2, y)}{c^{g(i_3,i_0)}V_{i_3,i_0}(x_3, y)}$$
$$= D_u \frac{c^{g(i_1,i_2)}V_{i_1,i_2}(x_1, x_2)}{c^{g(i_1,i_3)}V_{i_1,i_3}(x_1, x_3)c^{g(i_3,i_2)}V_{i_3,i_2}(x_3, x_2)}.$$

for all $i_1, i_2, i_3 \in \mathbb{Z}_m$ and $x_1, x_2, x_3 \in Y$. Thus (4.10) holds by setting $D = mD_u$. It follows from Lemma 4.2.2 that $W_{H,u,a}$ and $W'_{H,u,b}$ satisfy the type III condition (4.8), and hence they are spin models. Since $\delta(i, j) = \delta(j, i)$, $W'_{H,u,b}$ is symmetric.

Finally, we show that $W_{H,u,a}$ has index m. Since $a^{2m} = \eta$, we have $a^{\epsilon(i,j)-\epsilon(j,i)} = a^{2m(i-j)} = \eta^{i-j}$. So, $T_{ij} = \eta^{i-j}T_{ji}^T$ holds for all $i, j \in \mathbb{Z}_m$. From Lemma 4.2.3, $W_{H,u,a}$ has index m. This completes the proof of Theorem 4.1.1.

4.4 Properties of spin models in Theorem 4.1.1

For a positive integer r, we let u be a complex number satisfying (4.1).

Lemma 4.4.1. *If $r \leq 4$, then u is a root of unity. Otherwise, $|u| \neq 1$. If $r \geq 4$ or $r = 1$, then $u^4 > 0$.*

Proof. If u is a root of unity and $r > 1$, then $r = (u^2 + u^{-2})^2 \leq |u|^4 + 2 + |u|^{-4} = 4$. It is easy to see that u is indeed a root of unity if $r \leq 4$. If $r \geq 4$ or $r = 1$, then we have $u^4 > 0$ from (4.1). □

For a matrix $W \in Mat_X(\mathbb{C}^*)$, we define

$$E(W) = \{\frac{|W(x, y)|}{|W(x, x)|} \mid x, y \in X\} \subset \mathbb{R}_{>0}.$$

Then

$$E(W_1 \otimes W_2) = E(W_1)E(W_2) \tag{4.20}$$

holds for any matrices W_1, W_2 with nonzero entries.

For the remainder of this section, let $W_{H,u,a}$, $W'_{H,u,b}$ be the spin models given in Theorem 4.1.1(i) and (ii), respectively. This means that m is an even positive integer, a is a primitive $2m^2$-th root of unity, b is an m^2-th root of unity, and H is a Hadamard matrix of order r.

Lemma 4.4.2. *We have*

$$E(W_{H,u,a}) = E(W'_{H,u,b}) = \begin{cases} \{1, |u|^{-4}, |u|^{-3}\} & \text{if } r > 4, \\ \{1\} & \text{otherwise.} \end{cases}$$

Proof. Immediate from Theorem 4.1.1 and Lemma 4.4.1. □

Lemma 4.4.3. (i) *Suppose $r \geq 4$ or $r = 1$. Then the entries of $W_{H,u,a}$, $W'_{H,u,b}$ which have absolute value 1 are $2m^2$-th roots of unity, m^2-th roots of unity, respectively. Moreover, $W_{H,u,a}$ contains a primitive $2m^2$-th root of unity as one of its entries.*

(ii) *Suppose $r = 2$, and put $\nu = \mathrm{LCM}(2m^2, 16)$, $\nu' = \mathrm{LCM}(m^2, 16)$. Then the entries of $W_{H,u,a}$, $W'_{H,u,b}$ are ν-th roots of unity, ν'-th roots of unity, respectively. Moreover, $W_{H,u,a}$ contains a primitive ν-th root of unity as one of its entries.*

Proof. Firstly, suppose $r > 4$. From Lemma 4.4.1, the entries of $W_{H,u,a}$, $W'_{H,u,b}$ with absolute value 1 are

$$\begin{aligned} &\pm a^{2m(\ell-\ell')(i-j)+\epsilon(i,j)} && (i-j : \text{odd}), \\ &\pm \eta^{(\ell-\ell')(i-j)} b^{\delta(i,j)} && (i-j : \text{odd}), \end{aligned} \tag{4.21}$$

which are $2m^2$-th roots of unity, m^2-th roots of unity, respectively. Putting $i = 1$, $j = \ell = \ell' = 0$ in (4.21), we obtain a^{1+m} which is a primitive $2m^2$-th root of unity.

