Affine Weyl Groups and the Boundary Value Eigenvalue Problems of the Laplacian

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（アファイン・ワイル群とラプラシアンの境界値固有値問題）

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要約

二階偏微分方程式は、量子力学を始め、電磁気学、音響理論、電気工学、振動工学など、物理学や工学において広く一般的に用いられている。その中でも、特に境界値・固有値問題は、古くから研究されてきた分野である。(レーリー [22]、クーラン・ヒルベルト [7][8] など) 古くは 1930 年ころ、ワイル [26] が基礎を作り、それは物理や工学に広く応用されている。近年になり 1980 年に、ピンスキー [20] が正三角形領域における境界値・固有値問題の研究を行った。それに引き続き、同じく 1980 年、ベラール [2] は、アファイン・ワイル群の理論を使って、固有値、固有関数をワイル群を用いて一般式の形で表示した。

そののち、1982 年には浦川肇（４次元以上の領域についての等スペクトル領域の構成）[23]、最近になってゴルドン、ウエッブ、ウォルパートら [12]、チャップマン（折り紙を用いた等スペクトル２次元領域の構成）[5] など、多くの人々によって境界値問題が研究されてきた。1985 年「ビリヤード問題と境界値問題」[13] の仕事もある。 ベラールのアファイン・ワイル群を用いた境界値・固有値問題で得られた結果について、現在では特に、結晶物理学や、素粒子物理学での利用が目立つようになってきた。([1][17])

このような状況の中で、我々は [14] において、まず、２次元、３次元の場合に、ベラールの研究 [2] を発展させ、小部屋 $D(R)$ の形や結晶群の元を表現することにより、固有値、固有関数をより具体的な形で求めた。結果、ルート系、アファイン・ワイル群の理論（ブルバキの著書）により、アファイン・ワイル群の基本領域である小部屋 $D(R)$（$A_2, B_2, G_2, A_3, B_3, C_3$ 型）について、空間上の座標値による表示を行うことができた。更に、個々の $D(R)$ に対応する結晶群の元を行列の形に表現し、固有値、固有関数を求めた。 ディリクレ境界値問題の固有関数は、いずれも行列式の形で表すことができた。(固有関数の項の数は、結晶群の元の数（位数）に等しい。) ノイマン境界値問題の固有関数は、ディリクレ境界値問題の、行列式の各項の符号を全て正としたパーマネントで表した。

本研究では、ベラールの研究を発展させて、一般の ℓ 次元での小部屋 $D(R)$（領域）における固有値・固有関数について、行列式やパーマネントによる具体的表現を行う。更に境界値・固有値問題の応用として、熱核に関する計算を行う。また、固有関数の可視化を行う。

具体的方法としては、まず、一般次元（ℓ 次元）における小部屋 $D(R)$、固有値、固有関数を求める。([14] で行った $\ell = 2{,}3$ の場合の小部屋 $D(R)$、固有値、固有関数の計算を拡張したものである。) さらに、計算で得られた、小部屋 $D(R)$ 、固有値、固有関数をもとに、ディリクレ型、ノイマン型のそれぞれの熱核の具体的・明示的な公式を示す。また可視化については、固有関数（振動の様子）のコンピュータグラフィクスによる可視化を行う。

各章について詳述する。

１章は序論である。

２章では、結晶群とその基本領域について述べる。結晶群で構成される図形のうち、特に「鏡映によって空間を埋め尽くす図形」について述べる。すなわち、ルート系（ルート系の定義、基本ルート、双対ルートの定義、最大ルート、ワイル群、その中での鏡映、部屋など）や、アファイン・ワイル群（アファインワイル群の定義、その中での鏡映、小部屋など）について述べる。

次に、リーマン多様体上のラプラシアンの基礎事項や、その中でも特にユークリッド空間におけるラプラシアンについての基礎事項を述べ、ディリクレ境界値・固有値問題、ノイマン境界値・固有値問題に関する基本的事項を詳述する。さらに、第５章における境界値・固有値問題の熱拡散方程式への応用のための基礎を与える。具体的内容は、熱拡散方程式でのラプラス作用素、熱核の定義などである。

３章では、ピンスキーが正三角形領域において固有値、固有関数を明示的に求めたことについて記す。その中では、ディリクレ境界値問題、ノイマン境界値問題の両方について取り上げている。彼は、領域（正三角形）内での関数をもとに、領域外も含めた一般的な空間での関数を考え、その持つべき周期性から固有値、固有関数を明示的に定めた。

４章では、ベラールがブルバキのアファイン・ワイル群の理論を使って、結晶群における固有値、固有関数を一般式の形で表したことについて解説する。具体的には、結晶群を使った領域における、周期性、連続性を持った関数を拡張したものを考え、ディリクレ問題、ノイマン問題における固有値、固有関数について求めた方法である。（前述にもあるように、ベラールは、各型ごとに固有値、固有関数を明示的に表していない。）

５章では、我々の得た結果について述べる。ルート系、アファイン・ワイル群の理論を使いながら、小部屋 $D(R)$ の形や対称群の元を表現することにより、一般次元（ℓ 次元）における小部屋、固有値、固有関数の具体的な表示を行う。

(1) 既約ルート系 R について、空間上の座標値による表示を行うことで、アファイン・ワイル群の基本領域である小部屋 $D(R)$ を完全に決定する。
(2) 同様に, 各固有値についても, 具体的表現を得る。
(3) ディリクレ固有関数については行列式で表す。 ノイマン固有関数についても同様に, Perm( ) で表す。（ここで Perm( ) とは, 行列式において現れる符号をすべて, +1 に置き換えたものである。）

すなわち、次元 ℓ を決めることで、各型での小部屋（つまり境界値・固有値問題で扱う領域)、固有値、固有関数が具体的に定まるわけである。

６章では、５章で求めた一般式において、領域が２，３次元のときの固有値、固有関数をより具体的な形で求める。アファイン・ワイル群の基本領域である小部屋 $D(R)$（$A_2, B_2, G_2, A_3, B_3, C_3$ 型）について、空間上の座標値による表示を行う。更に個々の $D(R)$ に対応する固有値、固有関数を求める。ベラールは A_3〜C_3 の固有値については具体的に求めていなかったが、本研究ではそれを行う。

７章では、得られた結果（小部屋、固有値、固有関数）をもとに、熱拡散方程式にあてはめ、熱核の計算を行う。 熱核については、Poisson の和公式を求める。また熱核のトレース、さらにその $t \to 0$ での挙動を調べる。これらは、ディリクレ、ノイマン熱核では始めてであり、極めて興味ある公式である。

８章では、いくつかの固有関数についてのコンピュータグラフィクスによる可視化を行う。６章との関連も大きいが、２，３次元におけるさまざまな可視化、また有限要素法による可視化との比較など、今後研究がさらに深められる分野である。 ここでは、正方形、直角２等辺３角形、正三角形、一つの角が 30° の直角三角形についてのディリクレ境界値問題、ノイマン境界値問題におけるＢＡＳＩＣプログラムや数式処理ソフトＭａｐｌｅによる固有関数を示す。

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1 Introduction

The theory of second order partial differential equations has been applied broadly in physics and engineering: for example, in quantum physics, electromagnetism, theory of sound, electrical engineering, and vibration engineering. Among them, particularly, the boundary value eigenvalue problem is one of the fields which has been studied from the early days of the eighteenth century. (Lord Rayleigh [22], Courant and Hilbert [7], [8]). In 1930, H. Weyl [26] initiated such a study, and it has been developed and applied widely in physics and engineering. Recently, in 1980, a study of the boundary value eigenvalue problems of the equilateral triangle domain has been done by Pinsky [20], and also, in 1980, the eigenvalues and the eigenfunctions were formulated in terms of the affine Weyl group theory by Bérard [2].

The isospectral problem of the boundary value problem was solved by H. Urakawa [23] in the four-dimensional case in 1982. For the two-dimensional case, the isospectral examples were made by C. Gordon, D. Webb and S. Wolpert [12], and by S. J. Chapman [5]. The author here did some work which gives relations between the billiards problems and the boundary value problems [13] in 1985. The results which Bérard obtained for the boundary value eigenvalue problems in affine Weyl groups are used prominently nowadays, particularly in crystal physics and elementary particle physics ([1], [17]).

In the two- and three-dimensional cases, we gave precise expressions of the alcoves $D(R)$, and the affine Weyl groups (i.e., crystal graphic groups) $W_a(R)$, and gave the eigenvalues and the eigenfunctions explicitly. In other words, we gave the Euclidean coordinates expressions of types $A_2, B_2, G_2, A_3, B_3, C_3$, respectively. Also, we figured each element in the crystallographic groups for each $D(R)$ in terms of matrices, and calculated the eigenvalues and eigenfunctions explicitly.

We figured out the eigenfunctions for the Dirichlet boundary value problems in terms of the determinant, and also the ones for the Neumann boundary value problems as the permanent, which is by definition the quantity obtained by changing the sign of each term of the determinant into $+1$.

In our study, we develop Bérard's research, and give the explicit expression of the eigenvalues and eigenfunctions in terms of the determinant and permanent on each alcove $D(R)$ of general ℓ-dimension. Also, as an application, we give the explicit expression of the heat kernel. Finally, we visualize the eigenfunctions in the two-dimensional case.

We first calculate the alcove $D(R)$ and the eigenvalues and eigenfunctions in the general dimension. This is an extension of the calculation for alcove $D(R)$ and the eigenvalues and eigenfunctions in the case of $\ell = 2, 3$ ([14]). Furthermore, we show explicit formulas for each type of the Dirichlet and Neumann heat kernel by using alcove $D(R)$ and the eigenvalues and eigenfunctions. Also, we show computer graphics visualizations of the eigenfunctions.

We will now explain about each section in more detail.

Section 1 is the introduction.

In Section 2, we state the basic facts about crystallographic groups and their fundamental domains. We give the definitions of root, fundamental root, dual root, highest root, Weyl groups, reflection in them, Weyl chamber, affine Weyl group, and alcove.

Next, we give the basic facts about the Laplacian on a Riemannian manifold and in the Euclidean space, and the Dirichlet or Neumann boundary value eigenvalue problems, in more detail. Also, we give the basic materials of the heat diffusion equation for boundary value eigenvalue problems in Section 5. The contents are the Laplacian in the heat diffusion equation, and the definition of the heat kernel.

In Section 3, we explain the results of Pinsky [20], who calculated the eigenvalues and eigenfunctions explicitly on the equilateral triangle plane domain. In [20], he considered both the Dirichlet and Neumann boundary value problems. He also considered general space functions which are defined outside of the domain, and satisfy periodicity, and decided explicitly the eigenvalues and eigenfunctions by their periodicity.

In Section 4, we study Bérard's work [2], which expressed the eigenvalues and eigenfunctions on the fundamental domain of the crystallographic groups as general formulas by using the affine Weyl groups theory [4]. His method was to treat the extended function, which is periodic and smooth on the fundamental domain in the crystallographic groups, and he showed explicitly the eigenvalues and eigenfunctions for the Dirichlet problems and Neumann problems in terms of the Weyl group. But, he did not show concretely the eigenvalues and eigenfunctions for each type. This is the starting point of our study.

In Section 5, by using the root systems and affine Weyl groups, we figure out the alcove $D(R)$ and the Weyl groups, and also we show the eigenvalue and eigenfunction concretely in general ℓ-dimension.

(1) More precisely, for each irreducible root system R, we decide the alcove $D(R)$, which is the fundamental domain of affine Weyl groups, explicitly.
(2) We obtain the concrete expression for all the eigenvalues.
(3) We express the Dirichlet eigenfunctions in terms of the determinant.
The Neumann eigenfunction is shown similarly in terms of the permanent.

In Section 6, we show the eigenvalues and the eigenfunctions in the domain, which is of 2 or 3 dimensions, more concretely. We show the coordinate expression of all the alcoves $D(R)$ of type $A_2, B_2, G_2, A_3, B_3, C_3$ which are the fundamental domains in the affine Weyl groups. Also, we calculate the corresponding eigenvalues and eigenfunctions for each $D(R)$. Bérard did not calculate the eigenvalues for $A_3 \sim C_3$ concretely.

In Section 7, we obtain a concrete expression for the heat kernel. We obtain Poisson's summation formula for the heat kernel. Also, we observe the trace of the heat kernel and the behavior as $t \to 0$. These formulas are the first case of the Dirichlet and Neumann heat kernel, which are very interesting formulas.

In Section 8, we show visualizations of some eigenfunctions with computer graphics. This is related to Section 6, where we visualize many cases of 2 and 3 dimensions. In this section, we investigate the eigenfunctions for the Dirichlet and the Neumann boundary value problems on the square and right isosceles triangle, equilateral triangle, and the right triangle whose other angles are $30°$ and $60°$ with BASIC, and we investigate the eigenfunctions of the type $A_2,\ B_2,\ G_2$ for the Dirichlet and the Neumann boundary value problems with Maple.

2 Basic Materials

2.1 Summary

First, we describe the crystallographic groups, their fundamental domains, particularly, the root systems, and affine Weyl groups. Second, we show basic materials of the Laplacian, and the Dirichlet boundary value eigenvalue problems and the Neumann boundary value eigenvalue problems. Last, we show the boundary value problems for the heat equation.

2.2 The crystallographic groups and their fundamental domains

2.2.1 Aims

We describe the basic facts and theory of root systems and affine Weyl groups to calculate the eigenvalues and eigenfunctions of crystallographic domains and show Bérard's works below.

2.2.2 Basic facts of the root system

Let V be the ℓ dimension Euclidean space.

Definition 2.1 *Let R be the root systems on V. In other words, R is a finite set, which satisfies the following three conditions ([4] p.169 ∼ 170).*

Condition 1: $0 \notin R \subset V$, *and R generates V, that is,*

$$V = \left\{ \sum_{\alpha \in R} x_\alpha \alpha \,\middle|\, x_\alpha \in \mathbb{R} \right\}.$$

Condition 2: *For each $\alpha \in R$, there exists $\alpha^\vee \in V^*$(the dual space of V) which satisfies that*

$$\langle \alpha,\, \alpha^\vee \rangle \, (= \alpha^\vee(\alpha)) = 2.$$

Here, for each $f \in V^*$, $f(\mathbf{x}) = \langle \mathbf{x},\, f \rangle$ $({}^\forall \mathbf{x} \in V)$.

Condition 3: *For each $\alpha \in R$, $\alpha^\vee(R) \subset \mathbb{Z}$ (the set of all integers), i. e.,*

$$\textit{For each } \beta \in R,\ \alpha^\vee(\beta) = \langle \beta,\, \alpha^\vee \rangle \ \textit{ is an integer.}$$

Let $R^\vee = \{\alpha^\vee \,|\, \alpha \in R\ \}$. Each element in R is called a root. The rank of the root systems R is defined as a dimension of V, say ℓ.

We have

$$\alpha^{\vee\vee} = (\alpha^\vee)^\vee = \alpha \quad ({}^\forall\alpha \in R)$$

We call $R^\vee$ the dual root system or inversion root system of R.

Lemma 2.1 *For each $\alpha^\vee \in R^\vee$, it holds that $\alpha^\vee = \dfrac{2\alpha}{(\alpha|\alpha)}$. Here, $(\ |\ )$ is the inner product in V.*

Proof By the definition of $\alpha^\vee$, we have

$$\left\langle \alpha \,\middle|\, \frac{2\alpha}{(\alpha|\alpha)} \right\rangle = \left(\alpha \,\middle|\, \frac{2\alpha}{(\alpha|\alpha)} \right) = \frac{2(\alpha|\alpha)}{(\alpha|\alpha)} = 2.$$

Then, we have

$$\frac{2\alpha}{(\alpha|\alpha)} = \alpha^\vee. \qquad //$$

Lemma 2.2 *For each $f \in V^*$, there exists a unique $H_f \in V$ which satisfies*

$$f(\mathbf{x}) = (\mathbf{x} \,|\, H_f), \quad {}^\forall\mathbf{x} \in V. \tag{2.1}$$

By the correspondence $f \in V^* \leftrightarrow H_f \in V$, we identify V^* with V. Since $V^* \ni f \to H_f \in V$ is isomorphic, we may write f for H_f. We can write (2.1) as

$$f(\mathbf{x}) = (\mathbf{x} \,|\, f), \quad {}^\forall\mathbf{x} \in V.$$

Definition 2.2 *We define a linear transformation S_α of V to be $S_\alpha(\mathbf{x}) := \mathbf{x} - \langle \mathbf{x}, \alpha^\vee \rangle\, \alpha,\ \mathbf{x} \in V$, which is called a reflection with respect to L_α.*

Lemma 2.3 *The composition $(S_\alpha)^2$ is the identity operator. Namely,*

$$(S_\alpha)^2(\mathbf{x}) = \mathbf{x} \quad ({}^\forall\mathbf{x} \in V).$$

Proof We calculate $(S_\alpha)^2(\mathbf{x})$ by using the definition of $S_\alpha(\mathbf{x})$,

$$\begin{aligned}
(S_\alpha)^2(\mathbf{x}) &= (S_\alpha(S_\alpha(\mathbf{x})) = S_\alpha(\mathbf{x}) - \langle S_\alpha(\mathbf{x}), \alpha^\vee \rangle\, \alpha \\
&= \mathbf{x} - \langle \mathbf{x}, \alpha^\vee \rangle\, \alpha - \langle \mathbf{x} - \langle \mathbf{x}, \alpha^\vee \rangle\, \alpha, \alpha^\vee \rangle\, \alpha \\
&= \mathbf{x} - \langle \mathbf{x}, \alpha^\vee \rangle\, \alpha + \langle \mathbf{x}, \alpha^\vee \rangle\, \alpha \\
&= \mathbf{x} \quad ({}^\forall\mathbf{x} \in V). \qquad //
\end{aligned}$$

For each $\alpha \in R$, we can write $S_\alpha(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}, \alpha)\dfrac{2\alpha}{(\alpha|\alpha)}$, $\mathbf{x} \in V$, because $(\mathbf{x},\, \alpha^\vee) = \dfrac{2(x|\alpha)}{(\alpha|\alpha)}$.

We define the hyperplane which is written as L_α as follows:

$$\{\, \mathbf{x} \,\in\, V \,|\, S_\alpha(\mathbf{x}) = \mathbf{x} \,\} = \{\, \mathbf{x} \,\in\, V \,|\, (\, \mathbf{x} \,|\, \alpha) = 0 \,\}.$$

Here, S_α is the reflection with respect to L_α(We will prove this fact in Lemma 2.4.)

Definition 2.3 *The Weyl group $W(R)$ is by definition a finite group which is generated by $S_\alpha(\alpha \in R)$. In other words, an arbitrary element $w \in W(R)$ is written as $w = S_{\alpha_1} S_{\alpha_1} \cdots S_{\alpha_k}$, where $\alpha_1, \alpha_2, \cdots, \alpha_k \in R$.*

Definition 2.4 *We define a Weyl chamber as one of the connected components of complement of $\bigcup_\alpha \{L_\alpha \,|\, \alpha \,\in\, R\}$ in V. We denote it by $C(R)$. Here, L_α is a hyperplane of V which is defined by*

$$L_\alpha = \{\mathbf{x} \,\in\, V \,|\, (\alpha \,|\, \mathbf{x}) \ = 0\}$$

for each $\alpha \in R$.

Definition 2.5 *Among the set of all connected components of the complement of $\bigcup_{\alpha \in R, k \in Z} L_{\alpha,k}$ in V, we fix the one whose closure includes the origin o and which is included in $C(R)$ and denote it by $D(R)$. Here, for each $\alpha \in R$ and $k \in \mathbb{Z}$, the hyperplane $L_{\alpha,k}$ in V is defined by*

$$L_{\alpha,k} = \{\mathbf{x} \,\in\, V \,|\, \langle \alpha,\, \mathbf{x} \rangle = k\}\,.$$

All the connected components of the complement of $\bigcup_{\alpha \in R, k \in \mathbb{Z}} L_{\alpha,k}$ in V are called the alcoves.

Definition 2.6 *We define the discrete subgroup $Q(R)$ of V by*

$$Q(R) := \left\{ \sum_{\alpha \in R} m_\alpha \alpha \,\middle|\, m_\alpha \,\in\, \mathbb{Z} \quad (\alpha \,\in\, R) \right\}$$

which is generated by R, and also define a discrete subgroup $Q(R^\vee)$ of V by

$$Q(R^\vee) := \left\{ \sum_{\alpha \in R} m_\alpha \alpha^\vee \,\middle|\, m_\alpha \,\in\, \mathbb{Z} \quad (\alpha \,\in\, R) \right\} \,\subset\, V,$$

We define the set of all the weights of R by

$$P(R) := \left\{ \mathbf{x} \in V \mid (\mathbf{x} \mid \mathbf{y}^*) \in \mathbb{Z} \quad ({}^\forall \mathbf{y}^* \in Q(R^\vee)) \right\}.$$

Then, we also have

$$P(R) = \left\{ \mathbf{x} \in V \mid (\mathbf{x} \mid \alpha^\vee) \in \mathbb{Z} \quad ({}^\forall \alpha \in R) \right\}.$$

We call all the elements in $P(R)$ the weights of R.

2.2.3 Basic materials of affine Weyl groups

Definition 2.7 *The affine Weyl group $W_a(R)$ is by definition an infinite group which is generated by $S_{\alpha,k}$ ($\alpha \in R$, $k \in \mathbb{Z}$).*

Lemma 2.4 *$S_{\alpha,k}$ is a reflection with respect to a hyperplane $L_{\alpha,k}$ of V.*

Proof Recall that $S_{\alpha,k}(\mathbf{x}) := \mathbf{x} - \langle \alpha^\vee, \mathbf{x} \rangle \alpha + k\alpha^\vee$. To show that S_α is a reflection with respect to $L_{\alpha,k} = \{ \mathbf{x} \in V \mid \langle \alpha, \mathbf{x} \rangle = k \}$, it suffices to show the following two facts:

$$S_{\alpha,k}(\mathbf{x}) = \mathbf{x} \quad ({}^\forall \mathbf{x} \in L_{\alpha,k}) \tag{2.2}$$

$${S_{\alpha,k}}^2(\mathbf{x}) = \mathbf{x}, \quad \mathbf{x} \in V. \tag{2.3}$$

For (2.2), since it is for each $\mathbf{x} \in L_{\alpha,k}$, $\langle \alpha, \mathbf{x} \rangle = k$, then, $(\alpha \mid \mathbf{x}) = k$, by the definition of $S_{\alpha,k}(\mathbf{x})$. Then, we have

$$\begin{aligned} S_{\alpha,k}(\mathbf{x}) &= \mathbf{x} - \langle \alpha^\vee, \mathbf{x} \rangle \alpha + k\alpha^\vee \\ &= \mathbf{x} - \left(\frac{2\alpha}{(\alpha|\alpha)} \,\middle|\, \mathbf{x} \right) \alpha + \frac{2k\alpha}{(\alpha|\alpha)} \\ &= \mathbf{x} - \frac{2}{(\alpha|\alpha)} (\alpha \mid \mathbf{x}) \alpha + \frac{2k\alpha}{(\alpha|\alpha)} \\ &= \mathbf{x}. \end{aligned}$$

For (2.3),

$$\begin{aligned} S_{\alpha,k}(S_{\alpha,k}(\mathbf{x})) &= S_\alpha(S_\alpha(\mathbf{x}) + k\alpha^\vee) + k\alpha^\vee \\ &= S_{\alpha,k}^2(\mathbf{x}) + S_\alpha(k\alpha^\vee) + k\alpha^\vee \\ &= \mathbf{x} + k\alpha^\vee - \langle \alpha^\vee,\ k\alpha^\vee \rangle \alpha + k\alpha^\vee \\ &= \mathbf{x} + \frac{2k\alpha}{(\alpha|\alpha)} - \left(\frac{2}{(\alpha|\alpha)} \,\middle|\, \frac{2k\alpha}{(\alpha|\alpha)} \right) \alpha + \frac{2k\alpha}{(\alpha|\alpha)} \\ &= \mathbf{x}. \qquad\qquad // \end{aligned}$$

Let $S_{\alpha,k}$ be the reflection with respect to $L_{\alpha,k} = \{\mathbf{x} \in V \mid \langle \alpha, \mathbf{x} \rangle = k\}$. Then, we have

$$S_{\alpha,k}(\mathbf{x}) = S_\alpha(\mathbf{x}) + k\alpha^\vee.$$

Thus, $W_a(R)$ is a semi-direct product of both the groups $W(R)$ and $Q(R^\vee)$.

Lemma 2.5 *We have that*

$$(S_\alpha(\mathbf{x}) \mid S_\alpha(\mathbf{y})) = (\mathbf{x}|\mathbf{y}) \quad (\mathbf{x},\ \mathbf{y} \in V).$$

Then, S_α is an orthogonal transformation of V with respect to the inner product $(\ |\ )$.

Proof

$$\begin{aligned}(\mathbf{x} - (\alpha^\vee \mid \mathbf{x})\alpha,\ \mathbf{y} - (\alpha^\vee \mid \mathbf{y})\alpha)) &= \left(\mathbf{x} - \left(\frac{2\alpha}{(\alpha|\alpha)} \,\middle|\, \mathbf{x}\right)\alpha \,\middle|\, \mathbf{y} - \left(\frac{2\alpha}{(\alpha|\alpha)} \,\middle|\, \mathbf{y}\right)\alpha\right) \\ &= (\mathbf{x} \mid \mathbf{y}) - \left(\frac{2\alpha}{(\alpha|\alpha)} \,\middle|\, \mathbf{x}\right)(\alpha \mid \mathbf{y}) - \left(\frac{2\alpha}{(\alpha|\alpha)} \,\middle|\, \mathbf{y}\right)(\mathbf{x} \mid \alpha) \\ &\quad + \left(\frac{2\alpha}{(\alpha|\alpha)} \,\middle|\, \mathbf{x}\right)\left(\frac{2\alpha}{(\alpha|\alpha)} \,\middle|\, \mathbf{y}\right)(\alpha \mid \alpha) \\ &= (\mathbf{x} \mid \mathbf{y}).\end{aligned}$$

Therefore, S_α is an orthogonal transformation of V. //

Lemma 2.6 $\det(S_\alpha) = \pm 1$.

Proof We put $S_\alpha = \mathrm{T}$. Since T is an orthogonal transformation, $\det(\mathrm{T}) = \pm 1$. Because

$$1 = \det \mathrm{E} = \det\,({}^t\mathrm{T}\mathrm{T}) = \det\,({}^t\mathrm{T}) \det\,(\mathrm{T}) = \det\,(\mathrm{T})^2. \qquad //$$

Lemma 2.7 $\det(S_\alpha) = -1$.

Proof Let S_α be a reflection with respect to L_α. Here, $L_\alpha = \{\mathbf{x} \in V \mid (\mathbf{x}, \alpha) = 0\}$. We recall

$$S_\alpha(\mathbf{x}) = \mathbf{x} \quad ({}^\forall \mathbf{x} \in L_\alpha) \tag{2.4}$$

$$S_\alpha(\ell\alpha) = -\ell\alpha \quad (\mathit{for\ some}\ \ell \in \mathbb{R}). \tag{2.5}$$

Furthermore, any element in V is written as $\ell\alpha + \mathbf{x}$, where $\mathbf{x} \in L_\alpha$ and $\ell \in \mathbb{R}$. Then, we may take $v_1 = \alpha, \{v_2, \cdots, v_d\}$ to be a basis of L_α. Then, $\{v_i\}_{i=1}^{\ell}$ is a basis of V. So, S_α can be written as $S_\alpha(v_j) = \sum_{i=1}^{d} a_{ij} v_i$.

Then, let A $= (a_{ij})$ be an $\ell \times \ell$ matrix which is given by the above (2.4) and (2.5),

$$S_\alpha = \left(\begin{array}{c|ccc} -1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \cdot & & I_{\ell-1} & \\ \cdot & & & \\ 0 & & & \end{array} \right).$$

Here, $I_{\ell-1}$ is the identity matrix of degree $\ell - 1$. Thus, we have $\det(S_\alpha) = -1$. //

Lemma 2.8 *If $w = S_{\alpha_1, k_1} \cdots S_{\alpha_t, k_t} \in W_a(R)$, $\mathbf{x} \in D(R)$, then, we have*

$$\varepsilon(w) = (-1)^t.$$

Proof

$$\begin{aligned} (\det\, w) &= \det(S_{\alpha_1, k_1} \cdots S_{\alpha_t, k_t}) \\ &= \det(S_{\alpha_1, k_1}) \cdots \det(S_{\alpha_t, k_t}) \\ &= (-1)^t. \end{aligned}$$

Thus, we have $\varepsilon(w) = (-1)^t$. //

The above Lemma 2.8 means that the sign of w in $W_a(R)$ is decided uniquely, dependent on whether the element w is an odd permutation or an even one.

Furthermore, any element of $W_a(R)$ is written as $w = S_{\alpha_1, k_1} \cdots S_{\alpha_t, k_t}$, and $w(D(R))$ is also an alcove. Conversely, any alcove is written as $w(D(R))$.

Definition 2.8 *For any affine transformation $w(\mathbf{x}) = \mathrm{T}(\mathbf{x}) + \boldsymbol{v}$, $\mathbf{x} \in V$, we define the determinant* $\det\, w = (-1)^w$ *of w as follows: Here, $\boldsymbol{v} \in V$, and $\mathrm{T} : V \to V$ is a linear transformation. If we write w as*

$$w = \left(\begin{array}{ccc|c} & T & & \boldsymbol{v} \\ \hline 0 & \cdots & 0 & 1 \end{array} \right),$$

then we have

$$w \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \left(\begin{array}{ccc|c} & T & & \boldsymbol{v} \\ \hline 0 & \cdots & 0 & 1 \end{array} \right) \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} T\mathbf{x} + \boldsymbol{v} \\ 1 \end{pmatrix}.$$

Then, we can define

$$\det\, w := \det \left(\begin{array}{ccc|c} & T & & \boldsymbol{v} \\ \hline 0 & \cdots & 0 & 1 \end{array} \right) = \det\, T.$$

Here, we give a matrix representation of w in W_a. $S_{\alpha,k}(\mathbf{x})$ is given by

$$S_{\alpha,k}(\mathbf{x}) = S_\alpha(\mathbf{x}) + k\alpha^\vee \quad (\mathbf{x} \in V).$$

Therefore, if we put

$$S_{\alpha,k} = \left(\begin{array}{ccc|c} & S_\alpha & & k\alpha^\vee \\ \hline 0 & \cdots & 0 & 1 \end{array}\right),$$

then

$$\begin{aligned} S_{\alpha,k}\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} &= \left(\begin{array}{ccc|c} & S_\alpha & & k\alpha^\vee \\ \hline 0 & \cdots & 0 & 1 \end{array}\right)\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} S_\alpha(\mathbf{x}) + k\alpha^\vee \\ 1 \end{pmatrix}. \end{aligned}$$

Thus, we have

$$W_a(R) \ni w = S_{\alpha_1,k_1} \cdots S_{\alpha_t,k_t}.$$

Then we can write

$$\begin{aligned} w\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} &= \left(\begin{array}{ccc|c} & S_{\alpha_1} & & k_1\alpha_1^\vee \\ \hline 0 & \cdots & 0 & 1 \end{array}\right) \cdots \left(\begin{array}{ccc|c} & S_{\alpha_t} & & k_t\alpha_t^\vee \\ \hline 0 & \cdots & 0 & 1 \end{array}\right)\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \\ &= \left(\begin{array}{ccc|c} & T & & \boldsymbol{v} \\ \hline 0 & \cdots & 0 & 1 \end{array}\right)\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}. \end{aligned}$$

Here, $T = S_{\alpha_1} \cdots S_{\alpha_t}$. Then, we have

$$\det w = \det\,(S_{\alpha_1} \cdots S_{\alpha_t}) = \det\, S_{\alpha_1} \cdots \det\, S_{\alpha_t} = (-1)^t.$$

Also, we have

$$v = S_{\alpha_1}(S_{\alpha_2}(\cdots S_{\alpha_{t-1}}(k_t\alpha_t^\vee)\cdots)) + \cdots + S_{\alpha_1}(k_2\alpha_2^\vee) + k_1\alpha_1^\vee.$$

2.3 The Laplacian and the Boundary Value Eigenvalue Problems

2.3.1 Aims

Here, we define the Laplacian on a Riemannian manifold, and as a special case, we consider the Laplacian in the Euclidian space. We also consider the Dirichlet boundary value eigenvalue problems and Neumann boundary value eigenvalue problems.

2.3.2 The Laplacian on a Riemannian manifold

In this subsection, we define the Laplace operator, or Laplacian. We state several properties which we will need later. During this subsection, we denote an m-dimensional connected C^∞ Riemannian manifold by $M = (M, g)$, and we also assume that M is compact in what follows.

First, we prepare some facts from linear algebra. Let V be an m dimensional vector space, with the inner product $\langle \cdot, \cdot \rangle$. For $\wedge^p V$, by [16] Section 3,(5.8), the inner product is also defined, which is denoted by $\langle \cdot, \cdot \rangle$. Let $\{e^1, \cdots, e^m\}$ be an orthonormal basis of V. Then, $\{e^1 \wedge \cdots \wedge e^{j_p},\ j_1 < \cdots < j_p\}$ is an orthonormal basis of $\wedge^p V$.

Definition 2.9 *We define the linear mapping* $* : \wedge^p V \to \wedge^{m-p} V$ *to satisfy*

$$< *\omega, \tau > e^1 \wedge \cdots \wedge e^m = \omega \wedge \tau \quad (\forall \omega \in \wedge^p V,\ \forall \tau \in \wedge^{m-p} V). \tag{2.6}$$

We call this the **Hodge star operator**.

Lemma 2.9 *The operator* $*$ *is decided uniquely by* (2.6)*, and is a linear isomorphism of* V*. Futhermore, it satisfies*

$$*(e^{j_1} \wedge \cdots \wedge e^{j_p}) = \varepsilon \cdot e^{j_{p+1}} \wedge \cdots \wedge e^{j_m}, \tag{2.7}$$

where ε *stands for the sign of the permutation* $(1, \cdots, m) \mapsto (j_1, \cdots j_m)$*. Also,*

$$* * \omega = (-1)^{p(m-p)} \omega \quad (\forall \omega \in \wedge^p V), \tag{2.8}$$

$$* 1 = e^1 \wedge \cdots \wedge e^m, \quad *(e^1 \wedge \cdots \wedge e^m) = 1, \tag{2.9}$$

$$\omega \wedge *\tau =< \omega, \tau > e^1 \wedge \cdots \wedge e^m \quad (\forall \omega, \tau \in \wedge^p V). \tag{2.10}$$

Proof We notice that $\dim \wedge^p V = \dim \wedge^{m-p} V = \begin{pmatrix} m \\ p \end{pmatrix}$. We show the uniqueness of $*$ and its linear isomorphism. While keeping ω fixed, let

$$L_\omega(\tau) e^1 \wedge \cdots \wedge e^m = \omega \wedge \tau \quad (\forall \omega,\ \tau \in \wedge^{m-p} V) \tag{2.11}$$

where $L_\omega : \wedge^{m-p} V \to \mathbb{R}$ is a linear mapping. This is decided uniquely. On the other hand, $L_\omega \in (\wedge^{m-p} V)^*$. Then $*\omega \wedge^{m-p} V$ is decided uniquely, which satisfies

$$< *\omega, \tau >= L_\omega(\tau) \quad (\forall \omega,\ \tau \in \wedge^{m-p} V). \tag{2.12}$$

Thus, $*$ is decided uniquely. Furthermore, if $*\omega = 0$, then, for any $\tau \in \wedge^{m-p} V$, we have $\omega \wedge \tau = 0$, which implies that $\omega = 0$. Thus, $*$ is injective, so that it is an isomorphism. The proof of (2.7)∼(2.10) is direct, so we omit it. //

For an arbitrary $x \in M$, we put $V = T_x^*M$. Then, $*$ defines a mapping of $A^p(M)$ into $A^{m-p}(M)$ naturally. We also write it as $*$.

We can define the inner product on $A^p(M)$ by using of metric g of M. The fiberwise inner product$\langle \cdot, \cdot \rangle_x$ is defined. By using this, for $A^p(M)$, we define the inner product by

$$(\omega, \tau)_{L^2} := \int_M \langle \omega(x), \tau(x) \rangle_x \, dv_g(\mathbf{x}) \quad (\omega, \tau \in A^p(M)). \tag{2.13}$$

(See [16] (1.11).)

Lemma 2.10 *By using the exterior differentiation $d : A^{m-p}(M) \to A^{m-p+1}(M)$, we can define the linear operator $\delta : A^p(M) \to A^{m-p}(M)$ by*

$$\delta\omega := (-1)^p *^{-1} d * \omega \quad (\forall \omega \in A^p(M)). \tag{2.14}$$

Then, for every $\omega \in A^p(M)$, $\theta \in A^{p-1}(M)$, we have

$$(d\theta, \omega)_{L^2} = (\theta, \delta\omega)_{L^2}, \tag{2.15}$$

$$\delta(\delta\omega) = 0. \tag{2.16}$$

Proof From (2.10), we have

$$(\omega, \tau)_{L^2} := \int_M \omega \wedge *\tau. \tag{2.17}$$

Since (2.15) is shown by

$$\begin{aligned}
(d\theta, \omega)_{L^2} &= \int_M d\theta \wedge *\omega = \int_M d(\theta \wedge *\omega) + (-1)^p \int_M \theta \wedge d(*\omega) \\
&= (-1)^p \int_M \theta \wedge d(*\omega) \qquad (\textit{by } [16],\ \textit{Section } 3,\ \textit{Theorem } 9.2) \\
&= (-1)^p \int_M \theta \wedge *(*^{-1} d*)\omega = (-1)^p (\theta, (*^{-1} d*)\omega)_{L^2} \\
&= (\theta, \delta\omega)_{L^2}.
\end{aligned}$$

For (2.16), we have

$$\delta \circ \delta = \pm(*^{-1} d*) \circ (*^{-1} d*) = \pm *^{-1} \circ d \, d \circ * = 0. \qquad //$$

The property (2.15) means that δ is the formal adjoint operator of d.

Definition 2.10 *Le us define the linear operator $\Delta_p : A^p(M) \to A^p(M)$*

$$\Delta_p := d \circ \delta + \delta \circ d, \tag{2.18}$$

and we call it the **Laplace operator** *or* **Laplacian***. Δ_p is also written as Δ. Δ_p is the* **formal self-adjoint***, that is,*

$$(\Delta_p \omega, \tau)_{L^2} = (\omega, \Delta_p \tau)_{L^2} \qquad (\forall \omega, \tau \in A^p(M)). \tag{2.19}$$

We immediately have the following.

Lemma 2.11 *We take a coordinate on a local chart* $(x^1, \cdots, x^m)$ *for* M*, and we put* $g_{jk} := g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)$*, and* $G := \det(g_{jk})$*. Then, for every* $f \in A^0(M) = C^\infty(M)$*,* Δf *is given by*

$$\Delta f = -\sum_{j,k=1}^{m} \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^j}\left(\frac{1}{\sqrt{G}} g^{jk} \frac{\partial f}{\partial x^k}\right). \tag{2.20}$$

2.3.3 The Dirichlet or Neumann Boundary Value Eigenvalue Problems of the Laplacian of the Euclidian space

Let Δ be the Laplacian of V with respect to the inner product $(\ |\ )$. That is, if we put $\{e_j\}_{j=1}^{\ell}$ as an orthonormal basis of V, and put

$$V \ni \mathbf{x} = \sum_{j=1}^{\ell} \mathbf{x}_i e_i \mapsto (x_1, x_2, \cdots, x_d) \in \mathbb{R}^{\ell}$$

for coordinates of V, then the Laplacian is given by

$$\Delta = -\sum_{j=1}^{\ell} \frac{\partial^2}{\partial x_j^2}$$

(recall Lemma 2.11). We consider the eigenvalue problems of the vibrating membranes of the Laplacian.

From now on, we assume that Ω is a bounded domain in $\mathbb{R}^{\ell}$ whose boundary $\partial\Omega$ is piecewise smooth. Then, we can consider the boundary value eigenvalue problems on Ω as follows.

$$\text{(The Dirichlet problem)} \quad \begin{cases} \Delta u = \lambda u & (\text{on } \Omega) \\ u = 0 & (\text{on } \partial\Omega), \end{cases} \tag{2.21}$$

$$\text{(The Neumann problem)} \quad \begin{cases} \Delta v = \mu v & (\text{on } \Omega) \\ \frac{\partial v}{\partial \boldsymbol{n}} = 0 & (\text{on } \partial\Omega). \end{cases} \tag{2.22}$$

Here, $\dfrac{\partial v}{\partial \boldsymbol{n}}$ means the derivative of v with respect to the inward unit normal vector $\boldsymbol{n}$ at a smooth point on $\partial\Omega$. If there is a solution $u \not\equiv 0$, for some constant λ, then λ is called an **eigenvalue** for the Dirichlet eigenvalue problems on Ω, and u is called the **eigenfunction** for the eigenvalue λ. The space which consists of such u is called the **eigenspace** for the eigenvalue λ. We call its dimension the multiplicity of the eigenvalue λ. Similarly, we define the eigenvalues for the Neumann problems.

The set of all the eigenvalues counted with their multiplicities for both the Dirichlet and Neumann eigenvalue problems (2.21), (2.22) are at most countably infinite sets, and have no accumulation points. We denote all the eigenvalues and the linearly independent eigenfunctions which are orthogonal systems with respect to inner product (,) as follows, respectively.

$$\begin{array}{l} \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_i \leqq \cdots \ (\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_i \leqq \cdots) \\ u_1 \ , u_2 \ , \cdots \ , \ u_i \ , \cdots \ (v_1 \ , \ v_2 \ , \cdots \ , \ v_i \ , \cdots) \end{array}$$

Furthermore, $u_i \in C^\infty(\Omega) \quad (i = 1, 2, \cdots)$ and $\{u_i\}_{i=1}^\infty$ is a complete orthonormal basis of $L^2(\Omega)$ with respect to (,). That is, if $u \in L^2(\Omega)$, then we have

$$u = \sum_{i=1}^{\infty} (u, u_i) u_i \quad (L^2 \ \textit{convergent}).$$

If we put $u \in C_c^\infty(\Omega)$, we have, for each $x \in \Omega$,

$$(\Delta u)(x) = \sum_{i=1}^{\infty} \lambda_i (u, u_i) u_i(x) \quad (\textit{absolutely by convergent})$$

Similarly, $v_i \in C^\infty(\Omega) \quad (i = 1, 2, \cdots)$ and $\{v_i\}_{i=1}^\infty$ is a complete orthonormal basis of $L^2(\Omega)$ with respect to (,). That is, if $v \in L^2(\Omega)$, then we have

$$v = \sum_{i=1}^{\infty} (v, v_i) v_i \quad (L^2 \ \textit{convergent})$$

Particularly, if we put $v \in C_c^\infty(\Omega)$, then we have for each $x \in \Omega$,

$$(\Delta v)(x) = \sum_{i=1}^{\infty} \mu_i (v, v_i) v_i(x) \quad (\textit{absolutely convergent}).$$

Here, we take the alcove $D(R)$ of affine Weyl groups in the previous subsection as a domain Ω in (2.21) or (2.22). That is, we consider the boundary value eigenvalue problems on $D(R)$ as follows.

$$(\text{The Dirichlet problem}) \quad \begin{cases} \Delta f = \lambda f & (\text{on } D(R)) \\ \ \ f = 0 & (\text{on } \partial D(R)), \end{cases}$$

$$(\text{The Neumann problem}) \quad \begin{cases} \Delta f = \mu f & (\text{on } D(R)) \\ \frac{\partial f}{\partial \boldsymbol{n}} = 0 & (\text{on } D(R)). \end{cases}$$

Here, $\mathbf{n}$ is the inward unit normal vector field on $\partial D(R)$, and λ is called the eigenvalue, while $f \not\equiv 0$ is called the eigenfunction.