Next, suppose $r \leq 4$. Then the entries of $W_{H,u,a}$, $W'_{H,u,b}$ are given by

$$v a^{2m(\ell-\ell')(i-j)+\epsilon(i,j)} \qquad (v \in \{u^3, -u^{-1}, \pm 1\}), \tag{4.22}$$

$$v \eta^{(\ell-\ell')(i-j)} b^{\delta(i,j)} \qquad (v \in \{u^3, -u^{-1}, \pm 1\}), \tag{4.23}$$

respectively, all of which are roots of unity.

If $r = 4$ or 1, then from (4.1), $u^4 = 1$. From (4.22), (4.23), the entries of $W_{H,u,a}$, $W'_{H,u,b}$ are $2m^2$-th roots of unity, m^2-th roots of unity, respectively. Putting $i = 1$, $j = \ell = \ell' = 0$ in (4.22), we obtain a^{1+m} which is a primitive $2m^2$-th root of unity.

W	index	size	r	$\mu(W)$	$E(W)$
$W_{H,u,a}$	m	m^2r	$r=1$	$2m^2$	$\{1\}$
			$r=2$	$\mu(W)\|\operatorname{LCM}(2m^2,16)$	$\{1\}$
			$r=4$	$2m^2$	$\{1\}$
			$r>4$	$2m^2$	$\{1,\|u\|^{-4},\|u\|^{-3}\}$
$W'_{H,u,b}$	1	m^2r	$r=1$	$\mu(W)\|m^2$	$\{1\}$
			$r=2$	$\mu(W)\|\operatorname{LCM}(m^2,16)$	$\{1\}$
			$r=4$	$\mu(W)\|m^2$	$\{1\}$
			$r>4$	$\mu(W)\|m^2$	$\{1,\|u\|^{-4},\|u\|^{-3}\}$

Table 4.1: Summary of Properties

Finally, suppose $r=2$. Since u is a primitive 16-root of unity by (4.1), the expressions in (4.22), (4.23) are ν-th roots of unity, an ν'-th roots of unity, respectively. Putting $v=u^3$, $i=1$, $j=\ell=\ell'=0$ in (4.22), we obtain u^3a^{1+m} which is a primitive ν-th root of unity. □

For $S\in Mat_X(\mathbb{C}^*)$, we denote by $\mu(S)$ the least common multiple of the orders of the entries of S which have a finite order. If none of the entries of S has a finite order, then we define $\mu(S)=\infty$. For a nonzero complex number ζ, we denote by the same symbol $\mu(\zeta)$ the order of ζ if ζ has a finite order.

Lemma 4.4.4. *Suppose* $m\equiv 0 \pmod 4$. *Then for* $W=W_{H,u,a}$ *or* $W=W'_{H,u,b}$, *we have* $\mu(W)\mid 2m^2$.

Proof. Immediate from Lemma 4.4.3. □

In Table 4.1, we summarize the properties of $W=W_{H,u,a}$, $W'_{H,u,b}$ obtained from Lemmas 4.4.1, 4.4.2, and 4.4.4.

For $W\in Mat_X(\mathbb{C}^*)$ and for a permutation σ of X, we define W^σ by $W^\sigma(\alpha,\beta)=W(\sigma(\alpha),\sigma(\beta))$ for $\alpha,\beta\in X$. Observe that if W is a spin model, then W^σ is also a spin model. If W is a spin model, then from (4.7), (4.8), $-W$ and $\pm\sqrt{-1}W$ are also spin models. Two spin models W_1, W_2 are said to be *equivalent* if $cW_1^\sigma=W_2$ for some permutation σ of X and a complex number c with $c^4=1$.

Two Hadamard matrices are said to be *equivalent* if one can be obtained from the other by negating rows and columns, or and permuting rows and columns.

Lemma 4.4.5. *Let* H_1, $H_2\in Mat_Y(\mathbb{C}^*)$ *be equivalent Hadamard matrices. Then* $W_{H_1,u,a}$ *is equivalent to* $W_{H_2,u,a}$, *and* $W'_{H_1,u,b}$ *is equivalent to* $W'_{H_2,u,b}$.