We write

$$\Sigma_D(R) = \{0 < \lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots\}$$

for the spectrum of the Dirichlet problem for $D(R)$. We write

$$\Sigma_N(R) = \{0 = \mu_0 \leqq \mu_1 \leqq \mu_2 \leqq \cdots\}$$

for the spectrum of the Neumann problem for $D(R)$. Here, we count the eigenvalues with their multiplicities.

2.4 Basic facts of the heat kernel

2.4.1 Aims

Here, we write the basic materials for the application of the heat equation associated with the boundary value eigenvalue problems. That is, we give the expressions of the heat kernel in terms of the solutions (eigenvalues and eigenfunctions) which are calculated for the boundary value eigenvalue problems in affine Weyl groups.

2.4.2 The heat kernel

In this subsection, we consider the heat equation on a compact Riemanian manifold, and its application. We assume $M = (M, g)$ is a compact C^∞ Riemannian manifold, and put dim $M = m$. Also, we regard g as an inner product on the fiber of $\wedge^p T^*M$, which is also written as $\langle \cdot, \cdot \rangle$.

Definition 2.11 *We assume that*

$$\begin{aligned}&\omega \in C^\infty(M \times (0,\infty);\ \wedge^p T^*M) \cap C^0(M \times [0,\infty);\ \wedge^p T^*M)\\ &(\omega(x,t) \in \wedge^p T^*_x M,\ \forall (x,t) \in M \times [0,\infty)).\end{aligned}$$

That is, for the projection $f : M \times (0,\infty) \to M, (x,t) \mapsto x$, then $\omega \in \Gamma(f^*(\wedge^p T^*M))$, and it can be extended to $M \times \{0\}$ continuously. Here, C^0 means the space of all continuous sections. For each t, $\omega(\cdot,t) \in A^p(M)$. Then, we call the **heat equation** *with the* **initial condition** (2.24) *that which is given by, for each $\omega_0 \in A^p(M)$,*

$$\frac{\partial \omega}{\partial t} + \Delta\omega = 0, \tag{2.23}$$

$$\omega(x,0) = \omega_0(x), \tag{2.24}$$

where Δ is the Laplace operator with respect to the variable x.

Now, we notice that the following natural isomorphic relations hold by using the Riemann metric g of M,

$$\begin{aligned}\wedge^p T_x^* M \otimes \wedge^p T_y^* M &\simeq \wedge^p T_x^* M \otimes \wedge^p T_y M \\ &\simeq \mathrm{Hom}(\wedge^p T_y^* M, \wedge^p T_x^* M). \end{aligned} \tag{2.25}$$

Definition 2.12 *We assume that* $e^p \in C^\infty(M \times M \times (0,\infty);\ \wedge^p T^*M \otimes \wedge^p T^*M)$ $(e^p(x,y,t) \in \wedge^p T_x^* M \otimes \wedge^p T_x^* M,\ \forall (x,y,t) \in M \times M \times [0,\infty))$. *This means that if we consider the projection* $f : M \times M \times (0,\infty) \to M \times M, (x,y,t) \mapsto (x,y)$, *then* $e^p \in \Gamma\ (f^*(\wedge^p T^*M \otimes \wedge^p T^*M)$. *If* e^p *satisfies the equation below, we call it the* **fundamental solution** *or* **heat kernel** *for the heat equation* (2.23).

$$\left(\Delta_x + \frac{\partial}{\partial t}\right) e^p(x,y,t) = 0, \tag{2.26}$$

and for any $\omega, \theta \in A^p(M) = \Gamma(\wedge^p T^*M)$,

$$\begin{aligned} &\lim_{t\downarrow 0} \int_M \langle e^p(x,y,t),\, \omega(y)\rangle_y \, dv_g(y) = \omega(x) \qquad (\forall x \in M), \\ &\lim_{t\downarrow 0} \int_M \langle \theta(x),\, e^p(x,y,t)\rangle_x \, dv_g(x) = \theta(y) \qquad (\forall y \in M). \end{aligned} \tag{2.27}$$

Here, Δ_x *is the Laplace operator with respect to* x.

Definition 2.13 *The linear operator*

$$\begin{aligned} L := L_x := \Delta_x + \frac{\partial}{\partial t} :\ &C^\infty(M \times (0,\infty),\, \wedge^p T^*M) \\ &\to C^\infty(M \times (0,\infty),\, \wedge^p T^*M) \end{aligned} \tag{2.28}$$

is called the heat operator (*with respect to the variable* x).

Now, let us consider the case of $M = \mathbb{R}^m$, and $p = 0$. But $\mathbb{R}^m$ is not compact, so we must consider only bounded continuous functions ω, θ in (2.27). Also, we have $\Delta = \delta d = -\sum_{j=1}^m \frac{\partial^2}{\partial (x_j)^2}$. Then, $e(\mathbf{x},\mathbf{y},t)$, which is given by

$$e(\mathbf{x},\mathbf{y},t) := (4\pi t)^{-\frac{m}{2}} e^{-\frac{|x-y|^2}{4t}} \qquad \mathbf{x} = (x_1, \cdots x_m) \in \mathbb{R}^m,\ \ \mathbf{y} = (y_1, \cdots, y_m) \in \mathbb{R}^m, \tag{2.29}$$

becomes the fundamental solution of the heat equation (2.23). For a bounded continuous function ω_0, $\omega(\mathbf{x},t)$ given by

$$\omega(\mathbf{x},t) := \int_{\mathbb{R}^m} e(\mathbf{x},\mathbf{y},t)\, \omega_0(\mathbf{y})\, d\mathbf{y} \tag{2.30}$$

is a solution of (2.23) – (2.24).

Let us return to the case of a compact manifold M.

Lemma 2.12 *If a solution of the heat equation* (2.23) – (2.24) *exists, then it is unique.*

Proof We put $\omega = \omega(x,t)$ for a solution of (2.23). Then, for each $t > 0$, we put $\Omega(t)$ as follows:

$$\Omega(t) := \int_M \langle \omega(x,t),\, \omega(x,t)\rangle_x \, dv_g(x). \tag{2.31}$$

Then, we have

$$\begin{aligned}\frac{d\Omega}{dt}(t) &= 2\int_M \left\langle \frac{\partial\omega}{\partial t}(x,t),\, \omega(x,t)\right\rangle_x dv_g(x)\\ &= 2\int_M \langle \Delta\omega(x,t),\, \omega(x,t)\rangle_x \, dv_g(x)\\ &= -2\int_M \{\langle \delta\omega(x,t),\, \delta\omega(x,t)\rangle_x + \langle d\omega(x,t),\, d\omega(x,t)\rangle_x\} \, dv_g(x)\\ &\le 0.\end{aligned} \tag{2.32}$$

Thus, $\Omega(t)$ is a decreasing function. Now, we assume (2.23) – (2.24) has two solutions, and we set $\omega := \omega_1 - \omega_2$. Then, ω is a solution for (2.23) with the initial-value 0. Thus, we have that $\Omega(t) = 0$. Therefore, for an arbitrary $(x,t) \in M \times [0,\infty)$, we have $\omega(x,t) \equiv 0$, and $\omega_1(x,t) \equiv \omega_2(x,t)$. //

Lemma 2.13 (*Principle of Duhamel*) *We assume that* $u, v \in C^\infty(M \times (0,t);\, \wedge^p T^*M)$, *that is* $u(x,s), v(x,s) \in \wedge^p T^*_x M$, $\forall (x,s) \in M \times (0,t)$. *Then, if we define* $L := \Delta_z + \dfrac{\partial}{\partial s}$ *on each interval* $[\alpha,\beta] \subset (0,t)$, *it holds that*

$$\begin{aligned}&\int_M \{\langle u(z,t-\beta),\, v(z,\beta)\rangle_z - \langle u(z,t-\alpha),\, v(z,\alpha)\rangle_z\} \, dv_g(z)\\ &= \int_\alpha^\beta \int_M \{\langle (Lu)(z,t-s),\, v(z,s)\rangle_z - \langle u(z,t-s),\, (Lv)(z,s)\rangle_z\} \, dv_g(z).\end{aligned} \tag{2.33}$$

Proof We have

$$\begin{aligned}&\langle (Lu)(z,t-s),\, v(z,s)\rangle_z - \langle u(z,t-s),\, (Lv)(z,s)\rangle_z\\ &= \langle (\Delta_z u)(z,t-s),\, v(z,s)\rangle_z - \langle u(z,t-s),\\ &\quad (\Delta_z v)(z,s)\rangle_z + \frac{\partial}{\partial s}\langle u(z,t-s),\, v(z,s)\rangle_z .\end{aligned}$$

When we integrate both sides in z on M, both the first and second terms of the right hand side of the above vanish due to symmetry of Δ. And by integrating in s, we have the desired result. //

Theorem 2.1 *If the heat kernel $e^p = e^p(x, y, t)$ exists, it satisfies that*

$$\left\langle \langle e^p(x,y,t), \theta\rangle_y\,,\, \omega\right\rangle_x = \langle\langle e^p(x,y,t), \omega\rangle_x\,,\, \theta\rangle_y \tag{2.34}$$

for all $(x, y, t) \in M \times M \times (0, \infty)$, and $\omega \in \wedge^p T_x^* M$, $\forall\theta \in \wedge^p T_y^* M$, and it is unique.

Proof Let e_1^p, e_2^p be the heat kernels, and we put

$$u(z,s) := \langle e_1^p(z,x,s),\, \omega\rangle_x\,,\; v(z,s) := \langle e_2^p(z,y,s),\, \theta\rangle_y\,.$$

If we put $L := \Delta_z + \dfrac{\partial}{\partial s}$, then, since $Lu = Lv = 0$ by (2.26) and Lemma 2.13, we have

$$\int_M \langle u(z,t-\beta),\, v(z,\beta)\rangle_z\, dv_g(z) = \int_M \langle u(z,t-\alpha),\, v(z,\alpha)\rangle_z\, dv_g(z). \tag{2.35}$$

On the other hand, due to (2.27), we have

$$\begin{aligned}
&\lim_{\beta\uparrow t} \int_M \langle u(z,t-\beta),\, v(z,\beta)\rangle_z\, dv_g(z)\\
&= \lim_{\beta\uparrow t} \int_M \left\langle \langle e_1^p(z,x,t-\beta),\, \omega\rangle_x\,,\; \langle e_2^p(z,y,t),\, \theta\rangle_y \right\rangle dv_g(z)\\
&= \left\langle \langle e_2^p(z,y,t), \theta\rangle_y\,,\, \omega\right\rangle_x . \qquad (2.36)
\end{aligned}$$

Similarly, due to (2.27), we have

$$\lim_{\alpha\downarrow 0} \int_M \langle u(z,t-\alpha),\, v(z,\alpha)\rangle_z\, dv_g(z) = \langle\langle e_1^p(y,x,t),\; \omega\rangle_x\,,\, \theta\rangle_y\,. \tag{2.37}$$

By (2.35) $\sim$ (2.37), we have

$$\langle\langle e_1^p(y,x,t), \omega\rangle_x\,,\, \theta\rangle_y = \left\langle \langle e_2^p(x,y,t), \theta\rangle_y\,,\, \omega\right\rangle_x . \tag{2.38}$$

Particularly, if we put $e_1^p = e_2^p = e^p$, then we have (2.34). Also, by (2.38) and (2.34), we have

$$\begin{aligned}
\left\langle \langle e_1^p(x,y,t), \theta\rangle_y\,,\, \omega\right\rangle_x &= \langle\langle e_1^p(y,x,t), \omega\rangle_x\,,\, \theta\rangle_y\\
&= \left\langle \langle e_2^p(x,y,t), \theta\rangle_y\,,\, \omega\right\rangle_x .
\end{aligned}$$

Since θ and ω are arbitrary, we have $e_1^p = e_2^p$. //

Therefore, uniqueness of the heat kernel is proved. We will omit the proof of its existence.

Theorem 2.2 *The heat equation* (2.23) – (2.24) *has a unique solution* $\omega(x,t)$*, and, by using the heat kernel, it can be given by*

$$\omega(x,t) : = \int_M \langle e^p(x,y,t),\, \omega_0(y)\rangle_y \, dv_g(y) \quad (\forall (x,t) \in M \times (0,\infty)), \tag{2.39}$$

$$\omega(x,0) : = \lim_{t\downarrow 0} \omega(x,t) \quad (\forall x \in M). \tag{2.40}$$

Proof The uniqueness is proved by Lemma 2.12. The $\omega(x,t)$ given by (2.39) – (2.40) are the solutions of (2.26) and (2.27). //

Now, for each $t > 0$, we define the linear operator $e^{-t\Delta} : A^p(M) \to A^p(M)$.

$$(e^{-t\Delta}\omega_0)(x) := \int_M \langle e^p(x,y,t),\, \omega_0(y)\rangle_y \, dv_g(y) \quad (\forall x \in M,\ \forall \omega_0 \in A^p(M)). \tag{2.41}$$

Also, we write the L^2 completion of $A^p(M)$ as $L^2(A^p(M))$. Then, we have

Theorem 2.3 *For each* $t > 0$*,* $e^{-t\Delta}$ *is extended uniquely to a compact self-adjoint operator*

$$e^{-t\Delta} : L^2(A^p(M)) \;\to\; L^2(A^p(M)).$$

Furthermore, for an arbitrary $\omega_0 \in L^2(A^p(M))$*,* $\omega(x,t) := (e^{-t\Delta}\omega_0)(x)$ *is* C^∞ *on* $M \times (0,\infty)$*. Therefore, we have*

$$e^{-t\Delta}(L^2(A^p(M))) \subset (M) \quad (\forall t > 0).$$

Also, $\{e^{-t\Delta}\}_{t\geq 0}$ *is a semi-group of operators. That is,*

$$e^{-(t+s)\Delta} = e^{-t\Delta} \circ e^{-s\Delta} \quad (\forall t, s > 0), \tag{2.42}$$

$$\lim_{t\downarrow 0} e^{-t\Delta}\omega = \omega \quad (\forall \omega \in L^2(A^p(M))), \tag{2.43}$$

where the left hand side of (2.43) *is* L^2 *convergent.*

Proof For each $t > 0$, the integral kernel $e^p(x,y,t)$ of $e^{-t\Delta}$ is C^∞, and satisfies (2.34). The $e^{-t\Delta}$ can be extended to a compact self-adjoint operator $e^{-t\Delta} : L^2(A^p(M)) \to L^2(A^p(M))$ uniquely, and for each $\omega_0 \in L^2(A^p(M))$, $\omega(x,t) := (e^{-t\Delta}\omega_0)(x)$ is C^∞ on $M \times (0,\infty)$.

Next, we show (2.42). We take $\omega \in A^p(M)$. Then, by Theorem 2.2, both the $e^{-(t+s)\Delta}\omega$ and $e^{-t\Delta} \circ e^{-s\Delta}\omega$ are the solutions of the heat equation (2.23) whose initial-value is $e^{-s\Delta}\omega$ as $t = 0$. Thus, we have $e^{-(t+s)\Delta}\omega = e^{-t\Delta} \circ e^{-s\Delta}\omega$ by the uniqueness of the solution. On the other hand, $A^p(M)$ is dense in $L^2(A^p(M))$. Thus, we have (2.42). The (2.43) is shown as follows. First, if $\omega \in A^p(M)$ in (2.27), for each $x \in M$, it holds that $\lim_{t\downarrow 0}(e^{-t\Delta}\omega)(x) = \omega(x)$, uniformly on M.

Thus, (2.43) is L^2-convergent. Next, we take any $\omega \in L^2(A^p(M))$. By (2.32) (or Theorem 2.4 below), we have $\left\|e^{-t\Delta}\right\|_{L^2} \leq \|\omega\|_{L^2}$. On the other hand, for an arbitrary $\varepsilon > 0$, there exists $\theta \in A^p(M)$ which satisfies $\|\omega - \theta\|_{L^2} \leq \varepsilon$. Thus, we have

$$\begin{aligned}\left\|e^{-t\Delta}\omega - \omega\right\|_{L^2} &\leq \left\|e^{-t\Delta}(\omega - \theta)\right\|_{L^2} + \left\|e^{-t\Delta}\theta - \theta\right\|_{L^2} + \|\omega - \theta\|_{L^2} \\ &< 2\varepsilon + \left\|e^{-t\Delta}\theta - \theta\right\|_{L^2} \to 2\varepsilon \quad (t \downarrow 0).\end{aligned}$$

Since ε is arbitrary, we obtain (2.43). //

Theorem 2.4 (*Strum Liouville's resolution theorem*) *There exists a complete orthonormal basis* $\{\phi_0, \phi_1, \phi_2 \cdots\}$ *in* $L^2(A^p(M))$ *and a sequence of real numbers* $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$*, such that*

$$e^{-t\Delta}\phi_j = e^{-t\lambda_j}\phi_j \quad (\forall t > 0, \forall j = 0, 1, 2 \cdots), \tag{2.44}$$

$$\phi_j \in A^p(M) \quad (\forall j = 0, 1, 2 \cdots), \tag{2.45}$$

$$\textit{the multiplicity of each } \lambda_j \textit{ is finite.} \tag{2.46}$$

Here, (2.46) *means that the number of* λ_k *satisfying* $\lambda_k = \lambda_j$ *is finite for every* λ_j*.*

Proof For each $t > 0$, since $e^{-t\Delta}$ is compact and self-adjoint by Theorem 2.3, all the eigenvalues of $e^{-t\Delta}$, are given as

$$\lambda_0(t) \geq \lambda_1(t) \geq \cdots \downarrow 0,$$

and the corresponding eigenvectors are an orthonomal basis of $L^2(A^p(M))$, denoted by

$$\phi_0(t), \phi_1(t), \cdots .$$

Here, all the eigenvalues are non-negative, because

$$(e^{-t\Delta}\omega, \omega)_{L^2} = \left\|e^{-\frac{t}{2}\Delta}\omega\right\|_{L^2}^2 \geq 0 \tag{2.47}$$

by (2.42). We show that the diagonalization for each $e^{t\Delta}$ gives (2.44). First, by (2.42), for arbitrary $t > 0$ and $k \in \mathbb{N}$, we have

$$\lambda_j(kt) = (\lambda_j(t))^k, \ \phi_j(kt) = \phi_j(t).$$

This holds for each $k \in \mathbb{Q}$, and each real number $k > 0$, too. Because $e^{t\Delta}$ is continuous in t, this holds for arbitrary $t > 0$, and $k > 0$. For each $t > 0$,

$$\begin{cases} \lambda_j(t) = (\lambda_j(1))^t = e^{t(\log\lambda_j(1))}, \\ \phi_j(t) = \phi_j(1). \end{cases} \tag{2.48}$$

On the other hand, because $\frac{d}{dt}\left\|e^{-t\Delta}\omega\right\|^2_{L^2} \leqq 0$ by (2.32), we have that $\log\lambda_j(1) \leqq 0$ by (2.48). Then, if we put

$$\lambda_j := -\log\lambda_j(1),\ \ \phi_j := \phi_j(1),$$

we obtain the diagonalization (2.44). The (2.45) keeps holding follows immediately from the fact that ϕ_j are the eigenvectors of $e^{-t\Delta}$, and that the heat kernels e^p are C^∞. Also, $\{e^{-\lambda_j t}\}$ have no accumulation points aside from 0 (for each $t > 0$). On the other hand, because $e^{-\lambda_j t} > 0$, we obtain (2.46). //

Furthermore, we have

Corollary 2.1 *We assume that* $\{\lambda_j\}_{j\geq 0}$, $\{\phi_j\}_{j\geq 0}$ *are as in Theorem* 2.4. *Then,* ϕ_j *are the eigenvectors which correspond to eigenvalue* $-\lambda_j$ *of* Δ. *That is, they satisfy* (2.51). *Particularly, if we define* K :=*the space which is spanned by* ϕ_j *corresponding to* $\lambda_j = 0$, *then we have*

$$K = \text{Ker }(\Delta) := \{\omega \in A^p(M);\ \Delta\omega = 0\}. \tag{2.49}$$

Thus, particularly, we have

$$\dim \text{Ker }(\Delta) < \infty. \tag{2.50}$$

Proof If we operate $\Delta + \partial/\partial t$ on the both sides of (2.44), we have

$$\begin{aligned} 0 &= \left(\Delta + \frac{\partial}{\partial t}\right)(e^{-t\Delta}\phi_j) \\ &= e^{-\lambda_j t}\Delta\phi_j + \left(\frac{\partial}{\partial t}e^{-\lambda_j t}\right)\phi_j \\ &= e^{-\lambda_j t}\{\Delta\phi_j - \lambda_j\phi_j\}. \end{aligned}$$

Then, we have

$$\Delta\phi_j = \lambda_j\phi_j. \tag{2.51}$$

The proof of (2.49) is as follows. That $K \subset \text{Ker}(\Delta)$ is clear. We show that $K \supset \text{Ker}(\Delta)$. Let $\omega \in \text{Ker}(\Delta)$. That is, $\Delta\omega = 0$. Then, it holds that $(\Delta + \partial/\partial t)\,\omega = 0$. Thus, ω is a solution for (2.23) with the initial-value ω. Therefore, by Theorem 2.2, $\omega = e^{-t\Delta}\omega$. By this and Theorem 2.4, we have

$$\begin{aligned} \omega &= e^{-t\Delta}\left(\sum_{j=0}^{\infty}(\omega,\phi_j)_{L^2}\,\phi_j\right) = \sum_{j=0}^{\infty}(\omega,\phi_j)_{L^2}\,e^{-t\Delta}\phi_j \\ &= \sum_{j=0}^{\infty}e^{-\lambda_j t}(\omega,\phi_j)_{L^2}\,\phi_j, \end{aligned}$$

and then, we have

$$(\omega, \phi_j)_{L^2} = e^{-\lambda_j t}(\omega, \phi_j)_{L^2} \quad (\forall j = 0, 1, 2, \cdots).$$

Thus, we obtain $\omega \in K$. We have (2.49). (2.50) follows from this and (2.46). For arbitrary $\omega \in A^p(M)$, it holds that

$$\Delta\omega = \sum_{j=0}^{\infty}(\omega, \phi_j)_{L^2}\, \Delta\phi_j = \sum_{j=0}^{\infty} \lambda_j(\omega, \phi_j)_{L^2}\, \phi_j. \tag{2.52}$$

By this, we immediately have (2.49) . //

Corollary 2.2 *Let us denote by H the orthogonal projection : $A^p(M) \to \mathrm{Ker}(\Delta)$, (as the L^2 sense). If we denote by λ_k the k-th eigenvalue of Δ which is different from 0, then we have*

$$\left\|e^{t\Delta}\omega_0 - H\omega_0\right\|_{L^2} \leqq e^{-\lambda_k t}\left\|\omega_0 - H\omega_0\right\|_{L^2} \quad (\forall\omega_0 \in A^p(M),\ \forall t > 0). \tag{2.53}$$

Therefore, $e^{-t\Delta}\omega_0 \to H\omega_0$ $(t \uparrow \infty)$.

Proof By Theorem 2.4, we have

$$e^{-t\Delta}\omega_0 - H\omega_0 = \sum_{j\geq k}^{\infty} e^{-\lambda_j t}(\omega_0, \phi_j)_{L^2}\phi_j.$$

Thus, we have

$$\begin{aligned}\left\|e^{-t\Delta}\omega_0 - H\omega_0\right\|_{L^2}^2 &= \sum_{j\geq k}^{\infty}(e^{-\lambda_j t})^2(\omega_0, \phi_j)_{L^2}^2 \\ &\leqq (e^{-\lambda_k t})^2 \sum_{j\geq k}^{\infty}(\omega_0, \phi_j)_{L^2} \\ &= (e^{-\lambda_k t})^2 \left\|\omega_0 - H\omega_0\right\|_{L^2}^2 ,\end{aligned}$$

and we obtain (2.53). //

Theorem 2.5 *We assume that $\{\lambda_j\}_{j\geq 0}$, $\{\phi_j\}_{j\geq 0}$ are as in Theorem 2.4. Then the heat kernel e^p can be written as*

$$e^p(x, y, t) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\phi_j(x) \otimes \phi_j(y) \quad (\forall(x, y, t) \in M \times M \times (0, \infty)). \tag{2.54}$$

Here, the right hand side is the uniform convergence on $M \times M$ for each $t > 0$.

The proof is given by using Mercer's theorem, but it is omitted.

3 Pinsky's works on the eigenvalue problem on the equilateral triangle domain

3.1 Summary

In this section, we explain Pinsky's work [20], which is the Dirichlet boundary value eigenvalue problems, and Neumann boundary value eigenvalue problems, on the equilateral triangle domain.

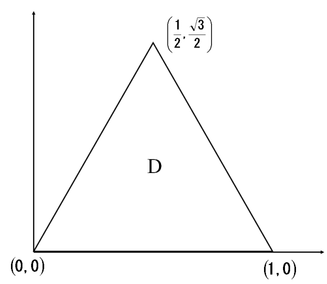
He extended the functions to the outside of the equilateral triangle domain, and explicitly calculated the eigenvalues and eigenfunctions on the equilateral triangle from the periodicity of the functions.

Pinsky's works were extended by Bérard, who considered the boundary value problems in general crystallographic domains. We show Pinsky's works on methods of calculation for the eigenvalues and the eigenfunctions.

3.2 The Dirichlet eigenvalue problems

We determine the eigenvalues of the Laplacian for the Dirichlet boundary value eigenvalue problems on the equilateral triangle domain

$$D = \left\{(x,\ y)\ :\ 0 < y < x\sqrt{3},\ y < \sqrt{3}(1-x)\right\}. \tag{3.1}$$



Theorem 3.1 *The eigenvalues of the Laplacian on D are*

$$\lambda_{mn} = \left(\frac{16\pi^2}{27}\right)(m^2+n^2+mn) \quad m,\, n = \pm 1,\ \pm 2, \cdots, \tag{3.2}$$

satisfying the following conditions:

$$\begin{aligned} &\text{(A)} \quad m+n \ \textit{is a multiple of}\ 3, \\ &\text{(B)} \quad m \neq 2n, \\ &\text{(C)} \quad n \neq 2m. \end{aligned} \tag{3.3}$$

The multiplicity of λ_{mn} is $\frac{1}{6}$ times the numbers which appear in the lattice (3.2). *The eigenfunctions are of the form*

$$f(x,y) = \sum_{(m,n)} \pm\exp\left(\frac{2\pi i}{3}\right)\left(nx + \frac{(2m-n)y}{\sqrt{3}}\right). \tag{3.4}$$

We remark that Pinsky **incorrectly** wrote

$$f(x, y) = \sum_{(m,n)} \pm\exp\left(\frac{2\pi i}{3}\right)\left(nx + \frac{(2n-m)y}{\sqrt{3}}\right).$$

In this sum, (m, n) *runs over the subset* $\mathfrak{S} \subseteq \mathbb{Z}^2$*, where* $|\,\mathfrak{S}\,| = 6$ *and the sign* $\pm$ *is determined by the following rules:*

$$(-n, m-n) \to (-n, -m) \to (n-m, -m) \to (n-m, n) \to (m, n) \to (m, m-n). \tag{3.5}$$

Each transition induces a change of sign in the $(m,\, n)$ *entry of* (3.4).

Notice that $|\,\mathfrak{S}\,| = 6$ which is the same value as the number of eigenvalues of combination of (m, n) that exist, that is, 6 kinds. We introduce the rotation operater by

$$R : (x, y) \mapsto \left(1 - \frac{x}{2} - \frac{y\sqrt{3}}{2}, \frac{x\sqrt{3}}{2} - \frac{y}{2}\right).$$

An eigenfunction is said to be symmetric if $R \circ f = f$. An eigenfunction is said to be complex if $R \circ f = \exp\left(\pm\frac{2\pi i}{3}\right) f$.

Corollary 3.1 *The eigenvalue* λ_{mn} *corresponds to a symmetric eigenfunction iff the following additional condition holds:*

$$\text{(D)} \ \ m \text{ is a multiple of } 3,$$

(*equivalently, the associated eigenfunction is periodic in* x *with period* 1). *The eigenvalue* λ_{mn} *belongs to a complex eigenfunction iff* m *is congruent to* $+1$ *or* -1, *modulo* 3. *In particular, each eigenvalue cannot belong simulataneously to both the complex eigenfunction and symmetric eigenfunction.*

Corollary 3.2 *The following are symmetric eigenfunctions and give a complete list of the simple eigenvalues.*

$$f_p(x, y) = sin(2\pi p\overline{d_1}) + sin(2\pi p\overline{d_2}) + sin(2\pi p\overline{d_3}) \quad p = 1, 2, \cdots$$
$$\left(\overline{d_1} = \frac{2y}{\sqrt{3}}, \ \overline{d_2} = x - \frac{y}{\sqrt{3}}, \ \overline{d_3} = 1 - x - \frac{y}{\sqrt{3}}\right) \tag{3.6}$$

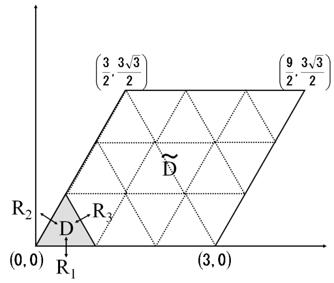
Here, $\overline{d_1}$, $\overline{d_2}$, $\overline{d_3}$ *are the normalized altitudes of the point* (x, y) *in the triangle* D. *They satisfy the normalization condition* $\overline{d_1} + \overline{d_2} + \overline{d_3} = 1$ *and the reflection laws* $R_i \circ \overline{d_i} = -\overline{d_i}$ $(i = 1, 2, 3)$. *The eigenvalues are obtained by the formula*

$$\lambda_{3p,0} = \left(\frac{16\pi^2}{27}\right)(9p^2).$$

Proof of Theorem 3.1

To prove these results, we introduce the parallelogram

$$\tilde{D} = \left\{ (x,y) : 0 < y < \frac{3\sqrt{3}}{2},\ \frac{y}{\sqrt{3}} < x < 3 + \frac{y}{\sqrt{3}} \right\},$$



and the reflection operators:

$$R_1 : (x,y) \to (x,\ -y)$$

$$R_2 : (x,y) \to \left(-\frac{x}{2} + \frac{y\sqrt{3}}{2},\ \frac{x\sqrt{3}}{2} + \frac{y}{2} \right)$$

$$R_3 : (x,y) \to \left(\frac{3}{2} - \frac{x}{2} + \frac{y\sqrt{3}}{2},\ \frac{y}{2} + \frac{\sqrt{3}}{2} + \frac{x\sqrt{3}}{2} \right).$$

There is a canonical isomorphism between $L^2(D)$ and the subspace H of $L^2(\tilde{D})$ defined by

$$H = \left\{ f \in L^2(\tilde{D}) :\ R_i \circ f = -f\ (i = 1, 2, 3) \right\} \tag{3.7}$$

which is given by $H \ni f \mapsto f$. Thus, any eigenfunction of the Laplacian on D can be obtained by solving the equation on H. The restriction to D will automatically satisfy the Dirichlet boundary conditions. By standard results, one can obtain a complete list of the eigenfunctions of the Laplacian on D, which are given by linear combinations of

$$\tilde{f}(x,y) = \exp\left\{ i\left(\alpha x + \beta y\right) \right\}, \tag{3.8}$$

where $(\alpha,\ \beta)$ are in the dual lattice. This requires that $3\alpha = 2n\pi$, $\frac{3\alpha}{2} + \frac{3\sqrt{3}\beta}{2} = 2m\pi$ (m, n are integers). Thus we have $\alpha = \frac{2n\pi}{3}$, $\beta = 2\pi\frac{(2m-n)}{3\sqrt{3}}$. It is readily verified that, corresponding to this,

$$\begin{aligned} \lambda_{mn} &= \alpha^2 + \beta^2 \\ &= \left(\frac{2n\pi}{3}\right)^2 + (2\pi)^2 \frac{(2m-n)^2}{27} \\ &= \left(\frac{16\pi^2}{27}\right)(m^2 + n^2 + mn). \end{aligned} \tag{3.9}$$

The eigenfunction is therefore of the form

$$\tilde{f} = \sum_{(m,n)} A_{mn} \exp\left\{ \left(\frac{2\pi i}{3}\right) \left(nx + \frac{(2m-n)y}{\sqrt{3}} \right) \right\}, \tag{3.10}$$

where the sum is over the set of (m,n) with $\lambda_{mn} = \lambda$. To satisfy the reflection conditions, we write

$$R_1 \circ \tilde{f} = \sum_{m',n'} A_{m',n'} \exp\left\{\left(\frac{2\pi i}{3}\right)\left(n'x - \frac{(2m'-n')y}{\sqrt{3}}\right)\right\}$$
$$= \sum_{m,n} A_{n-m,n} \exp\left\{\left(\frac{2\pi i}{3}\right)\left(nx + \frac{(2m-n)y}{\sqrt{3}}\right)\right\}.$$

We remark that Pinsky **incorrectly** wrote

$$\sum_{m,n} A_{m-n,n} \exp\left\{\left(\frac{2\pi i}{3}\right)\left(nx + \frac{(2m-n)y}{\sqrt{3}}\right)\right\}$$

for the final equation in his paper ($m' = n - m$, $n' = n$).

Hence $R_1 \circ \tilde{f} = -\tilde{f}$ requires that

$$A_{m',n'} = -A_{n-m,n}. \tag{3.11}$$

The second reflection operator is

$$R_2 \circ \tilde{f} = \sum_{m',n'} A_{m',n'} \exp\left\{\left(\frac{2\pi i}{3}\right)\left(n'\left(-\frac{x}{2} + \frac{y\sqrt{3}}{2}\right) + (2m'-n')\left(\frac{x}{2} + \frac{y}{2\sqrt{3}}\right)\right)\right\}$$
$$= \sum_{m',n'} A_{m',n'} \exp\left\{\left(\frac{2\pi i}{3}\right)\left((m'-n')x + \frac{(m'+n')y}{\sqrt{3}}\right)\right\}$$
$$= \sum_{m,n} A_{m,m-n} \exp\left\{\left(\frac{2\pi i}{3}\right)\left(nx + \frac{(2m-n)y}{\sqrt{3}}\right)\right\}.$$

We remark that Pinsky **incorrectly** wrote

$$\sum_{m',n'} A_{m',n'} \exp\left\{\left(\frac{2\pi i}{3}\right)\left((m-n')x + \frac{(m'+n')y}{\sqrt{3}}\right)\right\}$$

for the second equation.

Therefore, we must have

$$A_{m',n'} = -A_{m,m-n}. \tag{3.12}$$

The third reflection operator is

$$R_3 \circ \tilde{f} = \sum_{m',n'} A_{m',n'} \exp\left\{\left(\frac{2\pi i}{3}\right)\left(n'\left(\frac{3}{2} - \frac{x}{2} - \frac{y\sqrt{3}}{2}\right) + (2m'-n')\left(\frac{1}{2} - \frac{x}{2} + \frac{y}{2\sqrt{3}}\right)\right)\right\}$$
$$= \sum_{m',n'} A_{m',n'} \exp\left\{\left(\frac{2\pi i}{3}\right)(n'+m')\right\} \exp\left\{\left(\frac{2\pi i}{3}\right)\left(-m'x + \frac{(m'-2n')y}{\sqrt{3}}\right)\right\}$$
$$= \sum_{m,n} A_{-n,-m} \exp\left\{\left(\frac{2\pi i}{3}\right)(-m-n)\right\} \exp\left\{\left(\frac{2\pi i}{3}\right)\left(nx + \frac{(2m-n)y}{\sqrt{3}}\right)\right\}.$$

We remark that Pinsky **incorrectly** wrote.

$$\sum_{m,n} A_{-n,-m}\text{exp}\left\{\left(\frac{2\pi i}{3}\right)(-m-n)\right\}\text{exp}\left\{\left(\frac{2\pi i}{3}\right)\left(mx+\frac{(2m-n)y}{\sqrt{3}}\right)\right\}.$$

for the final equation.
Thus we must have

$$A_{m',n'} = -A_{-n,-m}\text{exp}\left\{\left(\frac{2\pi i}{3}\right)(-m-n)\right\}. \tag{3.13}$$

Now, if for a fixed (m_0, n_0) we have $A_{m_0,n_0} = A$, then by iterating (3.11) and (3.22) and referring to the graph (3.5), we see that $\text{exp}\left\{\left(\frac{2\pi i}{3}\right)(-m-n)\right\} = 1$, *i.e.*, $m+n$ is a multiple of 3, which proves the condition (A).
To prove the condition (B), assume to the contrary that $m_0 = 2n_0$. This is the same as $(m_0, n_0) = (m_0, m_0 - n_0)$. The property (3.12) therefore requires that $A = -A$, i.e., $A = 0$. Similary, to prove the condition (C), we note that $n_0 = 2m_0$ is the same as $(m_0, n_0) = (n_0 - m_0, n_0)$, which by (3.13) requires that $A = 0$. Conversely, one can verify directly that any sum of the form (3.4) satisfies the reflection conditions (3.7) and is therefore an eigenfunction. Thus we have proved the theorem. //

Proof of Corollary 3.1

We study the effect of the rotation operator on (3.10):

$$\begin{aligned} R\circ\tilde{f} &= \sum_{(m',n')} A_{m',n'}\text{exp}\left\{\left(\frac{2\pi i}{3}\right)(n')\right\}\text{exp}\left\{\left(\frac{2\pi i}{3}\right)\left((m'-n')x+\frac{(m'+n')y}{\sqrt{3}}\right)\right\} \\ &= \sum_{(m,n)} A_{n-m,-m}\text{exp}\left\{\left(\frac{2\pi i}{3}\right)(-m)\right\}\text{exp}\left\{\left(\frac{2\pi i}{3}\right)\left(nx+\frac{(2m-n)y}{\sqrt{3}}\right)\right\}. \end{aligned}$$

But (3.11) and (3.12) require that $A_{n-m,n} = A_{m,n}$. Therefore, we must have $\text{exp}\left\{\left(\frac{2\pi i}{3}\right)(-m)\right\} = 1$, i.e., m is a multiple of 3. Similarly, if m is congruent to ± 1 modulo 3, it is clear that $R\circ\tilde{f} = \text{exp}\left(\pm\frac{2\pi i}{3}\right)\ \tilde{f}$, i.e., f is a complex eigenfunction. //

We remark that Pinsky **incorrectly** wrote

$$\sum_{(m,n)} A_{n-m,-m}\text{exp}\left\{\left(\frac{2\pi i}{3}\right)(-m)\right\}\text{exp}\left\{\left(\frac{2\pi i}{3}\right)\left(mx+\frac{(2m-n)y}{\sqrt{3}}\right)\right\}$$

in the final equation.

Proof of Corollary 3.2

We first note that for the choice $m = 3p, n = 0$, the formula (3.4) yields the symmetric eigenfunction (3.6). One can now show that these correspond to the only simple eigenvalues of the Laplacian on D. Indeed, suppose that λ_{mn} is a simple eigenvalue corresponding to the set $\mathfrak{S}$ described in (3.5). If $(m, 0) \in \mathfrak{S}$ for some m, then by Theorem 3.1, $m = 3p$ and the result is proved. Therefore, one may suppose that $\mathfrak{S}$ contains no pair of the form $(m, 0)$. Thus one may write

$$\mathfrak{S} = \{(m_0, n_0), (m_0, m_0 - n_0), (-n_0, m_0 - n_0), (-n_0, -m_0), (n_0 - m_0, -m_0), (n_0 - m_0, n_0)\}.$$

By hypothesis one must have $n_0 \neq 0,\ m_0 \neq 0,\ n_0 \neq m_0$. Let $\overline{\mathfrak{S}}$ be the component of $\mathfrak{S}$. Then,

$$\overline{\mathfrak{S}} \equiv \{(n_0, m_0), (n_0, n_0 - m_0), (-m_0, n_0 - m_0), (-m_0, n_0 - m_0), (m_0 - n_0, n_0), (m_0 - n_0, m_0)\}.$$

Clearly $\mathfrak{S} \cap \overline{\mathfrak{S}} = \phi$, and therefore $\overline{\mathfrak{S}}$ can be used to manufacture a new eigenfunction according to formula (3.4) with the same eigenvalue, which is a contradiction. Therefore $(m, 0) \in \mathfrak{S}$ for some m and necessarily $m = 3p$ by Theorem 3.1. This again leads to formula (3.6), which was to be proved. //

3.3 The Neumann eigenvalue problem

The methods of 3.1 and 3.2 can also be applied to enumerate the eigenvalues of the problem

$$\Delta f + \lambda f = 0 \ \ (on\ \ D), \quad \left.\frac{\partial f}{\partial n}\right|_{\partial D} = 0 \ \ (on\ \partial D)$$

Indeed, given f on D, we lift f to a function $\tilde{f}$ on $\tilde{D}$ satisfying $R_i \circ \tilde{f} = +\tilde{f},\ i = 1, 2, 3;\ \tilde{f}|_D = f$. $\tilde{f}$ will still be an eigenfunction on D and hence a linear combination of (3.8) with the same values of (α, β). Applying the reflection operation, we have the following result.

Proposition 3.1 *The eigenvalue of Laplacian on on D with Neumann boundary conditions are given by the numbers*

$$\lambda_{mn} = \left(\frac{16\pi^2}{27}\right)(m^2 + n^2 + mn) \quad m, n = 0, \pm 1, \cdots (m + n \ \ is\ \ a\ \ multiple\ \ of\ \ 3).$$

The eigenfunctions are of the form

$$f = \sum_{(m,n)} A_{mn} \exp\left\{\left(\frac{2\pi i}{3}\right)\left(nx + \frac{(2m - n)y}{\sqrt{3}}\right)\right\},$$

where $(m,\ n)$ range over $\mathfrak{S} \subseteq Z^2$ with $|\mathfrak{S}| = 6$ determined by the transformation (3.5)*. One can prove that the number of combinations of $(m,\ n)$ corresponding to the same eigenvalues is* 6*, and then $|S| = 6$.*

3.4 Calculation of the eigenvalues and eigenfunctions on a torus

We calculate the eigenvalues and eigenfunctions of the Laplacian on the special torus which is a fundamental domain $\tilde{D}$ as follows. The L^2 function in $(x,\ y)$ on the torus which has a fundamental domain $\tilde{D}$ satisfies the periodicity conditions as follows.

$$\begin{aligned} \tilde{f}(x, : y) &= \tilde{f}(x+3,\ y), \\ \tilde{f}(x,\ y) &= \tilde{f}\left(x+\frac{3}{2},\ y+\frac{3\sqrt{3}}{2}\right). \end{aligned} \tag{3.14}$$

We show the form of $\tilde{f}(x,\ y)$ explicitly. If we put $\alpha = (3,\ 0),\ \beta = \left(\frac{3}{2},\ \frac{3\sqrt{3}}{2}\right)$, the lattice L which gives the fundamental domain $\tilde{D}$ is given by

$$L := \{m\alpha + n\beta;\ m,\ n = 0, \pm 1,\ \pm 2,\ \cdots\}.$$

Also, if we put $\xi = \left(\frac{1}{3},\ -\frac{1}{3\sqrt{3}}\right),\ \eta = \left(0,\ \frac{2}{3\sqrt{3}}\right)$, the dual lattice L^* of L is given by

$$L^* := \{p\xi + q\eta;\ p,\ q = 0, \pm 1,\ \pm 2,\ \cdots\}.$$

$(\alpha \,|\, \xi) = 1,\ (\beta \,|\, \eta) = 1, (\alpha \,|\, \eta) = 0$, and $(\beta \,|\, \xi) = 0$. Therefore,
$(p\xi + q\eta \,|\, m\alpha + n\beta) = pm + qn$ (integer).