Proof. Let $(W_1, W_2, c, g) = (W_{H_1,u,a}, W_{H_2,u,a}, a, \epsilon)$ or $(W'_{H_1,u,b}, W'_{H_2,u,b}, b, \delta)$.

If H_2 is obtained by a permutation of columns of H_1, then there exists a permutation π of Y such that $H_2(x, \pi(y)) = H_1(x, y)$ for all $x, y \in Y$. We define a permutation σ of X by

$$\sigma((i, \ell, x)) = \begin{cases} (i, \ell, \pi(x)) & \text{if } i \text{ is odd,} \\ (i, \ell, x) & \text{otherwise.} \end{cases}$$

Then for $\alpha = (i, \ell, x), \beta = (j, \ell', y) \in X$,

$$\begin{aligned} W_2^\sigma(\alpha, \beta) &= W_2(\sigma(\alpha), \sigma(\beta)) \\ &= \begin{cases} c^{g(i,j)} S_{ij}(\ell, \ell') A_u(x, y) & \text{if } i \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell, \ell') H_2(x, \pi(y)) & \text{if } i \equiv j + 1 \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell, \ell') H_2^T(\pi(x), y) & \text{if } i + 1 \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell, \ell') A_u(\pi(x), \pi(y)) & \text{if } i \equiv j \equiv 1 \pmod 2 \end{cases} \\ &= \begin{cases} c^{g(i,j)} S_{ij}(\ell, \ell') A_u(x, y) & \text{if } i \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell, \ell') H_1(x, y) & \text{if } i \equiv j + 1 \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell, \ell') H_1^T(x, y) & \text{if } i + 1 \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell, \ell') A_u(x, y) & \text{if } i \equiv j \equiv 1 \pmod 2 \end{cases} \\ &= W_1(\alpha, \beta). \end{aligned}$$

If H_2 is obtained by a permutation of rows of H_1, then there exists a permutation π' of Y such that $H_2(\pi'(x), y) = H_1(x, y)$ for all $x, y \in Y$. We define a permutation σ' of X by

$$\sigma'((i, \ell, x)) = \begin{cases} (i, \ell, \pi'(x)) & \text{if } i \text{ is even,} \\ (i, \ell, x) & \text{otherwise.} \end{cases}$$

Similar calculation shows $W_2^{\sigma'}(\alpha, \beta) = W_1(\alpha, \beta)$.

If H_2 is obtained by negating a column y_1 of H_1, then $H_2(x, y_1) = -H_1(x, y_1)$, $H_2(x, y) = H_1(x, y)$ for all $x \in Y$ and $y \in Y - \{y_1\}$. We define a permutation ρ of X by

$$\rho((i, \ell, x)) = \begin{cases} (i, \ell + \delta_{x,y_1} \frac{m}{2}, x) & \text{if } i \text{ is odd,} \\ (i, \ell, x) & \text{otherwise.} \end{cases}$$

Note that $S_{ij}(\ell, \ell') = (-1)^{i-j} S_{ij}(\ell + \frac{m}{2}, \ell') = (-1)^{i-j} S_{ij}(\ell, \ell' + \frac{m}{2})$. Thus for $\alpha = (i, \ell, x), \beta = (j, \ell', y) \in X$,

$$\begin{aligned} &W_2^\rho(\alpha, \beta) \\ &= W_2(\rho(\alpha), \rho(\beta)) \end{aligned}$$

$$
= \begin{cases} c^{g(i,j)} S_{ij}(\ell, \ell') A_u(x, y) & \text{if } i \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell, \ell' + \delta_{y,y_1} \frac{m}{2}) H_2(x, y) & \text{if } i \equiv j + 1 \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell + \delta_{x,y_1} \frac{m}{2}, \ell') H_2^T(x, y) & \text{if } i + 1 \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell + \delta_{x,y_1} \frac{m}{2}, \ell' + \delta_{y,y_1} \frac{m}{2}) A_u(x, y) & \text{if } i \equiv j \equiv 1 \pmod 2 \end{cases}
$$