Lemma 3.1 *$\tilde{f}(x, y)$ is given by a linear combination of continuous functions $\chi_{\boldsymbol{p},\boldsymbol{q}}(x,\ y)$, which are given by*

$$\chi_{\boldsymbol{p},\boldsymbol{q}}(x,\ y) = e^{2\pi i(x,\ p\xi+q\eta)} = \exp\left\{\left(\frac{2\pi i}{3}\right)\left(\frac{px}{3} + \frac{(-p+2q)y}{3\sqrt{3}}\right)\right\}.$$

Proof If we put $\chi_{\boldsymbol{p},\boldsymbol{q}}(x, y) = e^{2\pi i(x,\ p\xi+q\eta)} = \chi_{\boldsymbol{p},\boldsymbol{q}}(\mathbf{x})$, we have

$$\begin{aligned} \chi_{\boldsymbol{p},\boldsymbol{q}}(\mathbf{x} + m\alpha + n\beta) &= e^{2\pi i(\mathbf{x}+m\alpha+n\beta,\ p\xi+q\eta)} \\ &= e^{2\pi i\{(\mathbf{x},\ p\xi+q\eta)+(m\alpha+n\beta,\ p\xi+q\eta)\}} \\ &= e^{2\pi i(\mathbf{x},\ p\xi+q\eta)} \cdot e^{2\pi i(mp+nq)} \\ &= \chi_{\boldsymbol{p},\boldsymbol{q}}(\mathbf{x}) \quad (\mathbf{x} = (x, y)). \end{aligned}$$

Here, $mp + nq$ are integers. Thus, $\chi_{\boldsymbol{p},\boldsymbol{q}}$ satisfies (3.14). The set $\{\chi_{\boldsymbol{p},\boldsymbol{q}}; p, q = 0, \pm 1, \pm 2, \cdots\}$ is a basis of (3.14), and $\{\chi_{\boldsymbol{p},\boldsymbol{q}}; p, q = 0, \pm 1, \pm 2, \cdots\}$ is linearly independent, and any function $\tilde{f}$ which satisfies (3.14) can be written as

$$\tilde{f} = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (\tilde{f},\ \chi_{\boldsymbol{p},\boldsymbol{q}})\chi_{\boldsymbol{p},\boldsymbol{q}}.$$

Applying the Laplacian to $\chi_{\boldsymbol{p,q}}(x,\ y)$, we have

$$\Delta\chi_{\boldsymbol{p,q}}(x,y) = 4\pi^2 \left\{ \left(\frac{p}{3}\right)^3 + \left(\frac{-p+2q}{3\sqrt{3}}\right)^2 \right\} \chi_{\boldsymbol{p,q}}.$$

Here, $\Delta = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. //

By this lemma, we obtain the following theorem.

Theorem 3.2 *The functions $\chi_{\boldsymbol{p,q}}(x,y)$ exhaust all the eigenfunctions on the torus whose fundamental domain is $\tilde{D}$. The eigenvalues λ of the Laplacian on the torus whose fundamental domain is $\tilde{D}$ are*

$$\lambda = 4\pi^2 \left\{ \left(\frac{p}{3}\right)^3 + \left(\frac{-p+2q}{3\sqrt{3}}\right)^2 \right\} \quad (p,\ q = 0, \pm 1, \pm 2, \cdots),$$

and then, the eigenfunctions are $\chi_{\boldsymbol{p,q}}$, where

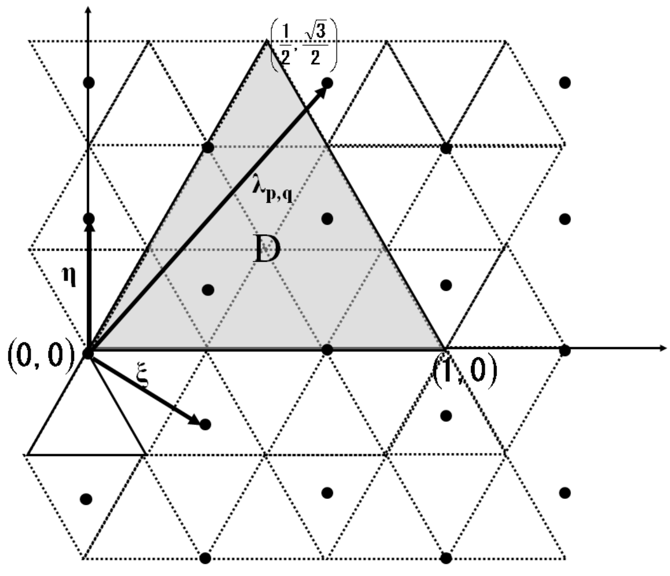
$$\chi_{\boldsymbol{p,q}}(x,\ y) = \exp\left\{2\pi i \left(\frac{px}{3} + \frac{(-p+2q)y}{3\sqrt{3}}\right)\right\}.$$

By using of this theorem, if we put $p = n, q = m$, we have

$$\chi_{\boldsymbol{p,q}}(x,\ y) = \chi_{\boldsymbol{p,q}}(x,\ y) = \tilde{f}(x,\ y) = \exp\left\{2\pi i \left(\frac{nx}{3} + \frac{(2m-n)y}{3\sqrt{3}}\right)\right\},$$

so that we have the eigenfunctions (3.4). We also have the eigenvalues λ (3.2).

In the figure below, if we take ξ, η as a basis for the domain, we have the lattice. Each point on the lattice corresponds to each eigenvalue $\lambda_{p,q}$.



4 Bérard's works

4.1 Summary

Bérard calculated the eigenvalues and the eigenfunctions of the Laplacian in crystallographic groups by using affine Weyl group theory. The features are as follows.

(1) He extended the problems to affine Weyl groups.

(2) He figured out $D(R)$, the eigenvalues, and the eigenfunctions, but his expressions are not explicit.

We show his theorem here.

Theorem 4.1 *The Dirichlet and Neumann eigenvalues are given as follows. Here, the Dirichlet boundary value eigenvalues run over* $\mathbf{p} \in P(R) \cap C(R)$, *and the Neumann boundary value eigenvalues run over* $\mathbf{p} \in P(R) \cap \overline{C(R)}$.

The eigenfunctions $f_{\mathbf{p}}(\mathbf{x}) = \exp(2\pi i(\mathbf{x}|\mathbf{p}))$ *are calculated as follows.*

All the Dirichlet and Neumann eigenvalues $\lambda_{\mathbf{p}}$ *on* $D(R)$ *are given by*

$$\lambda_{\mathbf{p}} = 4\pi^2(\mathbf{p}|\mathbf{p}). \tag{4.1}$$

The Dirichlet and Neumann eigenfunctions corresponding to the above eigenvalues are as follows.

(1) *The Dirichlet eigenfunctions are given by*

$$J_{\mathbf{p}}(\mathbf{x}) = \sum_{w \in W(R)} \varepsilon(w) f_{w(\mathbf{p})}(\mathbf{x}) \quad (\mathbf{x} \in V). \tag{4.2}$$

Here, we call $J_{\mathbf{p}}(\mathbf{x})$ *the* **anti-invariant** *element of* $W_a(R)$, *which satisfies*

$$J_{\mathbf{p}}(w(\mathbf{x})) = \varepsilon(w) J_{\mathbf{p}}(\mathbf{x}), \quad {}^{\forall} w \in W_a(R).$$

$J_{\mathbf{p}}(\mathbf{x})$ *exhaust all the anti-invariant elements.*

(2) *The Neumann eigenfunctions are given by*

$$S_{\mathbf{p}}(\mathbf{x}) = \sum_{w \in W(R)} f_{w(\mathbf{p})}(\mathbf{x}) \quad (\mathbf{x} \in V). \tag{4.3}$$

Here, we call $S_{\mathbf{p}}(\mathbf{x})$ *the* **invariant** *element of* $W_a(R)$, *which satisfies*

$$S_{\mathbf{p}}(w(\mathbf{x})) = S_{\mathbf{p}}(\mathbf{x}), \quad {}^{\forall} w \in W_a(R).$$

$S_{\mathbf{p}}(\mathbf{x})$ *exhaust all the invariant elements.*

4.2 Proof of Theorem 4.1

Let us consider two kinds of eigenvalue problems on $D(R)$ which are given as follows.

$$\text{(The Dirichlet problem)} \quad \begin{cases} \Delta f = \lambda f & (\text{on } D(R)) \\ f = 0 & (\text{on } \partial D(R)), \end{cases} \tag{4.4}$$

$$\text{(The Neumann problem)} \quad \begin{cases} \Delta f = \mu f & (\text{on } D(R)) \\ \frac{\partial f}{\partial \boldsymbol{n}} = 0 & (\text{on } D(R)). \end{cases} \tag{4.5}$$

Here, $\mathbf{n}$ is the unit normal vector on $\partial D(R)$.

If f is the eigenfunction with the Dirichlet problems for the eigenvalue λ in (4.4), one can extend f to a function $\tilde{f}$ on V as follows.

For all $\mathbf{x} \in V$, and all $w \in W_a(R)$, we extend $\tilde{f}$ in such a way that:

$$\begin{cases} \tilde{f}(w(x)) = \varepsilon(w)\tilde{f}(x), \\ \tilde{f}|_{D(R)} = f. \end{cases} \tag{4.6}$$

Lemma 4.1 $\tilde{f}|_{\partial D(R)} = 0.$

Proof

If $\mathbf{x} \in \partial D(R)$, we have $\mathbf{x} \in L_{\alpha,k}$ (for some $\alpha \in R$ and some integer k) and $S_{\alpha,k}(\mathbf{x}) = \mathbf{x}$. And we have $\tilde{f}(\mathbf{x}) = \tilde{f}(S_{\alpha,k}(\mathbf{x})) = (-1)\tilde{f}(\mathbf{x})$. Thus, we obtain $\tilde{f}(\mathbf{x}) = 0$. //

Lemma 4.2 *$\tilde{f}$ is C^∞ on V.*

Proof

We assume u is the solution of (4.4) (or (4.5)). We extend it to a $\tilde{u}$ on $\mathbb{R}^\ell$ which satisfies (4.6) for the Dirichlet boundary value eigenvalue problem. It is clear that $\tilde{u}$ is C^∞ except on a hyperplane with respect to reflection. By simple discussion of integration by parts (by Schwartz's reflective law), we show that $\tilde{u}$ is C^∞, with the exception of the subspace whose codimension is more than or equal to 2. Let us denote by O, such a point, and assume that $(d-j)$ is the dimension of the corresponding submanifold Y. (If the point O is at the summit, $d = j$, we need to take their ridge line, by orders increasing codimension.) We assume that $\mathbf{y} = (\mathrm{y}_1, \cdots, \mathrm{y}_{d-j})$ is the coordinate on $\mathbf{y}$, and $\mathbf{x} = (\mathrm{x}_1, \cdots, \mathrm{x}_j)$ is the coordinate on Y and orthogonal direction, where we set $Y = \{\mathbf{x} = 0\}$. We take a small sphere B inside of Y whose center is O and whose radius is small enough. We already have $\tilde{u} \in H^1(B)$ by classical theory. On the other hand, $\Delta\tilde{u} = \lambda\tilde{u}$ on $B \backslash Y$. We assume $\tilde{\chi} \in C^\infty(R_+)$. Here, we assume that $\tilde{\chi}$ satisfy the following

$$0 \leqq \tilde{\chi} \leqq 1, \quad \tilde{\chi} = 0 \left(t \leqq \frac{1}{2} \right), \quad \tilde{\chi} = 1 \ (t \geqq 1).$$

We put $\chi_\varepsilon(\mathbf{x}, \mathbf{y}) := \tilde{\chi}\left(\frac{|x|}{\varepsilon}\right)$. When $\varepsilon \to 0$, $\chi_\varepsilon \to 1$ almost anywhere. We can easily see that $\{\nabla\chi_\varepsilon\}$ is bounded in $L^2(B)$. χ_ε converges to 1 in $H^1(B)$ as $\varepsilon \to 0$ since $j \geqq 2$. We assume that f belongs to $C_0^\infty(B)$. Then we can show that $\int_B u\Delta f = \lambda \int_B uf$ ($\Delta u = \lambda u$ in the sense of distributions on B). We have

$$\int_B u\Delta f = \lim_{\varepsilon\to 0} \int_B u\Delta f \cdot \chi_\varepsilon.$$

Thus,

$$\int_B u\Delta f = \int_B \nabla u \cdot \Delta f \quad (\text{because } \lim_{\varepsilon\to 0} \int_B u\Delta f \cdot \nabla\chi_\varepsilon = 0).$$

On the other hand,

$$\int_B \nabla u \cdot \Delta f = \lambda \int_B u \cdot f \quad (\text{because } \lim_{\varepsilon\to 0} \int_B f \cdot \nabla u \cdot \nabla\chi_\varepsilon = 0).$$

Thus, we have

$$\int_B u\Delta f = \lambda \int_B uf.$$

Thus, we have that u is in C_0^∞(on B), which is what we wanted to show. //

By using this Lemma 4.2, we can consider the following.

Theorem 4.2 (1) *Since the Laplacian is invariant with respect to parallel displacements on V, we have $\Delta\tilde{f} = \lambda\tilde{f}$.*

(2) *Since $W_a(R)$ is a semi-direct product of both $W(R)$ and $Q(R^\vee)$, $\tilde{f}$ is invariant under $Q(R^\vee)$, that is,*

$$\tilde{f}(\mathbf{x} + \mathbf{d}) = \tilde{f}(\mathbf{x}) \quad (\mathbf{x} \in V, \ \mathbf{d} \in Q(R^\vee)).$$

Proof We notice

$$\tilde{f}(S_{\alpha,k}(\mathbf{x})) = \tilde{f}(S_\alpha(\mathbf{x}) + \mathbf{d}) = -\tilde{f}(\mathbf{x}).$$

Since det $S_\alpha = -1$. Here, we put $\mathbf{x} = S_\alpha(\mathbf{y})$ $(\mathbf{y} \in V)$. Since

$${S_\alpha}^2(\mathbf{y}) = \mathbf{y},$$

we have

$$-\tilde{f}(S_\alpha(\mathbf{y})) = \tilde{f}(S_\alpha(S_\alpha(\mathbf{y}) + \mathbf{d})) = \tilde{f}(\mathbf{y} + \mathbf{d}),$$
$$-(\det S_\alpha)\tilde{f}(\mathbf{y}) = \tilde{f}(\mathbf{y}).$$

Thus, we obtain $\tilde{f}(\mathbf{y}) = \tilde{f}(\mathbf{y}+\mathbf{d})$. //

Furthermore, we put

$$\tilde{f}_{\mathbf{d}}(\mathbf{y}) := \tilde{f}(\mathbf{y}+\mathbf{d}) \qquad (\mathbf{d} \in V),$$
$$\tilde{f}_w(\mathbf{x}) := \tilde{f}(w\mathbf{x}) \qquad (w \in W(R)).$$

Then, we have

$$\Delta(\tilde{f}_{\mathbf{d}})(\mathbf{x}) = (\Delta\,\tilde{f})_{\mathbf{d}}(\mathbf{x}),$$

and

$$\Delta(\tilde{f}_w)(\mathbf{x}) = (\Delta\,\tilde{f})_w(\mathbf{x}).$$

Thus, we obtain

$$\Delta(\tilde{f}_w)(\mathbf{x}) = (\Delta\tilde{f})_w(\mathbf{x}) \quad ({}^{\forall}w \in W_a(R),\ \mathbf{x} \in V).$$

If f is the solution of Dirichlet problems (2.9) on the domain $D(R)$, we extend f to everywhere on V as follows.

$$\tilde{f}(w(\mathbf{x})) = (\det\,w)\,\tilde{f}(\mathbf{x}).$$

We have $\Delta\tilde{f} = \lambda\tilde{f}$. Furthermore, we have that

$$\Delta(\tilde{f}_w) = (\Delta\tilde{f})_w = (\lambda\tilde{f})_w = \lambda\tilde{f}_w,$$
$$\tilde{f}(\mathbf{x}+\mathbf{d}) = \tilde{f}(\mathbf{x}) \quad (\mathbf{d} \in Q(R^{\vee})).$$

The function $\tilde{f}$ satisfies the conditions (4.7)∼(4.9):

$$\tilde{f} : C^{\infty} \ \ on\ \ V \tag{4.7}$$
$$\Delta\tilde{f} = \lambda\tilde{f}, \tag{4.8}$$
$$\tilde{f}(\mathbf{x}+\mathbf{d}) = \tilde{f}(\mathbf{x}). \tag{4.9}$$

Since $\Gamma = Q(R^{\vee})$ is a lattice of V, we have the torus $\mathbb{R}^{\ell}/\Gamma = V/Q(R^{\vee})$. Then, f is the eigenfunction for the eigenvalue λ on $\mathbb{R}^{\ell}/\Gamma = V/Q(R^{\vee})$. We have the eigenvalues λ and their eigenfunctions on the torus $\mathbb{R}^{\ell}/\Gamma = V/Q(R^{\vee})$ for the dual lattice $P(R)$ of $Q(R^{\vee})$. If we put

$$f_{\mathbf{p}}(\mathbf{x}) = \exp(2\pi i(\mathbf{x}|\mathbf{p})), \quad \mathbf{x} \in V,\ \mathbf{p} \in P(R),$$

the eigenvalues λ are

$$\lambda_{\mathbf{p}} = 4\pi^2(\mathbf{p}|\mathbf{p}).$$

For the eigenfunctions, we have

$$\begin{aligned} f_{\mathbf{p}}(\mathbf{x}+\mathbf{d}) &= \exp(2\pi i(\mathbf{x}+\mathbf{d}|\mathbf{p})) \\ &= \exp(2\pi i(\mathbf{x}|\mathbf{p}))\exp(2\pi i(\mathbf{d}|\mathbf{p})). \end{aligned}$$

Since $P(R)$ is the dual lattice of $Q(R^\vee)$, $(\mathbf{d}|\mathbf{p})$ are integers. Thus, we have

$$\begin{aligned} f_{\mathbf{p}}(\mathbf{x}+\mathbf{d}) &= \exp(2\pi i(\mathbf{x}|\mathbf{p})) \quad ({}^\forall \mathbf{d} \in Q(R^\vee)) \\ &= f_{\mathbf{p}}(\mathbf{x}). \end{aligned}$$

Let us define the function $\tilde{f}$ from $f_{\mathbf{p}}$ by

$$\tilde{f}(w(\mathbf{x})) = \varepsilon(w)\tilde{f}(\mathbf{x}) \quad ({}^\forall w \in W_a(R),\ \mathbf{x} \in V). \tag{4.10}$$

Since $W_a(R)$ is a semi-direct product of both $W(R)$ and $Q(R^\vee)$, $\tilde{f}$ satisfies (4.6). //

Proposition 4.1 *The function $\tilde{f}$ which satisfies* (4.10) *is given by*

$$f(\mathbf{x}) = \sum_{w\in W(R)} \varepsilon(w) f_{\mathbf{p}}(w(\mathbf{x})).$$

Proof

$$\begin{aligned} \sum_{w'\in W(R)} \varepsilon(w')\ f_{\mathbf{p}}(w'(w(\mathbf{x}))) &= \sum_{w'\in W(R)} \varepsilon(w')\ f_{\mathbf{p}}(w'w(\mathbf{x})) \quad (\text{we put } w'w = w'') \\ &= \sum_{w'\in W(R)} \varepsilon(w''w^{-1})\ f_{\mathbf{p}}(w''(\mathbf{x})),\ \ \varepsilon(w^{-1}) = \varepsilon(w) \\ &= \varepsilon(w) \sum_{w''\in W(R)} \varepsilon(w'')\ f_{\mathbf{p}}(w''(\mathbf{x})). \end{aligned}$$

Here, we used the fact $W(R) = \{w'' \,|\, w'' \in W(R)\} = \{w'w \,|\, w',\ w \in W(R)\}$. //

Lemma 4.3 *It holds that* $(w(\mathbf{x}) \,|\, \mathbf{p}) = (\mathbf{x} \,|\, w^{-1}(\mathbf{p}))$. *Furthermore, we have*

$$\begin{aligned} f_p(w(\mathbf{x})) &= \exp(2\pi i(w(\mathbf{x})) \,|\, \mathbf{p})) \quad (\mathbf{x} \in V) \\ &= \exp(2\pi i(\mathbf{x} \,|\, w^{-1}(\mathbf{p})) \\ &= f_{w^{-1}(\mathbf{p})}(\mathbf{x}). \end{aligned}$$

Proof If we put $\mathbf{p} = w(\mathbf{q})$, then $\mathbf{q} = w^{-1}(\mathbf{p})$. Thus, we have

$$\begin{aligned} (w(\mathbf{x}) \,|\, \mathbf{p}) &= (w(\mathbf{x}) \,|\, w(\mathbf{q})) \\ &= (\mathbf{x} \,|\, \mathbf{q}) \\ &= (\mathbf{x} \,|\, w^{-1}(\mathbf{p})). \qquad // \end{aligned}$$

Lemma 4.4 *It holds that* $\varepsilon(w^{-1}) = \varepsilon(w)$ $({}^{\forall}w \in W_a(R))$.

Proof If we assume for $w \in W(R)$, $w = S_{\alpha_1,k_1} \cdots S_{\alpha_t,k_t}$, then

$w^{-1} = S_{\alpha_t,k_t} \cdots S_{\alpha_1,k_1}$. Thus, $\varepsilon(w^{-1}) = (-1)^t = \varepsilon(w)$. //

Lemma 4.5 *We have* $\displaystyle\sum_{w\in W(R)} \varepsilon(w) f_{\mathbf{p}}(w(\mathbf{x})) = J_{\mathbf{p}}(\mathbf{x}),\ \mathbf{x} \in V$.
Here, $\displaystyle J_{\mathbf{p}}(\mathbf{x}) = \sum_{w\in W(R)} \varepsilon(w) f_{w(\mathbf{p})}(\mathbf{x})\ \ (\mathbf{x} \in V)$.

Proof By using Lemma 4.3, we have

$$\begin{aligned}\sum_{w\in W(R)} \varepsilon(w) f_{\mathbf{p}}(w(\mathbf{x})) &= \sum_{w\in W(R)} \varepsilon(w) f_{w^{-1}(\mathbf{p})}(\mathbf{x}) \\ &= \sum_{w'\in W(R)} \varepsilon(w')^{-1} f_{w'(\mathbf{p})}(\mathbf{x}) \\ &= \sum_{w'\in W(R)} \varepsilon(w') f_{w'(\mathbf{p})}(\mathbf{x}) = J_{\mathbf{p}}(\mathbf{x}), \quad (\mathbf{x} \in V).\end{aligned}$$

Here, we put $w' = w^{-1}$. //

Lemma 4.6 *The set* $J_{\mathbf{p}},\ \mathbf{p} \in P(R) \cap C(R)$ *gives a basis of the space of all anti-invariant elements* (*See [4],* p.220).

Here, $f(\mathbf{x})$, which is called the **anti-invariant** element of $W_a(R)$, satisfies $f(w(\mathbf{x})) = \varepsilon(w) f(\mathbf{x}), {}^{\forall}w \in W_a(R)$. All the $J_{\mathbf{p}}(\mathbf{x})$ are anti-invariant elements.

Proposition 4.2 *The function* $\tilde{f}$ *with the condition that* $\tilde{f}(w(\mathbf{x})) = \varepsilon(w)\tilde{f}(\mathbf{x})$ $({}^{\forall}w \in W_a(R),\ \mathbf{x} \in V)$ *satisfies the Dirichlet boundary value condition* (*See [4],* p.219). *Here, we put*

$$\begin{aligned} w \in W_a(R) \qquad & (w(\mathbf{x}) = S_{\alpha_1,k_1} \cdots S_{\alpha_t,k_t}(\mathbf{x}) = \mathrm{T}(\mathbf{x}) + v\ (\mathbf{x} \in V)), \\ w(\mathbf{x}) = w_1(\mathbf{x}) + \mathbf{d} \qquad & (w_1 \in W(R),\ \mathbf{d} \in Q(R^{\vee})), \\ \tilde{f}(w_1(\mathbf{x})) = \varepsilon(w_1)\tilde{f}(\mathbf{x}) \qquad & (\mathbf{x} \in V,\ \mathbf{d} \in Q(R^{\vee})), \quad ({}^{\forall}w_1 \in W(R),\ \mathbf{x} \in V).\end{aligned}$$

Proof We can prove this from the fact that $\tilde{f}(S_{\alpha}(\mathbf{x})) = -\tilde{f}(\mathbf{x})$ $({}^{\forall}\alpha \in R)$. //

Lemma 4.7 *If we put* $\tilde{f}(\mathbf{x}) = \sum_{w \in W(R)} \varepsilon(w) f_{\mathbf{p}}(w(\mathbf{x}))$, *we have*

$$\begin{aligned}\tilde{f}(\mathbf{x}+\mathbf{d}) &= \sum_{w \in W(R)} \varepsilon(w) f_{\mathbf{p}}(w(\mathbf{x}+\mathbf{d})) \\ &= \sum_{w \in W(R)} \varepsilon(w) f_{\mathbf{p}}(w(\mathbf{x})) \\ &= \tilde{f}(\mathbf{x}).\end{aligned}$$

Proof Since for each $w \in W(R)$, $\mathbf{d} \in Q(R^\vee)$, $\mathbf{d}' = w(\mathbf{d}) \in Q(R^\vee)$, and then it holds that $f_{\mathbf{p}}(\mathbf{x}+\mathbf{d}') = f_{\mathbf{p}}(\mathbf{x})$. //

We have (1) of Theorem 4.2. //

We extend f to be a function $\tilde{f}$ on V as follows.

$$\tilde{f}(w(\mathbf{x})) = \tilde{f}(\mathbf{x}),\ (^\forall \mathbf{x} \in V,\ ^\forall w \in W_a(R)),\ \textit{and}\ \tilde{f}\,|_{D(R)} = f.$$

Lemma 4.8 *The set of all* $S_{\mathbf{p}}$, $\mathbf{p} \in P(R) \cap \overline{C(R)}$ *gives a basis of the space of all invariant elements* (*See [4],* p.220).

The function $f(\mathbf{x})$, which satisfies $f(w(\mathbf{x})) = f(\mathbf{x}), ^\forall w \in W_a(R)$ is called the **invariant** element of $W_a(R)$. All the $S_{\mathbf{p}}(\mathbf{x})$ are the invariant elements (See [4], p.222).

Proposition 4.3 *The function* $\tilde{f}$ *which satisfies that* $\tilde{f}(w(\mathbf{x})) = \tilde{f}(\mathbf{x})$ *satisfies the Neumann boundary value conditions.*

Proof Since $\tilde{f}$ satisfies $\tilde{f}(S_\alpha(\mathbf{x})) = \tilde{f}(\mathbf{x})$ for all $\alpha \in R$, it holds that
$\dfrac{\partial f}{\partial \mathbf{n}}(\mathbf{x}) = 0$ *on* $\partial D(R)$. //

We have (2) of Theorem 4.2. //

5 Explicit formulas for the eigenvalues and eigenfunctions

5.1 Summary

In this section, we figure out the alcove $D(R)$ and the elements in symmetric groups by using root systems and affine Weyl group theory in Section 2, and we show the eigenvalues and eigenfunctions in general ℓ-dimension, concretely.

(1) We determine explicitly the crystallographic Euclidean domains $D(R)$, namely, the fundamental domains of affine Weyl groups of irreducible root systems R.

(2) We write down all the Dirichlet eigenvalues and Neumann eigenvalues of $D(R)$ explicitly.

(3) In the cases of types $A_\ell,\ B_\ell,\ C_\ell,\ D_\ell$ and G_2, we determine the Dirichlet eigenfunctions explicitly in terms of the determinant, and the Neumann eigenfunctions in terms of the permanent Perm( ). Here, the permanent, Perm( ), is by definition given by changing all the signs in the definition of the determinant into $+1$. For the other four cases $E_6,\ E_7,\ E_8$ and F_4, we cannot give the formulas because their Weyl groups are too complicated.

5.2 Decision of the alcove $D(R)$

Now, we show our results in the following.

We first show the explicit expression of the alcove $D(R)$ for each irreducible root system R. In the theory of root systems and affine Weyl groups, we explicitly determined the fundamental domains, $D(R)$ of the affine Weyl groups. The Weyl chamber $C(R)$ of a root system R is by definition given as follows.

$$C(R)=\left\{\sum_{i=1}^{\ell} x_i\omega_i \,\middle|\, x_1>0,\cdots,x_\ell>0\right\},$$

where $\omega_i(i=1,\cdots,\ell)$ are the fundamental weights of $P(R)$. Then, one can determine the alcove $D(R)$ as follows.

Lemma 5.1

$$\begin{aligned}D(R)&=\left\{\sum_{i=1}^{\ell} x_i\omega_i \,\middle|\, x_1>0,\cdots,x_\ell>0,\ \sum_{i=1}^{\ell}(x_i\,|\,\tilde{\alpha})<1\right\}\\&=\left\{\omega=\sum_{i=1}^{\ell} x_i\omega_i \,\middle|\, (\omega\,|\,\alpha_j^\vee)>0\ (j=1,\cdots,\ell),\ 1>(\omega\,|\,\tilde{\alpha})\right\},\end{aligned}$$

where $\widetilde{\alpha}$ is the highest root of R.

Proof (1) The definition of $C(R)$ and $D(R)$ are as follows.

$$C(R) = \left\{ \sum_{i=1}^{\ell} x_i \omega_i \;\middle|\; (x,\, \alpha) > 0 \ \ (\alpha \,\in\, R) \right\},$$
$$D(R) = \left\{ \sum_{i=1}^{\ell} x_i \omega_i \;\middle|\; 1 > (x,\, \alpha) > 0 \ \ (\alpha \,\in\, R) \right\}.$$

Here, for an arbitrary $0 < \alpha \,\in\, R,\ \alpha = \sum_{i=1}^{\ell} m_i \alpha_i$, here, $m_i\,(i = 1, \cdots, \ell)$ are 0 or positive integers, and some m_j are positive. Thus, we have

$$C(R) = \left\{ \sum_{i=1}^{\ell} x_i \omega_i \;\middle|\; x_i > 0 \ \ (i = 1, \cdots, \ell) \right\},$$
$$D(R) = \left\{ \sum_{i=1}^{\ell} x_i \omega_i \;\middle|\; x_i > 0 \ \ (i = 1, \cdots, \ell) \text{ and } 1 > \langle x,\, \alpha \rangle \ \ (\alpha \,\in\, R) \right\}.$$

Here, by [4], p.197, Proposition 27, the highest root $\tilde{\alpha} = \sum_{i=1}^{\ell} n_i \alpha_i \in R$ exists, and for all $\alpha \in R, \alpha = \sum_{i=1}^{\ell} m_i \alpha_i > 0$, we have $n_i \geq m_i$ $(\forall i = 1, \cdots, \ell)$. Then, for $\boldsymbol{x} \in C(R)$,

$$1 > \langle \tilde{\alpha},\, \boldsymbol{x} \rangle \quad \Leftrightarrow \quad 1 > \langle \alpha,\, \boldsymbol{x} \rangle \quad (\alpha \,\in\, R,\ \alpha > 0). \tag{5.1}$$

Because $(\Leftarrow)$ is consistent, for $(\Rightarrow)$, if we assume $\alpha \in R, \alpha > 0$, for $\alpha = \sum_{i=1}^{\ell} m_i \alpha_i$, we have

$$1 > \langle \tilde{\alpha}, \boldsymbol{x} \rangle = \sum_{i=1}^{\ell} n_i \langle \alpha_i, \boldsymbol{x} \rangle \geq \sum_{i=1}^{\ell} m_i \langle \alpha_i, \boldsymbol{x} \rangle = \langle \alpha, \boldsymbol{x} \rangle.$$

(2) By the basic feature of root systems,

$$(\omega_i \,|\, \alpha_j^{\vee}) = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

because of

$$(\omega \,|\, \alpha_j^{\vee}) = \left(\sum_{i=1}^{\ell} x_i \omega_i \;\middle|\; \alpha_j^{\vee} \right) = \sum_{i=1}^{\ell} x_i (\omega_i \,|\, \alpha_j^{\vee}) = x_j,$$

so (2) is proved //

We calculate the alcove $D(R)$ for each irreducible root system. Notice here that our ω_i $(i = 1, \cdots, \ell)$ are different from the ϖ_i $(i = 1, \cdots, \ell)$ in the Corollary of Proposition 5, $n^\circ 2$, Section 2, Chapter 6, [4] which are the dual basis of $\{\alpha_i\}_{i=1}^{\ell}$, but our ω_i are the ones of $\{\alpha_i^\vee\}_{i=1}^{\ell}$.

Theorem 5.1 (1) *In the case of type* A_ℓ $(\ell \geq 1)$*, the alcove* $D(R)$ *is the* ℓ *-dimensional polyhedron whose vertices are the origin* o *and the fundamental weights* $\omega_1, \omega_2, \cdots, \omega_\ell$ *given as follows:*

$$\omega_i = \sum_{s=1}^{i} \varepsilon_s - \frac{i}{\ell+1} \sum_{s=1}^{\ell+1} \varepsilon_s. \tag{5.2}$$

Here, we denote by $\varepsilon_s = {}^t(0, \cdots, \overset{s}{1}, \cdots, 0)$ *the column vector of degree* $\ell+1$ *consisting of the* s*-th component as* 1 *and the others as* 0 *for each* s $(s = 1, \cdots, \ell+1)$.

(2) *In the case of type* B_ℓ $(\ell \geq 2)$*, and when* $\ell = 2$*, the alcove* $D(R)$ *is the* 2*-dimensional polyhedron which has three vertices,* ω_1, ω_2 *and the origin* o*. When* $\ell \geq 3$*,* $D(R)$ *is the* ℓ *dimensional polygon which has* $\ell+1$ *vertices,* $\omega_1,\ -\frac{1}{2}\omega_2, \cdots, \frac{1}{2}\omega_{\ell-1}$, ω_ℓ *and the origin* o*. The fundamental weights are given as follows:*

$$\omega_i = \sum_{j=1}^{i} \varepsilon_j \ (1 \leq i \leq \ell - 1),\ \omega_\ell = \frac{1}{2} \sum_{j=1}^{\ell} \varepsilon_j. \tag{5.3}$$

(3) *In the case of type* C_ℓ $(\ell \geq 2)$*, the alcove* $D(R)$ *is the* ℓ*-dimensional polyhedron which has* ℓ *vertices,* $\frac{1}{2}\omega_1,\ \frac{1}{2}\omega_2, \cdots,\ \frac{1}{2}\omega_\ell$ *and the origin* o*. The fundamental weights are given as follows:*

$$\omega_i = \sum_{j=1}^{i} \varepsilon_j \ (1 \leq i \leq \ell) \tag{5.4}$$

(4) *In the case of type* D_ℓ $(\ell \geq 3)$*, the alcove* $D(R)$ *is the* ℓ*-dimensional polyhedron which has* $\ell+1$ *vertices,* $\omega_1,\ \frac{1}{2}\omega_2, \cdots,\ \frac{1}{2}\omega_{\ell-2},\ \omega_{\ell-1},\ \omega_\ell$ *and the origin* o*. The fundamental weights are given as follows:*

$$\omega_i = \sum_{j=1}^{i} \varepsilon_j \ (1 \leq i \leq \ell - 2),\ \omega_{\ell-1} = \frac{1}{2}\left(\sum_{i=1}^{\ell-1} \varepsilon_i - \varepsilon_\ell\right),\ \omega_\ell = \frac{1}{2}\left(\sum_{i=1}^{\ell} \varepsilon_i\right). \tag{5.5}$$

(5) *In the case of type* E_6*, the alcove* $D(R)$ *is the* 6*-dimensional polyhedron which has* 7 *vertices,* $\omega_1,\ \frac{1}{2}\omega_2,\ \frac{1}{2}\omega_3,\ \frac{1}{3}\omega_4, \frac{1}{2}\omega_5,\ \omega_6$ *and the origin* o*. The fundamental weights are given as follows:*

$$\omega_1 = \frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6),\ \omega_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 - \varepsilon_8),$$
$$\omega_3 = \frac{5}{6}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5),$$
$$\omega_4 = \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8,$$
$$\omega_5 = \frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_4 + \varepsilon_5,\ \omega_6 = \frac{1}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_5.$$

(6) *In the case of type E_7, the alcove $D(R)$ is the 7-dimensional polyhedron which has 8 vertices, $\frac{1}{2}\omega_1,\ \frac{1}{2}\omega_2,\ \frac{1}{3}\omega_3,\ \frac{1}{4}\omega_4,\ \frac{1}{3}\omega_5,\ \frac{1}{2}\omega_6,\ \omega_7$ and the origin o. The fundamental weights are as follows:*

$$\omega_1 = \varepsilon_8 - \varepsilon_7, \omega_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - 2\varepsilon_7 + 2\varepsilon_8),$$
$$\omega_3 = \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - 3\varepsilon_7 + 3\varepsilon_8),$$
$$\omega_4 = \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + 2(\varepsilon_8 - \varepsilon_7),\ \omega_5 = \frac{1}{2}(2\varepsilon_4 + 2\varepsilon_5 + 2\varepsilon_6 + 3(\varepsilon_8 - 3\varepsilon_7)),$$
$$\omega_6 = \varepsilon_5 + \varepsilon_6 - \varepsilon_7 + \varepsilon_8,\ \omega_7 = \varepsilon_6 + \frac{1}{2}(\varepsilon_8 - \varepsilon_7).$$

(7) *In the case of type E_8, the alcove $D(R)$ is the 8-dimensional polyhedron which has 9 vertices, $\frac{1}{2}\omega_1,\ \frac{1}{3}\omega_2,\ \frac{1}{4}\omega_3,\ \frac{1}{6}\omega_4,\ \frac{1}{5}\omega_5,\ \frac{1}{4}\omega_6,\ \frac{1}{3}\omega_7,\ \frac{1}{2}\omega_8$ and the origin o. The fundamental weights are as follows:*

$$\omega_1 = 2\varepsilon_8,\ \omega_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 5\varepsilon_8),$$
$$\omega_3 = \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 7\varepsilon_8),$$
$$\omega_4 = \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 5\varepsilon_8,\ \omega_5 = \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 4\varepsilon_8,$$
$$\omega_6 = \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + 3\varepsilon_8,\ \omega_7 = \varepsilon_6 + \varepsilon_7 + 2\varepsilon_8,\ \omega_5 = \varepsilon_7 + \varepsilon_8.$$

(8) *In the case of type F_4, the alcove $D(R)$ is the 4-dimensional polyhedron which has 5 vertices, $\frac{1}{2}\omega_1,\ \frac{1}{3}\omega_2,\ \frac{1}{2}\omega_3,\ \omega_4$ and the origin o. The fundamental weights are given as follows:*

$$\omega_1 = \varepsilon_1 + \varepsilon_2,\ \omega_2 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3,\ \omega_3 = \frac{1}{2}(3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4),\ \omega_4 = \varepsilon_1.$$

(9) *In the case of type G_2, $D(R)$ is the 2 dimensional polyhedron which has 3 vertices, $\frac{1}{3}\omega_1,\ \frac{1}{6}\omega_2$ and the origin o. The fundamental weights are given as follows:*

$$\omega_1 = -\varepsilon_2 + \varepsilon_3,\ \omega_2 = -\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3. \tag{5.6}$$

Proof (1) In the case of type $A_\ell (\ell \geq 1)$,

$$\tilde{\alpha} = \varepsilon_1 - \varepsilon_{\ell+1} = \omega_1 + \omega_\ell = \alpha_1 + \cdots + \alpha_\ell = {\alpha_1}^\vee + \cdots + \alpha_\ell^\vee$$

$$(\omega \,|\, \tilde{\alpha}) = \left(\sum_{i=1}^{\ell} x_i \omega_i \,\middle|\, \alpha_1^\vee + \cdots + \alpha_\ell^\vee \right) = x_1 + \cdots + x_\ell.$$

Therefore,

$$D(R) = \left\{ \omega = \sum_{i=1}^{\ell} x_i \omega_i \,\middle|\, x_j > 0 \ (j = 1, \cdots, \ell), \ x_1 + \cdots + x_\ell \leq 1 \right\}.$$

When $1 \leq j \leq \ell$ and $x_j > 0$, on the boundary in the alcove $D(R)$, $x_1 + \cdots + x_\ell = 1$. When $1 \leq j \leq \ell$ and $x_j = 1$,

$$\sum_{k=1, k \neq j}^{\ell} x_k = 0.$$

Therefore, the alcove $D(R)$ is the ℓ dimensional polyhedron which has $\ell + 1$ vertices $\omega_1, \omega_2, \cdots, \omega_\ell$ and the origin o.

In the remaining cases from (2) B_ℓ to (9) G_2, we can prove them in the same manner. //

5.3 Calculation of the eigenvalues

We calculate explicitly every eigenvalue for alcove $D(R)$ of each irreducible root system R. The elements of $P(R)$ are called weights of R, and $C(R)$ is called the Weyl chamber.

$$P(R) \cap C(R) = \left\{ \sum_{i=1}^{\ell} n_i \omega_i \,\middle|\, n_1, \cdots, n_\ell > 0 \text{ integer} \right\},$$
$$P(R) \cap \overline{C(R)} = \left\{ \sum_{i=1}^{\ell} n_i \omega_i \,\middle|\, n_1, \cdots, n_\ell \geq 0 \text{ integer} \right\}.$$

For each

$$\mathbf{p} = \sum_{i=1}^{\ell} n_i \omega_i \in P(R) \cap \overline{C(R)},$$

we have

$$\lambda = 4\pi^2 (\mathbf{p}|\mathbf{p}) = 4\pi^2 \sum_{i,j=1}^{\ell} n_i\, n_j\, (\omega_i \,|\, \omega_j).$$

Here, in the case of the Dirichlet eigenvalue problems, each integer n_i runs over the set of all the $\ell-$tuples $(n_1, \cdots, n_\ell)$ which satisfy the conditions: $n_1 > 0, \cdots,\ n_\ell > 0$, and in the case of the Neumann eigenvalue problems, each integer runs over the set of all the $\ell-$tuples $(n_1, \cdots, n_\ell)$ which satisfy the conditions: $n_1 \geq 0, \cdots,\ n_\ell \geq 0$.