$$
= \begin{cases} c^{g(i,j)} S_{ij}(\ell, \ell') A_u(x, y) & \text{if } i \equiv j \equiv 0 \pmod 2, \\ (-1)^{\delta_{y,y_1}} c^{g(i,j)} S_{ij}(\ell, \ell') H_2(x, y) & \text{if } i \equiv j + 1 \equiv 0 \pmod 2, \\ (-1)^{\delta_{x,y_1}} c^{g(i,j)} S_{ij}(\ell, \ell') H_2^T(x, y) & \text{if } i + 1 \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell, \ell') A_u(x, y) & \text{if } i \equiv j \equiv 1 \pmod 2 \end{cases}
$$

$$
= \begin{cases} c^{g(i,j)} S_{ij}(\ell, \ell') A_u(x, y) & \text{if } i \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell, \ell') H_1(x, y) & \text{if } i \equiv j + 1 \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell, \ell') H_1^T(x, y) & \text{if } i + 1 \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell, \ell') A_u(x, y) & \text{if } i \equiv j \equiv 1 \pmod 2 \end{cases}
$$

$$
= W_1(\alpha, \beta).
$$

If H_2 is obtained by negating a row x_1 of H_1, then $H_2(x_1, y) = -H_1(x_1, y)$, $H_2(x, y) = H_1(x, y)$ for all $x \in Y - \{x_1\}$ and $y \in Y$. We define a permutation ρ' of X by

$$
\rho'((i, \ell, x)) = \begin{cases} (i, \ell + \delta_{x,x_1} \frac{m}{2}, x) & \text{if } i \text{ is even}, \\ (i, \ell, x) & \text{otherwise}. \end{cases}
$$

Similar calculation shows $W_2^{\rho'}(\alpha, \beta) = W_1(\alpha, \beta)$. □

4.5 Decomposability

Lemma 4.5.1. *Let S_1, S_2 be finite subsets of positive real numbers. Suppose $1 \in S_1 \cap S_2$ and $|S_1 S_2| = 3$. Then*

$$
(|S_1|, |S_2|) \in \{(2, 2), (1, 3), (3, 1)\}.
$$

If $|S_1| = |S_2| = 2$, then $S_1 S_2 = \{1, a, a^2\}$ or $\{1, a, a^{-1}\}$ for some positive real number $a \neq 1$.

Proof. By way of contradiction, we prove that if $|S_1| \geq 3$ and $|S_2| \geq 2$ then $|S_1 S_2| > 3$. Since $S_1 \cup S_2 \subset S_1 S_2$, we obtain $S_2 \subset S_1 = S_1 S_2$. Let $S_1 = \{1, \lambda, \mu\}$ $(\lambda, \mu \neq 1, \lambda \neq \mu)$. Then we may put $S_2 = \{1, \lambda\}$ without loss of generality. Then we have $\lambda^2 \in S_1 S_2 = S_1$, so $\mu = \lambda^2$ and $S_1 S_2 = \{1, \lambda, \lambda^2, \lambda^3\}$. This implies $|S_1 S_2| = 4$, a contradiction.

Suppose $|S_1| = |S_2| = 2$. Then $S_1 = \{1, a\}$, $S_2 = \{1, b\}$ for some $a, b \neq 1$. Then $|S_1 S_2| = 3$ implies $a = b$ or $a = b^{-1}$. □

Lemma 4.5.2. *Let $A \in Mat_{Z_1}(\mathbb{C}^*)$ be a matrix all of whose entries are roots of unity. Let $B \in Mat_{Z_2}(\mathbb{C}^*)$ be a matrix which satisfies $\mu(B) < \infty$. Then $\mu(A \otimes B)$ is a divisor of* $\mathrm{LCM}(\mu(A), \mu(B))$.

Proof. Let $Z_2' = \{(x_2, y_2) \in Z_2 \times Z_2 \mid o(B(x_2, y_2)) < \infty\}$. Then

$$\mu(A \otimes B) = \mathrm{LCM}(\{o(A(x_1, y_1)B(x_2, y_2)) \mid x_1, y_1 \in Z_1, (x_2, y_2) \in Z_2'\}),$$

which is a divisor of $\mathrm{LCM}(\mu(A), \mu(B))$. □

Some examples of spin models are listed in Section 1, i.e., Potts model, non-symmetric Hadamard models, and Hadamard models. We remark that non-symmetric Hadamard models and Hadamard models are special cases of spin models given in Theorem 4.1.1 (i), (ii), respectively. In addition to these examples, the following spin models are known.