We calculate each eigenvalue λ by substituting of calculation of ω_i as follows.

Theorem 5.2 (1) *In the case of type* $A_\ell (\ell \geq 1)$,

$$\lambda = 4\pi^2 \left\{ \sum_{i=1}^{\ell} i n_i^2 + 2 \sum_{1 \leq i < j \leq \ell} i n_i n_j - \frac{1}{\ell+1} \left(\sum_{i=1}^{\ell} i n_i \right)^2 \right\}.$$

Here, $(\omega_i \,|\, \omega_j) = \min(i,\, j) - \dfrac{i \cdot j}{\ell + 1}$.

(2) *In the case of type* $B_\ell (\ell \geq 2)$,

$$\lambda = 4\pi^2 \left\{ \sum_{i=1}^{\ell-1} i n_i^2 + 2 \sum_{1 \leq i < j \leq \ell-1} i n_i n_j + n_\ell \sum_{i=1}^{\ell-1} i n_i + \frac{1}{4} \ell n_\ell^2 \right\}.$$

Here,
$(\omega_i \,|\, \omega_j) = \min(i,\, j)\ (1 \leq i,\, j \leq \ell - 1), (\omega_i \,|\, \omega_\ell) = \dfrac{i}{2}\ (1 \leq i \leq \ell - 1), (\omega_\ell \,|\, \omega_\ell) = \dfrac{\ell}{4}$.

(3) *In the case of type* $C_\ell (\ell \geq 3)$,

$$\lambda = 4\pi^2 \left\{ \sum_{i=1}^{\ell} i n_i^2 + 2 \sum_{1 \leq i < j \leq \ell} i n_i n_j \right\}. \quad (\omega_i \,|\, \omega_j) = \min(i,\, j).$$

(4) *In the case of type* $D_\ell (\ell \geq 3)$,

$$\frac{\lambda}{4\pi^2} = \sum_{1 \leq i < j \leq \ell-2}^{\ell} i n_i n_j + (n_{\ell-1} + n_\ell) \sum_{i=1}^{\ell-2} i n_i + \frac{1}{4} \ell (2 n_{\ell-1}^2 + n_\ell^2) + \frac{1}{2} (\ell - 2) n_{\ell-1} n_\ell.$$

Here,

$$(\omega_i \mid \omega_j) = \min(i,\, j)\ (1 \leq i,\, j \leq \ell - 2),$$
$$(\omega_i \mid \omega_{\ell-1}) = (\omega_i \mid \omega_\ell) = \frac{1}{2} i\ (1 \leq i \leq \ell - 2),$$
$$(\omega_{l-1} \mid \omega_{\ell-1}) = \frac{1}{2}\ell,\ (\omega_{l-1} \mid \omega_\ell) = \frac{1}{4}(\ell - 2),\ (\omega_l \mid \omega_l) = \frac{1}{4}\ell.$$

(5) *In the case of type* E_6,

$$\begin{aligned}\frac{\lambda}{4\pi^2} &= 2n_1^2 + 4n_2^2 + \frac{10}{3}n_3^2 + 6n_4^2 + \frac{10}{3}n_5^2 + \frac{4}{3}n_6^2 \\ &\quad + 4n_1n_2 + \frac{10}{3}n_1n_3 + 4n_1n_4 + \frac{8}{3}n_1n_5 + \frac{4}{3}n_1n_6 \\ &\quad + \frac{5}{3}n_2n_3 3n_2n_4 + 6n_2n_5 + 2n_2n_6 \\ &\quad + 8n_3n_4 + \frac{16}{3}n_3n_5 + \frac{8}{3}n_3n_6 + 8n_4n_5 + 4n_4n_6 + \frac{10}{3}n_5n_6,\end{aligned}$$

$$(\omega_i \,|\, \omega_j) = \begin{pmatrix} 2 & 2 & \frac{5}{3} & 2 & \frac{4}{3} & \frac{2}{3} \\ 2 & 4 & \frac{5}{4} & \frac{3}{2} & 3 & 1 \\ \frac{5}{3} & \frac{5}{4} & \frac{10}{3} & 4 & \frac{8}{3} & \frac{4}{3} \\ 2 & \frac{3}{2} & 4 & 6 & 4 & 2 \\ \frac{4}{3} & 3 & \frac{8}{3} & 4 & \frac{10}{3} & \frac{5}{3} \\ \frac{2}{3} & 1 & \frac{4}{3} & 2 & \frac{5}{3} & \frac{4}{3} \end{pmatrix}.$$

(6) *In the case of type* E_7,

$$\begin{aligned}\frac{\lambda}{4\pi^2} &= 2n_1^2 + \frac{7}{2}n_2^2 + 6n_3^2 + 12n_4^2 + 6n_5^2 + 4n_6^2 + \frac{3}{2}n_7^2 \\ &\quad + 4n_1n_2 + 6n_1n_3 + 8n_1n_4 + 6n_1n_5 + 4n_1n_6 + 2n_1n_7 \\ &\quad + 8n_2n_3 + 12n_2n_4 + 9n_2n_5 + 6n_2n_6 + 3n_2n_7 \\ &\quad + 16n_3n_4 + 12n_3n_5 + 8n_3n_6 + 4n_3n_7 \\ &\quad + 18n_4n_5 + 12n_4n_6 + 6n_4n_7 + 10n_5n_6 + 5n_5n_7 + 4n_6n_7,\end{aligned}$$

$$(\omega_i \,|\, \omega_j) = \begin{pmatrix} 2 & 2 & 3 & 4 & 3 & 2 & 1 \\ 2 & \frac{7}{2} & 4 & 6 & \frac{9}{2} & 3 & \frac{3}{2} \\ 3 & 4 & 6 & 8 & 6 & 4 & 2 \\ 4 & 6 & 8 & 12 & 9 & 6 & 3 \\ 3 & \frac{9}{2} & 6 & 9 & 6 & 5 & \frac{5}{2} \\ 2 & 3 & 4 & 6 & 5 & 4 & 2 \\ 1 & \frac{3}{2} & 2 & 3 & \frac{5}{2} & 2 & \frac{3}{2} \end{pmatrix}.$$

(7) *In the case of type* E_8,

$$
\begin{aligned}
\frac{\lambda}{4\pi^2} &= 4n_1^2 + 8n_2^2 + 14n_3^2 + 30n_4^2 + 20n_5^2 + 12n_6^2 + 6n_7^2 + 2n_8^2 \\
&\quad + 10n_1n_2 + 14n_1n_3 + 20n_1n_4 + 16n_1n_5 + 12n_1n_6 + 8n_1n_7 + 4n_1n_8 \\
&\quad + 20n_2n_3 + 30n_2n_4 + 24n_2n_5 + 18n_2n_6 + 16n_2n_7 + 6n_2n_8 \\
&\quad + 40n_3n_4 + 32n_3n_5 + 24n_3n_6 + 16n_3n_7 + 8n_3n_8 \\
&\quad + 48n_4n_5 + 36n_4n_6 + 24n_4n_7 + 12n_4n_8 \\
&\quad + 30n_5n_6 + 20n_5n_7 + 10n_5n_8 + 16n_6n_7 + 6n_6n_8 + 40n_7n_8,
\end{aligned}
$$

$$
(\omega_i \,|\, \omega_j) = \begin{pmatrix}
4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\
5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\
7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\
10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\
8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\
6 & 9 & 12 & 18 & 15 & 12 & 8 & 3 \\
4 & 6 & 8 & 12 & 10 & 8 & 6 & 2 \\
2 & 3 & 4 & 6 & 5 & 3 & 2 & 2
\end{pmatrix}.
$$

(8) *In the case of type* F_4,

$$
\frac{\lambda}{4\pi^2} = 2n_1^2 + 6n_2^2 + 3n_3^2 + n_3^2 + 6n_1n_2 + 4n_1n_3 + 2n_1n_4 + 8n_2n_3 + 4n_2n_4 + 3n_3n_4,
$$

$$
(\omega_i \,|\, \omega_j) = \begin{pmatrix}
2 & 3 & 2 & 1 \\
3 & 6 & 4 & 2 \\
2 & 4 & 3 & \frac{3}{2} \\
1 & 3 & \frac{3}{2} & 1
\end{pmatrix}.
$$

(9) *In the case of type* G_2,

$$
\lambda = 4\pi^2 \left\{ 2{n_1}^2 + 6n_1n_2 + 6{n_2}^2 \right\}, \qquad (\omega_i \,|\, \omega_j) = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}.
$$

5.4 Calculation of the eigenfunctions

For an irreducible root system R, we express both the Dirichlet eigenfunctions on $D(R)$ in terms of the determinant and the Neumann eigenfunctions on $D(R)$ in terms of the permanent, respectively.

By Berard's work ([2]), the Dirichlet eigenfunctions are given by (3.2), and the Neumann eigenfunctions are given by (3.3), respectively. Then, we have

Theorem 5.3 (Dirichlet eigenfunctions) *For* $\mathbf{p} = \sum_{i=1}^{\ell} n_i\,\omega_i \in P(R) \cap C(R)$ *where* $(n_1, n_2, \cdots, n_\ell > 0\, integers)$, *we define*

$$D_\ell^{\mathbf{p}}(x_1, \cdots, x_\ell) := \mathrm{Det}\Big((e^{2\pi i\,(n_s + \cdots + n_{\ell-1})\,x_t})_{1\le s,\,t\le \ell} \Big). \tag{5.7}$$

Here, Det( ) *is the determinant of degree* ℓ. *And for each* $(s,t) = (\ell, t)$, $(t = 1, \cdots, \ell)$, $e^{2\pi i\,(n_s + \cdots + n_{\ell-1})\,x_t}$ *is* 1 *in the* (s,t)*-component in the right hand side of* (5.7). *Then, we have the following:*

(1) *In the case of type* A_ℓ,

$$J_{\mathbf{p}}(\mathbf{x}) = D_{\ell+1}^{\mathbf{p}}(x_1, \cdots, x_{\ell+1}). \tag{5.8}$$

(2) *In the case of type* B_ℓ,

$$J_{\mathbf{p}}(\mathbf{x}) = \sum_{\varepsilon_1, \cdots, \varepsilon_\ell = \pm 1} \varepsilon_1 \cdots \varepsilon_\ell\; e^{\pi i n_\ell (\varepsilon_1 x_1 + \cdots + \varepsilon_\ell x_\ell)}\; D_\ell^{\mathbf{p}}(\varepsilon_1 x_1, \cdots, \varepsilon_\ell x_\ell). \tag{5.9}$$

(3) *In the case of type* C_ℓ,

$$J_{\mathbf{p}}(\mathbf{x}) = \sum_{\varepsilon_1, \cdots, \varepsilon_\ell = \pm 1} \varepsilon_1 \cdots \varepsilon_\ell\; e^{2\pi i n_\ell (\varepsilon_1 x_1 + \cdots + \varepsilon_\ell x_\ell)}\; D_\ell^{\mathbf{p}}(\varepsilon_1 x_1, \cdots, \varepsilon_\ell x_\ell). \tag{5.10}$$

(4) *In the case of type* D_ℓ,

$$J_{\mathbf{p}}(\mathbf{x}) = \sum_{\varepsilon_1, \cdots, \varepsilon_\ell = \pm 1,\; \prod_{i=1}^{\ell} \varepsilon_i = 1} e^{\pi i (n_\ell - n_{\ell-1})(\varepsilon_1 x_1 + \cdots + \varepsilon_\ell x_\ell)}\; D_\ell^{\mathbf{p}}(\varepsilon_1 x_1, \cdots, \varepsilon_\ell x_\ell). \tag{5.11}$$

(5) *In the case of type* G_2, $J_{\mathbf{p}}(\mathbf{x})$ *is as follows:*

$$\begin{aligned} J_{\mathbf{p}}(\mathbf{x}) &= \sum_{\varepsilon = \pm 1} \varepsilon \begin{vmatrix} e^{2\pi i \varepsilon (n_1 + 2n_2)(2x_1 + x_2)} & e^{2\pi i \varepsilon (-n_2)(2x_1 + x_2)} & e^{2\pi i \varepsilon (-n_1 - n_2)(2x_1 + x_2)} \\ e^{2\pi i \varepsilon (n_1 + 2n_2)(x_1 + 2x_2)} & e^{2\pi i \varepsilon (-n_2)(x_1 + 2x_2)} & e^{2\pi i \varepsilon (-n_1 - n_2)(x_1 + 2x_2)} \\ 1 & 1 & 1 \end{vmatrix} \\ &= \sum_{\varepsilon = \pm 1} \varepsilon\; e^{-6\pi i \varepsilon (n_1 + n_2)(x_1 + x_2)} \begin{vmatrix} e^{2\pi i \varepsilon (2n_1 + 3n_2)(2x_1 + x_2)} & e^{2\pi i \varepsilon (2n_1 + 3n_2)(2x_1 + x_2)} & 1 \\ e^{2\pi i \varepsilon (n_1)(2x_1 + x_2)} & e^{2\pi i \varepsilon (n_1)(x_1 + 2x_2)} & 1 \\ 1 & 1 & 1 \end{vmatrix}. \end{aligned} \tag{5.12}$$

Theorem 5.4 (Neumann eigenfunctions) *For* $\mathbf{p} = \sum_{i=1}^{\ell} n_i\,\omega_i \in P(R) \cap \overline{C(R)}$ *where* $(n_1, n_2, \cdots, n_\ell \geq 0$ *integers*), *we define*

$$\mathrm{Perm}_\ell^{\mathbf{p}}(x_1, \cdots, x_\ell) := \mathrm{Perm}\Big((e^{2\pi i\,(n_s + \cdots + n_{\ell-1})\,x_t})_{1 \leq s,\,t \leq \ell} \Big), \tag{5.13}$$

where Perm( ) *is the permanent, which is given by changing all the signs in the definition of the determinant into* $+1$.

In the right hand side of (5.13), *for each* $(s,t) = (\ell, t),\ (t = 1, \cdots, \ell)$, $e^{2\pi i\,(n_s + \cdots + n_{\ell-1})\,x_t} = 1$ *for the* $(s,t)-$*component. Then, we have:*

(1) *In the case of type* A_ℓ,

$$S_{\mathbf{p}}(\mathbf{x}) = \mathrm{Perm}_{\ell+1}^{\mathbf{p}}(x_1, \cdots, x_{\ell+1}). \tag{5.14}$$

(2) *In the case of type* B_ℓ,

$$S_{\mathbf{p}}(\mathbf{x}) = \sum_{\varepsilon_1, \cdots, \varepsilon_\ell = \pm 1} e^{\pi i n_\ell (\varepsilon_1 x_1 + \cdots + \varepsilon_\ell x_\ell)}\, \mathrm{Perm}_\ell^{\mathbf{p}}(\varepsilon_1 x_1, \cdots, \varepsilon_\ell x_\ell). \tag{5.15}$$

(3) *In the case of type* C_ℓ,

$$S_{\mathbf{p}}(\mathbf{x}) = \sum_{\varepsilon_1, \cdots, \varepsilon_\ell = \pm 1} e^{2\pi i n_\ell (\varepsilon_1 x_1 + \cdots + \varepsilon_\ell x_\ell)}\, \mathrm{Perm}_\ell^{\mathbf{p}}(\varepsilon_1 x_1, \cdots, \varepsilon_\ell x_\ell). \tag{5.16}$$

(4) *In the case of type* D_ℓ,

$$S_{\mathbf{p}}(\mathbf{x}) = \sum_{\varepsilon_1, \cdots, \varepsilon_\ell = \pm 1,\ \prod_{i=1}^{\ell} \varepsilon_i = 1} e^{\pi i (n_\ell - n_{\ell-1})(\varepsilon_1 x_1 + \cdots + \varepsilon_\ell x_\ell)}\, \mathrm{Perm}_\ell^{\mathbf{p}}(\varepsilon_1 x_1, \cdots, \varepsilon_\ell x_\ell). \tag{5.17}$$

(5) *In the case of type* G_2, $S_{\mathbf{p}}(\mathbf{x})$ *is as follows:*

$$\begin{aligned} S_{\mathbf{p}}(\mathbf{x}) &= \sum_{\varepsilon = \pm 1} \mathrm{Perm} \begin{pmatrix} e^{2\pi i \varepsilon (n_1 + 2n_2)(2x_1 + x_2)} & e^{2\pi i \varepsilon (-n_2)(2x_1 + x_2)} & e^{2\pi i \varepsilon (-n_1 - n_2)(2x_1 + x_2)} \\ e^{2\pi i \varepsilon (n_1 + 2n_2)(x_1 + 2x_2)} & e^{2\pi i \varepsilon (-n_2)(x_1 + 2x_2)} & e^{2\pi i \varepsilon (-n_1 - n_2)(x_1 + 2x_2)} \\ 1 & 1 & 1 \end{pmatrix} \\ &= \sum_{\varepsilon = \pm 1} e^{-6\pi i \varepsilon (n_1 + n_2)(x_1 + x_2)}\, \mathrm{Perm} \begin{pmatrix} e^{2\pi i \varepsilon (2n_1 + 3n_2)(2x_1 + x_2)} & e^{2\pi i \varepsilon (2n_1 + 3n_2)(2x_1 + x_2)} & 1 \\ e^{2\pi i \varepsilon (n_1)(2x_1 + x_2)} & e^{2\pi i \varepsilon (n_1)(x_1 + 2x_2)} & 1 \\ 1 & 1 & 1 \end{pmatrix}. \end{aligned} \tag{5.18}$$

Now, we will give proofs of the Theorems 5.3 and 5.4.

Proof of Theorem 5.3 (1) In the case of type A_ℓ, it suffices to show

$$J_{\mathbf{p}}(\mathbf{x}) = \begin{vmatrix} e^{2\pi i(n_1+n_2+\cdots+n_\ell)x_1} & \cdots & e^{2\pi i(n_1+n_2+\cdots+n_\ell)x_{\ell+1}} \\ e^{2\pi i(n_2+\cdots+n_\ell)x_1} & \cdots & e^{2\pi i(n_2+\cdots+n_\ell)x_{\ell+1}} \\ \vdots & & \vdots \\ e^{2\pi i(n_\ell)x_1} & \cdots & e^{2\pi i(n_\ell)x_{\ell+1}} \\ 1 & \cdots & 1 \end{vmatrix} \tag{5.19}$$

for all $\mathbf{x} = (x_1, x_2, \cdots, x_\ell, x_{\ell+1}) \in \mathbb{R}^{\ell+1}$.

By Chapter 6, section 4, $n°7$ of [4], page 207, the Weyl group $W(R)$ of A_ℓ type ($\ell \geq 1$) is isomorphic to the symmetric group $\mathfrak{S}_{\ell+1}$, and the isomorphism is given as follows. Let $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{\ell+1}\}$ be the standard basis of $\mathbb{R}^{\ell+1}$, and we put

$$V = \{(x_1, x_2, \cdots, x_{\ell+1}) \in \mathbb{R}^{\ell+1} |\, x_1 + x_2 + \cdots + x_{\ell+1} = 0\}.$$

For an arbitary automorphism g of V, let $\varphi(g)$ be an automorphism of $\mathbb{R}^{\ell+1}$ which is an extension of g, and makes $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{\ell+1}$ invariant. If g is the restriction to V of the orthogonal reflection $s_{\varepsilon_i - \varepsilon_j}$ (cf. section 2) which exchanges ε_i and ε_j of $\mathbb{R}^{\ell+1}$, we have that $\varphi(g) = s_{\varepsilon_i - \varepsilon_j}$. If we put $X := \{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{\ell+1}\}$, then the mapping $g \mapsto \varphi(g)|_X$ gives an isomorphism between $W(R)$ and the symmetric group of X. Here, if we put

$$w(\mathbf{p}) = \sum_{i=1}^{\ell+1} \mathbf{p}_{\sigma(i)}\varepsilon_i \qquad (w \in W(R),\ \mathbf{p} \in P(R) \cap C(R)),$$

we have

$$(\mathbf{x}|w(\mathbf{p})) = \left(\sum_{i=1}^{\ell+1} x_i \varepsilon_i \middle| \sum_{j=1}^{\ell+1} p_{\sigma(j)}\varepsilon_j \right) = \sum_{i=1}^{\ell+1} x_j p_{\sigma(j)} \qquad (\mathbf{x} \in \mathbb{R}^{\ell+1}).$$

Then,

$$J_{\mathbf{p}}(\mathbf{x}) = \sum_{\sigma \in \sigma_{\ell+1}} \varepsilon(\sigma) e^{2\pi i \sum_{j=1}^{\ell+1} x_j p_{\sigma(j)}} = \begin{vmatrix} e^{2\pi i x_1 p_1} & \cdots & e^{2\pi i x_{\ell+1} p_1} \\ \vdots & & \vdots \\ e^{2\pi i x_1 p_{\ell+1}} & \cdots & e^{2\pi i x_{\ell+1} p_{\ell+1}} \end{vmatrix}. \tag{5.20}$$

By (5.2) in Theorem 5.1 (1) , each $\mathbf{p} = \sum_{i=1}^{\ell} n_i \omega_i$, can be expressed as

$$\begin{aligned} \mathbf{p} &= \sum_{i=1}^{\ell} n_i \left(\sum_{s=1}^{i} \varepsilon_s - \frac{i}{\ell+1} \sum_{s=1}^{\ell+1} \varepsilon_s \right) \\ &= \sum_{s=1}^{\ell} \left\{ \sum_{i=s}^{\ell} n_i - \frac{1}{\ell+1} \sum_{i=1}^{\ell} n_i i \right\} \varepsilon_s - \frac{1}{\ell+1} \left(\sum_{i=1}^{\ell} n_i i \right) \varepsilon_{\ell+1}. \end{aligned} \tag{5.21}$$

Therefore, if we put $\mathbf{p} = \sum_{i=1}^{\ell} p_i\varepsilon_i$, we have

$$\left\{\begin{aligned} p_1 &= \sum_{i=1}^{\ell} n_i - \frac{i}{\ell+1}\sum_{i=1}^{\ell} n_i i = \frac{i}{\ell+1}\sum_{i=1}^{\ell}(\ell+1-i)n_i \\ p_2 &= \sum_{i=2}^{\ell} n_i - \frac{i}{\ell+1}\sum_{i=1}^{\ell} n_i i = \frac{i}{\ell+1}\left\{n_1 + \sum_{i=2}^{\ell}(\ell+1-i)n_i\right\} \\ &\vdots \\ p_{\ell-1} &= n_{\ell-1} + n_\ell - \frac{1}{\ell+1}\sum_{i=1}^{\ell} n_i i \\ p_\ell &= n_\ell - \frac{1}{\ell+1}\sum_{i=1}^{\ell} n_i i \\ p_{\ell+1} &= \frac{-1}{\ell+1}\left(\sum_{i=1}^{\ell} n_i i\right) = \sum_{i=1}^{\ell}(-i)n_i. \end{aligned}\right. \tag{5.22}$$

Therefore, we have the determinant, which is the same as the right hand side of (5.20), as follows:

$$\begin{vmatrix} e^{\frac{2\pi i}{\ell+1}(\ell n_1+(\ell-1)n_2+(\ell-2)n_3+\cdots+n_\ell)x_1} & \cdots & e^{\frac{2\pi i}{\ell+1}(\ell n_1+(\ell-1)n_2+(\ell-2)n_3+\cdots+n_\ell)x_{\ell+1}} \\ e^{\frac{2\pi i}{\ell+1}((-1)n_1+(\ell-1)n_2+(\ell-2)n_3+\cdots+n_\ell)x_1} & \cdots & e^{\frac{2\pi i}{\ell+1}((-1)n_1+(\ell-1)n_2+(\ell-2)n_3+\cdots+n_\ell)x_{\ell+1}} \\ e^{\frac{2\pi i}{\ell+1}((-1)n_1+(-2)n_2+(\ell-2)n_3+\cdots+n_\ell)x_1} & \cdots & e^{\frac{2\pi i}{\ell+1}((-1)n_1+(-2)n_2+(\ell-2)n_3+\cdots+n_\ell)x_{\ell+1}} \\ \vdots & & \vdots \\ e^{\frac{2\pi i}{\ell+1}((-1)n_1+(-2)n_2+(-3)n_3+\cdots+(-\ell)n_\ell)x_1} & \cdots & e^{\frac{2\pi i}{\ell+1}((-1)n_1+(-2)n_2+(-3)n_3+\cdots+(-\ell)n_\ell)x_{\ell+1}} \end{vmatrix}. \tag{5.23}$$

Here, $x_1 + x_2 + \cdots x_{\ell+1} = 0$, and

$$e^{\frac{2\pi i}{\ell+1}((-1)n_1x_1+(-1)n_1x_2+\cdots+(-1)n_1x_{\ell+1})} = e^{\frac{2\pi i}{\ell+1}(-1)n_1(x_1+x_2+\cdots+x_{\ell+1})} = 1.$$

We have (5.23) which is calculated from (5.19).

Therefore, we proved (1) by the formula of $J_{\mathbf{p}}(\mathbf{x})$. //

(2) In the case of type B_ℓ, we only have to prove

$$\begin{aligned} J_{\mathbf{p}}(\mathbf{x}) \quad &= \sum_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_\ell=\pm1} \varepsilon_1\varepsilon_2\cdots\varepsilon_\ell\, e^{\pi i n_\ell(\varepsilon_1x_1+\varepsilon_2x_2+\cdots+\varepsilon_\ell x_\ell)} \times \\ &\times \begin{vmatrix} e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_1x_1} & \cdots & e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_1x_1} & \cdots & e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ \vdots & & \vdots \\ e^{2\pi i(n_{\ell-1})\varepsilon_1x_1} & \cdots & e^{2\pi i(n_{\ell-1})\varepsilon_\ell x_\ell} \\ 1 & \cdots & 1 \end{vmatrix}, \end{aligned} \tag{5.24}$$

for all $\mathbf{x} = (x_1, x_2, \cdots, x_\ell) \in \mathbb{R}^\ell$.

Similarly, by Chapter 6, section 4, $n°5$ in [4] p. 241, the Weyl group, $W(R)$, of B_ℓ type ($\ell \geq 2$) is isomorphic to the semi-direct product of the symmetric group $\mathfrak{S}_\ell$ and $(\mathbb{Z}/2\mathbb{Z})^\ell$, and the correspondence is given as follows. In $\mathbb{R}^\ell$, the orthogonal reflection $s_{\varepsilon_i - \varepsilon_j} (i \neq j)$ induces a group G_1 which is isomorphic to the symmetric group $\mathfrak{S}_\ell$. And the orthogonal reflection s_{ε_i} exchanges ε_i and $-\varepsilon_i$, and makes ε_k invariant for all $k \neq i$. Therefore, $s_{\varepsilon_i} (1 \leq i \leq \ell)$ induces a group G_2 which isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\ell$, and the Weyl group $W(R)$ is the semi-direct product of G_1 and G_2. Therefore,

$$J_{\mathbf{p}}(\mathbf{x}) = \sum_{\sigma \in G_1} \sum_{u \in G_2} \varepsilon(\sigma)\varepsilon(u) e^{2\pi i (\mathbf{x}|\sigma(p))}. \tag{5.25}$$

For every element $u \in G_2$, it holds that

$$\begin{cases} (u(\mathbf{x}), u(\mathbf{x})) = (\mathbf{x}|\mathbf{x}) \qquad (\mathbf{x} \in \mathbb{R}^\ell), \\ \quad u(u(\varepsilon_i)) = u(-\varepsilon_i) = \varepsilon_i, \end{cases}$$

thus we have

$$\begin{aligned} (\mathbf{x} \,|\, u(\sigma(\mathbf{p})) &= (u(\mathbf{x}) \,|\, u(u(\sigma(\mathbf{p}))) \\ &= (u(\mathbf{x}) \,|\, \sigma(\mathbf{p})) \\ &= \left(\sum_{i=1} u_i x_i \varepsilon_i \,\middle|\, \sum_{i=1} p_{\sigma(j)} \varepsilon_j \right) \\ &= \sum_{i=1} x_i u_i p_{\sigma(j)}. \end{aligned}$$

Therefore, we have

$$e^{2\pi i (\mathbf{x}|u(\sigma(\mathbf{p})))} = e^{2\pi i \left(\sum_{i=1}^{\ell} x_i u_i p_{\sigma(j)} \right)}. \tag{5.26}$$

The sign of the transformation u of $\mathbb{R}^\ell$, which is defined by $\varepsilon_1 \mapsto u_1 \varepsilon_1$, $\varepsilon_2 \mapsto u_2 \varepsilon_2$, $\cdots$, and $\varepsilon_\ell \mapsto u_\ell \varepsilon_\ell$ (where, $u_i = \pm 1$), is equal to $(-1)^u = \prod_{i=1}^{\ell} u_i$. Then, we have

$$\begin{aligned} J_{\mathbf{p}}(\mathbf{x}) &= \sum_{u \in G_2} \prod_{i=1}^{\ell} u_i \sum_{\sigma \in \sigma_1} \varepsilon(\sigma) \; e^{2\pi i x_1 u_{\sigma(1)} p_{\sigma(1)}} \cdots e^{2\pi i x_\ell u_{\sigma(\ell)} p_{\sigma(\ell)}} \\ &= \sum_{u \in G_2} \prod_{i=1}^{\ell} u_i \begin{vmatrix} e^{2\pi i x_1 u_1 p_1} \cdots e^{2\pi i x_\ell u_\ell p_1} \\ \vdots \qquad \vdots \\ e^{2\pi i x_1 u_1 p_\ell} \cdots e^{2\pi i x_\ell \varepsilon_\ell p_\ell} \end{vmatrix} \\ &= \sum_{\varepsilon_1, \varepsilon_2, \cdots \varepsilon_\ell = \pm 1} \prod_{i=1}^{\ell} \varepsilon_i \begin{vmatrix} e^{2\pi i x_1 \varepsilon_1 p_1} \cdots e^{2\pi i x_\ell \varepsilon_\ell p_1} \\ \vdots \qquad \vdots \\ e^{2\pi i x_1 \varepsilon_1 p_\ell} \cdots e^{2\pi i x_\ell \varepsilon_\ell p_\ell} \end{vmatrix}. \end{aligned} \tag{5.27}$$

In the case of B_ℓ type, each $\mathbf{p} = \sum_{i=1}^{\ell} n_i\omega_i$, can be expressed as in (5.3) in Theorem 5.1. So, if we put $\mathbf{p} = \sum_{i=1}^{\ell} p_i\varepsilon_i$, then we have,

$$\begin{cases} p_1 = n_1 + n_2 + \cdots + n_{\ell-1} + \dfrac{n_\ell}{2} \\ p_2 = n_2 + \cdots + n_{\ell-1} + \dfrac{n_\ell}{2} \\ \quad\vdots \\ p_{\ell-1} = n_{\ell-1} + \dfrac{n_\ell}{2} \\ p_\ell = \dfrac{n_\ell}{2}. \end{cases} \tag{5.28}$$

Therefore (5.27) is equal to the equation which is given as follows:

$$\sum_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_\ell=\pm 1} \varepsilon_1\varepsilon_2\cdots\varepsilon_\ell \begin{vmatrix} e^{2\pi i\varepsilon_1(n_1+n_2+\cdots+\frac{1}{2}n_\ell)x_1} & \cdots & e^{2\pi i\varepsilon_\ell(n_1+n_2+\cdots+\frac{1}{2}n_\ell)x_\ell} \\ e^{2\pi i\varepsilon_1(n_2+n_3+\cdots+\frac{1}{2}n_\ell)x_1} & \cdots & e^{2\pi i\varepsilon_\ell(n_2+n_3+\cdots+\frac{1}{2}n_\ell)x_\ell} \\ e^{2\pi i\varepsilon_1(n_3+n_4+\cdots+\frac{1}{2}n_\ell)x_1} & \cdots & e^{2\pi i\varepsilon_\ell(n_3+n_4+\cdots+\frac{1}{2}n_\ell)x_\ell} \\ \vdots & & \vdots \\ e^{2\pi i\varepsilon_1(\frac{1}{2}n_\ell)x_1} & \cdots & e^{2\pi i\varepsilon_\ell(\frac{1}{2}n_\ell)x_\ell} \end{vmatrix}. \tag{5.29}$$

Therefore, we have

$$\begin{aligned} J_{\mathbf{p}}(\mathbf{x}) &= \sum_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_\ell=\pm 1} \varepsilon_1\varepsilon_2\cdots\varepsilon_\ell e^{2\pi i\varepsilon_1\frac{1}{2}n_\ell x_1} e^{2\pi i\varepsilon_2\frac{1}{2}n_\ell x_1}\cdots e^{2\pi i\varepsilon_\ell(\frac{1}{2}n_\ell x_\ell)} \times \\ &\quad \times \begin{vmatrix} e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_1x_1} & \cdots & e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_1x_1} & \cdots & e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ \vdots & & \vdots \\ e^{2\pi i(n_{\ell-1})\varepsilon_1x_1} & \cdots & e^{2\pi i(n_{\ell-1})\varepsilon_\ell x_\ell} \\ 1 & \cdots & 1 \end{vmatrix} \\ &= \sum_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_\ell=\pm 1} \varepsilon_1\varepsilon_2\cdots\varepsilon_\ell e^{\pi i n_\ell(\varepsilon_1x_1+\varepsilon_2x_2,\cdots,\varepsilon_\ell x_\ell)} \times \\ &\quad \times \begin{vmatrix} e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_1x_1} & \cdots & e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_1x_1} & \cdots & e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ \vdots & & \vdots \\ e^{2\pi i(n_{\ell-1})\varepsilon_1x_1} & \cdots & e^{2\pi i(n_{\ell-1})\varepsilon_\ell x_\ell} \\ 1 & \cdots & 1 \end{vmatrix} \end{aligned}$$

Therefore, we obtain the required formula for $J_{\mathbf{p}}(\mathbf{x})$, and (2). //

(3) In the case of type C_ℓ, we only have to prove

$$J_{\mathbf{p}}(\mathbf{x}) = \sum_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_\ell=\pm 1} \varepsilon_1\varepsilon_2\cdots\varepsilon_\ell e^{2\pi i n_\ell(\varepsilon_1 x_1+\varepsilon_2 x_2+\cdots+\varepsilon_\ell x_\ell)} \times$$
$$\times \begin{vmatrix} e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_1 x_1} & \cdots & e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_1 x_1} & \cdots & e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ \vdots & & \vdots \\ e^{2\pi i(n_{\ell-1})\varepsilon_1 x_1} & \cdots & e^{2\pi i(n_{\ell-1})\varepsilon_\ell x_\ell} \\ 1 & \cdots & 1 \end{vmatrix}$$

for all $\mathbf{x} = (x_1, x_2, \cdots, x_\ell) \in \mathbb{R}^\ell$. Similarly, by Chapter 6, section 4, $n°7$ in [4], p. 243, the Weyl group, $W(R)$, of type $C_\ell (\ell \geq 2)$ is isomorphic to type B_ℓ. Therefore, it is a similar calculation as for the eigenfunctions of type B_ℓ. By Theorem 5.1 (3), every $\mathbf{p} = \sum_{i=1}^{\ell} n_i\omega_i$ can be expressed as (5.4).

So, if we put $\mathbf{p} = \sum_{i=1}^{\ell} p_i\varepsilon_i$, we have

$$\begin{cases} p_1 = n_1 + n_2 + \cdots + n_\ell \\ p_2 = n_2 + \cdots + n_\ell \\ \quad \vdots \\ p_\ell = n_\ell. \end{cases} \tag{5.30}$$

We calculate $J_{\mathbf{p}}(\mathbf{x})$ similarly to type B_ℓ by using equation (5.30). Thus, we obtain the required formula for $J_{\mathbf{p}}(\mathbf{x})$ and (3). //

(4) In the case of type D_ℓ, we only have to prove

$$J_{\mathbf{p}}(\mathbf{x}) = \sum_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_\ell=\pm 1} \varepsilon_1\varepsilon_2\cdots\varepsilon_\ell e^{\pi i(n_\ell-n_{\ell-1})(\varepsilon_1 x_1+\varepsilon_2 x_2,\cdots,\varepsilon_\ell x_\ell)} \times$$
$$\times \begin{vmatrix} e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_1 x_1} & \cdots & e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_1 x_1} & \cdots & e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ \vdots & & \vdots \\ e^{2\pi i(n_{\ell-1})\varepsilon_1 x_1} & \cdots & e^{2\pi i(n_{\ell-1})\varepsilon_\ell x_\ell} \\ 1 & \cdots & 1 \end{vmatrix} \tag{5.31}$$

for all $\mathbf{x} = (x_1, x_2, \cdots, x_\ell) \in \mathbb{R}^\ell$.

Similarly, by Chapter 6, section 4, $n°\,9$ in [4] p.248, the Weyl group $W(R)$ of type $D_\ell (\ell \geq 3)$ is isomorphic to the semi-direct product of the symmetric group $\mathfrak{S}_\ell$ and $(\mathbb{Z}/2\mathbb{Z})^{\ell-1}$, and the isomorphism is given as follows. In $\mathbb{R}^\ell$, all orthogonal reflections $s_{\varepsilon_i-\varepsilon_j} (i \neq j)$ induce a group G_1 which is isomorphic to the symmetric group $\mathfrak{S}_\ell$.

On the other hand, $s_{ij} = s_{\varepsilon_i-\varepsilon_j} s_{\varepsilon_i+\varepsilon_j}$ sends ε_i to $-\varepsilon_i$, and ε_j to $-\varepsilon_j$, respectively, and makes ε_k invariant for each $k \neq i, j$. The group G_2 which is generated by all of the s_{ij} which consists of all the automorphisms u of $\mathbb{R}^\ell$ satisfying the condition that $u(\varepsilon_i) = (-1)^{\nu_i}\varepsilon_i$, $\displaystyle\prod_{i=1}^{\ell}(-1)^{\nu_i} = 1$. The group G_2 is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\ell-1}$, and $W(R)$ is the semi-direct product of G_1 and G_2.