Spin models on finite abelian groups. Bannai-Bannai-Jaeger [4] gives solutions to modular invariance equation for finite abelian groups, and every solution gives a spin model. Let U be a finite abelian group, and $e = \exp(U)$ denote the exponent of U. Let $\{\chi_a \mid a \in U\}$ be the set of characters of U with indices chosen so that $\chi_a(b) = \chi_b(a)$ for all $a, b \in U$. Let $U = U_1 \oplus \cdots \oplus U_h$ be a decomposition of U into a direct sum of cyclic groups $U_1, U_2, \ldots, U_h$. For each $i \in \{1, 2, \ldots, h\}$ let a_i be a generator and n_i be the order of the cyclic group U_i. For each $x \in U$, we define the matrix $A_x \in Mat_U(\mathbb{C})$ by

$$A_x(\alpha, \beta) = \delta_{x, \beta - \alpha} \quad (\alpha, \beta \in U).$$

For any $x = \sum_{i=1}^{h} x_i a_i$ $(0 \leq x_i < n_i)$, let

$$t_x = t_0 \prod_{i=1}^{h} \eta_i^{x_i} \chi_{a_i}(a_i)^{\frac{x_i(x_i-1)}{2}} \prod_{1 \leq \ell < k \leq h} \chi_{a_\ell}(a_k)^{x_\ell x_k}, \tag{4.24}$$

where $\eta_i^{n_i} = \chi_{a_i}(a_i)^{-\frac{n_i(n_i-1)}{2}}$ and

$$t_0^2 = D^{-1} \sum_{x \in U} \prod_{j=1}^{h} \eta_j^{-x_j} \chi_{a_j}(a_j)^{-\frac{x_j(x_j-1)}{2}} \prod_{1 \leq \ell < k \leq h} \chi_{a_\ell}(a_k)^{-x_\ell x_k}, \tag{4.25}$$

where $D^2 = |U|$. Let $\theta_x = t_x / t_0$ for any $x \in U$. Then, for any $x \in U$, θ_x is a root of unity and $\theta_x^{2e} = 1$. Especially, we get

$$\theta_x^{2|U|} = 1. \tag{4.26}$$

The matrix

$$W = \sum_{x \in U} t_x A_x. \tag{4.27}$$

is a spin model.

Jaeger's Higman-Sims model. In [24], F. Jaeger constructed a spin model W_J on the Higman-Sims graph of size 100. We denote by A the adjacency matrix of the Higman-Sims graph. We put $W_J = -\tau^5 I - \tau A + \tau^{-1}(J - A - I)$, where τ satisfies $\tau^2 + \tau^{-2} = 3$. Then W_J is a symmetric spin model.

Now every known spin model belongs to one of the following five families:

(a) A_u: Potts model of size $r \geq 2$. If $r = 2$, then $\mu(A_u) = 16$. If $r = 4$, then $\mu(A_u) = 2$ or 4. If $r = 2, 4$, then $E(A_u) = \{1\}$. If $r > 4$, then $E(A_u) = \{1, |u|^{-4}\}$, and hence $|E(A_u)| = 2$.

(b) W_U: spin model on a finite abelian group U. We have various kinds of indices and $E(W_U) = \{1\}$.

(c) W_J: Jaeger's Higman-Sims model of size 100. We have $E(W_J) = \{1, \tau^{-4}, \tau^{-6}\}$ with $\tau^2 + \tau^{-2} = 3$. and hence $|E(W_J)| = 3$.

(d) $W_{H,u,a}$: spin models given in Theorem 4.1.1(i).

(e) $W'_{H,u,b}$: spin models given in Theorem 4.1.1(ii).

By way of contradiction, we now give a proof of Theorem 4.1.3. Let H be a Hadamard matrix of order $r > 4$. Let s be a positive integer and a a primitive 2^{2s+1}-th root of unity. For the remainder of this section, we denote by W the spin model $W_{H,u,a}$ given in Theorem 4.1.1 (i) of index 2^s. By Lemma 4.4.2 we obtain

$$E(W) = \{1, |u|^{-4}, |u|^{-3}\}. \tag{4.28}$$

We assume that

$$W = W_1 \otimes W_2 \otimes \cdots \otimes W_v, \tag{4.29}$$

where each of $W_1, W_2, \ldots, W_v$ is a known spin model listed in (a)–(e) and their sizes are not equal to 1. Since $|E(W)| = 3$ from (4.28), using Lemma 4.5.1 we may assume without loss of generality

$$(|E(W_1)|, |E(W_2)|, \ldots, |E(W_v)|) = (1, \ldots, 1, 2, 2) \text{ or } (1, \ldots, 1, 3).$$

A known spin model W' with $|E(W')| = 1$ belongs to the family (b) or to the families (a), (d) and (e) with $r \leq 4$. Therefore, (4.29) can be reduced to the following cases:

$$W = W_1 \otimes W_2 \otimes W_3 \text{ with } E(W_1) = \{1\}, |E(W_2)| = |E(W_3)| = 2, \tag{4.30}$$
$$W = W_1 \otimes W_2 \text{ with } E(W_1) = \{1\}, |E(W_2)| = 3, \tag{4.31}$$

where in (4.30), (4.31), W_1 is a tensor product of spin models on finite abelian groups and spin models in the families (a), (d) and (e) with $r \leq 4$. Note that W_1 could possibly be of size 1 in (4.30).

First, we treat the case (4.30). Then Lemma 4.5.1 implies $E(W_2 \otimes W_3) = \{1, \beta, \beta^2\}$, or $\{1, \beta, \beta^{-1}\}$ for some β. On the other hand, $E(W_2 \otimes W_3) = E(W_1)E(W_2 \otimes W_3) = E(W) = \{1, |u|^{-4}, |u|^{-3}\}$ by (4.28). This is a contradiction.

Next, we treat the case (4.31). We have $E(W_2) = E(W_1)E(W_2) = E(W) = \{1, |u|^{-4}, |u|^{-3}\}$ from (4.28). Since $\{1, |u|^{-4}, |u|^{-3}\} \neq \{1, \tau^{-4}, \tau^{-6}\}$, W_2 cannot be the spin model (c). Therefore, W_2 belongs to the family (d) or (e). This means $W_2 = W_{H',u',a'}$ or $W_2 = W'_{H',u',b'}$, where H' is a Hadamard matrix of order $r' = (u'^2 + u'^{-2})^2$. Since $|E(W_2)| = 3$, Lemma 4.4.2 implies $r' > 4$ and $E(W_2) = \{1, |u'|^{-4}, |u'|^{-3}\}$. Then we have $|u'| = |u|$, as $E(W) = E(W_2)$. Now the second part of Lemma 4.4.1 implies $u^4 > 0$ and $u'^4 > 0$, hence

$$u^4 = u'^4, \tag{4.32}$$

and further $r = r'$ by (4.1). Therefore the size of W_2 is $2^{2s'}r$ for some integer s' with $0 < s' < s$, and the size of W_1 is $2^{2(s-s')}$. In particular, we obtain $s > 1$.

Since the tensor product of spin models on finite abelian groups is also a spin model on a finite abelian group, we may suppose that

$$W_1 = W_{11} \otimes W_{12} \otimes W_{13}, \tag{4.33}$$

where W_{11} is a spin model on a finite abelian group U, W_{12} is a tensor product of spin models in the family (a) with $r \leq 4$, and W_{13} is a tensor product of spin models in the families (d) and (e) with $r \leq 4$.

We put $|U| = 2^{n_1}$. Since the size 2^{n_1} of W_{11} cannot exceed that of W_1, we have $n_1 \leq 2(s - s')$. Then the size of $W_{12} \otimes W_{13}$ is $2^{2(s-s')-n_1}$. The diagonal entry of W_{11} is a complex number t_0 given by (4.25). The diagonal entries of W_{12}, W_{13} are 16-th roots of unity. We denote by κ_2, κ_3 the diagonal entries of W_{12}, W_{13}, respectively. Comparing the diagonal entries of (4.33), we have $u^3 = t_0\kappa_2\kappa_3 u'^3$, thus

$$W = (t_0^{-1}W_{11}) \otimes (\kappa_2^{-1}W_{12}) \otimes (\kappa_3^{-1}W_{13}) \otimes (u^3u'^{-3}W_2). \tag{4.34}$$