In addition, by the equality we have

$$(\mathbf{x} \mid u(\sigma(\mathbf{p}))) = (u(\mathbf{x}) \mid \sigma(\mathbf{p})) = \left(\sum_{i=1}^{\ell} x_i u_i \varepsilon_i \,\middle|\, \sum_{j=1}^{\ell} p_{\sigma(j)}\varepsilon_j\right) = \sum_{i=1}^{\ell} x_i u_i p_{\sigma(j)},$$

and

$$e^{2\pi i(\mathbf{x}|u(\sigma(\mathbf{p})))} = e^{2\pi i \sum_{i=1}^{\ell} x_i u_i p_{\sigma(j)}}.$$

Therefore, we have

$$\begin{aligned}
J_{\mathbf{p}}(\mathbf{x}) &= \sum_{\sigma\in\sigma_\ell}\sum_{u\in G_2} \varepsilon(\sigma)\varepsilon(u) e^{2\pi i(\mathbf{x}|u(\sigma(\mathbf{p})))} \\
&= \sum_{\sigma\in\sigma_\ell}\sum_{u\in G_2} \varepsilon(\sigma) e^{2\pi i \sum_{i=1}^{\ell} x_i(-1)^{\nu_i} p_{\sigma(i)}} \\
&= \sum_{u\in G_2}\sum_{\sigma\in\sigma_\ell} \varepsilon(\sigma) e^{2\pi i x_1(-1)^{\nu_1} p_{\sigma(1)}} \cdots e^{2\pi i x_\ell(-1)^{\nu_\ell} p_{\sigma(\ell)}} \\
&= \sum_{\nu_1,\cdots,\nu_\ell=0,1,\ \prod_{i=1}^{\ell}(-1)^{\nu_\ell}=1} \begin{vmatrix} e^{2\pi i x_1(-1)^{\nu_1}p_1} & \dots & e^{2\pi i x_\ell(-1)^{\nu_\ell}p_1} \\ \vdots & & \vdots \\ e^{2\pi i x_1(-1)^{\nu_1}p_\ell} & \dots & e^{2\pi i x_\ell(-1)^{\nu_\ell}p_\ell} \end{vmatrix} \\
&= \sum_{\varepsilon_1,\cdots,\varepsilon_\ell=\pm1,\ \prod_{i=1}^{\ell}\varepsilon_i=1} \begin{vmatrix} e^{2\pi i\varepsilon_1 x_1 p_1} & \dots & e^{2\pi i\varepsilon_\ell x_\ell p_1} \\ \vdots & & \vdots \\ e^{2\pi i\varepsilon_1 x_1 p_\ell} & \dots & e^{2\pi i\varepsilon_\ell x_\ell p_\ell} \end{vmatrix}.
\end{aligned} \tag{5.32}$$

In the case of type D_ℓ, by using Theorem 5.1 (4), $\mathbf{p} = \displaystyle\sum_{i=1}^{\ell} n_i\omega_i$ can be expressed as in (5.5). If we put $\mathbf{p} = \displaystyle\sum_{i=1}^{\ell} p_i\varepsilon_i$, then

$$\begin{cases} p_1 = n_1 + n_2 + \cdots + \dfrac{n_{\ell-1}}{2} + \dfrac{n_\ell}{2} \\ p_2 = n_2 + \cdots + \dfrac{n_{\ell-1}}{2} + \dfrac{n_\ell}{2} \\ \quad \vdots \\ p_{\ell-2} = n_{\ell-2} + \dfrac{n_{\ell-1}}{2} + \dfrac{n_\ell}{2} \\ p_{\ell-1} = \dfrac{n_{\ell-1}}{2} + \dfrac{n_\ell}{2} \\ p_\ell = -\dfrac{n_{\ell-1}}{2} + \dfrac{n_\ell}{2}. \end{cases} \tag{5.33}$$

Therefore, (5.32) is equal to the following:

$$\sum_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_\ell=\pm1,\ \prod_{i=1}^{\ell}\varepsilon_i=1} \begin{vmatrix} e^{2\pi i\varepsilon_1(n_1+n_2+\cdots+\frac{1}{2}n_{\ell-1}+\frac{1}{2}n_\ell)x_1} & \cdots & e^{2\pi i\varepsilon_\ell(n_1+n_2+\cdots+\frac{1}{2}n_{\ell-1}+\frac{1}{2}n_\ell)x_\ell} \\ e^{2\pi i\varepsilon_1(n_2+n_3+\cdots+\frac{1}{2}n_{\ell-1}+\frac{1}{2}n_\ell)x_1} & \cdots & e^{2\pi i\varepsilon_\ell(n_2+n_3+\cdots+\frac{1}{2}n_{\ell-1}+\frac{1}{2}n_\ell)x_\ell} \\ e^{2\pi i\varepsilon_1(n_3+n_4+\cdots+\frac{1}{2}n_{\ell-1}+\frac{1}{2}n_\ell)x_1} & \cdots & e^{2\pi i\varepsilon_\ell(n_3+n_4+\cdots+\frac{1}{2}n_{\ell-1}+\frac{1}{2}n_\ell)x_\ell} \\ \vdots & & \vdots \\ e^{2\pi i\varepsilon_1(\frac{1}{2}n_{\ell-1}+\frac{1}{2}n_\ell)x_1} & \cdots & e^{2\pi i\varepsilon_\ell(\frac{1}{2}n_{\ell-1}+\frac{1}{2}n_\ell)x_\ell} \\ e^{2\pi i\varepsilon_1(-\frac{1}{2}n_{\ell-1}+\frac{1}{2}n_\ell)x_1} & \cdots & e^{2\pi i\varepsilon_\ell(-\frac{1}{2}n_{\ell-1}+\frac{1}{2}n_\ell)x_\ell} \end{vmatrix}.$$

Furthermore, we make this into a simpler form as follows:

$$\begin{aligned} J_{\mathbf{p}}(\mathbf{x}) &= \sum_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_\ell=\pm1} \varepsilon_1\varepsilon_2\cdots\varepsilon_\ell e^{2\pi i\frac{1}{2}n_\ell(\sum_{i=1}^{\ell}\varepsilon_i x_i)} e^{-2\pi i\frac{1}{2}n_{\ell-1}(\sum_{i=1}^{\ell}\varepsilon_i x_i)} \times \\ &\quad \times \begin{vmatrix} e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_1 x_1} & \cdots & e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_1 x_1} & \cdots & e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ \vdots & & \vdots \\ e^{2\pi i(n_{\ell-1})\varepsilon_1 x_1} & \cdots & e^{2\pi i(n_{\ell-1})\varepsilon_\ell x_\ell} \\ 1 & \cdots & 1 \end{vmatrix} \\ &= \sum_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_\ell=\pm1} \varepsilon_1\varepsilon_2\cdots\varepsilon_\ell e^{\pi i(n_\ell-n_{\ell-1})\sum_{i=1}^{\ell}\varepsilon_i x_i} \times \\ &\quad \times \begin{vmatrix} e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_1 x_1} & \cdots & e^{2\pi i(n_1+n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_1 x_1} & \cdots & e^{2\pi i(n_2+\cdots+n_{\ell-1})\varepsilon_\ell x_\ell} \\ \vdots & & \vdots \\ e^{2\pi i(n_{\ell-1})\varepsilon_1 x_1} & \cdots & e^{2\pi i(n_{\ell-1})\varepsilon_\ell x_\ell} \\ 1 & \cdots & 1 \end{vmatrix}. \end{aligned}$$

Therefore, we obtain the required formula for $J_{\mathbf{p}}(\mathbf{x})$ and (4). //

(5) In the case of type G_2, since the angle between α_1 and α_2 is equal to $\frac{5\pi}{6}$, $W(R)$ is isomorphic to the dihedral group of order 12 ([4]). Since $\mathbf{p} = n_1\omega_1 + n_2\omega_2 = (-n_2, -n_1 - n_2, n_1 + 2n_2)$, we have

$$
\begin{aligned}
J_{\mathbf{p}}(\mathbf{x}) &= \sum_{w\in W(R)} \varepsilon(w) e^{2\pi i(\mathbf{x}|w(\mathbf{p}))} \\
&= \sum_{\varepsilon=\pm 1} \varepsilon \begin{vmatrix} e^{2\pi i\varepsilon(-n_2)x_1} & e^{2\pi i\varepsilon(-n_2)x_2} & e^{2\pi i\varepsilon(-n_2)x_3} \\ e^{2\pi i\varepsilon(-n_1-n_2)x_1} & e^{2\pi i\varepsilon(-n_1-n_2)x_2} & e^{2\pi i\varepsilon(-n_1-n_2)x_3} \\ e^{2\pi i\varepsilon(n_1+2n_2)x_1} & e^{2\pi i\varepsilon(n_1+2n_2)x_2} & e^{2\pi i\varepsilon(n_1+2n_2)x_3} \end{vmatrix} \\
&= \sum_{\varepsilon=\pm 1} \varepsilon\; e^{2\pi i\varepsilon(-n_2)(-x_1-x_2)} e^{2\pi i\varepsilon(-n_1-n_2)(-x_1-x_2)} e^{2\pi i\varepsilon(n_1+2n_2)(-x_1-x_2)} \times \\
&\quad \times \begin{vmatrix} e^{2\pi i\varepsilon(n_1+2n_2)(2x_1+x_2)} & e^{2\pi i\varepsilon(-n_2)(2x_1+x_2)} & e^{2\pi i\varepsilon(-n_1-n_2)(2x_1+x_2)} \\ e^{2\pi i\varepsilon(n_1+2n_2)(x_1+2x_2)} & e^{2\pi i\varepsilon(-n_2)(x_1+2x_2)} & e^{2\pi i\varepsilon(-n_1-n_2)(x_1+2x_2)} \\ 1 & 1 & 1 \end{vmatrix}.
\end{aligned}
$$

Therefore, $J_{\mathbf{p}}(\mathbf{x})$ is equal to the sum which is given by

$$
\begin{aligned}
J_{\mathbf{p}}(\mathbf{x}) &= \sum_{\varepsilon=\pm 1} \varepsilon \begin{vmatrix} e^{2\pi i\varepsilon(n_1+2n_2)(2x_1+x_2)} & e^{2\pi i\varepsilon(-n_2)(2x_1+x_2)} & e^{2\pi i\varepsilon(-n_1-n_2)(2x_1+x_2)} \\ e^{2\pi i\varepsilon(n_1+2n_2)(x_1+2x_2)} & e^{2\pi i\varepsilon(-n_2)(x_1+2x_2)} & e^{2\pi i\varepsilon(-n_1-n_2)(x_1+2x_2)} \\ 1 & 1 & 1 \end{vmatrix} \\
&= \sum_{\varepsilon=\pm 1} \varepsilon\; e^{-6\pi i\varepsilon(n_1+n_2)(x_1+x_2)} \begin{vmatrix} e^{2\pi i\varepsilon(2n_1+3n_2)(2x_1+x_2)} & e^{2\pi i\varepsilon(2n_1+3n_2)(2x_1+x_2)} & 1 \\ e^{2\pi i\varepsilon(n_1)(2x_1+x_2)} & e^{2\pi i\varepsilon\; n_1(x_1+2x_2)} & 1 \\ 1 & 1 & 1 \end{vmatrix}.
\end{aligned}
$$

Therefore, we obtain the required formula for $J_{\mathbf{p}}(\mathbf{x})$ and (5). //

Thus, we obtain Theorem 5.3. //

Proof of Theorem 5.2.

(1) In the case of type A_ℓ, we only have to prove

$$
S_{\mathbf{p}}(\mathbf{x}) \;=\; \mathrm{Perm}\begin{pmatrix} e^{2\pi i(n_1+n_2+\cdots+n_\ell)x_1} & \cdots & e^{2\pi i(n_1+n_2+\cdots+n_\ell)x_{\ell+1}} \\ e^{2\pi i(n_2+\cdots+n_\ell)x_1} & \cdots & e^{2\pi i(n_2+\cdots+n_\ell)x_{\ell+1}} \\ \vdots & & \vdots \\ e^{2\pi i(n_\ell)x_1} & \cdots & e^{2\pi i(n_\ell)x_{\ell+1}} \\ 1 & \cdots & 1 \end{pmatrix} \tag{5.34}
$$

for all $\mathbf{x} = (x_1, x_2, \cdots, x_\ell, x_{\ell+1}) \in \mathbb{R}^{\ell+1}$.

We come back to the situation of the proof of Theorem 5.3. We have

$$
\begin{aligned}
S_{\mathbf{p}}(\mathbf{x}) &= \sum_{\sigma\in\mathfrak{S}_{\ell+1}} e^{2\pi i\sum_{j=1}^{\ell+1} x_j p_{\sigma(j)}} \\
&= \mathrm{Perm}\begin{pmatrix} e^{2\pi i x_1 p_1} & \cdots & e^{2\pi i x_{\ell+1} p_1} \\ \vdots & & \vdots \\ e^{2\pi i x_1 p_{\ell+1}} & \cdots & e^{2\pi i x_{\ell+1} x_{\ell+1}} \end{pmatrix}.
\end{aligned} \tag{5.35}
$$

Then, we have to substitute (5.22) into (5.35). To carry this out, we should summarize the fundamental properties of the permanent Perm(A) for a matrix $A = (a_{ij})$ of degree n as follows. By definition, the permanent Perm(A) is given by

$$\mathrm{Perm}(A) := \sum_{\sigma \in \mathfrak{S}_n} a_{\sigma(1)1}\, a_{\sigma(2)2} \cdots a_{\sigma(n)n}.$$

(i) For every scalar c,

$$\mathrm{Perm}(A_{\bullet 1} \cdots , c\, A_{\bullet j}, \cdots , A_{\bullet n}) = c\ \mathrm{Perm}(A).$$

(ii)

$$\begin{aligned}
&\mathrm{Perm}(\mathrm{A}_{\bullet 1}, \cdots , \mathrm{A}_{\bullet j-1}, \mathrm{A}_{\bullet j}^{(1)} + \mathrm{A}_{\bullet j}^{(2)}, \mathrm{A}_{\bullet j+1}, \cdots , \mathrm{A}_{\bullet n}) \\
&\quad = \mathrm{Perm}(\mathrm{A}_{\bullet 1}, \cdots , \mathrm{A}_{\bullet j-1}, \mathrm{A}_{\bullet j}^{(1)}, \mathrm{A}_{\bullet j+1}, \cdots , \mathrm{A}_{\bullet n}) \\
&\qquad + \mathrm{Perm}(\mathrm{A}_{\bullet 1}, \cdots , \mathrm{A}_{\bullet j-1}, \mathrm{A}_{\bullet j}^{(2)}, \mathrm{A}_{\bullet j+1}, \cdots , \mathrm{A}_{\bullet n})
\end{aligned}$$

(iii) For every permutation τ of $\{1, 2, \cdots , n\}$,

$$\mathrm{Perm}(A_{\bullet \tau(1)}, \cdots , A_{\bullet \tau(n)}) = \mathrm{Perm}(A).$$

(iv) For the transpose matrix ${}^tA$ of A,

$$\mathrm{Perm}({}^tA) = \mathrm{Perm}(A).$$

Here, $A_{\bullet j}$ stands for the j-th column vector of $A (j = 1, \dots , n)$. By substituting (5.22) into (5.35), and by making use of only (i) and (iv) in the above properties of the permanent, we have (5.34).

In the same way, we have (2), (3), (4) and (5) in Theorem 5.4. We obtain Theorem 5.4. //

6 The calculation of the eigenvalues and eigenfunctions of 2 and 3 dimension

6.1 Summary

We calculate each alcove $D(R)$ (type $A_2, B_2, G_2, A_3, B_3, C_3$) which is the fundamental domain of the Affine Weyl groups by using root systems and Affine Weyl groups theory [4]. And we have an expression with coordinates in the space for each domain explicitly. Furthermore, we calculate corresponding eigenvalues and eigenfunctions for each $D(R)$. Bérard [2] did not calculate eigenvalues and eigenfunctions of type $A_3 \sim C_3$ explicitly, so we show these have.

6.2 Figure of $D(R)$

We prepare several notions for calculation of alcove $D(R)$. We calculate basis α_i and fundamental weights ω_i in root systems, hyperplane $L_{\alpha,k}$, and obtain alcove $D(R)$ by Weyl chamber $C(R)$. We do this with type $A_2, B_2, G_2, A_3, B_3, C_3$. The methods of calculation of alcove $D(R)$ are as follows.

(1) Define the space V by
$V = \mathbb{R}^2 = \{(x, y) \mid x,\ y \in \mathbb{R}\}$ (in the case of B_2, C_2),
$V = \{(x, y, z) \mid x,\ y,\ z \in \mathbb{R},\ x + y + z = 0\}$ (in the case of type A_2, G_2).

(2) Calculate fundamental root α_i, $(\alpha_i \mid \alpha_j)$, $\alpha_i^\vee$.

(3) Calculate fundamental weights ω_i, inner product$(w_i \mid w_i)$.

(4) Select the highest root positive root set.

(5) Determine the Weyl chamber $C(R)$,
$C(R) = \{x_1\omega_1 + x_2\omega_2 \mid x_1 > 0,\ x_2 > 0\}$.

(6) Calculate the alcove $D(R)$,
$D(R) = \{x_1\omega_1 + x_2\omega_2 \mid x_1 > 0,\ x_2 > 0,\ (x_1\omega_1 + x_2\omega_2 \mid \tilde{\alpha}) < 1\}$.

We show the coordinate expression of vertices for each $D(R)$.

Definition

The Weyl chamber $C(R)$ is defined as one of the connected components of the open set $V \backslash (\bigcup_{\alpha \in R} L_\alpha)$. For each $\alpha \in R$, we define $L_\alpha := \{\mathbf{x} \in V \mid (\alpha \mid \mathbf{x}) = 0\}$. The alcove $D(R)$ is one of the connected components of the open set $V \backslash (\bigcup_{\alpha,k} L_{\alpha,k})$. We choose it such that it is inside $C(R)$, whose closure includes origin o. Here the hyperplane $L_{\alpha,k} (\alpha \in R,\ k \in \mathbb{Z})$ is defined by $L_{\alpha,k} := \{\mathbf{x} \in V \mid (\alpha \mid \mathbf{x}) = k\}$.

We show the calculation of $D(R)$ as follows.

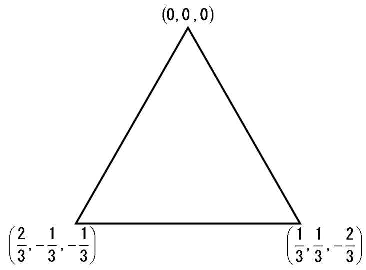
Type A_2

We use standard basis $e_1 = (1,\ 0,\ 0),\ e_2 = (0,\ 1,\ 0),\ e_3 = (0,\ 0,\ 1)$.
Then, $V = \{(x,\ y,\ z) \,|\, x + y + z = 0\}$.

The fundamental root is given by $\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3$.
We have $(\alpha_1 \,|\, \alpha_1) = 2,\ (\alpha_2 \,|\, \alpha_2) = 2$, then,

$$\alpha_1^\vee = \frac{2\alpha_1}{(\alpha_1 \,|\, \alpha_1)},\ \alpha_2^\vee = \frac{2\alpha_2}{(\alpha_2 \,|\, \alpha_2)}.$$

The fundamental weights are



$$\omega_1 = e_1 - \frac{1}{3}(e_1 + e_2 + e_3) = \left(\frac{2}{3},\ -\frac{1}{3},\ -\frac{1}{3}\right),$$
$$\omega_2 = e_1 + e_2 - \frac{2}{3}(e_1 + e_2 + e_3) = \left(\frac{1}{3},\ \frac{1}{3},\ -\frac{2}{3}\right).$$

Also, the modulus of both ω_1 and ω_2 are $\sqrt{\frac{2}{3}}$, and it is at angle of 60 degrees between $\omega_1$and ω_2, that is,

$$|\omega_1|^2 = \frac{2}{3}\left(= \frac{4}{9} + \frac{1}{9} + \frac{1}{9} = \frac{6}{9}\right), \quad |\omega_2|^2 = \frac{2}{3}\left(= \frac{1}{9} + \frac{1}{9} + \frac{4}{9} = \frac{6}{9}\right),$$
$$(\omega_1, \omega_2) = \frac{1}{3}\left(= \frac{2}{9} - \frac{1}{9} + \frac{2}{9} = \frac{3}{9}\right)$$

The Weyl chamber is $C(R) = \{(x_1\omega_1 + x_2\omega_2) \,|\, x_1 > 0,\ x_2 > 0\}$.

The set of **positive roots** is $\{e_1 - e_2,\ e_2 - e_3,\ e_1 - e_3\} = \{\alpha_1,\ \alpha_2,\ \alpha_1 + \alpha_2\}$.

The highest root $\tilde{\alpha}$ is $\tilde{\alpha} = e_1 - e_3 = \alpha_1 + \alpha_2$.
We calculate alcove $D(R)$.

$$(\omega \,|\, \alpha_1 + \alpha_2) = (x_1\omega_1 + x_2\omega_2 \,|\, \alpha_1^\vee + \alpha_2^\vee) = x_1 + x_2.$$

The alcove $D(R)$ is given by

$$\begin{aligned} D(R) &= \{x_1\omega_1 + x_2\omega_2 \,|\, 0 \leqq x_1,\ 0 \leqq x_2,\ x_1 + x_2 \leqq 1\} \\ &= \{x_1\omega_1 + x_2\omega_2 \,|\, 0 \leqq x_1 \leqq 1,\ 0 \leqq x_2 \leqq 1,\ x_1 + x_2 \leqq 1\}. \end{aligned}$$

We take a straight line in order to figure out the $D(R)$ as follows. If $x_1 + x_2 = 1$, then, we have $x_1\omega_1 + (1 - x_1)\omega_2 = x_1(\omega_1 - \omega_2) + \omega_2$. That is a straight line which includes vectors, ω_1, ω_2. Thus, $D(R)$ is the equilateral triangle which has 3 vertices $(1,\ 0,\ 0), \left(\frac{2}{3},\ -\frac{1}{3},\ -\frac{1}{3}\right), \left(\frac{1}{3},\ \frac{1}{3},\ -\frac{2}{3}\right)$.

Type B_2

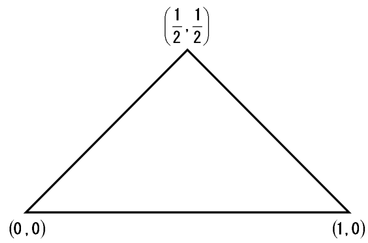
We use the standard basis $e_1 = (1,\ 0),\ e_2 = (0,\ 1)$.

Then, $V = \mathbb{R}^2 = \{(x, y)\,|\,x, y \in \mathbb{R}\}$.

The fundamental root is given by $\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2$.

We have $(\alpha_1\,|\,\alpha_1) = 1,\ (\alpha_2\,|\,\alpha_2) = 1$, then,

$$\alpha_1^\vee = \frac{2\alpha_1}{(\alpha_1\,|\,\alpha_1)} = \alpha_1, \quad \alpha_2^\vee = \frac{2\alpha_2}{(\alpha_2\,|\,\alpha_2)} = \alpha_2.$$



The fundamental weights are

$\omega_1 = e_1 = (1,\ 0),\ \omega_2 = (e_1 + e_2) = (1,\ 1)$.

Also, we have

$$|\omega_1|^2 = 1,\ |\omega_2|^2 = \frac{1}{2},\ (\omega_1,\ \omega_2) = \frac{1}{2},$$

$$\cos\theta = \frac{(\omega_1|\omega_2)}{|\omega_1||\omega_2|} = \frac{\frac{1}{2}}{1 \times \frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}.$$

The Weyl chamber is $C(R) = \{(x_1\omega_1 + x_2\omega_2)\,|\,x_1 > 0,\ x_2 > 0\}$.

The set of **positive roots** is

$$\{e_1,\ e_2,\ e_1 + e_2,\ e_1 - e_2\} = \{\alpha_1 + \alpha_2,\ \alpha_2,\ \alpha_1 + 2\alpha_2,\ \alpha_1\}.$$

The highest root $\tilde{\alpha}$ is $\tilde{\alpha} = e_1 + e_2 = \alpha_1 + 2\alpha_2$.

We calculate the alcove $D(R)$. We have

$$D(R) = \{x_1\omega_1 + x_2\omega_2\,|\,x_1 > 0,\ x_2 > 0,\ (x_1\omega_1 + x_2\omega_2\,|\,\alpha) < 1, {}^\forall\alpha(\textit{positive root})\},$$

$$\begin{aligned}(\omega\,|\,\alpha_1 + 2\alpha_2) &= (x_1\omega_1 + x_2\omega_2\,|\,\alpha_1 + 2\alpha_2)\\ &= (x_1\omega_1 + x_2\omega_2\,|\,\alpha_1^\vee + \alpha_2^\vee) = x_1 + x_2 < 1.\end{aligned}$$

The alcove $D(R)$ is given by

$$D(R) = \{x_1\omega_1 + x_2\omega_2\,|\,x_1 > 0,\ x_2 > 0,\ x_1 + x_2 < 1\}.$$

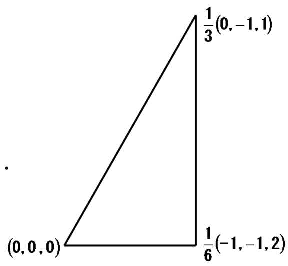
We take a straight line for imaging $D(R)$ as follows. If $x_1 + x_2 = 1$, then $x_1\omega_1 + x_2\omega_2 = x_1\omega_1 + (1 - x_1)\omega_2 = x_1(\omega_1 - \omega_2) + \omega_2$. That is a straight line which includes vectors $\omega_1,\ \omega_2$. Thus, $D(R)$ is the right isosceles triangle which has 3 vertices $(0, 0), (1, 0), \left(\frac{1}{2},\ \frac{1}{2}\right)$.

Type G_2

We use the standard basis $e_1 = (1,\ 0,\ 0),\ e_2 = (0,\ 1,\ 0),\ e_3 = (0,\ 0,\ 1)$.
Then, $V = \{(x,\ y,\ z) \mid x + y + z = 0\}$.

The fundamental root is given by $\alpha_1 = e_1 - e_2,\ \alpha_2 = -2e_1 + e_2 + e_3$.
We have $(\alpha_1 \mid \alpha_1) = 2,\ (\alpha_1 \mid \alpha_2) = -3,\ (\alpha_2 \mid \alpha_2) = 6$, then

$$\alpha_1^\vee = \frac{2\alpha_1}{(\alpha_1 \mid \alpha_1)} = \alpha_1,\ \alpha_2^\vee = \frac{2\alpha_2}{(\alpha_2 \mid \alpha_2)} = \frac{1}{3}\alpha_1.$$



The fundamental weights are

$\omega_1 = 2\alpha_1 + \alpha_2 = 2(1,\ -1,\ 0) + (-2,\ 1, 1) = (0,\ -1,\ 1)$
$\omega_2 = 3\alpha_1 + 2\alpha_2 = 3(1,\ -1,\ 0) + 2(-2,\ 1,\ 1) = (-1,\ -1,\ 2)$.

Also, the modulus of ω_1 is $\sqrt{2}$, the modulus of ω_2 is $\sqrt{6}$, and it is at angle of 30 degrees between ω_1 and ω_2, that is,

$$|\omega_1|^2 = 2,\ |\omega_2|^2 = 6,\ (\omega_1, \omega_2) = 3,\ \cos\theta = \frac{(\omega_1|\omega_2)}{|\omega_1||\omega_2|} = \frac{3}{\sqrt{2} \times \sqrt{6}} = \frac{\sqrt{3}}{2}.$$

The Weyl chamber is $C(R) = \{(x_1\omega_1 + x_2\omega_2) \mid x_1 > 0,\ x_2 > 0\}$.

The set of **positive roots** is

$$\begin{aligned}&\{e_1 - e_2,\ -2e_1 + e_2 + e_3,\ -e_1 + e_3, -e_2 + e_3,\ e_1 - 2e_2 + e_3, -e_1 - e_2 + 2e_3\}\\ &\quad = \{\alpha_1, \alpha_2,\ \alpha_2 + \alpha_1,\ 2\alpha_1 + \alpha_2,\ 3\alpha_1 + \alpha_2,\ 3\alpha_1 + 2\alpha_2\}\\ &\quad = \{\alpha_1^\vee, 3\alpha_2^\vee,\ \alpha_1^\vee + 3\alpha_2^\vee,\ 2\alpha_1^\vee + 3\alpha_2^\vee,\ 3\alpha_1^\vee + 3\alpha_2^\vee,\ 3\alpha_1^\vee + 6\alpha_2^\vee\}.\end{aligned}$$

The highest root $\tilde{\alpha}$ is $\tilde{\alpha} = -e_1 - e_2 + 2e_3 = 3\alpha_1 + 2\alpha_2$.

We calculate the alcove $D(R) = \{x_1\omega_1 + x_2\omega_2 \mid x_1 > 0,\ x_2 > 0,$ $(x_1\omega_1 + x_2\omega_2 \mid \alpha) < 1,\ ^\forall\alpha$, positive root$\}$. Here, we use the highest root as α.

We have

$$\begin{aligned}(\omega \mid 3\alpha_1 + 2\alpha_2\ ) &= (x_1\omega_1 + x_2\omega_2 \mid 3\alpha_1 + 2\alpha_2) = (x_1\omega_1 + x_2\omega_2 \mid 3\alpha_1^\vee + 6\alpha_2^\vee)\\ &= 3x_1 + 6x_2 < 1,\ x_2 < \frac{1}{6}(1 - 3x_1).\end{aligned}$$

The alcove $D(R)$ is given by

$$D(R) = \{x_1\omega_1 + x_2\omega_2 \mid x_1 > 0,\ x_2 > 0,\ x_2 < \frac{1}{6}(1 - 3x_1)\}.$$

We take a straight line for imaging $D(R)$ as follows. If $x_2 = \dfrac{1}{6}(1 - 3x_1)$, then

$$x_1\omega_1 + x_2\omega_2 = x_1\omega_1 + \frac{1}{6}(1 - 3x_1)\omega_2 = \frac{1}{6}\omega_2 + 3x_1\left(\frac{1}{3}\omega_1 - \frac{1}{6}\omega_2\right)\ (x_1 \text{ is arbitrary}).$$

That is a straight line which includes the vectors $\dfrac{1}{3}\omega_1,\ \dfrac{1}{6}\omega_2$. Thus, $D(R)$ is the right triangle with 30 and 60 degrees which has 3 vertices $(0,\ 0,\ 0),\ \dfrac{1}{6}(-1,\ -1,\ 2),\ \dfrac{1}{3}(0,\ -1,\ 1)$.

Type A_3

We use the standard basis $e_1 = (1, 0, 0, 0),\ e_2 = (0, 1, 0, 0),\ e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 1, 0)$.

Then $V = \{(x, y, z, w) \,|\, x + y + z + w = 0\}$.

The fundamental root is given by $\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3 - e_4$.

We have $(\alpha_1 \,|\, \alpha_1) = 2,\ (\alpha_1 \,|\, \alpha_2) = 2,\ (\alpha_3 \,|\, \alpha_3) = 2$, and then

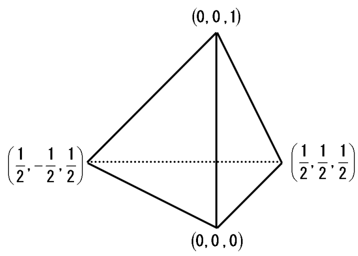
$$\alpha_1^\vee = \frac{2\alpha_1}{(\alpha_1 \,|\, \alpha_1)} = \alpha_1,\ \alpha_2^\vee = \frac{2\alpha_2}{(\alpha_2 \,|\, \alpha_2)} = \alpha_2,\ \alpha_3^\vee = \frac{2\alpha_3}{(\alpha_3 \,|\, \alpha_3)} = \alpha_3.$$

The fundamental weights are

$$\begin{aligned}
\omega_1 &= e_1 - \frac{1}{4}(e_1 + e_2 + e_3 + e_4) = \left(\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right),\\
\omega_2 &= e_1 + e_2 - \frac{2}{4}(e_1 + e_2 + e_3 + e_4) = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right),\\
\omega_3 &= e_1 + e_2 + e_3 - \frac{3}{4}(e_1 + e_2 + e_3 + e_4) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}\right).
\end{aligned}$$

Also, the modulus of ω_1 is $\frac{\sqrt{3}}{2}$, modulus of ω_2 is 1, modulus of ω_3 is $\frac{\sqrt{3}}{2}$, that is,

$$(\omega_1 \,|\, \omega_1) = \frac{3}{4},\quad (\omega_1 \,|\, \omega_2) = \frac{1}{2},\quad (\omega_1 \,|\, \omega_3) = \frac{1}{4},$$
$$(\omega_2 \,|\, \omega_2) = 1,\quad (\omega_2 \,|\, \omega_3) = \frac{1}{2},\quad (\omega_3 \,|\, \omega_3) = \frac{3}{4}.$$



The Weyl chamber is $C(R) = \{(x_1\omega_1 + x_2\omega_2 + x_3\omega_3) \,|\, x_1 > 0,\ x_2 > 0,\ x_3 > 0\}$.

The set of **positive roots** is
$\{e_1 - e_2,\ e_1 - e_3,\ e_1 - e_4,\ e_2 - e_3,\ e_2 - e_4,\ e_3 - e_4\}$
$= \{\alpha_1, \alpha_2,\ \alpha_1 + \alpha_2,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2,\ \alpha_2 + \alpha_3,\ \alpha_3\}$,

The highest root $\tilde{\alpha}$ is $\tilde{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$.

The alcove $D(R)$ is given by

$$\begin{aligned}
D(R) = &\{(x_1\omega_1 + x_2\omega_2 + x_3\omega_3) \,|\, x_1 > 0,\ x_2 > 0,\ x_3 > 0,\\
&(x_1\omega_1 + x_2\omega_2 + x_3\omega_3 \,|\, \alpha_1 + \alpha_2 + \alpha_3) = x_1 + x_2 + x_3 < 1\}.
\end{aligned}$$

We take a plane for imaging $D(R)$ as follows. If $x_1 + x_2 + x_3 = 1$, then, $x_1\omega_1 + x_2\omega_2 + x_3\omega_3 = x_1\omega_1 + x_2\omega_2 + (1 - x_1 - x_2)\omega_3 = \omega_3 + x_1(\omega_1 - \omega_3) + x_2(\omega_2 - \omega_3)$. That is a plane which includes the vectors $\omega_1,\ \omega_2,\ \omega_3$. Thus, $D(R)$ is the tetrahedron which has 4 vertices $(0, 0, 0),\ \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right),\ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), (0, 0, 1)$.

Type B_3

We use the standard basis $e_1 = (1,\, 0,\, 0),\ e_2 = (0,\, 1,\, 0),\ e_3 = (0,\, 0,\, 1)$.

Then $V = \mathbb{R}^3$.

The fundamental root is given by $\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3$.

We have $(\alpha_1 \,|\, \alpha_1) = 2,\ (\alpha_2 \,|\, \alpha_2) = 2,\ (\alpha_3 \,|\, \alpha_3) = 1$, then

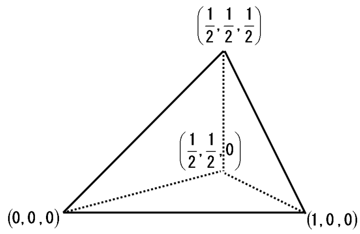
$$\alpha_1^\vee = \frac{2\alpha_1}{(\alpha_1 \,|\, \alpha_1)} = \alpha_1,\ \alpha_2^\vee = \frac{2\alpha_2}{(\alpha_2 \,|\, \alpha_2)} = \alpha_2,\ \alpha_3^\vee = \frac{2\alpha_3}{(\alpha_3 \,|\, \alpha_3)} = 2\alpha_3.$$

The fundamental weights are

$$\omega_1 = e_1 = (1,\, 0,\, 0),\ \omega_2 = e_1 + e_2 = (1,\, 1,\, 0),\ \omega_3 = \frac{1}{2}(e_1 + e_2 + e_3) = \left(\frac{1}{2},\, \frac{1}{2},\, \frac{1}{2}\right).$$

Also, the modulus of ω_1 is 1, the modulus of ω_2 is $\sqrt{2}$, the modulus of ω_3 is $\dfrac{\sqrt{3}}{2}$, and it is at angle of 45 degrees between ω_1 and ω_2, that is,

$$\begin{aligned}
&(\omega_1 \,|\, \omega_1) = 1, \quad (\omega_1 \,|\, \omega_2) = 1, \quad (\omega_1 \,|\, \omega_3) = \frac{1}{2},\\
&(\omega_2 \,|\, \omega_1) = 1, \quad (\omega_2 \,|\, \omega_2) = 2, \quad (\omega_2 \,|\, \omega_3) = 1,\\
&(\omega_3 \,|\, \omega_1) = \frac{1}{2}, \quad (\omega_3 \,|\, \omega_2) = 1, \quad (\omega_3 \,|\, \omega_3) = \frac{3}{4}.
\end{aligned}$$



The Weyl chamber is $C(R) = \{(x_1\omega_1 + x_2\omega_2 + x_3\omega_3) \,|\, x_1 > 0,\ x_2 > 0,\ x_3 > 0\}$.

The set of **positive roots** is

$\{e_1,\ e_2,\ e_3,\ e_1 - e_2,\ e_1 - e_3,\ e_2 - e_3,\ e_1 + e_2,\ e_1 + e_3,\ e_2 + e_3\}$
$= \{\alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_3,\ \alpha_3,\ \alpha_1,\ \alpha_1 + \alpha_2,\ \alpha_2,\ \alpha_1 + 2(\alpha_2 + \alpha_3),$
$\alpha_1 + \alpha_2 + 2\alpha_3,\ \alpha_2 + 2\alpha_3\}$.

The highest root $\tilde{\alpha}$ is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 = \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee = e_1 + e_2$.

The alcove $D(R)$ is given by

$$\begin{aligned}
D(R) = \{&x_1\omega_1 + x_2\omega_2 + x_3\omega_3 \,|\, x_1 > 0,\ x_2 > 0,\ x_3 > 0,\\
&(x_1\omega_1 + x_2\omega_2 + x_3\omega_3 \,|\, \alpha_1 + 2\alpha_2 + 2\alpha_3) < 1\}\\
= \{&x_1\omega_1 + x_2\omega_2 + x_3\omega_3 \,|\, x_1 > 0,\ x_2 > 0,\ x_3 > 0,\\
&(x_1\omega_1 + x_2\omega_2 + x_3\omega_3 \,|\, \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee) < 1\}\\
= \{&x_1\omega_1 + x_2\omega_2 + x_3\omega_3 \,|\, x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ x_1 + 2x_2 + x_3 < 1\}.
\end{aligned}$$

We take a plane for imaging $D(R)$ as follows. If $x_1 + 2x_2 + x_3 = 1$, then

$$x_1\omega_1 + x_2\omega_2 + x_3\omega_3 = x_1\omega_1 + 2x_2\frac{\omega_2}{2} + x_3\omega_3 = \frac{\omega_2}{2} + x_1\left(\omega_1 - \frac{\omega_2}{2}\right) + x_3\left(\omega_3 - \frac{\omega_2}{2}\right).$$

That is a plane which includes the vectors $\omega_1,\ \dfrac{\omega_2}{2},\ \omega_3$. Thus, $D(R)$ is the tetrahedron which has 4 vertices $(0,\, 0,\, 0), (1,\, 0,\, 0),\ \left(\frac{1}{2},\, \frac{1}{2},\, 0\right),\ \left(\frac{1}{2},\, \frac{1}{2},\, \frac{1}{2}\right)$.

Type C_3

We use the standard basis $e_1 = (1,\ 0,\ 0),\ e_2 = (0,\ 1,\ 0),\ e_3 = (0,\ 0,\ 1)$.
Then $V = \mathbb{R}^3$.

The fundamental root is given by $\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = 2e_3$.
We have $(\alpha_1 \,|\, \alpha_1) = 2,\ (\alpha_2 \,|\, \alpha_2) = 2,\ (\alpha_3 \,|\, \alpha_3) = 4$, then

$$\alpha_1^\vee = \frac{2\alpha_1}{(\alpha_1 \,|\, \alpha_1)} = \alpha_1,\ \alpha_2^\vee = \frac{2\alpha_2}{(\alpha_2 \,|\, \alpha_2)} = \alpha_2,\ \alpha_3^\vee = \frac{2\alpha_3}{(\alpha_3 \,|\, \alpha_3)} = \frac{1}{2}\alpha_3.$$

The fundamental weights are

$$\omega_1 = e_1 = (1,\ 0,\ 0),\ \omega_2 = e_1 + e_2 = (1,\ 1,\ 0),\ \omega_3 = \frac{1}{2}(e_1 + e_2 + e_3) = (1,\ 1,\ 1)$$

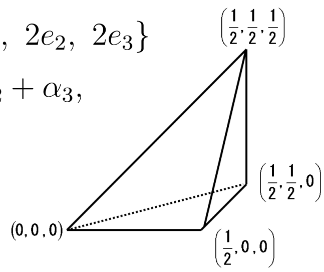
Also, the modulus of ω_1 is 1, the modulus of ω_2 is $\sqrt{2}$, the modulus of ω_3 is $\sqrt{3}$, and it is at angle of 45 degrees between ω_1 and ω_2, that is,

$$(\omega_1 \,|\, \omega_1) = 1,\ (\omega_1 \,|\, \omega_2) = 1,\ (\omega_1 \,|\, \omega_3) = 1,\ (\omega_2 \,|\, \omega_2) = 2,\ (\omega_2 \,|\, \omega_3) = 2,\ (\omega_3 \,|\, \omega_3) = 3.$$

The Weyl chamber is $C(R) = \{(x_1\omega_1 + x_2\omega_2 + x_3\omega_3) \,|\, x_1 > 0,\ x_2 > 0,\ x_3 > 0\}$.

The set of **positive roots** is

$$\{e_1 - e_2,\ e_1 - e_3,\ e_2 - e_3,\ e_1 + e_2,\ e_1 + e_3,\ e_2 + e_3,\ 2e_1,\ 2e_2,\ 2e_3\}$$
$$= \{\alpha_1,\ \alpha_1 + \alpha_2,\ \alpha_2,\ \alpha_1 + 2\alpha_2 + \alpha_3,\ \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3,$$
$$2(\alpha_1 + \alpha_2) + \alpha_3,\ 2\alpha_2 + \alpha_3,\ \alpha_3\}.$$



The highest root $\tilde{\alpha}$ is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3$
$= \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee = e_1 + e_2$.

The alcove $D(R)$ is given by

$$\begin{aligned} D(R) &= \{x_1\omega_1 + x_2\omega_2 + x_3\omega_3 \,|\, x_1 > 0,\ x_2 > 0,\ x_3 > 0, \\ &\quad (x_1\omega_1 + x_2\omega_2 + x_3\omega_3 \,|\, 2\alpha_1 + 2\alpha_2 + 2\alpha_3) < 1\} \\ &= \{x_1\omega_1 + x_2\omega_2 + x_3\omega_3 \,|\, x_1 > 0,\ x_2 > 0,\ x_3 > 0, \\ &\quad (x_1\omega_1 + x_2\omega_2 + x_3\omega_3 \,|\, 2\alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee) < 1\} \\ &= \{x_1\omega_1 + x_2\omega_2 + x_3\omega_3 \,|\, x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ 2x_1 + 2x_2 + 2x_3 < 1\}. \end{aligned}$$

We take a plane for imaging $D(R)$ as follows. If $2x_1 + 2x_2 + 2x_3 = 1$, then

$$\begin{aligned} x_1\omega_1 + x_2\omega_2 + x_3\omega_3 &= 2x_1\frac{\omega_1}{2} + 2x_2\frac{\omega_2}{2} + 2x_3\frac{\omega_3}{2} = 2x_1\frac{\omega_1}{2} + 2x_2\frac{\omega_2}{2} + (1 - 2x_1 - 2x_2)\frac{\omega_3}{2} \\ &= \frac{\omega_3}{2} + 2x_1\left(\omega_1 - \frac{\omega_3}{2}\right) + 2x_2\left(\frac{\omega_2}{2} - \frac{\omega_3}{2}\right). \end{aligned}$$

That is a plane which includes the vectors $\frac{\omega_1}{2},\ \frac{\omega_2}{2},\ \frac{\omega_3}{2}$. Thus, $D(R)$ is the tetrahedron which has 4 vertices $(0, 0, 0), \left(\frac{1}{2},\ 0,\ 0\right),\ \left(\frac{1}{2},\ \frac{1}{2},\ 0\right),\ \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$.

6.3 Calculation of the eigenvalues and eigenfunctions

We calculate the eigenvalues by the fundamental weights ω. In addition, we figure out the eigenfunctions, which are based on the calculation of elements ω of $W(R)$ by Bérard's formula of eigenfunctions.

The eigenvalues on the domain $D(R)$ are determined.

In the case of Dirichlet conditions, $4\pi^2(\mathbf{p}|\mathbf{p})$,

In the case of Neumann conditions, $4\pi^2(\mathbf{p}|\mathbf{p})$.

The methods of calculation of the eigenvalues and eigenfunctions are as follows.

(1) Calculate $\alpha_i,\ (\alpha_i\,|\,\alpha_j),\ \alpha_i^\vee$.

(2) Calculate $\omega_i,\ (\omega_i\,|\,\omega_j)$.

(3) Calculate the eigenvalues (in the case of 2 dimensions).

$$\begin{aligned}&4\pi^2(m\omega_1+n\omega_2\,|\,m\omega_1+n\omega_2)\\&=4\pi^2(m^2(\omega_1\,|\,\omega_1)\ +2mn(\omega_1\,|\,\omega_2)+n^2(\omega_2\,|\,\omega_2)),\end{aligned}$$

(where Dirichlet eigenvalues $(m,\,n=1,2,\cdots)$,
Neumann eigenvalues $(m,\,n=0,1,\cdots)$).

(4) Calculate the eigenfunctions.

The Dirichlet eigenfunctions are

$$J_{\mathbf{p}}(\mathbf{x})=\sum_{w\in W(R)}\varepsilon(w)f_{w(\mathbf{p})}(\mathbf{x})=\sum_{w\in W(R)}\varepsilon(w)\exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\},$$

where $\mathbf{p}\in P(R)\cap C(R)$.

The Neumann eigenfunctions are

$$S_{\mathbf{p}}(\mathbf{x})=\sum_{w\in W(R)}f_{w(\mathbf{p})}(\mathbf{x})\ =\sum_{w\in W(R)}\exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\},$$

where $\mathbf{p}\in P(R)\cap\overline{C(R)}$.

We show the methods of calculation of the eigenvalues and eigenfunctions as follows.

The eigenvalues of Dirichlet eigenvalues problems on $D(R)$ are

$$4\pi^2(\mathbf{p}|\mathbf{p}), \quad \mathbf{p} \in P(R) \cap C(R).$$

The multiplicity of eigenvalues $4\pi^2(\mathbf{p}|\mathbf{p})$ are $\#\{4\pi^2(\mathbf{q}|\mathbf{q})|\, 4\pi^2(\mathbf{q}|\mathbf{q}) = 4\pi^2(\mathbf{p}|\mathbf{p})\}$.

The eigenvalues of Neumann eigenvalues problems on $D(R)$ are

$$4\pi^2(\mathbf{p}|\mathbf{p}), \quad \mathbf{p} \in P(R) \cap \overline{C(R)}.$$

Here, $\overline{C(R)}$ is the closure of $C(R)$.

The multiplicity of eigenvalues $4\pi^2(\mathbf{p}|\mathbf{p})$ are $\#\{4\pi^2(\mathbf{q}|\mathbf{q})|\, 4\pi^2(\mathbf{q}|\mathbf{q}) = 4\pi^2(\mathbf{p}|\mathbf{p})\}$.