From (4.26), we have

$$\mu(t_0^{-1}W_{11}) \mid 2^{n_1+1}. \tag{4.35}$$

From (a), we have

$$\mu(\kappa_2^{-1}W_{12}) \mid 2^4. \tag{4.36}$$

From (a) and Lemma 4.4.4, we have

$$\mu(\kappa_3^{-1}W_{13}) \mid 2^{2(s-s')-n_1+1}. \tag{4.37}$$

Since W_2 is a spin model belonging to the family (d) or (e), Lemma 4.4.3 and (4.32) imply

$$\mu(u^3u'^{-3}W_2) \mid 2^{2s'+1}. \tag{4.38}$$

From (4.34)–(4.38) and Lemma 4.5.2, we have

$$\mu(W) \mid \mathrm{LCM}(2^{n_1+1}, 2^4, 2^{2(s-s')-n_1+1}, 2^{2s'+1}).$$

Since $n_1 < 2s$, we have $\max(n_1+1, 4, 2(s-s')-n_1+1, 2s'+1) \le 2s$. This implies $\mu(W) \mid 2^{2s}$, which contradicts Lemma 4.4.3 (i).

4.6 Spin models in Theorem 4.1.1 with $r \le 4$

In this section, we treat the case of $r \le 4$ in Theorem 4.1.3. We show that if $r = 1, 4$ in Theorem 4.1.3, then $W_{H,u,a}$ is not new.

If $r = 4$ in Theorem 4.1.1 (i), then $W_{H,u,a}$ is a tensor product of a Hadamard matrix of order 4 and $W_{(1),u,a}$. Indeed, up to equivalence, there is a unique Hadamard matrix of order $r = 4$. By Lemma 4.4.5, we may assume without loss of generality

$$H = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

Then $A_u = u^3H$ with $(u^2+u^{-2})^2 = 4$. Therefore we have $W_{H,u,a} = H \otimes W_{(1),u,a}$. Similarly, a spin model $W'_{H,u,b}$ in Theorem 4.1.1 (ii) can be decomposed as $H \otimes W'_{(1),u,b}$.

Lemma 4.6.1. *Let $m \equiv 0 \pmod 4$. Let $W_{(1),u,a}$ be a spin model given in Theorem 4.1.1 of index m, where $u^4 = 1$ and a is a primitive $2m^2$-th root of unity. Then $W_{(1),u,a}$ is equivalent to $W_{(1),1,au^3}$.*

Proof. First we assume that $u = -1$. Then $a^{\epsilon(i,j)}(-1)^{i-j-1} = -(-a)^{\epsilon(i,j)}$ holds for all $i, j \in \mathbb{Z}_{m^2}$. From this, we have $W_{(1),-1,a} = -W_{(1),1,-a}$. Therefore $W_{(1),-1,a}$ is equivalent to $W_{(1),1,-a}$.

Next we assume that $u^2 = -1$. Since $m \equiv 0 \pmod 4$, we have

$$u(au^3)^{\epsilon(i,j)} = \begin{cases} a^{\epsilon(i,j)}u & \text{if } i-j \text{ is even,} \\ a^{\epsilon(i,j)} & \text{if } i-j \text{ is odd.} \end{cases}$$

From this, we have $uW_{(1),1,au^3} = W_{(1),u,a}$. Therefore $W_{(1),u,a}$ is equivalent to $W_{(1),1,au^3}$. □

Lemma 4.6.2. *Let m be even, and ξ be a primitive $2m^2$-th root of unity. Then we have*

$$\sum_{x=0}^{m^2-1} \xi^{-x(x-m)} = m. \tag{4.39}$$

Proof. If (4.39) holds for $\xi = \exp(2\pi\sqrt{-1}/(2m^2))$, then by considering the action of the Galois group, we see that (4.39) holds for any primitive $2m^2$-th root of unity ξ. Therefore we may assune $\xi = \exp(2\pi\sqrt{-1}/(2m^2))$ without loss of generality. Since m is even, we may write $m = 2k$. Then

$$\begin{aligned}
\sum_{x=0}^{m^2-1} \xi^{-x(x-m)} &= \sum_{x=0}^{m^2-1} \xi^{-((x-k)^2-k^2)} \\
&= \xi^{k^2} \sum_{x=0}^{m^2-1} \xi^{-(x-k)^2} \\
&= \frac{\xi^{k^2}}{2} \sum_{x=0}^{m^2-1} (\xi^{-(x-k)^2} + \xi^{-(x-k+m^2)^2}) \\
&= \frac{\exp(\pi\sqrt{-1}/4)}{2} \sum_{x=0}^{2m^2-1} \xi^{-(x-k)^2} \\
&= \frac{1+\sqrt{-1}}{2\sqrt{2}} \sum_{x=0}^{2m^2-1} \xi^{-x^2}.
\end{aligned}$$