The set of weights of R, $P(R)$, is given by $\left\{\sum_{i=1}^{l} m_i\omega_i \,|\, m_i \text{ are integers}\right\}$. Here, $\omega_1, \omega_2, \cdots, \omega_l$ are the fundamental weights. Then, $\{\omega_1, \omega_2, \cdots, \omega_l\}$ are the dual basis of $\alpha_i^\vee (i = 1, 2, \cdots, l)$. $\alpha_i^\vee = \dfrac{2\alpha_i}{(\alpha_i|\alpha_i)}$, $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ are the fundamental roots. Then, we have

$$\begin{array}{lll} \omega \in P(R) \ \text{ and } \ \omega \in C(R) \leftrightarrow & m_i = 1, 2, \cdots & (i = 1, 2, \cdots, \ l), \\ \omega \in P(R) \ \text{ and } \ \omega \in \overline{C(R)} \leftrightarrow & m_i = 0, 1, \cdots & (i = 1, 2, \cdots, \ l). \end{array}$$

Thus, we obtain

$$P(R) \cap C(R) = \left\{\omega = \sum_{i=1}^{l} m_i\omega_i \,|\, m_i \text{ are positive integers}\right\},$$

$$P(R) \cap \overline{C(R)} = \left\{\omega = \sum_{i=1}^{l} m_i\omega_i \,|\, m_i \text{ are non} - \text{negative integers}\right\}.$$

We will take positive for left-handed direction.

We will use the following symbols:

$$\begin{cases} E \text{ means ''identity transformation''}, \\ RE \text{ means ''reflection with respect to''}, \\ RO \text{ means ''rotation''}, \\ I \text{ means ''inversion''}. \end{cases}$$

We will calculate for each type of $A_2 \sim C_3$, the eigenvalues and eigenfunctions.

Type A_2(the eigenvalues)

We use the standard basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, and

$$\begin{aligned} V &= \{(x_1, x_2, x_3) \,|\, x_1 + x_2 + x_3 = 0 \} \\ &= \left\{ x\tfrac{1}{\sqrt{2}}(1, -1, 0) + y\tfrac{1}{\sqrt{2}}(0, 1, -1) \,\middle|\, \text{x, y are real numbers} \right\}. \end{aligned}$$

The set of roots is $\{\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_1 - e_3)\}$.

$$\begin{aligned} C(R) &= \{(x_1, x_2, x_3) \,|\, x_1 + x_2 + x_3 = 0,\ x_1 - x_2 > 0 \text{ and } x_2 - x_3 > 0\} \\ &= \{(x_1, x_2, x_3) \,|\, x_1 + x_2 + x_3 = 0,\ x_1 > x_2 > x_3\}. \end{aligned}$$

In the previous calculation of $D(R)$, we have $\omega_1 = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$, $\omega_2 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$, then

$$(\omega_1, \omega_1) = \frac{2}{3}, \quad (\omega_1, \omega_2) = \frac{1}{3}, \quad (\omega_2, \omega_1) = \frac{1}{3}, \quad (\omega_2, \omega_2) = \frac{2}{3}.$$

We will calculate the eigenvalues.

Because of $P(R) \cap C(R) = \{m\omega_1 + n\omega_2 \,|\, m, n > 0 \text{ integer}\}$, then the eigenvalues are

$$\begin{aligned} 4\pi^2(\mathbf{p}|\mathbf{p}) &= 4\pi^2(m\omega_1 + n\omega_2 \,|\, m\omega_1 + n\omega_2) \\ &= 4\pi^2(m^2(\omega_1 \,|\, \omega_1) + 2mn(\omega_1 \,|\, \omega_2) + n^2(\omega_2 \,|\, \omega_2)) \\ &= 4\pi^2\left(m^2\tfrac{2}{3} + 2mn\tfrac{1}{3} + n^2\tfrac{2}{3}\right) \\ &= \frac{8}{3}\pi^2(m^2 + mn + n^2), \end{aligned}$$

where $\mathbf{p} = m\omega_1 + n\omega_2$.

We can also write

$$4\pi^2(\mathbf{p}|\mathbf{p}) = \frac{16}{9}\pi^2(k^2 + \ell^2 + k\ell).$$

The first Dirichlet eigenvalue λ_1 on $D(R)$ is $8\pi^2$.

The second Neumann eigenvalue λ_2 on $D(R)$ is $\dfrac{8}{3}\pi^2$.

Calculation of the eigenvalues

In the case of type A_2, the Dirichlet eigenvalues are

$$\lambda_{kl} = \frac{16}{9}\pi^2(k^2 + \ell^2 + k\ell), \quad k, l = 1, 2, \cdots.$$

In [20], Pinsky showed that

$$\lambda_{mn} = \frac{16}{27}\pi^2(m^2 + mn + n^2), \quad m + n \text{ is multiple of 3.}$$

Type $\boldsymbol{A_2}$(The Dirichlet eigenfunction)

$W(R)$ consists of 6 symmetric operations which have an axis in the (1, 1, 1) direction, as follows.

$$\begin{pmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{pmatrix}(E), \quad \begin{pmatrix} 0\,1\,0 \\ 1\,0\,0 \\ 0\,0\,1 \end{pmatrix}(RE\ y=x), \quad \begin{pmatrix} 0\,0\,1 \\ 0\,1\,0 \\ 1\,0\,0 \end{pmatrix}(RE\ x=z),$$
$$\begin{pmatrix} 1\,0\,0 \\ 0\,0\,1 \\ 0\,1\,0 \end{pmatrix}(RE\ z=y), \begin{pmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 1\,0\,0 \end{pmatrix}(RO\ -120^\circ), \begin{pmatrix} 0\,0\,1 \\ 1\,0\,0 \\ 0\,1\,0 \end{pmatrix}(RO\ 120^\circ).$$

If $\mathbf{p} = m\omega_1 + n\omega_2 = \dfrac{1}{3}(2m+n,\ -m+n,\ -m-2n)$, then

$$\begin{aligned}
J_{\mathbf{p}}(\mathbf{x}) &= \sum_{w \in W(R)} \varepsilon(w) f_{w(\mathbf{p})}(\mathbf{x}) \ = \sum_{w \in W(R)} \varepsilon(w) \exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\} \\
&= \exp\left[\frac{2\pi i}{3}\{(2m+n)x + (-m+n)y + (-m-2n)z\}\right] \\
&\quad - \exp\left[\frac{2\pi i}{3}\{(-m+n)x + (2m+n)y + (-m-2n)z\}\right] \\
&\quad - \exp\left[\frac{2\pi i}{3}\{(-m-2n)x + (-m+n)y + (2m+n)z\}\right] \\
&\quad - \exp\left[\frac{2\pi i}{3}\{(2m+n)x + (-m+2n)y + (-m+n)z\}\right] \\
&\quad + \exp\left[\frac{2\pi i}{3}\{(-m+n)x + (-m+2n)y + (2m+n)z\}\right] \\
&\quad + \exp\left[\frac{2\pi i}{3}\{(-m-2n)x + (2m+n)y + (-m+n)z\}\right] \\
&= \begin{vmatrix} e^{\frac{2\pi i}{3}(2m+n)x} & e^{\frac{2\pi i}{3}(2m+n)y} & e^{\frac{2\pi i}{3}(2m+n)z} \\ e^{\frac{2\pi i}{3}(-m+n)x} & e^{\frac{2\pi i}{3}(-m+n)y} & e^{\frac{2\pi i}{3}(-m+n)z} \\ e^{\frac{2\pi i}{3}(-m-2n)x} & e^{\frac{2\pi i}{3}(-m-2n)y} & e^{\frac{2\pi i}{3}(-m-2n)z} \end{vmatrix},
\end{aligned}$$

provided $x+y+z=0$. Here, $\mathbf{x} = (x,\ y,\ z)$.

Type $\boldsymbol{A_2}$(The Dirichlet eigenfunction) (The alternative way)

If we put $\mathbf{p} = m\omega_1 + n\omega_2,\ \mathbf{x} = (x,\ y,\ z)$, then,

$$J_{\mathbf{p}}(\mathbf{x}) = \begin{vmatrix} e^{\frac{2\pi i}{3}(2m+n)x} & e^{\frac{2\pi i}{3}(2m+n)y} & e^{\frac{2\pi i}{3}(2m+n)z} \\ e^{\frac{2\pi i}{3}(-m+n)x} & e^{\frac{2\pi i}{3}(-m+n)y} & e^{\frac{2\pi i}{3}(-m+n)z} \\ e^{\frac{2\pi i}{3}(-m-2n)x} & e^{\frac{2\pi i}{3}(-m-2n)y} & e^{\frac{2\pi i}{3}(-m-2n)z} \end{vmatrix}.$$

Here, $x+y+z=0$, Then

$$
\begin{aligned}
J_{\mathbf{p}}(\mathbf{x}) &= e^{\frac{2\pi i}{3}(2m+n)(-x-y)}e^{\frac{2\pi i}{3}(-m+n)(-x-y)}e^{\frac{2\pi i}{3}(-m-2n)(-x-y)} \\
&\quad \times \begin{vmatrix} e^{\frac{2\pi i}{3}(2m+n)x}e^{\frac{2\pi i}{3}(2m+n)(x+y)} & e^{\frac{2\pi i}{3}(2m+n)y}e^{\frac{2\pi i}{3}(2m+n)(x+y)} & 1 \\ e^{\frac{2\pi i}{3}(-m+n)x}e^{\frac{2\pi i}{3}(2m+n)(x+y)} & e^{\frac{2\pi i}{3}(-m+n)y}e^{\frac{2\pi i}{3}(2m+n)(x+y)} & 1 \\ e^{\frac{2\pi i}{3}(-m-2n)x}e^{\frac{2\pi i}{3}(2m+n)(x+y)} & e^{\frac{2\pi i}{3}(-m-2n)y}e^{\frac{2\pi i}{3}(2m+n)(x+y)} & 1 \end{vmatrix}. \\
&= \begin{vmatrix} e^{\frac{2\pi i}{3}(2m+n)(2x+y)} & e^{\frac{2\pi i}{3}(2m+n)(x+2y)} & 1 \\ e^{\frac{2\pi i}{3}(-m+n)(2x+y)} & e^{\frac{2\pi i}{3}(-m+n)(x+2y)} & 1 \\ e^{\frac{2\pi i}{3}(-m-2n)(2x+y)} & e^{\frac{2\pi i}{3}(-m-2n)(x+2y)} & 1 \end{vmatrix}.
\end{aligned}
$$

Type A_2(The Neumann eigenfunction)

$$
\begin{aligned}
S_{\mathbf{p}}(\mathbf{x}) &= \sum_{w\in W(R)} f_{w(\mathbf{p})}(\mathbf{x}) \;=\; \sum_{w\in W(R)} \exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\} \\
&= \exp\left[\frac{2\pi i}{3}\{(2m+n)x+(-m+n)y+(-m-2n)z\}\right] \\
&\quad + \exp\left[\frac{2\pi i}{3}\{(-m+n)x+(2m+n)y+(-m-2n)z\}\right] \\
&\quad + \exp\left[\frac{2\pi i}{3}\{(-m-2n)x+(-m+n)y+(2m+n)z\}\right] \\
&\quad + \exp\left[\frac{2\pi i}{3}\{(2m+n)x+(-m+2n)y+(-m+n)z\}\right] \\
&\quad + \exp\left[\frac{2\pi i}{3}\{(-m+n)x+(-m+2n)y+(2m+n)z\}\right] \\
&\quad + \exp\left[\frac{2\pi i}{3}\{(-m-2n)x+(2m+n)y+(-m+n)z\}\right] \\
&= e^{\frac{2\pi i}{3}(2m+n)x}e^{\frac{2\pi i}{3}(-m+n)y}e^{\frac{2\pi i}{3}(-m-2n)z} + e^{\frac{2\pi i}{3}(2m+n)x}e^{\frac{2\pi i}{3}(-m+2n)y}e^{\frac{2\pi i}{3}(-m+n)z} \\
&\quad + e^{\frac{2\pi i}{3}(-m+n)x}e^{\frac{2\pi i}{3}(2m+n)y}e^{\frac{2\pi i}{3}(-m-2n)z} + e^{\frac{2\pi i}{3}(-m+n)x}e^{\frac{2\pi i}{3}(-m+2n)y}e^{\frac{2\pi i}{3}(2m+n)z} \\
&\quad + e^{\frac{2\pi i}{3}(-m-2n)x}e^{\frac{2\pi i}{3}(2m+n)y}e^{\frac{2\pi i}{3}(-m+n)z} + e^{\frac{2\pi i}{3}(-m-2n)x}e^{\frac{2\pi i}{3}(-m+n)y}e^{\frac{2\pi i}{3}(2m+n)z} \\
&= \operatorname{Perm}\begin{pmatrix} e^{\frac{2\pi i}{3}(2m+n)x} & e^{\frac{2\pi i}{3}(2m+n)y} & e^{\frac{2\pi i}{3}(2m+n)z} \\ e^{\frac{2\pi i}{3}(-m+n)x} & e^{\frac{2\pi i}{3}(-m+n)y} & e^{\frac{2\pi i}{3}(-m+n)z} \\ e^{\frac{2\pi i}{3}(-m-2n)x} & e^{\frac{2\pi i}{3}(-m-2n)y} & e^{\frac{2\pi i}{3}(-m-2n)z} \end{pmatrix},
\end{aligned}
$$

provided $x+y+z=0$. Here, $\mathbf{x}=(x,\ y,\ z)$.

Type B_2(The eigenvalues)

We use the standard basis $e_1 = (1,\ 0),\ e_2 = (0,\ 1)$.

$V = \mathbb{R}^2 = \{(x,\ y)\,|\,\text{x, y are real numbers}\}$.

In previous calculation of $D(R),\ \omega_1 = (1,\ 0),\ \omega_2 = \dfrac{1}{2}(1,\ 1),$ then

$$(\omega_1, \omega_1) = 1, \quad (\omega_1, \omega_2) = \frac{1}{2}, \quad (\omega_2, \omega_1) = \frac{1}{2}, \quad (\omega_2, \omega_2) = \frac{1}{2}.$$

We calculate eigenvalues as follows.

$\mathbf{p} = m\omega_1 + n\omega_2$, then the Dirichlet eigenvalues are

$$\begin{aligned} 4\pi^2(\mathbf{p}|\mathbf{p}) &= 4\pi^2(m\omega_1 + n\omega_2\,|\,m\omega_1 + n\omega_2) \\ &= 4\pi^2\left(m^2 + \tfrac{1}{2}\cdot 2mn + \tfrac{n^2}{2}\right) \\ &= 4\pi^2\left(m^2 + mn + \tfrac{n^2}{2}\right),\ (m, n = 1, 2, \cdots). \end{aligned}$$

Similarly, Neumann eigenvalues are

$$4\pi^2\left(m^2 + mn + \tfrac{n^2}{2}\right),\ (m, n = 1, 2, \cdots)$$

The first Dirichlet eigenvalue λ_1 on $D(R)$ is $10\pi^2$.
The second Neumann eigenvalue λ_2 on $D(R)$ is $2\pi^2$.

Type B_2(The Dirichlet eigenfunction)

$W(R)$ consists of 8 symmetric operations which have an axis in the (1, 1, 1) direction, as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (E), \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (RE\ \ x\ \ axis),$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (RE\ \ y\ \ axis), \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (RE\ \ y = x),$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad (RE\ \ to\ \ y = -x), \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad (RO\ \ 180^\circ),$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (RO\ \ 90^\circ), \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (RO\ \ -90^\circ).$$

$\mathbf{p} = m\omega_1 + n\omega_2 = \left(m + \tfrac{1}{2}n,\ \tfrac{1}{2}n\right)$, then

$$
\begin{aligned}
J_{\mathbf{p}}(\mathbf{x}) &= \sum_{w\in W(R)} \varepsilon(w) f_{w(\mathbf{p})}(\mathbf{x}) \;=\; \sum_{w\in W(R)} \varepsilon(w)\exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\} \\
&= \exp\left[2\pi i\{(m+\frac{1}{2}n)x+\frac{1}{2}ny\}\right] - \exp\left[2\pi i\{(m+\frac{1}{2}n)x-\frac{1}{2}ny\}\right] \\
&\quad - \exp\left[2\pi i\{-(m+\frac{1}{2}n)x+\frac{1}{2}ny\}\right] - \exp\left[2\pi i\{\frac{1}{2}nx+(m+\frac{1}{2}n)y\}\right] \\
&\quad - \exp\left[2\pi i\{-\frac{1}{2}nx-(m+\frac{1}{2}n)y\}\right] + \exp\left[2\pi i\{-(m+\frac{1}{2}n)x-\frac{1}{2}ny\}\right] \\
&\quad + \exp\left[2\pi i\{-\frac{1}{2}nx+(m+\frac{1}{2}n)y\}\right] + \exp\left[2\pi i\{\frac{1}{2}nx-(m+\frac{1}{2}n)y\}\right] \\
&= \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \varepsilon_1\varepsilon_2 \begin{vmatrix} e^{2\pi i\varepsilon_1(m+\frac{1}{2}n)x} & e^{2\pi i\varepsilon_1(m+\frac{1}{2}n)y} \\ e^{2\pi i\varepsilon_2(\frac{1}{2}n)x} & e^{2\pi i\varepsilon_2(\frac{1}{2}n)y} \end{vmatrix}.
\end{aligned}
$$

Here, $\mathbf{x} = (x,\, y)$.

Type B_2(The Neumann eigenfunction)

$$
\begin{aligned}
S_{\mathbf{p}}(\mathbf{x}) &= \sum_{w\in W(R)} f_{w(\mathbf{p})}(\mathbf{x}) \;=\; \sum_{w\in W(R)} \exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\} \\
&= \exp\left[2\pi i\{(m+\frac{1}{2}n)x+\frac{1}{2}ny\}\right] + \exp\left[2\pi i\{(m+\frac{1}{2}n)x-\frac{1}{2}ny\}\right] \\
&\quad + \exp\left[2\pi i\{-(m+\frac{1}{2}n)x+\frac{1}{2}ny\}\right] + \exp\left[2\pi i\{\frac{1}{2}nx+(m+\frac{1}{2}n)y\}\right] \\
&\quad + \exp\left[2\pi i\{-\frac{1}{2}nx-(m+\frac{1}{2}n)y\}\right] + \exp\left[2\pi i\{-(m+\frac{1}{2}n)x-\frac{1}{2}ny\}\right] \\
&\quad + \exp\left[2\pi i\{-\frac{1}{2}nx+(m+\frac{1}{2}n)y\}\right] + \exp\left[2\pi i\{\frac{1}{2}nx-(m+\frac{1}{2}n)y\}\right] \\
&= \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \left\{ e^{2\pi i\varepsilon_1(m+\frac{1}{2}n)x} e^{2\pi i\varepsilon_2(\frac{1}{2}n)y} + e^{2\pi i\varepsilon_2(\frac{1}{2}n)x} e^{2\pi i\varepsilon_1(m+\frac{1}{2}n)y} \right\} \\
&= \sum_{\varepsilon_1,\varepsilon_2=\pm 1} \operatorname{Perm}\begin{pmatrix} e^{2\pi i\varepsilon_1(m+\frac{1}{2}n)x} & e^{2\pi i\varepsilon_1(m+\frac{1}{2}n)y} \\ e^{2\pi i\varepsilon_2(\frac{1}{2}n)x} & e^{2\pi i\varepsilon_2(\frac{1}{2}n)y} \end{pmatrix}.
\end{aligned}
$$

Here, $\mathbf{x} = (x,\, y)$.

Type G_2(The eigenvalues)

We use the standard basis $e_1 = (1,\ 0,\ 0),\ e_2 = (0,\ 1,\ 0),\ e_3 = (0,\ 0,\ 1)$.

$V = \{(x,\ y,\ z) \,|\, x+y+z=0\} = \{x(e_1-e_3)+y(e_2-e_3) \,|\, x,\ y \text{ are real numbers}\}$

In the previous calculation of $D(R)$, $\omega_1 = (0,\ -1,\ 1)$, $\omega_2 = (-1,\ -1,\ 2)$, then

$$(\omega_1 \,|\, \omega_1) = 2, \quad (\omega_1 \,|\, \omega_2) = 3, \quad (\omega_2 \,|\, \omega_2) = 6$$

We calculate the eigenvalues. If $\mathbf{p} = m\omega_1 + n\omega_2$, then
the Dirichlet eigenvalues on $D(R)$ are

$$4\pi^2(m\omega_1 + n\omega_2 \,|\, m\omega_1 + n\omega_2) = 4\pi^2(2m^2 + 6mn + 6n^2),\ m, n > 0,$$

and the Neumann eigenvalues on $D(R)$ are

$$4\pi^2(m\omega_1 + n\omega_2 \,|\, m\omega_1 + n\omega_2) = 4\pi^2(2m^2 + 6mn + 6n^2),\ m, n \geqq 0.$$

The first Dirichlet eigenvalues λ_1 on $D(R)$ is $56\pi^2$.
The second Neumann eigenvalues λ_2 on $D(R)$ is $8\pi^2$.

Type G_2(The Dirichlet eigenfunction)

$W(R)$ consists of 12 symmetric operations which have an axis in the (1, 1, 1) direction, as follows. Here, $\varepsilon = \pm 1$.

$$\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \quad (E \text{ and } I), \qquad \begin{pmatrix} 0 & \varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \quad (RE\ y = x \text{ and } I),$$

$$\begin{pmatrix} 0 & 0 & \varepsilon \\ 0 & \varepsilon & 0 \\ \varepsilon & 0 & 0 \end{pmatrix} \quad (RE\ x = z \text{ and } I), \qquad \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon \\ 0 & \varepsilon & 0 \end{pmatrix} \quad (RE\ z = y \text{ and } I),$$

$$\begin{pmatrix} 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \\ \varepsilon & 0 & 0 \end{pmatrix} \quad (RO\ -120^\circ \text{ and } I), \qquad \begin{pmatrix} 0 & 0 & \varepsilon \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \end{pmatrix} \quad (RO\ 120^\circ \text{ and } I).$$

$\mathbf{p} = m\omega_1 + n\omega_2 = (-n, -m-n, m+2n)$, then

$$
\begin{aligned}
J_{\mathbf{p}}(\mathbf{x}) &= \sum_{w\in W(R)} \varepsilon(w) f_{w(\mathbf{p})}(\mathbf{x}) \;=\; \sum_{w\in W(R)} \varepsilon(w)\exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\} \\
&= \exp\left[2\pi i\{(-n)x+(-m-n)y+(m+2n)z\}\right] \\
&\quad - \exp\left[2\pi i\{(-m-n)x+(-n)y+(m+2n)z\}\right] \\
&\quad - \exp\left[2\pi i\{(m+2n)x+(-m-n)y+(-n)z\}\right] \\
&\quad - \exp\left[2\pi i\{(-n)x+(m+2n)y+(-m-n)z\}\right] \\
&\quad + \exp\left[2\pi i\{(-m-n)x+(m+2n)y+(-n)z\}\right] \\
&\quad + \exp\left[2\pi i\{(m+2n)x+(-n)y+(-m-n)z\}\right] \\
&\quad - \exp\left[-2\pi i\{(-n)x+(-m-n)y+(m+2n)z\}\right] \\
&\quad + \exp\left[-2\pi i\{(-m-n)x+(-n)y+(m+2n)z\}\right] \\
&\quad + \exp\left[-2\pi i\{(m+2n)x+(-m-n)y+(-n)z\}\right] \\
&\quad + \exp\left[-2\pi i\{(-n)x+(m+2n)y+(-m-n)z\}\right] \\
&\quad - \exp\left[-2\pi i\{(-m-n)x+(m+2n)y+(-n)z\}\right] \\
&\quad - \exp\left[-2\pi i\{(m+2n)x+(-n)y+(-m-n)z\}\right] \\
&= \sum_{\varepsilon=\pm 1} \varepsilon \begin{vmatrix} e^{2\pi i\varepsilon(-n)x} & e^{2\pi i\varepsilon(-n)y} & e^{2\pi i\varepsilon(-n)z} \\ e^{2\pi i\varepsilon(-m-n)x} & e^{2\pi i\varepsilon(-m-n)y} & e^{2\pi i\varepsilon(-m-n)z} \\ e^{2\pi i\varepsilon(m+2n)x} & e^{2\pi i\varepsilon(m+2n)y} & e^{2\pi i\varepsilon(m+2n)z} \end{vmatrix}.
\end{aligned}
$$

provided, $x+y+z=0$. Here, $\mathbf{x}=(x,\,y,\,z)$.

Type G_2(The Dirichlet eigenfunction) (The alternative way)

If we put $\mathbf{p}=m\omega_1+n\omega_2$, $\mathbf{x}=(x,\,y,\,z)$, we have

$$
J_{\mathbf{p}}(\mathbf{x}) = \sum_{\varepsilon=\pm 1} \varepsilon \begin{vmatrix} e^{2\pi i\varepsilon(-n)x} & e^{2\pi i\varepsilon(-n)y} & e^{2\pi i\varepsilon(-n)z} \\ e^{2\pi i\varepsilon(-m-n)x} & e^{2\pi i\varepsilon(-m-n)y} & e^{2\pi i\varepsilon(-m-n)z} \\ e^{2\pi i\varepsilon(m+2n)x} & e^{2\pi i\varepsilon(m+2n)y} & e^{2\pi i\varepsilon(m+2n)z} \end{vmatrix}.
$$

Here, $x+y+z=0$. Then,

$$
\begin{aligned}
J_{\mathbf{p}}(\mathbf{x}) &= \sum_{\varepsilon=\pm 1} \varepsilon\; e^{2\pi i\varepsilon(-n)(-x-y)} e^{2\pi i\varepsilon(-m-n)(-x-y)} e^{2\pi i\varepsilon(m+2n)(-x-y)} \\
&\quad \times \begin{vmatrix} e^{2\pi i\varepsilon(-n)x} e^{2\pi i\varepsilon(-n)(x+y)} & e^{2\pi i\varepsilon(-n)y} e^{2\pi i\varepsilon(-n)(x+y)} & 1 \\ e^{2\pi i\varepsilon(-m-n)x} e^{2\pi i\varepsilon(-m-n)(x+y)} & e^{2\pi i\varepsilon(-m-n)y} e^{2\pi i\varepsilon(-m-n)(x+y)} & 1 \\ e^{2\pi i\varepsilon(m+2n)x} e^{2\pi i\varepsilon(m+2n)(x+y)} & e^{2\pi i\varepsilon(m+2n)y} e^{2\pi i\varepsilon(m+2n)(x+y)} & 1 \end{vmatrix} \\
&= \begin{vmatrix} e^{2\pi i\varepsilon(-n)(x+y)} & e^{2\pi i\varepsilon(-n)(x+2y)} & 1 \\ e^{2\pi i\varepsilon(-m-n)(2x+y)} & e^{2\pi i\varepsilon(-m-n)(x+2y)} & 1 \\ e^{2\pi i\varepsilon(m+2n)(2x+y)} & e^{2\pi i\varepsilon(m+2n)(2x+y)} & 1 \end{vmatrix}
\end{aligned}
$$

Type G_2(The Neumann eigenfunction)

$$\begin{aligned}
S_{\mathbf{p}}(\mathbf{x}) &= \sum_{w\in W(R)} f_{w(\mathbf{p})}(\mathbf{x}) \ = \sum_{w\in W(R)} \exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\} \\
&= \exp\left[2\pi i\{(-n)x + (-m-n)y + (m+2n)z\}\right] \\
&\quad + \exp\left[2\pi i\{(-m-n)x + (-n)y + (m+2n)z\}\right] \\
&\quad + \exp\left[2\pi i\{(m+2n)x + (-m-n)y + (-n)z\}\right] \\
&\quad + \exp\left[2\pi i\{(-n)x + (m+2n)y + (-m-n)z\}\right] \\
&\quad + \exp\left[2\pi i\{(-m-n)x + (m+2n)y + (-n)z\}\right] \\
&\quad + \exp\left[2\pi i\{(m+2n)x + (-n)y + (-m-n)z\}\right] \\
&\quad + \exp\left[-2\pi i\{(-n)x + (-m-n)y + (m+2n)z\}\right] \\
&\quad + \exp\left[-2\pi i\{(-m-n)x + (-n)y + (m+2n)z\}\right] \\
&\quad + \exp\left[-2\pi i\{(m+2n)x + (-m-n)y + (-n)z\}\right] \\
&\quad + \exp\left[-2\pi i\{(-n)x + (m+2n)y + (-m-n)z\}\right] \\
&\quad + \exp\left[-2\pi i\{(-m-n)x + (m+2n)y + (-n)z\}\right] \\
&\quad + \exp\left[-2\pi i\{(m+2n)x + (-n)y + (-m-n)z\}\right] \\
&= \sum_{\varepsilon=\pm1} [e^{2\pi i\varepsilon(-n)x} e^{2\pi i\varepsilon(-m-n)y} e^{2\pi i\varepsilon(m+2n)z} + e^{2\pi i\varepsilon(-n)x} e^{2\pi i\varepsilon(m+2n)y} e^{2\pi i\varepsilon(-m-n)z} \\
&\quad + e^{2\pi i\varepsilon(-m-n)x} e^{2\pi i\varepsilon(m+2n)y} e^{2\pi i\varepsilon(-n)z} + e^{2\pi i\varepsilon(-m-n)x} e^{2\pi i\varepsilon(-n)y} e^{2\pi i\varepsilon(m+2n)z} \\
&\quad + e^{2\pi i\varepsilon(m+2n)x} e^{2\pi i\varepsilon(-n)y} e^{2\pi i\varepsilon(-m-n)z} + e^{2\pi i\varepsilon(m+2n)x} e^{2\pi i\varepsilon(-m-n)y} e^{2\pi i\varepsilon(-n)z}] \\
&= \sum_{\varepsilon=\pm1} \operatorname{Perm} \begin{pmatrix} e^{2\pi i\varepsilon(-n)x} & e^{2\pi i\varepsilon(-n)y} & e^{2\pi i\varepsilon(-n)z} \\ e^{2\pi i\varepsilon(-m-n)x} & e^{2\pi i\varepsilon(-m-n)y} & e^{2\pi i\varepsilon(-m-n)z} \\ e^{2\pi i\varepsilon(m+2n)x} & e^{2\pi i\varepsilon(m+2n)y} & e^{2\pi i\varepsilon(m+2n)z} \end{pmatrix}.
\end{aligned}$$

provided, $x+y+z=0$. Here, $\mathbf{x}=(x,\,y,\,z)$.

Type A_3(The eigenvalues)

We use the standard basis $e_1=(1,\,0,\,0,\,0),\ e_2=(0,\,1,\,0,\,0),\ e_3=(0,\,0,\,1,\,0),$ $e_4=(0,\,0,\,0,\,1)$.

In the previous calculation of $D(R)$, $\omega_1=\left(\frac{3}{4},\,-\frac{1}{4},\,-\frac{1}{4},\,-\frac{1}{4}\right)$, $\omega_2=\left(\frac{1}{2},\,\frac{1}{2},\,\frac{1}{2},\,-\frac{1}{2}\right)$, $\omega_3=\left(\frac{1}{4},\,\frac{1}{4},\,\frac{1}{4},\,-\frac{3}{4}\right)$,

$$(\omega_1\,|\,\omega_1)=\frac{3}{4},\quad (\omega_1\,|\,\omega_2)=\frac{1}{2},\quad (\omega_1\,|\,\omega_3)=\frac{1}{4},$$
$$(\omega_2\,|\,\omega_2)=1,\quad (\omega_2\,|\,\omega_3)=\frac{1}{2},\quad (\omega_3\,|\,\omega_3)=\frac{3}{4}.$$

We calculate the eigenvalues. If $\mathbf{p} = \ell\omega_1 + m\omega_2 + n\omega_3$, then

$$\begin{aligned} 4\pi^2(\mathbf{p}|\mathbf{p}) &= 4\pi^2(\ell\omega_1 + m\omega_2 + n\omega_3 \,|\, \ell\omega_1 + m\omega_2 + n\omega_3) \\ &= 4\pi^2\{\ell^2(\omega_1\,|\,\omega_1) + m^2(\omega_2\,|\,\omega_2) + n^2(\omega_3\,|\,\omega_3) + 2\ell m(\omega_1\,|\,\omega_2) + 2mn(\omega_2\,|\,\omega_3) \\ &\quad + 2n\ell(\omega_3\,|\,\omega_1)\} \\ &= 4\pi^2\left(\tfrac{3}{4}\ell^2 + m^2 + \tfrac{3}{4}n^2 + \ell m + mn + \tfrac{1}{2}n\ell\right). \end{aligned}$$

In the case of Dirichlet eigenvalues, $\ell, m,$ and n run over the set of positive integers. In the case of the Neumann eigenvalues, $\ell, m,$ and n run over the set of non-negative integers.

The first Dirichlet eigenvalue λ_1 on $D(R)$ is $20\pi^2$.

The second Neumann eigenvalue λ_2 on $D(R)$ is $3\pi^2$.

Type A_3(The Dirichlet eigenfunction)

$W(R)$ consists of 24 symmetric operations which have an axis in the (1, 1, 1) direction, as follows.

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} (RE\ z = w), \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix},$$

$$\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{pmatrix},$$

$$\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&1&0&0\\0&0&0&1\\1&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&0&0&0\\0&0&1&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}.$$

$$\begin{aligned} \mathbf{p} &= \ell\omega_1 + m\omega_2 + n\omega_3 \\ &= \left(\tfrac{3}{4}\ell + \tfrac{1}{2}m + \tfrac{1}{4}n,\ -\tfrac{1}{4}\ell + \tfrac{1}{2}m + \tfrac{1}{4}n,\ -\tfrac{1}{4}\ell - \tfrac{1}{2}m + \tfrac{1}{4}n,\ -\tfrac{1}{4}\ell - \tfrac{1}{2}m - \tfrac{3}{4}n\right). \end{aligned}$$

Then,

$$J_{\mathbf{p}}(\mathbf{x}) = \sum_{w\in W(R)} \varepsilon(w) f_{w(\mathbf{p})}(\mathbf{x}) \ = \sum_{w\in W(R)} \varepsilon(w)\exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\}$$

$$= \begin{vmatrix} e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x} & e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y} & e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z} & e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w} \\ e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x} & e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y} & e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z} & e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w} \\ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)x} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)y} & e^{2\pi i-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)z} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)w} \\ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)x} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)y} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)z} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)w} \end{vmatrix}.$$

provided $x+y+z+w=0$. Here, $\mathbf{x}=(x,\,y,\,z,\,w)$.

Type A_3(The Dirichlet eigenfunction) (the alternative way)

If we put $\mathbf{p}=\ell\omega_1+m\omega_2+n\omega_3,\ \mathbf{x}=(x,\,y,\,z,\,w)$, then,

$$J_{\mathbf{p}}(\mathbf{x}) = \begin{vmatrix} e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x} & e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y} & e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z} & e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w} \\ e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x} & e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y} & e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z} & e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w} \\ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)x} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)y} & e^{2\pi i-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)z} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)w} \\ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)x} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)y} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)z} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)w} \end{vmatrix}.$$

Here, $x+y+z+w=0$. Then,

$$J_{\mathbf{p}}(\mathbf{x}) = \begin{vmatrix} e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)(2x+y+z)} & e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)(x+2y+z)} & e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)(x+y+2z)} & 1 \\ e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)(2x+y+z)} & e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)(x+2y+z)} & e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)(x+y+2z)} & 1 \\ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)(2x+y+z)} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)(x+2y+z)} & e^{2\pi i-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)(x+y+2z)} & 1 \\ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)(2x+y+z)} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)(x+2y+z)} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)(x+y+2z)} & 1 \end{vmatrix}.$$

Type A_3(The Neumann eigenfunction)

$$S_{\mathbf{p}}(\mathbf{x}) = \sum_{w\in W(R)} f_{w(\mathbf{p})}(\mathbf{x}) \ = \sum_{w\in W(R)} \exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\}$$

$$\begin{aligned} &= e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x} e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y} e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)z} e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)w} \\ &\quad + e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x} e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y} e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)z} e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)w} \\ &\quad + e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x} e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)y} e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z} e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)w} \\ &\quad + e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x} e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)y} e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)z} e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w} \\ &\quad + e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x} e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)y} e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z} e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)w} \\ &\quad + e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x} e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)y} e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)z} e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w} \end{aligned}$$

$$
\begin{aligned}
&+ e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)z}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)z}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)y}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)y}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w}
\end{aligned}
$$

$$
\begin{aligned}
&+ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)x}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)x}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)z}e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)x}e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)x}e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)z}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)x}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)y}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)x}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)y}e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w}
\end{aligned}
$$

$$
\begin{aligned}
&+ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)x}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)x}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)x}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)x}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)x}e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w}\\
&+ e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)x}e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y}e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z}e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)w}
\end{aligned}
$$

$$
= \mathrm{Perm}\begin{pmatrix}
e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x} & e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y} & e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z} & e^{2\pi i(\frac{3}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w}\\
e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)x} & e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)y} & e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)z} & e^{2\pi i(-\frac{1}{4}\ell+\frac{1}{2}m+\frac{1}{4}n)w}\\
e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)x} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)y} & e^{2\pi i-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)z} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m+\frac{1}{4}n)w}\\
e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)x} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)y} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)z} & e^{2\pi i(-\frac{1}{4}\ell-\frac{1}{2}m-\frac{3}{4}n)w}
\end{pmatrix}.
$$

provided, $x+y+z+w=0$. Here, $\mathbf{x}=(x,\, y,\, z,\, w)$.

Type B_3(The eigenvalues)

We use the standard basis $e_1=(1,\,0,\,0),\ e_2=(0,\,1,\,0),\ e_3=(0,\,0,\,1)$.

In the previous calculation of $D(R)$, $\omega_1=(1,0,0)$, $\omega_2=(1,1,0)$, $\omega_3=\left(\frac{1}{2},\,\frac{1}{2},\,\frac{1}{2}\right)$, then

$$
\begin{aligned}
&(\omega_1\,|\,\omega_1)=1, \quad (\omega_1\,|\,\omega_2)=1, \quad (\omega_1\,|\,\omega_3)=\frac{1}{2},\\
&(\omega_2\,|\,\omega_1)=1, \quad (\omega_2\,|\,\omega_2)=2, \quad (\omega_2\,|\,\omega_3)=1,
\end{aligned}
$$

$$(\omega_3\,|\,\omega_1) = \frac{1}{2}, \quad (\omega_3\,|\,\omega_2) = 1, \quad (\omega_3\,|\,\omega_3) = \frac{3}{4}.$$

We calculate the eigenvalues. If $\mathbf{p} = \ell\omega_1 + m\omega_2 + n\omega_3$, then

$$\begin{aligned} 4\pi^2(\mathbf{p}|\mathbf{p}) &= 4\pi^2(\ell\omega_1 + m\omega_2 + n\omega_3\,|\,\ell\omega_1 + m\omega_2 + n\omega_3) \\ &= 4\pi^2\{\ell^2(\omega_1\,|\,\omega_1) + m^2(\omega_2\,|\,\omega_2) + n^2(\omega_3\,|\,\omega_3) + 2\ell m(\omega_1\,|\,\omega_2) + 2mn(\omega_2\,|\,\omega_3) \\ &\quad + 2n\ell(\omega_3\,|\,\omega_1)\} \\ &= 4\pi^2\left(\ell^2 + 2m^2 + \tfrac{3}{4}n^2 + 2\ell m + 2mn + nl\right). \end{aligned}$$

In the case of the Dirichlet eigenvalues, $\ell, m,$ and n run over the set of positive integers. In the case of the Neumann eigenvalues, $\ell, m,$ and n run over the set of non-negative integers.

The first Dirichlet eigenvalue λ_1 on $D(R)$ is $35\pi^2$.
The second Neumann eigenvalue λ_2 on $D(R)$ is $3\pi^2$.

Type B_3(The Dirichlet eigenfunction)

$W(R)$ consists of 48 symmetric operations which have an axis in the (1, 1, 1) direction, as follows. Each of these 6 matrixes includes reflections with respect to the xy-plane and yz-plane and zx-plane independently as follows. Here, $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$.

$$\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} (E), \quad \begin{pmatrix} 0 & \varepsilon_1 & 0 \\ \varepsilon_2 & 0 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} (REy = x), \begin{pmatrix} 0 & 0 & \varepsilon_1 \\ 0 & \varepsilon_2 & 0 \\ \varepsilon_3 & 0 & 0 \end{pmatrix} (REx = z),$$

$$\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 \\ 0 & \varepsilon_3 & 0 \end{pmatrix} (REz = y), \begin{pmatrix} 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \\ \varepsilon_3 & 0 & 0 \end{pmatrix} (RO{-}120^\circ), \begin{pmatrix} 0 & 0 & \varepsilon_1 \\ \varepsilon_2 & 0 & 0 \\ 0 & \varepsilon_3 & 0 \end{pmatrix} (RO120^\circ).$$

Let $\mathbf{p} = \ell\omega_1 + m\omega_2 + n\omega_3 = \left(\ell + m + \frac{1}{2}n,\ m + \frac{1}{2}n,\ \frac{1}{2}n\right)$. Then,

$$\begin{aligned} J_{\mathbf{p}}(\mathbf{x}) &= \sum_{w \in W(R)} \varepsilon(w) f_{w(\mathbf{p})}(\mathbf{x}) \ = \sum_{w \in W(R)} \varepsilon(w) \exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\} \\ &= \pm \begin{vmatrix} e^{\pm 2\pi i(l+m+\frac{1}{2}n)x} & e^{\pm 2\pi i(l+m+\frac{1}{2}n)y} & e^{\pm 2\pi i(l+m+\frac{1}{2}n)z} \\ e^{\pm 2\pi i(m+\frac{1}{2}n)x} & e^{\pm 2\pi i(m+\frac{1}{2}n)y} & e^{\pm 2\pi i(m+\frac{1}{2}n)z} \\ e^{\pm 2\pi i(\frac{1}{2}n)x} & e^{\pm 2\pi i(\frac{1}{2}n)y} & e^{\pm 2\pi i(\frac{1}{2}n)z} \end{vmatrix}. \\ &= \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3 = \pm 1} \varepsilon_1\varepsilon_2\varepsilon_3 \begin{vmatrix} e^{2\pi i\varepsilon_1(l+m+\frac{1}{2}n)x} & e^{2\pi i\varepsilon_1(l+m+\frac{1}{2}n)y} & e^{2\pi i\varepsilon_1(l+m+\frac{1}{2}n)z} \\ e^{2\pi i\varepsilon_2(m+\frac{1}{2}n)x} & e^{2\pi i\varepsilon_2(m+\frac{1}{2}n)y} & e^{2\pi i\varepsilon_2(m+\frac{1}{2}n)z} \\ e^{2\pi i\varepsilon_3(\frac{1}{2}n)x} & e^{2\pi i\varepsilon_3(\frac{1}{2}n)y} & e^{2\pi i\varepsilon_3(\frac{1}{2}n)z} \end{vmatrix}. \end{aligned}$$

Here, $\mathbf{x} = (x,\ y,\ z)$.

Type B_3(The Neumann eigenfunction)

$$S_{\mathbf{p}}(\mathbf{x}) = \sum_{w\in W(R)} f_{w(\mathbf{p})}(\mathbf{x}) \ = \sum_{w\in W(R)} \exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\}$$
$$= \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm1} \mathrm{Perm}\begin{pmatrix} e^{2\pi i\varepsilon_1(l+m+\frac{1}{2}n)x} & e^{2\pi i\varepsilon_1(l+m+\frac{1}{2}n)y} & e^{2\pi i\varepsilon_1(l+m+\frac{1}{2}n)z} \\ e^{2\pi i\varepsilon_2(m+\frac{1}{2}n)x} & e^{2\pi i\varepsilon_2(m+\frac{1}{2}n)y} & e^{2\pi i\varepsilon_2(m+\frac{1}{2}n)z} \\ e^{2\pi i\varepsilon_3(\frac{1}{2}n)x} & e^{2\pi i\varepsilon_3(\frac{1}{2}n)y} & e^{2\pi i\varepsilon_3(\frac{1}{2}n)z} \end{pmatrix}.$$

Here, $\mathbf{x} = (x,\ y,\ z)$.

Type C_3(The eigenvalues)

We use the standard basis $e_1 = (1,\ 0,\ 0),\ e_2 = (0,\ 1,\ 0),\ e_3 = (0,\ 0,\ 1)$.

In the previous calculation of $D(R)$, $\omega_1 = (1,0,0),\ \omega_2 = (1,1,0),\ \omega_3 = (1,1,1)$, then

$$\begin{aligned} &(\omega_1\,|\,\omega_1) = 1, \quad (\omega_1\,|\,\omega_2) = 1, \quad (\omega_1\,|\,\omega_3) = 1, \\ &(\omega_2\,|\,\omega_1) = 1, \quad (\omega_2\,|\,\omega_2) = 2, \quad (\omega_2\,|\,\omega_3) = 2, \\ &(\omega_3\,|\,\omega_1) = 1, \quad (\omega_3\,|\,\omega_2) = 2, \quad (\omega_3\,|\,\omega_3) = 3. \end{aligned}$$

We calculate the eigenvalues. If $\mathbf{p} = \ell\omega_1 + m\omega_2 + n\omega_3$, then

$$\begin{aligned} 4\pi^2(\mathbf{p}|\mathbf{p}) &= 4\pi^2(\ell\omega_1 + m\omega_2 + n\omega_3\,|\,\ell\omega_1 + m\omega_2 + n\omega_3) \\ &= 4\pi^2\{\ell^2(\omega_1\,|\,\omega_1) + m^2(\omega_2\,|\,\omega_2) + n^2(\omega_3\,|\,\omega_3) + 2\ell m(\omega_1\,|\,\omega_2) + 2mn(\omega_2\,|\,\omega_3) \\ &\quad + 2n\ell(\omega_3\,|\,\omega_1)\} \\ &= 4\pi^2(\ell^2 + 2m^2 + 3n^2 + 2\ell m + 4mn + 2nl). \end{aligned}$$

In the case of the Dirichlet eigenvalues, ℓ, m, and n run over the set of positive integers. In the case of the Neumann eigenvalues, ℓ, m, and n run over the set of non-negative integers.

The first Dirichlet eigenvalue λ_1 on $D(R)$ is $56\pi^2$.
The socond Neumann eigenvalue λ_2 on $D(R)$ is $4\pi^2$.

Type C_3(The Dirichlet eigenfunction)

$W(R)$ consists of 48 symmetric operations which have an axis in the (1, 1, 1) direction, as follows. Each of these 6 matrixes includes reflections with respect to xy-plane and yz-plane and zx-plane independently as follows. Here, $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3 = \pm1$.

$$\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}(E), \qquad \begin{pmatrix} 0 & \varepsilon_1 & 0 \\ \varepsilon_2 & 0 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}(REy=x), \begin{pmatrix} 0 & 0 & \varepsilon_1 \\ 0 & \varepsilon_2 & 0 \\ \varepsilon_3 & 0 & 0 \end{pmatrix}(REx=z),$$

$$\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 \\ 0 & \varepsilon_3 & 0 \end{pmatrix}(REz=y), \begin{pmatrix} 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \\ \varepsilon_3 & 0 & 0 \end{pmatrix}(RO{-}120°), \begin{pmatrix} 0 & 0 & \varepsilon_1 \\ \varepsilon_2 & 0 & 0 \\ 0 & \varepsilon_3 & 0 \end{pmatrix}(RO120°).$$

$$\mathbf{p} = \ell\omega_1 + m\omega_2 + n\omega_3 = (\ell+m+n, m+n, n)$$

$$\begin{aligned} J_{\mathbf{p}}(\mathbf{x}) &= \sum_{w\in W(R)} \varepsilon(w) f_{w(\mathbf{p})}(\mathbf{x}) \ = \sum_{w\in W(R)} \varepsilon(w)\exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\} \\ &= \pm \begin{vmatrix} e^{\pm 2\pi i(l+m+n)x} & e^{\pm 2\pi i(l+m+n)y} & e^{\pm 2\pi i(l+m+n)z} \\ e^{\pm 2\pi i(m+n)x} & e^{\pm 2\pi i(m+n)y} & e^{\pm 2\pi i(m+n)z} \\ e^{\pm 2\pi i(n)x} & e^{\pm 2\pi i(n)y} & e^{\pm 2\pi i(n)z} \end{vmatrix}. \\ &= \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm 1} \varepsilon_1\varepsilon_2\varepsilon_3 \begin{vmatrix} e^{2\pi i\varepsilon_1(l+m+n)x} & e^{2\pi i\varepsilon_1(l+m+n)y} & e^{2\pi i\varepsilon_1(l+m+n)z} \\ e^{2\pi i\varepsilon_2(m+n)x} & e^{2\pi i\varepsilon_2(m+n)y} & e^{2\pi i\varepsilon_2(m+n)z} \\ e^{2\pi i\varepsilon_3(n)x} & e^{2\pi i\varepsilon_3(n)y} & e^{2\pi i\varepsilon_3(n)z} \end{vmatrix}. \end{aligned}$$

Here, $\mathbf{x} = (x,\ y,\ z)$.

Type C_3(The Neumann eigenfunction)

$$S_{\mathbf{p}}(\mathbf{x}) = \sum_{w\in W(R)} f_{w(\mathbf{p})}(\mathbf{x}) \ = \sum_{w\in W(R)} \exp\{2\pi i(\mathbf{x}|w(\mathbf{p}))\}$$

$$= \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm 1} \mathrm{Perm} \begin{pmatrix} e^{2\pi i\varepsilon_1(l+m+n)x} & e^{2\pi i\varepsilon_1(l+m+n)y} & e^{2\pi i\varepsilon_1(l+m+n)z} \\ e^{2\pi i\varepsilon_2(m+n)x} & e^{2\pi i\varepsilon_2(m+n)y} & e^{2\pi i\varepsilon_2(m+n)z} \\ e^{2\pi i\varepsilon_3(n)x} & e^{2\pi i\varepsilon_3(n)y} & e^{2\pi i\varepsilon_3(n)z} \end{pmatrix}.$$

Here, $\mathbf{x} = (x,\ y,\ z)$.

7 Poisson's summation formula for the heat kernels

7.1 Summary

(1) In this section, we give Poisson's summation formulas for the Dirichlet and Neumann heat kernels on $D(R)$.
(2) We also give motion of trace $Z_D(t), Z_N(t)$ at $t \to 0$ for the Dirichlet and the Neumann heat kernels on the alcove $D(R)$.

7.2 The Dirichlet and Neumann heat kernels

First, we prepare to decide the Dirichlet and Neumann heat kernels.
For this, we consider the boundary value problems in the domain which belong to affine Weyl groups.

We assume that $\Omega \subset \mathbb{R}^\ell$ is a bounded domain, and $\partial\Omega$ is piecewise C^∞.

The Dirichlet eigenvalue problems are

$$\begin{cases} \Delta u = \lambda u & (\text{ on } \Omega) \\ u = 0 & (\text{on } \partial\Omega). \end{cases}$$

Here, we denote the Dirichlet eigenvalues by

$$0 \ < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_i \leqq \cdots ,$$

and the corresponding eigenfunctions by

$$u_1,\ u_2, \cdots , u_i, \cdots .$$

The Neumann eigenvalue problems are

$$\begin{cases} \Delta v = \mu v & (\text{ on } \Omega) \\ v = 0 & (\text{on } \partial\Omega). \end{cases}$$

Here, we denote the Neumann eigenvalues by

$$0 \ < \mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_i \leqq \cdots ,$$

and the corresponding eigenfunctions by

$$v_1,\ v_2, \cdots , v_i, \cdots .$$

Here, we can assume that

$$(u_i, u_j) = \int_\Omega u_i(\mathbf{x}) u_j(\mathbf{x}) d\mathbf{x} = \delta_{ij},$$
$$(v_i, v_j) = \int_\Omega v_i(\mathbf{x}) v_j(\mathbf{x}) d\mathbf{x} = \delta_{ij}.$$

Here, if we put $d\mathbf{x} = dx_1 \cdots dx_\ell$, then the Dirichlet and Neumann heat kernels can be written on Ω as follows:

$$e_D(\mathbf{x}, \mathbf{y}, t) = \sum_{i=1}^{\infty} e^{-t\lambda_i} u_i(\mathbf{x}) u_i(\mathbf{y}), \quad (t > 0,\ \mathbf{x},\ \mathbf{y}\ \in \Omega), \tag{7.1}$$
$$e_N(\mathbf{x}, \mathbf{y}, t) = \sum_{i=1}^{\infty} e^{-t\mu_i} v_i(\mathbf{x}) v_i(\mathbf{y}), \quad (t > 0,\ \mathbf{x},\ \mathbf{y}\ \in \Omega). \tag{7.2}$$

These are called the **Dirichlet heat heat kernel** and **Neumann heat heat kernel** on Ω.

These heat kernel have the following features.

Lemma 7.1 (1) *Both the Dirichlet heat heat kernel* (7.1) *and Neumann heat heat kernel* (7.2) *are absolutely convergent series, which are* C^∞ *with three variables* $\mathbf{x}, \mathbf{y}, t$. *For each* $\mathbf{y} \in \Omega$*, the heat equation satisfies*

$$\frac{\partial e}{\partial t} + \Delta_x e = 0 \quad (\textit{on}\ \Omega \times \mathbb{R}^+ = \{(\mathbf{x}, t) \,|\, t > 0,\ \mathbf{x}\ \in \Omega\}).$$

(2) *For each* $\mathbf{y}\ \in \overline{\Omega}\ (= \Omega \cup \partial\Omega)$*, the heat equation satisfies*

$$\begin{cases} e_D(\mathbf{x}, \mathbf{y}, t) & = 0, \quad (t > 0,\ \mathbf{x}\ \in \partial\overline{\Omega}\ ), \\ \frac{\partial}{\partial \mathbf{n}} e_N(\mathbf{x}, \mathbf{y}, t) & = 0, \quad (t > 0,\ \mathbf{x}\ \in \partial\overline{\Omega}\ ). \end{cases}$$

(3) *For each continuous function* f *on* Ω*, we have*

$$\lim_{t \downarrow 0} \int_\Omega e_D(\mathbf{x}, \mathbf{y}, t) f(\mathbf{y}) d\mathbf{y} = f(\mathbf{x}), \quad (\mathbf{x}\ \in \Omega),$$
$$\lim_{t \downarrow 0} \int_\Omega e_N(\mathbf{x}, \mathbf{y}, t) f(\mathbf{y}) d\mathbf{y} = f(\mathbf{x}), \quad (\mathbf{x}\ \in \Omega).$$

Proof For the absolute convergence of series in (1), see [9], p.690~p.692. Also, (2) is proved by the definition immediately.

The proof of (3). We put $f(\mathbf{x}) = \sum_{j=1}^{\infty} (f, u_j) u_j(\mathbf{x})\ (\mathbf{x} \in \Omega)$. Here, $(f, u_j) = \int_\Omega f(\mathbf{x}) u_j(\mathbf{x}) d\mathbf{x}$. We have

$$\int_\Omega e_D(\mathbf{x},\mathbf{y},t)f(\mathbf{y})d\mathbf{y} = \int_\Omega \sum_{i=1}^\infty \sum_{j=1}^\infty e^{-t\lambda_i}\, u_i(\mathbf{x})u_i(\mathbf{y})(f,u_j)u_j(\mathbf{y})d\mathbf{y}$$
$$= \sum_{i=1}^\infty \sum_{j=1}^\infty e^{-t\lambda_i}\, u_i(\mathbf{x})(f,\, u_j)\int_\Omega u_i(\mathbf{y})u_j(\mathbf{y})d\mathbf{y}$$
$$= \sum_{i=1}^\infty e^{-t\lambda_i}\, u_i(\mathbf{x})(f,\, u_j)\delta_{ij}$$
$$= \sum_{i=1}^\infty e^{-t\lambda_i}\, u_i(\mathbf{x})(f,\, u_i)\ ({}^\forall \mathbf{x} \in \overline{\Omega}\ ).$$

Thus, we obtain

$$\lim_{t\downarrow 0}\int_\Omega e_D(\mathbf{x},\mathbf{y},t)f(\mathbf{y})d\mathbf{y} = \sum_{i=1}^\infty (f,\, u_i)u_i(\mathbf{x}) = f(\mathbf{x}).$$

The proof of (1). We have

$$\frac{\partial}{\partial t}e_D(\mathbf{x},\mathbf{y},t) + \Delta_\mathbf{x} e_D(\mathbf{x},\mathbf{y},t)$$
$$= \frac{\partial}{\partial t}\sum_{i=1}^\infty e^{-t\lambda_i}\, u_i(\mathbf{x})u_j(\mathbf{y}) \ + \Delta_\mathbf{x}\sum_{i=1}^\infty e^{-t\lambda_i}\, u_i(\mathbf{x})u_j(\mathbf{y})$$
$$= \sum_{i=1}^\infty \left(\frac{\partial}{\partial t}e^{-t\lambda_i}\right) u_i(\mathbf{x})u_j(\mathbf{y}) \ + \sum_{i=1}^\infty e^{-t\lambda_i}\, (\Delta_\mathbf{x} u_i(\mathbf{x}))u_i(\mathbf{y})$$
$$= \sum_{i=1}^\infty (-\lambda_i + \lambda_i)\, e^{-t\lambda_i}\, u_i(\mathbf{x})u_j(\mathbf{y}) \quad (\textit{by}\ \ \Delta_\mathbf{x} u_i(\mathbf{x}) = \lambda_i u_i(\mathbf{x}))$$
$$= 0.$$

Similarly, we have $\frac{\partial}{\partial t}e_N(\mathbf{x},\mathbf{y},t) + \Delta_\mathbf{x} e_N(\mathbf{x},\mathbf{y},t) = 0.$ //

We consider the asymptotic expansion of the Dirichlet and Neumann heat equations. We write that Ω is a crystallographic domain $D(R)$ as follows. We have a proposition as follows.

Proposition 7.1 *The heat kernel $P(\mathbf{x},\mathbf{y},t)$ for each flat torus $(\mathbb{R}^\ell/\Gamma,\, g_\Gamma)$ is given as follows:*

$$P(\mathbf{x},\mathbf{y},t) = \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)}\sum_{\mathbf{p}\in P(R)} e^{-4\pi^2(\mathbf{p}|\mathbf{p})t}\, e^{2\pi i(\mathbf{x}-\mathbf{y}|\mathbf{p})} \quad (t>0,\ \ \mathbf{x},\ \mathbf{y}\ \in \mathbb{R}^\ell).$$

Here, we express about the symbol in Proposition (7.1) and flat torus. For each $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^\ell$, if $\mathbf{x} \sim \mathbf{x}' \in \mathbb{R}^\ell$, then

$$\mathbf{x} - \mathbf{x}' \in \Gamma.$$

Then, we say that $\mathbf{x}$ and $\mathbf{x}'$ satisfy the equivalence relation. We put $\mathbb{R}^\ell/\Gamma$ for the set of all equivalent classes under the equivalence relation. That is,

$$\mathbb{R}^\ell/\Gamma = \{[\mathbf{x}] \mid \mathbf{x} \in \mathbb{R}^\ell\}.$$

Here, $[\mathbf{x}]$ means the equivalence class to which $\mathbf{x} \in \mathbb{R}^\ell$ belongs.

If $\mathbf{x}' \in [\mathbf{x}]$, then

$$\mathbf{x}' - \mathbf{x} \in \Gamma,$$

and also, if $\mathbf{x}_1,\ \mathbf{x}_2 \in [\mathbf{x}]$, then we have $\mathbf{x}_1 - \mathbf{x}_2 = (\mathbf{x}_1 - \mathbf{x}) - (\mathbf{x}_1 - \mathbf{x}) \in \Gamma$.

We call the projection for the mapping $\pi : \mathbb{R}^\ell \ni \mathbf{x} \mapsto [\mathbf{x}] \in \mathbb{R}^\ell/\Gamma$. Also, the topology of $\mathbb{R}^\ell/\Gamma$ is given as follows: If $U \subset \mathbb{R}^\ell/\Gamma$ is open, $\pi^{-1}(U) \subset \mathbb{R}^\ell$ is also open.

If f is a function on $\mathbb{R}^\ell/\Gamma$, then $\tilde{f}$ is a function on $\mathbb{R}^\ell$ satisfying that

$$\tilde{f}(\mathbf{x} + \gamma) = \tilde{f}(\mathbf{x}) \quad (\mathbf{x} \in \mathbb{R}^\ell,\ \gamma \in \Gamma). \tag{7.3}$$

We put $f([\mathbf{x}]) = \tilde{f}(\mathbf{x})$ $(\mathbf{x} \in \mathbb{R}^\ell)$. When f is a C^∞ function on $\mathbb{R}^\ell/\Gamma$, $\tilde{f}$ is a C^∞function on $\mathbb{R}^\ell$ which satisfies the periodicity condition (7.3).

The C^∞ function $\tilde{f}$ on $\mathbb{R}^\ell$ which satisfies the condition (7.3) is written as follows. Here, $\Gamma = Q(R^\vee) = \sum_{\alpha \in R} \mathbb{Z}\alpha^\vee = \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i^\vee$, we write the set $P(R)$ of weights of a root system R by

$$P(R) = \left\{ \sum_{i=1}^{l} m_i \omega_1 \;\middle|\; m_i \in \mathbb{Z}\ (i = 1, \cdots, l) \right\}.$$

Here, $(\omega_i \mid \alpha_j^\vee) = \delta_{ij}$. We say $\{\omega_i\}_{i=1}^{l}$ is a dual lattice of $\{\alpha_j^\vee\}_{j=1}^{l}$.

The periodic function $\tilde{f}(\mathbf{x})$ is given as follows.

$$\tilde{f}(\mathbf{x}) = \sum_{\mathbf{p} \in P(R)} a_{\mathbf{p}}\, e^{-4\pi^2 i(\mathbf{x}|\mathbf{p})} \quad ({}^\forall \mathbf{x} \in \mathbb{R}^\ell). \tag{7.4}$$

If $\sum_{\mathbf{p} \in P(R)} |a_{\mathbf{p}}| < \infty$, then, this (complex) series is absolutely convergent. We have

$$\left| \sum_{\mathbf{p} \in P(R)} a_{\mathbf{p}}\, e^{-4\pi^2 i(\mathbf{x}|\mathbf{p})} \right| \leqq \sum_{\mathbf{p} \in P(R)} |a_{\mathbf{p}}||e^{-4\pi^2 i(\mathbf{x}|\mathbf{p})}| = \sum_{\mathbf{p} \in P(R)} |a_{\mathbf{p}}| < \infty.$$

Lemma 7.2 *The periodic condition*

$$\tilde{f}(\mathbf{x}+\gamma) = \tilde{f}(\mathbf{x}) \quad (\mathbf{x} \in \mathbb{R}^\ell,\ \gamma \in \Gamma) \tag{7.5}$$

is equivalent to the following condition.

$$\sum_{\mathbf{p}\in P(R)} a_{\mathbf{p}}\, e^{-2\pi i(\mathbf{x}+\gamma|\mathbf{p})} = \sum_{\mathbf{p}\in P(R)} a_{\mathbf{p}}\, e^{-2\pi i(\mathbf{x}|\mathbf{p})}.$$

Proof

$$\begin{aligned}\sum_{\mathbf{p}\in P(R)} a_{\mathbf{p}}\, e^{-2\pi i(\mathbf{x}+\gamma|\mathbf{p})} &= \sum_{\mathbf{p}\in P(R)} a_p\, e^{-2\pi i(\mathbf{x}|\mathbf{p})}\, e^{-2\pi i(\gamma|\mathbf{p})} \\ &= \sum_{\mathbf{p}\in P(R)} a_{\mathbf{p}}\, e^{-2\pi i(\mathbf{x}|\mathbf{p})}\end{aligned}$$

since $(\gamma|\mathbf{p})$ are integers. The reason is, when ${}^\forall\gamma = \sum_{j=1}^{\ell} n_i\alpha_j^\vee \in \Gamma$, ${}^\forall\mathbf{p} = \sum_{j=1}^{\ell} m_i\omega_i \in P(R)$, because of

$$(\gamma|\mathbf{p}) = \sum_{i,j=1}^{\ell} m_i n_i(\omega_i \,|\, \alpha_j^\vee) = \sum_{i=1}^{\ell} m_i n_i \ \in \ \mathbb{Z}. \quad //$$

The function $\tilde{f}(\mathbf{x})$ given by (7.4) satisfies the condition (7.5). Conversely, any continuous function $\tilde{f}$ on $\mathbb{R}^\ell$ which satisfies the periodic condition (7.5) is figured out as follows. The series of the left hand side of (7.1) is called the **Fourier series**, and $a_{\mathbf{p}}$ $(\mathbf{p}\in P(R))$ are called the **Fourier coefficients**. Let us define

$$F(R) = \left\{ \sum_{i=1}^{\ell} x_i\alpha_j^\vee \;\middle|\; 0 \leqq x_i \leqq 1 \ (i = 1, \cdots, l) \right\}.$$

If $\mathbf{p} = \sum_{j=1}^{l} m_j\omega_j \in P(R)$, $\mathbf{p}' = \sum_{j=1}^{l} m_j'\omega_j \in P(R)$, we have

$$\begin{aligned}\int_{F(R)} e^{2\pi i(\mathbf{x}|\mathbf{p})}\, \overline{e^{2\pi i(\mathbf{x}|\mathbf{p}')}} d\mathbf{x} &= \int_{F(R)} e^{2\pi i(\mathbf{x}|\mathbf{p}-\mathbf{p}')} d\mathbf{x} \\ &= \int_{F(R)} e^{2\pi i(\sum_{i=1}^{l} x_i\alpha_i^\vee \,|\, \sum_{i=1}^{l}(m_j-m_j')\omega_j)} d\mathbf{x} \\ &= \int_{F(R)} e^{2\pi i \sum_{i,j=1}^{l} x_i(m_j-m_i')\delta_{ij}} d\mathbf{x} \\ &= \int_{F(R)} e^{2\pi i \sum_{i=1}^{l}(m_i-m_i')x_i} d\mathbf{x} \\ &= C \int_0^1 \int_0^1 e^{2\pi i \sum_{i=1}^{l}(m_i-m_i')x_i} dx_1 \cdots dx_l.\end{aligned}$$

Furthermore, we have

$$\mathbf{p} \neq \mathbf{p}' \;\leftrightarrow\; {}^{\exists}i\ (1 \leqq l),\ m_i \neq m_i'.$$

Here, we calculate the positive constant C for later use. Then, for simplicity, we put

$$\begin{cases} m_i \neq m_i' & (1 \leqq {}^{\forall}i \leqq k) \\ m_j = m_j' & (k+1 \leqq {}^{\forall}j \leqq l). \end{cases}$$

Then, the above equation is calculated as follows.

$$\begin{aligned} \int_{F(R)} e^{2\pi i(\mathbf{x}|\mathbf{p})}\, \overline{e^{2\pi i(\mathbf{x}|\mathbf{p}')}} d\mathbf{x} &= C \int_0^1 \int_0^1 e^{2\pi i \sum_{i=1}^{l}(m_i - m_i')x_i} dx_1 \cdots dx_l \\ &= C \prod_{i=1}^{k} \int_0^1 e^{2\pi i(m_i - m_i')x_i} dx_i \\ &= C \prod_{i=1}^{k} \left[\frac{1}{2\pi i(m_i - m_i')}\, e^{2\pi i(m_i - m_i')x_i} \right]_{x_i=0}^{x_i=1} \\ &= C \prod_{i=1}^{k} \frac{1}{2\pi i(m_i - m_i')} \left\{ e^{2\pi i(m_i - m_i')1} - e^{2\pi i(m_i - m_i')0} \right\} \\ &= 0. \end{aligned}$$

Here, in the equation

$$\int_{F(R)} e^{-2\pi i(\mathbf{x}|\mathbf{p})}\, \overline{e^{2\pi i(\mathbf{x}|\mathbf{p}')}} d\mathbf{x} = C \int_0^1 \cdots \int_0^1 e^{2\pi i \sum_{i=1}^{l}(m_i - m_i')x_i} dx_1 \cdots dx_l,$$

we put $\mathbf{p} = \mathbf{p}'$. Then, the left hand side is the volume of $\int_{F(R)} 1\, d\mathbf{x} = \mathrm{Vol}(F(R))$, and the right hand side is $C \int_0^1 \cdots \int_0^1 1\, dx_1 \cdots dx_l = C$. We have a constant C as follows:

$$C = \mathrm{Vol}(F(R)) = \mathrm{Vol}(F(R)) = \mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\, g_\gamma).$$

Thus,

$$\left\{ \frac{1}{\sqrt{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma, g_\gamma)}}\, e^{2\pi i(\mathbf{x}|\mathbf{p})} \;\middle|\; \mathbf{p} \in P(R) \right\}$$

is an orthogonal system about the Hermitian inner product

$$(\varphi,\, \varphi') = \int_{F(R)} \varphi(\mathbf{x}) \overline{\varphi'(\mathbf{x})} d\mathbf{x}$$

on $\mathbb{R}^\ell/\Gamma$, that is, if $\mathbf{p} \neq \mathbf{p}'$, $(\varphi_{\mathbf{p}},\, \varphi_{\mathbf{p}}') = 0$, if $\mathbf{p} = \mathbf{p}'$, $(\varphi_{\mathbf{p}},\, \varphi_{\mathbf{p}}) = 1$.

The Fourier coefficients of a continuous function $\tilde{f}(\mathbf{x})$ are calculated as follows: Let

$$\tilde{f}(\mathbf{x}) = \sum_{\mathbf{p}\in P(R)} b_{\mathbf{p}}\varphi_{\mathbf{p}}(\mathbf{x}) = \sum_{p\in P(R)} b_{\mathbf{p}} \frac{1}{\sqrt{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)}}\, e^{2\pi i(\mathbf{x}|\mathbf{p})}.$$

Then, we have

$$b_{\mathbf{p}} \frac{1}{\sqrt{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)}} = a_{\mathbf{p}}.$$

Furthermore,

$$(\tilde{f},\, \varphi') = \left(\sum_{\mathbf{p}\in P(R)} b_{\mathbf{p}}\varphi_{\mathbf{p}},\ \varphi'_{\mathbf{p}} \right) = \sum_{\mathbf{p}\in P(R)} b_{\mathbf{p}}(\varphi_{\mathbf{p}},\ \varphi'_{\mathbf{p}}) = b'_{\mathbf{p}} \quad (\mathbf{p}' \in P(R)),$$

and we have

$$\begin{aligned}
\tilde{f}(\mathbf{x}) &= \sum_{\mathbf{p}\in P(R)} (\tilde{f}, \varphi_{\mathbf{p}})\varphi_{\mathbf{p}}(\mathbf{x}) \\
&= \sum_{\mathbf{p}\in P(R)} \left(\tilde{f},\, \frac{1}{\sqrt{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)}} e^{2\pi i(\mathbf{p}|\cdot)} \right) \frac{e^{2\pi i(\mathbf{p}|\mathbf{x})}}{\sqrt{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)}} \\
&= \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \sum_{\mathbf{p}\in P(R)} \left(\tilde{f}, e^{2\pi i(\mathbf{p}|\cdot)} \right) e^{2\pi i(\mathbf{p}|\mathbf{x})}
\end{aligned}$$

Here,

$$\left(\tilde{f},\, \frac{1}{\sqrt{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)}}\, e^{2\pi i(\mathbf{p}|\cdot)} \right) = \int_{F(R)} \tilde{f}(\mathbf{x})\, \frac{1}{\sqrt{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)}}\, e^{2\pi i(\mathbf{p}|\mathbf{x})} d\mathbf{x}.$$

Thus, a_p is given by

$$a_p = \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \left(\tilde{f}, e^{2\pi i(\mathbf{p}|\cdot)} \right), \quad \mathbf{p} \in P(R).$$

We summarize the above as follows:
Every continuous function $\tilde{f}(x)$ on $\mathbb{R}^\ell$ which satisfies

$$\tilde{f}(\mathbf{x} + \gamma) = \tilde{f}(\mathbf{x}) \quad (\mathbf{x} \in \mathbb{R}^\ell,\ \gamma \in \Gamma), \tag{7.6}$$

is given by

$$\tilde{f}(\mathbf{x}) = \sum_{\mathbf{p}\in P(R)} a_p\, e^{2\pi i(\mathbf{p}|\mathbf{x})} \quad (\mathbf{x} \in \mathbb{R}^\ell). \tag{7.7}$$

This is called the **Fourier series expansion** of $\tilde{f}$. Here, $a_{\mathbf{p}}$ is given by

$$a_{\mathbf{p}} = \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \left(\tilde{f}, e^{2\pi i(\mathbf{p}|\cdot)}\right), \quad \mathbf{p} \in P(R),$$

which is called the **Fourier coefficient** of $\tilde{f}$.

$\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\gamma) = \mathrm{Vol}(F(R))$, and the Hermitian inner product of $\varphi_{\mathbf{p}}$ and $\varphi_{\mathbf{p}'}$ is given by

$$(\varphi,\ \varphi') = \int_{F(R)} \varphi(\mathbf{x})\, \overline{\varphi'(\mathbf{x})} d\mathbf{x}.$$

Here, $d\mathbf{x}$ is a standard Lebesgue measure on $\mathbb{R}^\ell$. The converse is also true. If $\sum_{\mathbf{p}\in P(R)} |a_{\mathbf{p}}| < \infty$, then, $\tilde{f}$, which is defined by (7.7), satisfies the condition (7.6) for every continuous function on $\mathbb{R}^\ell$. If $(\tilde{f},\ \tilde{f}') < \infty$, then we have

$$(\tilde{f},\ \tilde{f}') = \left(\sum_{\mathbf{p}\in P(R)} a_{\mathbf{p}}\, e^{2\pi i(\cdot|\mathbf{p})},\ \sum_{\mathbf{p}\in P(R)} a'_{\mathbf{p}}\, e^{2\pi i(\cdot|\mathbf{p}')} \right) = \sum_{\mathbf{p}\in P(R)} a_{\mathbf{p}}\, \overline{a'_{\mathbf{p}}}\, (e^{2\pi i(\cdot|\mathbf{p})},\ e^{2\pi i(\cdot|\mathbf{p}')}),$$

$$(e^{2\pi i(\cdot|\mathbf{p})},\ e^{2\pi i(\cdot|\mathbf{p}')}) = \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\gamma)} \delta_{\mathbf{p}\mathbf{p}'}.$$

Here, we put $\delta_{\mathbf{p}\mathbf{p}'} = \begin{cases} 1 & \mathbf{p} = \mathbf{p}' \\ 0 & \mathbf{p} \neq \mathbf{p}' \end{cases}$. Therefore, we have

$$(\tilde{f},\ \tilde{f}') = \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\gamma)} \sum_{\mathbf{p}\in P(R)} |a_{\mathbf{p}}|^2 < \infty.$$

Next, we want to adjust the periodic function $\tilde{f}(\mathbf{x}, t)$ ($t > 0$ and $\mathbf{x} \in \mathbb{R}^\ell$) so that it satisfies the heat equation

$$\frac{\partial \tilde{f}}{\partial t} + \Delta_{\mathbf{x}} \tilde{f} = 0. \tag{7.8}$$

The function $\tilde{f}(\mathbf{x}, t)$ is given by

$$\tilde{f}(\mathbf{x}, t) = \sum_{\mathbf{p}\in P(R)} a_{\mathbf{p}}(t)\, e^{2\pi i(\mathbf{x}|\mathbf{p})}.$$

Here, $a_{\mathbf{p}}(t)$ is a function of $t > 0$, and we have

$$\frac{\partial \tilde{f}}{\partial t} + \Delta_{\mathbf{x}} \tilde{f} = \sum_{\mathbf{p}\in P(R)} a'_{\mathbf{p}}(t)\, e^{2\pi i(\mathbf{x}|\mathbf{p})} + \sum_{\mathbf{p}\in P(R)} a_{\mathbf{p}}(t) \Delta_{\mathbf{x}} e^{2\pi i(\mathbf{x}|\mathbf{p})}.$$

Here, $\mathbf{x} = \sum_{i=1}^{\ell} x_i \alpha_i^\vee$, $\mathbf{y} = \sum_{i=1}^{\ell} y_i \alpha_i^\vee$, $\Delta = -\sum_{i=1}^{\ell} \frac{\partial^2}{\partial y_i^2}$, and $\{e_1, \cdots, e_l\}$ is an orthogonal basis of $\mathbb{R}^\ell$. Since $\{\alpha_i^\vee, \cdots, \alpha_l^\vee\}$ is also a basis of $\mathbb{R}^\ell$, we have

$$\mathbf{x} = \sum_{i=1}^{\ell} x_i \alpha_i^\vee = \sum_{j=1}^{\ell} y_j e_j = \sum_{j=1}^{\ell} y_j \sum_{i=1}^{\ell} a_{ij} \alpha_i^\vee = \sum_{i=1}^{\ell} \left(\sum_{j=1}^{\ell} y_j a_{ij} \right) \alpha_i^\vee.$$

Here, if we put $e_j = \sum_{i=1}^{l} a_{ij} \alpha_i^\vee$, then we have $x_i = \sum_{j=1}^{l} y_j a_{ij}$. If $\alpha_i^\vee = \sum_{j=1}^{l} b_{ji} e_j$, $y_j = \sum_{j=1}^{l} x_i b_{ji}$, $B = (b_{ij})$, then we have

$$\sum_{i=1}^{\ell} x_i \sum_{j=1}^{\ell} b_{ji} e_j = \sum_{j=1}^{\ell} \left(\sum_{i=1}^{\ell} x_i b_{ji} \right) e_j = \sum_{j=1}^{\ell} y_j e_j,$$

$$(\alpha_k^\vee \mid \alpha_l^\vee) = \left(\sum_{s=1}^{\ell} b_{sk} e_s \,\middle|\, \sum_{t=1}^{\ell} b_{tl} e_t \right) = \sum_{s,t=1}^{\ell} b_{sk} b_{tl} \delta_{st} = \sum_{s=1}^{\ell} b_{sk} b_{sl} = ({}^t BB)_{kl},$$

$$\Delta = -\sum_{j=1}^{\ell} \frac{\partial^2}{\partial y_j^2} = \sum_{j=1}^{\ell} \sum_{s,t=1}^{\ell} a_{sj} a_{tj} \frac{\partial^2}{\partial x_s \partial x_t} = \sum_{s,t=1}^{\ell} \left(\sum_{j=1}^{\ell} a_{sj} a_{tj} \right) \frac{\partial^2}{\partial x_s \partial x_t}$$

$$= -\sum_{s,t=1}^{\ell} ({}^t AA) \frac{\partial^2}{\partial x_s \partial x_t} = -\sum_{s,t=1}^{\ell} ({}^t BB)^{-1}_{st} \frac{\partial^2}{\partial x_s \partial x_t}.$$

We know the relation between $A = (a_{ij})$ and $B = (a_{ij})$.

$$\frac{\partial f}{\partial y_i} = \sum_{k=1}^{\ell} \frac{\partial f}{\partial x_k} \frac{\partial x_k}{\partial y_j} = a_{kj} \sum_{k=1}^{\ell} \frac{\partial f}{\partial x_k} \quad \left(\text{by } \frac{\partial x_k}{\partial y_j} = a_{kj} \right).$$

Thus, we have

$$\frac{\partial f}{\partial y_i} = a_{kj} \sum_{k=1}^{\ell} \frac{\partial f}{\partial x_k}.$$

Here,

$$x_i = \sum_{j=1}^{l} y_j a_{ij} = \sum_{j=1}^{\ell} \left(\sum_{k=1}^{\ell} x_k b_{ik} \right) a_{ij} = \sum_{k=1}^{\ell} \left(\sum_{j=1}^{\ell} b_{jk} a_{ij} \right) x_k,$$

$$\sum_{j=1}^{l} a_{ij} b_{jk} = (AB)_{ik} = \sum_{j=1}^{\ell} b_{jk} a_{ij} = \delta_{ki}.$$

Thus, we have

$$AB = I_l, \quad A = B^{-1}, \quad {}^t AA = B^{-1t} B^{-1} = ({}^t BB)^{-1}.$$

If we put $(w_{st})_{1 \le s,t \le l}$ as the inverse matrix of $((\alpha_i^\vee | \alpha_j^\vee))_{1 \le i,j \le l}$, then Δ is shown as $\Delta = -\sum_{s,t=1}^{l} w_{st} \frac{\partial^2}{\partial x_s \partial x_t}$. Here, $\mathbf{x} = \sum_{i=1}^{l} x_i \alpha_i^\vee$.

Lemma 7.3 *We have $w_{st} = (w_s \,|\, w_t)$.*

Proof

If we put $w_i = \sum_{k=1}^{l} \xi_{ki}\, \alpha_k^\vee$, then

$$\delta_{ij} = (w_i \,|\, \alpha_j^\vee) = \left(\sum_{k=1}^{l} \xi_{ki}\, \alpha_k^\vee \,\middle|\, \alpha_j^\vee \right) = \sum_{k=1}^{l} \xi_{ki}\, (\alpha_k^\vee \,|\, \alpha_j^\vee).$$

Thus, $\xi_{ki} = w_{ik}$. Therefore, $w_i = \sum_{k=1}^{l} w_{ik}\, \alpha_k^\vee$. Thus, we have

$$(w_s \,|\, w_t) = \left(\sum_{p=1}^{l} w_{sp}\, \alpha_p^\vee \,\middle|\, w_t \right) = \sum_{p=1}^{l} w_{sp}\, \delta_{pt} = w_{st}. \qquad //$$

Then we obtain Proposition 7.2 as follows:

Proposition 7.2

$$\Delta = -\sum_{s,t=1}^{l} w_{st} \frac{\partial^2}{\partial x_s \partial x_t}.$$

Here, $(w_{st})_{1 \leq s,t \leq l}$ is the inverse matrix of $((\alpha_i^\vee | \alpha_j^\vee))_{1 \leq i,j \leq l}$, we put $\mathbf{x} = \sum_{i=1}^{l} x_i \alpha_i^\vee$.

We calculate (7.8) by using Proposition 7.2 as follows.

By $\mathbf{x} = \sum_{i=1}^{l} x_i \alpha_i^\vee,\ \mathbf{p} = \sum_{j=1}^{l} m_j w_j,\ (\alpha_i^\vee \,|\, w_j) = \delta_{ij}$,

$$(\mathbf{x}|\mathbf{p}) = \left(\sum_{i=1}^{l} x_i \alpha_i^\vee \,\middle|\, \sum_{j=1}^{l} m_j w_j \right) = \sum_{i,j=1}^{l} x_i m_j (\alpha_i^\vee \,|\, w_j) = \sum_{i=1}^{l} x_i m_i.$$

By this, we have

$$\begin{aligned} \frac{\partial^2}{\partial x_s \partial x_t} e^{2\pi i \sum_{i=1}^{l} x_i m_i} &= \{(2\pi i)^2\, m_s m_t\}\, e^{2\pi i \sum_{i=1}^{l} x_i m_i} \\ &= (4\pi^2\, m_s m_t)\, e^{2\pi i \sum_{i=1}^{l} x_i m_i}. \end{aligned}$$

Since,

$$\begin{aligned} \Delta_x e^{2\pi i (\mathbf{x}|\mathbf{p})} &= -\sum_{s,t=1}^{l} (w_s \,|\, w_t) \frac{\partial^2}{\partial x_s \partial x_t}\, e^{2\pi i \sum_{i=1}^{l} x_i m_i} \\ &= -\sum_{s,t=1}^{l} (w_s \,|\, w_t)(4\pi^2\, m_s m_t)\, e^{2\pi i \sum_{i=1}^{l} x_i m_i} \\ &= 4\pi^2 \left(\sum_{s=1}^{l} m_s w_s \,\middle|\, \sum_{t=1}^{l} m_t w_t \right) e^{2\pi i (\mathbf{x}|\mathbf{p})} \\ &= 4\pi^2 (\mathbf{p}|\mathbf{p})\, e^{2\pi i (\mathbf{x}|\mathbf{p})} \qquad \left(\sum_{s=1}^{l} m_s w_s = \mathbf{p},\ \sum_{t=1}^{l} m_t w_t = \mathbf{p} \right). \end{aligned}$$

That is, we have

$$\Delta_{\mathbf{x}} e^{2\pi i(\mathbf{x}|\mathbf{p})} = 4\pi^2(\mathbf{p}|\mathbf{p})\, e^{2\pi i(\mathbf{x}|\mathbf{p})}.$$

By using this,

$$\begin{aligned} 0 = \frac{\partial \tilde{f}}{\partial t} + \Delta_{\mathbf{x}} \tilde{f} &= \sum_{\mathbf{p}\in P(R)} a'_{\mathbf{p}}(t)\, e^{2\pi i(\mathbf{x}|\mathbf{p})} + \sum_{\mathbf{p}\in P(R)} a_{\mathbf{p}}(t) \Delta_{\mathbf{x}} e^{2\pi i(\mathbf{x}|\mathbf{p})} \\ &= \sum_{\mathbf{p}\in P(R)} \{a'_{\mathbf{p}}(t) + a_{\mathbf{p}}(t)\, 4\pi^2(\mathbf{p}|\mathbf{p})\}\Delta_{\mathbf{x}} e^{2\pi i(\mathbf{x}|p)}. \end{aligned}$$

Then

$$a'_{\mathbf{p}}(t) + a_{\mathbf{p}}(t)\, 4\pi^2(\mathbf{p}|\mathbf{p}) = 0.$$

Since, $a_{\mathbf{p}}(t)$ is given by:

$$a_{\mathbf{p}}(t) = C_{\mathbf{p}}\, e^{4\pi^2(\mathbf{p}|\mathbf{p})t}.$$

Thus, we obtain

$$\tilde{f}(\mathbf{x},\, t) = \sum_{\mathbf{p}\in P(R)} C_{\mathbf{p}}\, e^{-4\pi^2(\mathbf{p}|\mathbf{p})t + 2\pi i(\mathbf{x}|\mathbf{p})}.$$

Here, we have

$$C_{\mathbf{p}} = \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)}\, e^{-2\pi i(\mathbf{y}|\mathbf{p})} \quad \mathbf{p} \in P(R).$$

Then, we can define:

Definition 7.1

$$\begin{aligned} P(\mathbf{x}, \mathbf{y}, t) &= \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \sum_{\mathbf{p}\in P(R)} e^{-2\pi i(\mathbf{y}|\mathbf{p})}\, e^{-4\pi^2(\mathbf{p}|\mathbf{p})t}\, e^{2\pi i(\mathbf{x}|\mathbf{p})} \\ &= \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \sum_{\mathbf{p}\in P(R)} e^{-4\pi^2(\mathbf{p}|\mathbf{p})t}\, e^{2\pi i(\mathbf{x}-\mathbf{y}|\mathbf{p})}. \end{aligned} \tag{7.9}$$

Here, for an arbitrary $s > 0$, we have

$$\sum_{\mathbf{p}\in P(R)} \left| e^{-4\pi^2(\mathbf{p}|\mathbf{p})t} e^{2\pi i(\mathbf{x}-\mathbf{y}|\mathbf{p})} \right|^s = \sum_{\mathbf{p}\in P(R)} \left| e^{-4\pi^2(\mathbf{p}|\mathbf{p})(st)} \right| < \infty.$$

$P(\mathbf{x}, \mathbf{y}, t)$ is C^∞ for each $t > 0$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\ell$.