Now the result follows from [35, Theorem 99]. □

Of particular interest among spin models on finite abelian groups are spin models on finite cyclic groups. The spin model defined below is a special case of spin models on finite cyclic groups constructed by [2]. Let m be even, and a be a primitive $2m^2$-th root of unity. We restrict (4.24) and (4.25) to $\mathbb{Z}_{m^2}$, that is, $h = 1$. In (4.24) and (4.25), we put $\eta_1 = a^{-m+1}$, $\chi_{a_1}(a_1) = a^2$. Then (4.24) and (4.25) become

$$t_x = t_0 a^{x(x-m)} \quad (x \in \mathbb{Z}_{m^2}), \tag{4.40}$$

$$t_0^2 = m^{-1} \sum_{x=0}^{m^2-1} a^{-x(x-m)} = 1, \tag{4.41}$$

respectively, where we used Lemma 4.6.2 in (4.41). Thus we may take $t_0 = 1$. Then the matrix W given in (4.27) has entries

$$W(\alpha, \beta) = a^{(\beta-\alpha)(\beta-\alpha-m)} \quad (\alpha, \beta \in \mathbb{Z}_{m^2}). \tag{4.42}$$

We note that this spin model W on $\mathbb{Z}_{m^2}$ was constructed originally in [3, Theorem 2].

Proposition 4.6.3. *Let* $m \equiv 0 \pmod 4$. *Let* $W_{(1),u,a}$ *be a spin model given in Theorem* 4.1.1 (i) *of index* m*, where* $u^4 = 1$ *and* a *is a primitive* $2m^2$*-th root of unity. Then* $W_{(1),u,a}$ *is equivalent to* W *defined in* (4.42).

Proof. From Lemma 4.6.1 it is sufficient to prove that $W_{(1),1,au^3}$ is equivalent to W. By assumption, $m = 4k$ for some positive integer k. Since a^{8k^2} is a primitive 4-th root of unity, there exists $t \in \mathbb{Z}_4$ such that $u^3 = a^{8k^2 t}$. We define a bijection $\psi : \mathbb{Z}_m^2 \to \mathbb{Z}_{m^2}$ by

$$\psi(i, \ell) = (4k^2 t + 1)i + 4k\ell$$

for $(i, \ell) \in \mathbb{Z}_m^2$. Then for all $i, j, \ell, \ell' \in \mathbb{Z}_m$,

$$\begin{aligned}
&(\psi(j, \ell') - \psi(i, \ell))(\psi(j, \ell') - \psi(i, \ell) - m) \\
&= ((4k^2 t + 1)(j - i) + 4k(\ell' - \ell))((4k^2 t + 1)(j - i) + 4k(\ell' - \ell) - 4k) \\
&= (8k^2 t + 1)(8k(\ell - \ell')(i - j) + (i - j)^2 + 4k(i - j)) \\
&\quad + 32k^2 \left(-kt(j - i)(l' - l) + \frac{kt(j - i)(kt(j - i) + 1)}{2} \right. \\
&\qquad \left. + \frac{(l' - l)(l' - l - 1)}{2} \right) \\
&\equiv (8k^2 t + 1)(8k(\ell - \ell')(i - j) + (i - j)^2 + 4k(i - j)) \pmod{32k^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
W(\psi(i, \ell), \psi(j, \ell')) &= a^{(\psi(j,\ell')-\psi(i,\ell))(\psi(j,\ell')-\psi(i,\ell)-m)} \\
&= a^{(8k^2 t+1)(8k(\ell-\ell')(i-j)+(i-j)^2+4k(i-j))} \\
&= (au^3)^{2m(\ell-\ell')(i-j)+(i-j)^2+m(i-j)} \\
&= W_{(1),1,au^3}((i, \ell, 1), (j, \ell', 1)),
\end{aligned}$$

and we conclude that W is equivalent to $W_{(1),1,au^3}$. □

To conclude the paper, we note that the decomposability and identification with known spin models are yet to be determined for the following cases.

(1) $W_{H,u,a}$: $r = 1$, $m \equiv 2 \pmod 4$,

(2) $W'_{H,u,b}$: $r = 1$,

(3) $W_{H,u,a}$ and $W'_{H,u,b}$: $r = 2$,

(4) $W_{H,u,a}$ and $W'_{H,u,b}$: $r > 4$ and m is not a power of 2.

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