We obtain the theorem as follows.

Theorem 7.1 (1) *For each* $y \in \mathbb{R}^\ell$*, the function* $P = P(\mathbf{x}, \mathbf{y}, t)$ *in* (7.9) *satisfies* $\frac{\partial}{\partial t}P + \Delta_x P = 0$.

(2) *The following equations hold.*

$$P(\mathbf{x}+\gamma, \mathbf{y}, t) = P(\mathbf{x}, \mathbf{y}, t), \quad P(\mathbf{x}, \mathbf{y}+\gamma, t) = P(\mathbf{x}, \mathbf{y}, t) \quad (\gamma \in \Gamma).$$

(3) *All continuous functions* $\tilde{f}$ *on* $\mathbb{R}^\ell$ *with the condition*

$$(*)\tilde{f}(\mathbf{x}+\gamma) = \tilde{f}(\mathbf{x}), \quad \forall\gamma \in \Gamma, \quad \mathbf{x} \in \mathbb{R}^\ell$$

satisfy that

$$\lim_{t\downarrow 0} \int_{F(R)} P(\mathbf{x}, \mathbf{y}, t)\tilde{f}(\mathbf{y})d\mathbf{y} = \tilde{f}(\mathbf{y}).$$

Proof We have to show only (3). We have

$$\begin{aligned}
\lim_{t\downarrow 0} \int_{F(R)} P(\mathbf{x}, \mathbf{y}, t)\tilde{f}(\mathbf{y})d\mathbf{y} &= \lim_{t\downarrow 0} \int_{F(R)} \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \times \\
&\qquad \times \sum_{\mathbf{p}\in P(R)} e^{-4\pi^2(\mathbf{p}|\mathbf{p})t}\, e^{2\pi i(\mathbf{x}-\mathbf{y}|\mathbf{p})}\tilde{f}(\mathbf{y})d\mathbf{y} \\
&= \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \sum_{\mathbf{p}\in P(R)} e^{2\pi i(\mathbf{x}|\mathbf{p})} \int_{F(R)} e^{-2\pi i(\mathbf{y}|\mathbf{p})}\tilde{f}(\mathbf{y})d\mathbf{y}.
\end{aligned}$$

If we put

$$\lim_{t\downarrow 0} \int_{F(R)} e^{-2\pi i(\mathbf{y}|\mathbf{p})}\tilde{f}(\mathbf{y})d\mathbf{y} = (\tilde{f},\ e^{-2\pi i(\mathbf{p}|\cdot)}),$$

we have

$$\begin{aligned}
\lim_{t\downarrow 0} \int_{F(R)} P(\mathbf{x}, \mathbf{y}, t)\tilde{f}(\mathbf{y})d\mathbf{y} &= \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \sum_{\mathbf{p}\in P(R)} (\tilde{f},\ e^{-2\pi i(\mathbf{p}|\cdot)})\, e^{2\pi i(\mathbf{x}|\mathbf{p})} \\
&= \tilde{f}(\mathbf{x}). \qquad //
\end{aligned}$$

7.3 The explicit formula of the heat kernel

We have

Theorem 7.2 *The Dirichlet heat kernel* $e_D(\mathbf{x}, \mathbf{y}, t)$ *and the Neumann heat kernel* $e_N(\mathbf{x}, \mathbf{y}, t)$ *on* $D(R)$ *are given as follows:*

$$(1)\ e_D(\mathbf{x},\mathbf{y},t) = \sum_{w\in W(R)} \varepsilon(w)\, P(\mathbf{x}, w(\mathbf{y}), t)$$

$$= \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \sum_{w\in W(R), \mathbf{p}\in P(R)} \varepsilon(w) e^{-4\pi^2(\mathbf{p}|\mathbf{p})t}\, e^{2\pi i(\mathbf{x}-w(\mathbf{y})|\mathbf{p})}.$$

$$(2)\ e_N(\mathbf{x},\mathbf{y},t) = \sum_{w\in W(R)} P(\mathbf{x}, w(\mathbf{y}), t)$$

$$= \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \sum_{w\in W(R), \mathbf{p}\in P(R)} e^{-4\pi^2(\mathbf{p}|\mathbf{p})t}\, e^{2\pi i(\mathbf{x}-w(\mathbf{y})|\mathbf{p})}.$$

Proof See [2], p.186∼p.187. //

7.4 Main theorem (Poisson's summation formula)

The heat kernel of the flat torus $(\mathbb{R}^\ell/\Gamma, g_\gamma)$, denoted by $P(\mathbf{x},\mathbf{y},t)$, is given as follows:

$$P(\mathbf{x},\mathbf{y},t) = \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \sum_{p\in P(R)} e^{-4\pi^2(\mathbf{p}|\mathbf{p})t}\, e^{2\pi i(\mathbf{x}-\mathbf{y}|\mathbf{p})}.$$

Here, we have

$$\Gamma = \sum_{\alpha\in R} \mathbb{Z}\alpha^\vee = \left\{ \sum_{i=1}^{l} n_i \alpha_i^\vee \,\middle|\, n_1, \cdots, n_l \in \mathbb{Z} \right\},$$

$$P(R) = \left\{ \sum_{i=1}^{l} m_i \omega_i \,\middle|\, m_i \in \mathbb{Z}\ (i = 1, \cdots, l) \right\}.$$

Thus, we obtain the Dirichlet heat kernel and Neumann heat kernel by Theorem 7.1 and Theorem 7.2.

Theorem 7.3 (1) *The Dirichlet heat kernel* $e_D(\mathbf{x},\mathbf{y},t)$ *is given by*

$$e_D(\mathbf{x},\mathbf{y},t) = \sum_{w\in W(R)} \varepsilon(w)\, P(\mathbf{x}, w(\mathbf{y}), t)$$

$$= \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{\gamma\in\Gamma} \sum_{w\in W(R)} \varepsilon(w)\, e^{-\frac{|\mathbf{x}-w(\mathbf{y})-\gamma|^2}{4t}}. \tag{7.10}$$

(2) *The Neumann heat kernel* $e_N(\mathbf{x},\mathbf{y},t)$ *is given by*

$$e_N(\mathbf{x},\mathbf{y},t) = \sum_{w\in W(R)} P(\mathbf{x}, w(\mathbf{y}), t)$$

$$= \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{\gamma\in\Gamma} \sum_{w\in W(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{y})-\gamma|^2}{4t}}. \tag{7.11}$$

Theorem 7.4 (1) *For the Dirichlet heat kernel* $e_D(\mathbf{x},\mathbf{y},t)$*, if we put* $e_D(\mathbf{x},\mathbf{y},t)=\sum_{i=1}^{\infty} e^{-\lambda_i t}\, u_i(\mathbf{x})u_i(\mathbf{y})$*, then we have*

$$\int_{D(R)} e_D(\mathbf{x},\mathbf{x},t)d\mathbf{x}=\sum_{i=1}^{\infty} e^{-\lambda_i t}.$$

Furthermore, we define $Z_D(t)=\int_{D(R)} e_D(\mathbf{x},\mathbf{x},t)d\mathbf{x}$*. Then, we have*

$$\begin{aligned} Z_D(t) &= \int_{D(R)} \sum_{w\in W(R)} \varepsilon(w) \sum_{\gamma\in\Gamma} \frac{1}{(4\pi t)^{\frac{\ell}{2}}}\, e^{-\frac{|\mathbf{x}-w(\mathbf{x})-\gamma|^2}{4t}} d\mathbf{x} \\ &= \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{\gamma\in\Gamma}\sum_{w\in W(R)} \varepsilon(w) \int_{D(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{x})-\gamma|^2}{4t}} d\mathbf{x}. \end{aligned} \tag{7.12}$$

(2) *For the Neumann heat kernel* $e_N(\mathbf{x},\mathbf{y},t)$*, if we put* $e_N(\mathbf{x},\mathbf{y},t)=\sum_{i=1}^{\infty} e^{-\mu_i t}\, u_i(\mathbf{x})u_i(\mathbf{y})$*, then we have*

$$\int_{D(R)} e_N(\mathbf{x},\mathbf{x},t)d\mathbf{x}=\sum_{i=1}^{\infty} e^{-\mu_i t}.$$

Furthermore, we define $Z_N(t)=\int_{D(R)} e_N(\mathbf{x},\mathbf{x},t)d\mathbf{x}$*. Then, we have*

$$\begin{aligned} Z_N(t) &= \int_{D(R)} \sum_{w\in W(R)} \sum_{\gamma\in\Gamma} \frac{1}{(4\pi t)^{\frac{\ell}{2}}}\, e^{-\frac{|\mathbf{x}-w(\mathbf{y})-\gamma|^2}{4t}} d\mathbf{x} \\ &= \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{\gamma\in\Gamma}\sum_{w\in W(R)} \int_{D(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{y})-\gamma|^2}{4t}} d\mathbf{x}. \end{aligned} \tag{7.13}$$

Here, we prepare the following: Let

$$\Gamma=\sum_{\alpha\in R}\mathbb{Z}\alpha^\vee=\sum_{i=1}^{\ell}\mathbb{Z}\gamma_{\boldsymbol{i}}^\vee.$$

Here, $\{\gamma_i\}_{i=1}^{\ell}$ is the set of fundamental root systems of R, which are the simple root systems of R. Then, Γ is a lattice of $\mathbb{R}^\ell$, and we have a flat torus $(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)$.

Here, let us recall Poisson's summation formula for heat kernel $P(\mathbf{x},\mathbf{y},t)$ on our flat torus $(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)$ as follows.

Lemma 7.4

$$P(\mathbf{x},\mathbf{y},t) = \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \sum_{\mathbf{p}\in P(R)} e^{-4\pi^2(\mathbf{p}|\mathbf{p})t}\, e^{2\pi i(\mathbf{x}-\mathbf{y}|\mathbf{p})}$$
$$= \sum_{\gamma\in\Gamma} \frac{1}{(4\pi t)^{\frac{\ell}{2}}}\, e^{-\frac{|\mathbf{x}-\mathbf{y}-\gamma|^2}{4t}} \qquad (t>0).$$

We obtain Theorem 7.5 for the Dirichlet heat kernel and Neumann heat kernel by applying Lemma 7.4 to $e_D(\mathbf{x},\mathbf{y},t), e_N(\mathbf{x},\mathbf{y},t)$.

Theorem 7.5 (Poisson's summation formula) *For both the Dirichlet heat kernel and the Neumann heat kernel for $D(R)$, it holds that*

$$(1)\ \ e_D(\mathbf{x},\mathbf{y},t) = \sum_{\gamma\in\Gamma} \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{w\in W(R)} \varepsilon(w)\, e^{-\frac{|\mathbf{x}-w(\mathbf{y})-\gamma|^2}{4t}},$$
$$(2)\ \ e_N(\mathbf{x},\mathbf{y},t) = \sum_{\gamma\in\Gamma} \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{w\in W(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{y})-\gamma|^2}{4t}}.$$

By Theorem 7.5, we have

Theorem 7.6 *Let us consider the zeta functions $Z_D(t),\ Z_N(t)$ for the Dirichlet and Neumann eigenvalue problems of the Laplacian for $D(R)$. Then, we define*

$$Z_D(t) = \sum_{i=1}^{\ell} e^{-t\lambda_i} = \int_{D(R)} e_D(\mathbf{x},\mathbf{x},t)d\mathbf{x},$$
$$Z_N(t) = \sum_{i=1}^{\ell} e^{-t\mu_i} = \int_{D(R)} e_N(\mathbf{x},\mathbf{x},t)d\mathbf{x}.$$

Then we have:

$$(1)\ \ Z_D(t) = \int_{D(R)} \sum_{w\in W(R)} \varepsilon(w)\, P(\mathbf{x},w(\mathbf{x}),t)d\mathbf{x}$$
$$= \int_{D(R)} \sum_{w\in W(R)} \varepsilon(w) \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \sum_{p\in P(R)} e^{-4\pi^2(\mathbf{p}|\mathbf{p})t}\, e^{2\pi i(\mathbf{x}-w(\mathbf{x})|\mathbf{p})} d\mathbf{x}$$
$$= \int_{D(R)} \sum_{\gamma\in\Gamma} \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{w\in W(R)} \varepsilon(w)\, e^{-\frac{|\mathbf{x}-w(\mathbf{x})-\gamma|^2}{4t}} d\mathbf{x}.$$

$$(2)\ \ Z_N(t) = \int_{D(R)} \sum_{w\in W(R)} P(\mathbf{x},w(\mathbf{x}),t)d\mathbf{x}$$
$$= \int_{D(R)} \sum_{w\in W(R)} \frac{1}{\mathrm{Vol}(\mathbb{R}^\ell/\Gamma,\ g_\Gamma)} \sum_{p\in P(R)} e^{-4\pi^2(\mathbf{p}|\mathbf{p})t}\, e^{2\pi i(\mathbf{x}-w(\mathbf{x})|\mathbf{p})} d\mathbf{x}$$
$$= \int_{D(R)} \sum_{\gamma\in\Gamma} \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{w\in W(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{x})-\gamma|^2}{4t}} d\mathbf{x}.$$

For both the Dirichlet eigenvalue problem and the Neumann eigenvalue problem for $D(R)$, we have

Theorem 7.7 (Poisson's summation formula) *It holds that*

$$(1)\ Z_D(t) = \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{\gamma \in \Gamma} \sum_{w \in W(R)} \varepsilon(w) \int_{D(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{x})-\gamma|^2}{4t}} d\mathbf{x}.$$

$$(2)\ Z_N(t) = \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{\gamma \in \Gamma} \sum_{w \in W(R)} \int_{D(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{x})-\gamma|^2}{4t}} d\mathbf{x}.$$

Here,

$$Z_D(t) = \int_\Omega e_D(\mathbf{x}, \mathbf{x}, t) d\mathbf{x} = \sum_{i=1}^{\ell} e^{-\lambda_i t}, \quad Z_N(t) = \int_{D(R)} e_N(\mathbf{x}, \mathbf{x}, t) d\mathbf{x} = \sum_{i=1}^{\ell} e^{-\mu_i t}$$

are the **trace** *of the Dirichlet heat kernel $e_D(\mathbf{x}, \mathbf{x}, t)$ and the Neumann heat kernel $e_N(\mathbf{x}, \mathbf{x}, t)$ for the bounded domain $\Omega = D(R)$.*

Then, the following thorem is known (see [18], p.45, p.53).

Theorem 7.8 (*The asymptotic expansion for the trace of the Dirichlet and Neumann heat kernels*)

$$Z_D(t) \sim \frac{1}{(4\pi t)^{\frac{\ell}{2}}} (a_0 + b_1 t^{\frac{1}{2}} + a_1 t + b_2 t^{\frac{3}{2}} + a_2 t^2 + b_3 t^{\frac{5}{2}} + \cdots) \quad (t \downarrow 0), \tag{7.14}$$

$$Z_N(t) \sim \frac{1}{(4\pi t)^{\frac{\ell}{2}}} (c_0 + d_1 t^{\frac{1}{2}} + c_1 t + d_2 t^{\frac{3}{2}} + c_2 t^2 + d_3 t^{\frac{5}{2}} + \cdots) \quad (t \downarrow 0). \tag{7.15}$$

Here, $a_0 = c_0 = \mathrm{Vol}(\Omega),\ b_1 = -\frac{\sqrt{4\pi}}{4} \mathrm{Vol}_{\ell-1}(\partial\Omega),\ d_1 = \frac{\sqrt{4\pi}}{4} \mathrm{Vol}_{\ell-1}(\partial\Omega).$

It is a very interesting problem to study the asymptotic behaviors of $Z_D(t)$ and $Z_N(t)$ when $t \to 0+$ to calculate the above integrals in (6.9) and (6.10) over $D(R)$. We first show that only a finite number of elements in the set Γ contribute to the asymptotic behaviors of $Z_D(t)$ and $Z_N(t)$. To do this, let us consider the following open subset $\mathcal{F}(R)$:

Definition 7.2 *Let us define*

$$\mathcal{F}(R) := \{\mathbf{x} - w(\mathbf{x}) \mid \mathbf{x} \in D(R),\ w \in W(R)\}, \tag{7.16}$$

and also

$$a(R) := \max_{\mathbf{z} \in \mathcal{F}(R)} |\mathbf{z}|. \tag{7.17}$$

Furtheremore, let us consider the finite subset Γ_0 of Γ which is defined by

$$\Gamma_0 := \{\gamma \in \Gamma \mid |\gamma| < 3\,a(R)\}. \tag{7.18}$$

Then, we have:

Proposition 7.3

$$\int_{D(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{x})-\gamma|^2}{4t}} d\mathbf{x} \le \mathrm{Vol}(D(R))\; e^{-\frac{|\gamma|^2}{16t}-\frac{3\,a(R)^2}{16t}}. \tag{7.19}$$

To prove this proposition, we only need the following lemma.

Lemma 7.5 *For every* $\mathbf{z}\in\mathcal{F}$ *and* $\gamma\in\Gamma$ *outside* Γ_0*, it holds that*

$$|\mathbf{z}-\gamma|^2 \ge \frac{|\gamma|^2}{4} + \frac{3\,a(R)^2}{4}. \tag{7.20}$$

Proof Indeed, for every $\mathbf{z}\in\mathcal{F}$ and $\gamma\notin\Gamma_0$, we have

$$\begin{aligned}
|\boldsymbol{z}-\gamma|^2 &\ge |\gamma|^2 - 2\,|\gamma|\,|\boldsymbol{z}| + |\boldsymbol{z}|^2\\
&\ge |\gamma|^2 - 2a(R)\,|\gamma|\\
&= \frac{1}{4}\,|\gamma|^2 + \frac{3}{4}\,|\gamma|^2 - 2a(R)\,|\gamma|\\
&= \frac{1}{4}\,|\gamma|^2 + \frac{3}{4}\left(|\gamma|^2 - \frac{8}{3}a(R)\,|\gamma|\right)\\
&= \frac{1}{4}\,|\gamma|^2 + \frac{3}{4}\left(|\gamma| - \frac{4}{3}a(R)\right)^2 - \frac{4}{3}a(R)^2\\
&\ge \frac{1}{4}\,|\gamma|^2 + \frac{3}{4}\left(3a(R) - \frac{4}{3}a(R)\right)^2 - \frac{4}{3}a(R)^2\\
&= \frac{1}{4}\,|\gamma|^2 + \frac{3}{4}\left(\frac{5}{3}a(R)\right)^2 - \frac{4}{3}a(R)^2\\
&= \frac{1}{4}\,|\gamma|^2 + \frac{3}{4}a(R)^2. \qquad //
\end{aligned}$$

We have

$$\int_{D(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{x})-\gamma|^2}{4t}} d\mathbf{x} \le \mathrm{Vol}(D(R))\, e^{-\frac{|\gamma|^2}{4t}-\frac{1}{4t}\frac{3}{4}a(R)^2} \quad (^{\forall}\gamma \in \Gamma,\ |\gamma| \gg 3a(R)). \tag{7.21}$$

If we put $\Gamma_0 = \{\gamma\in\Gamma \mid |\gamma| < 3a(R)\}$, $\Gamma\backslash\Gamma_0 = \{\gamma\in\Gamma \mid |\gamma| \ge 3a(R)\}$, then we have

$$\begin{aligned}
Z_D(t) &= \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{\gamma\in\Gamma_0} \sum_{w\in W(R)} \varepsilon(w) \int_{D(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{x})-\gamma|^2}{4t}} d\mathbf{x}\\
&\quad + \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{\gamma\in\Gamma\backslash\Gamma_0} \sum_{w\in W(R)} \varepsilon(w) \int_{D(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{x})-\gamma|^2}{4t}} d\mathbf{x}.
\end{aligned} \tag{7.22}$$

Here, we can estimate the second term of this form.

$$
\begin{aligned}
|\text{second term}| &\leqq \#\,[W(R)]\, \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{\gamma \in \Gamma \setminus \Gamma_0} \mathrm{Vol}(D(R))\, e^{-\frac{|\gamma|^2}{4t} - \frac{1}{4t}\frac{3}{4}a(R)^2} \\
&= \#\,[W(R)]\, \mathrm{Vol}(D(R))\, e^{-\frac{1}{4t}\frac{3}{4}a(R)^2} \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{\gamma \in \Gamma \setminus \Gamma_0} e^{-\frac{|\gamma|^2}{4t}}. \qquad (7.23)
\end{aligned}
$$

Notice here that the infinite sum is of the form $\dfrac{1}{\text{polynomial in t}} e^{-\frac{a^2}{4t}}$, so it can be estimated from above by $e^{-\frac{a^2}{4t}}$ for some positive number a which is rapidly decreasing when $t \to 0+$. In a similar way as for $Z_N(t)$, we obtain the following theorem:

Theorem 7.9

$$
Z_D(t) \sim_{t \to 0+} \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{\gamma \in \Gamma_0} \sum_{w \in W(R)} \varepsilon(w) \int_{D(R)} e^{-\frac{|\mathbf{x} - w(\mathbf{x}) - \gamma|^2}{4t}} d\mathbf{x} \quad (\textit{the finite sum}), \qquad (7.24)
$$

$$
Z_N(t) \sim_{t \to 0+} \frac{1}{(4\pi t)^{\frac{\ell}{2}}} \sum_{\gamma \in \Gamma_0} \sum_{w \in W(R)} \int_{D(R)} e^{-\frac{|\mathbf{x} - w(\mathbf{x}) - \gamma|^2}{4t}} d\mathbf{x} \quad (\textit{the finite sum}). \qquad (7.25)
$$

As applications, we have the following well known facts (see [6], pp. 172-173). As a special case of Theorem (7.9), we have:

Theorem 7.10 (1) *For the Dirichlet zeta function for $D(R)$,*

$$
Z_D(t) \sim \frac{\mathrm{Vol}(D(R))}{(4\pi t)^{\ell/2}} \quad (\textit{as } t \to 0+). \qquad (7.26)
$$

(2) *For the Neumann zeta function for $D(R)$,*

$$
Z_N(t) \sim \frac{\mathrm{Vol}(D(R))}{(4\pi t)^{\ell/2}} \quad (\textit{as } t \to 0+). \qquad (7.27)
$$

Proof Let us recall W_a is a semi-direct product of $W(R)$ and Γ, so we denote

$$
W_a = \{ g = (\gamma, w) \mid \gamma \in \Gamma,\ w \in W(R) \},
$$

whose action on $\mathbb{R}^\ell$ is given by

$$
g\mathbf{x} = w\mathbf{x} + \gamma, \quad (\mathbf{x} \in \mathbb{R}^\ell,\ w \in W(R),\ \gamma \in \Gamma).
$$

The closure $\overline{D(R)}$ of $D(R)$ is a fundamental domain for W_a, and W_a acts simply transitively on the set of all the alcoves. In particular, for each $g \in W_a (g \neq e)$, the action of g has no fixed point in $D(R)$.

For an arbitrary small $\varepsilon > 0$, there exists a positive number $\delta > 0$ such that the δ-neighborhood from the boundary $\partial D(R)$ in $\overline{D(R)}$, the set

$$N_\delta := \{\mathbf{x} \in \overline{D(R)} \,|\, d(\mathbf{x}, \partial D(R)) < \delta\},$$

has $\mathrm{Vol}(N_\delta)$ which is smaller than ε.

Then, for each $g \in W_a (g \neq e)$, since

$$|\mathbf{x} - g\mathbf{x}| > 0 \quad (\forall\, \mathbf{x} \in \overline{D(R)} \backslash N_\delta)$$

and $\overline{D(R)} \backslash N_\delta$ is compact, there exists a positive constant $c(g, \delta) > 0$ such that

$$|\mathbf{x} - g\mathbf{x}|^2 \geq c\,(g, \delta) \quad (\forall\, \mathbf{x} \in \overline{D(R)} \backslash N_\delta).$$

Now we have, for each fixed $g = (\gamma, w) \in W_a (g \neq e)$,

$$\begin{aligned}\int_{D(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{x})-\gamma|^2}{4t}} d\mathbf{x} &= \int_{D(R)} e^{-\frac{|\mathbf{x}-g\mathbf{x}|^2}{4t}} d\mathbf{x} \\ &= \int_{N_\delta} e^{-\frac{|\mathbf{x}-g\mathbf{x}|^2}{4t}} d\mathbf{x} + \int_{D(R)\backslash N_\delta} e^{\frac{|\mathbf{x}-g\mathbf{x}|^2}{4t}} d\mathbf{x} \\ &\leq \mathrm{Vol}(N_\delta) + \mathrm{Vol}(D(R))\, e^{-\frac{c(g,\delta)}{4t}} \\ &< \varepsilon + \mathrm{Vol}(D(R))\, e^{-\frac{c(g,\delta)}{4t}},\end{aligned}$$

which implies that

$$\limsup_{t \to 0+} \int_{D(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{x})-\gamma|^2}{4t}} d\mathbf{x} \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have that

$$\int_{D(R)} e^{-\frac{|\mathbf{x}-w(\mathbf{x})-\gamma|^2}{4t}} d\mathbf{x} \to 0 \quad (as\ t \to 0+).$$

This together with Theorem 7.11, gives us the desired result (1) in Theorem 7.12 because the set $\{g = (\gamma, w) \,|\, \gamma \in \Gamma_0,\ w \in W(R)\}$ is a finite one, and for $g = e = (0, 1) \in W_a$,

$$\int_{D(R)} e^{-\frac{|\mathbf{x}-g\mathbf{x}|^2}{4t}} d\mathbf{x} = \mathrm{Vol}(D(R)).$$

For the Neumann heat zeta function, we can prove Theorem 7.12 (2) in a similar way. //

8 Visualization for eigenfunctions

8.1 Visualization by BASIC

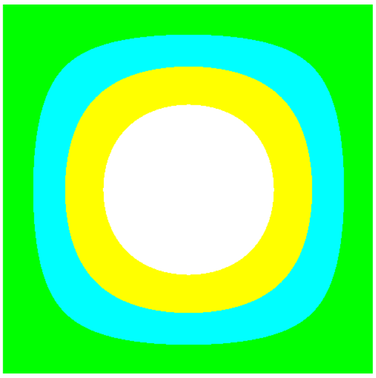
We show the Dirichlet and the Neumann boundary eigenfunctions on the square, right isosceles triangle, equilateral triangle, and right triangle which has also 30° and 60° angles, as follows.

The value of the function becomes bright as it increases positively (green, sky blue, yellow, white), and becomes dark as it decreases negatively (pink, red, blue, black) when they are negative.

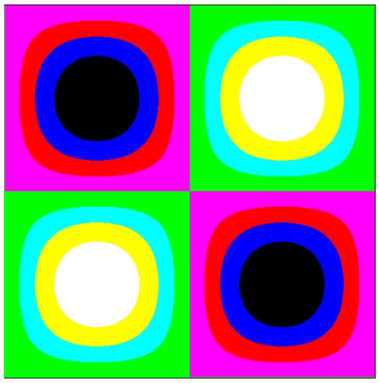
The Dirichlet eigenvalue problem

The square

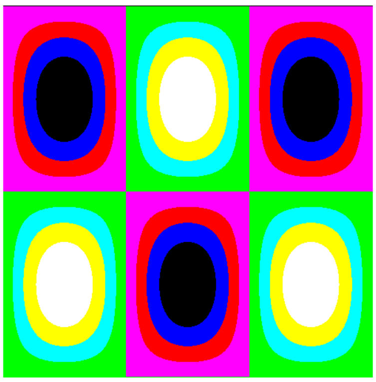
$$\psi = \sin\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi y}{a}\right)$$



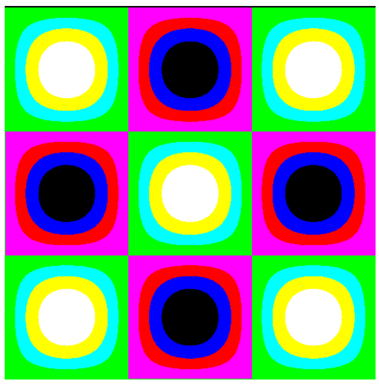
m = 1, n = 1



m = 2, n = 2



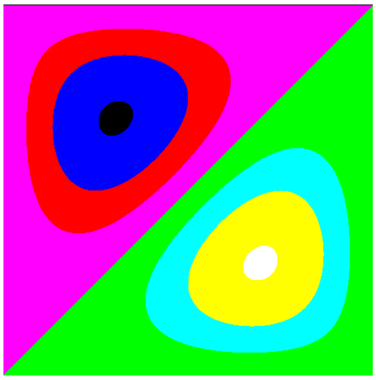
m = 3, n = 2



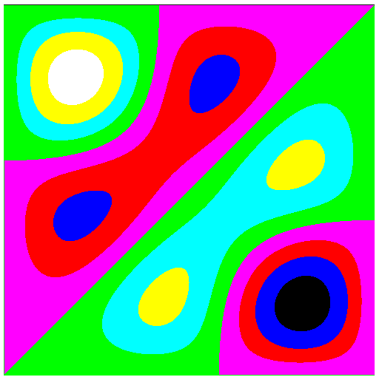
m = 3, n = 3.

The right isosceles triangle

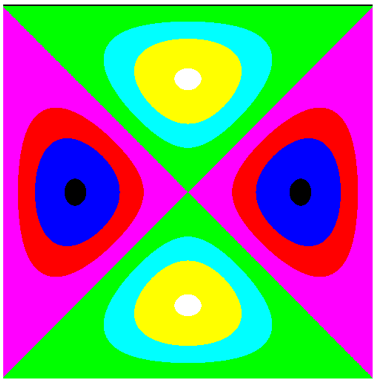
$$\psi = \sin\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi y}{a}\right) - \sin\left(\frac{n\pi x}{a}\right) \cdot \sin\left(\frac{m\pi y}{a}\right)$$



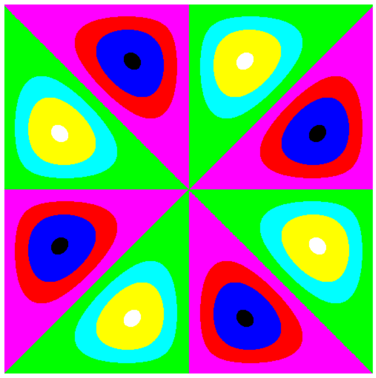
m = 1, n = 2



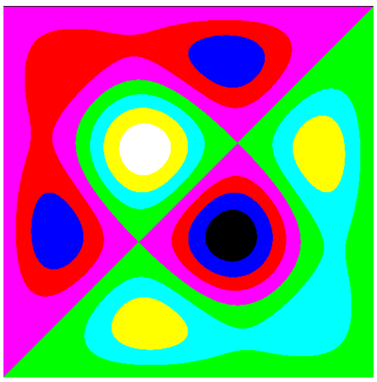
m = 2, n = 3



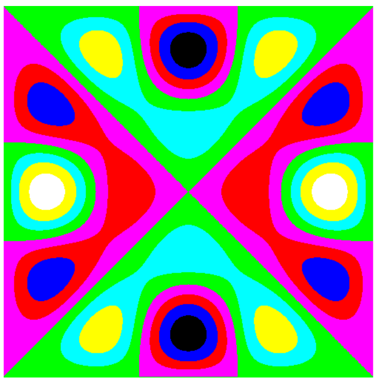
m = 1, n = 3



m = 2, n = 4



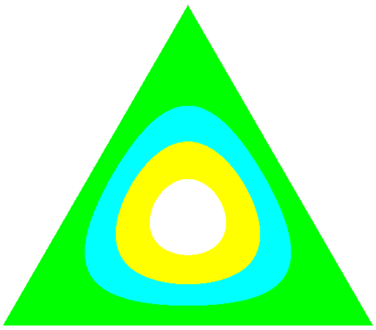
m = 3, n = 4



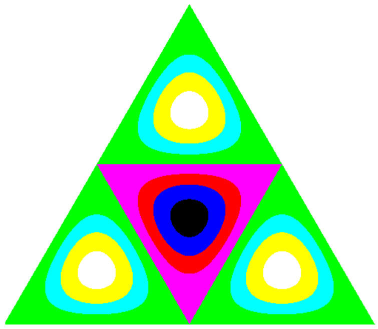
m = 3, n = 5.

The equilateral triangle

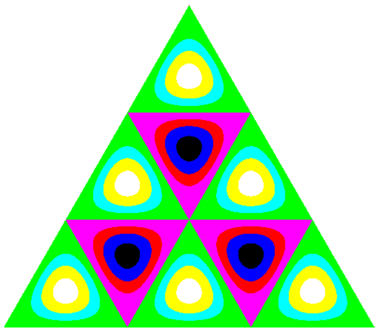
$$\psi = \sin\left(\frac{2\pi}{a}\cdot p\cdot\frac{2y}{\sqrt{3}}\right) + \sin\left(\frac{2\pi}{a}\cdot p\cdot\left(x-\frac{y}{\sqrt{3}}\right)\right) + \sin\left(\frac{2\pi}{a}\cdot p\cdot\left(-x-\frac{y}{\sqrt{3}}\right)\right)$$



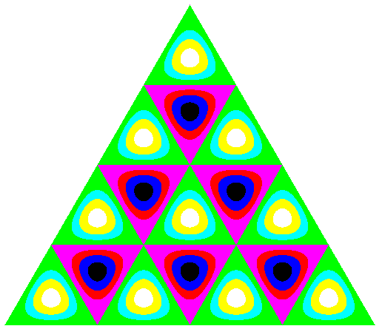
p = 1



p = 2



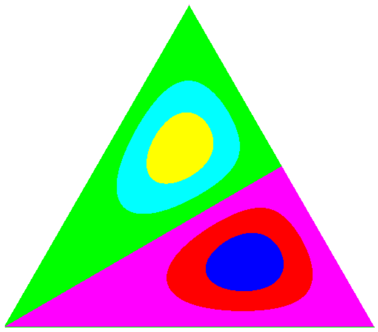
p = 3



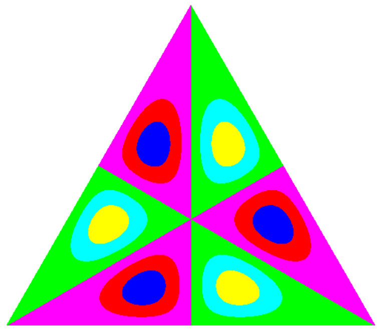
p = 4.

The right triangle which has 30°and 60° angles

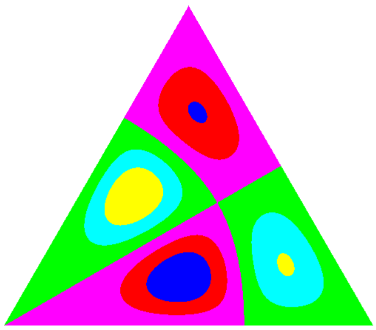
$$\psi = \psi_1 + \psi_2 + \psi_3$$
$$\psi_1 = \sin\left(\frac{2\pi}{3a}\cdot(2m+n)x\right)\cdot\sin\left(\frac{2\pi}{3a}\cdot 3n\cdot\frac{y}{\sqrt{3}}\right)$$
$$\psi_2 = \sin\left(\frac{2\pi}{3a}\cdot(m-n)x\right)\cdot\sin\left(\frac{2\pi}{3a}\cdot 3(m+n)\cdot\frac{y}{\sqrt{3}}\right)$$
$$\psi_3 = \sin\left(\frac{2\pi}{3a}\cdot(m+2n)x\right)\cdot\sin\left(\frac{2\pi}{3a}\cdot(-3m)\cdot\frac{y}{\sqrt{3}}\right)$$



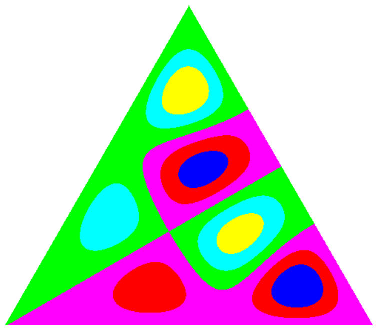
m = 1, n = 2



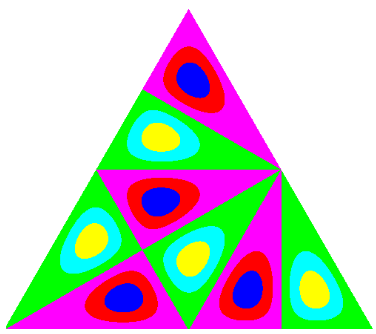
m = 1, n = 4



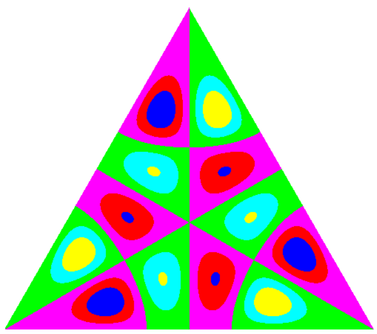
m = 1, n = 3



m = 2, n = 3



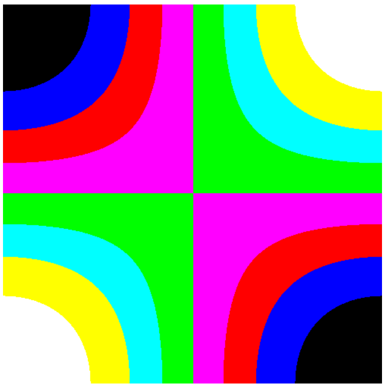
m = 2, n = 4



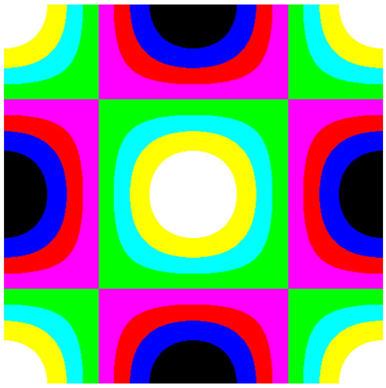
m = 2, n = 5.

The Neumann eigenvalue problem
The square

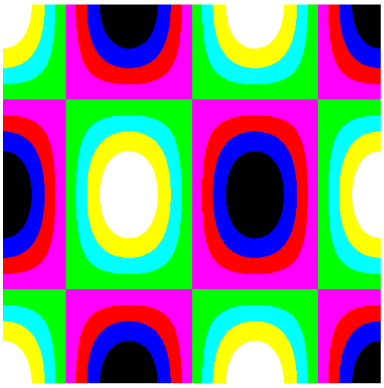
$$\psi = \cos\left(\frac{m\pi x}{a}\right) \cdot \cos\left(\frac{n\pi y}{a}\right)$$



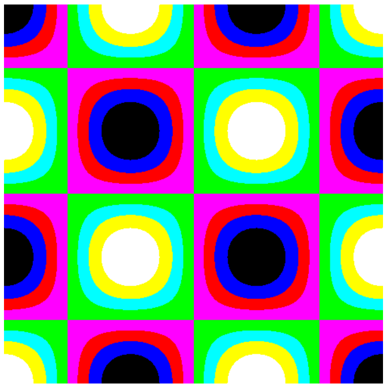
m = 1,   n = 1



m = 2,   n = 2



m = 3,   n = 2



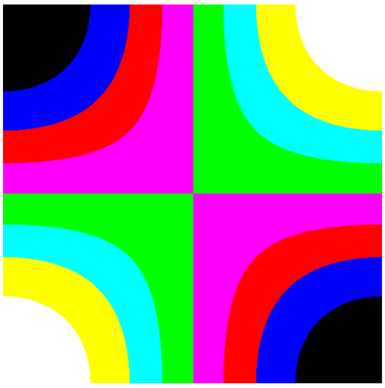
m = 3,   n = 3.

The right isosceles triangle

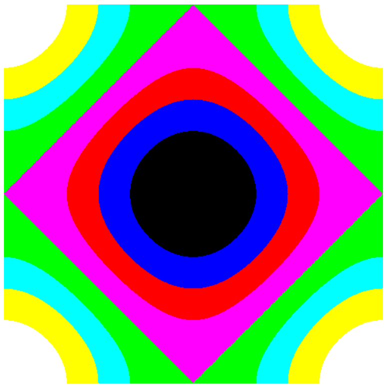
$$\psi = \cos\left(\frac{m\pi x}{a}\right) \cdot \cos\left(\frac{n\pi y}{a}\right) + \cos\left(\frac{n\pi x}{a}\right) \cdot \cos\left(\frac{m\pi y}{a}\right)$$

m = 0,   n = 1

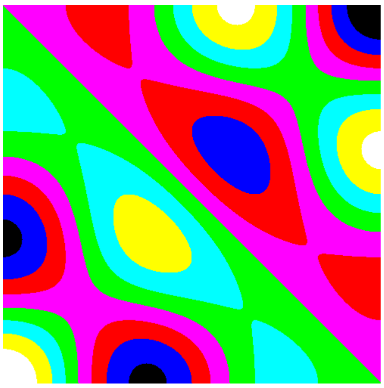
m = 1,   n = 2



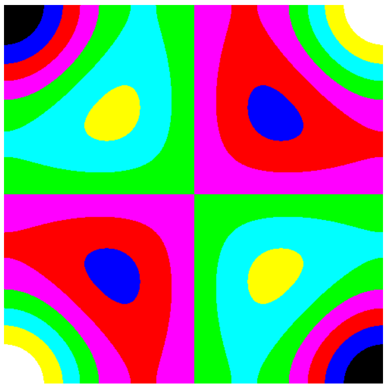
m = 1,   n = 1



m = 0,   n = 2



m = 2,   n = 3

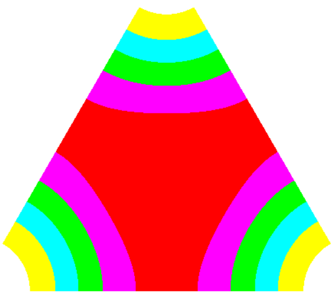


m = 1,   n = 3.

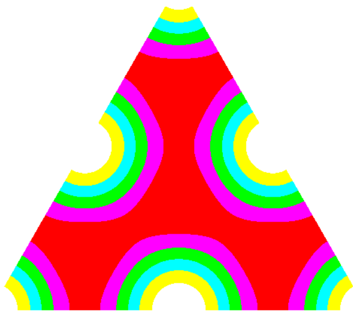
The equilateral triangle
(Same as the Dirichlet eigenvalue problem)

The right triangle which has 30°and 60° angles

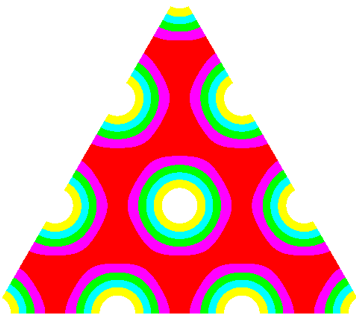
$$\psi = \psi_1 + \psi_2 + \psi_3$$
$$\psi_1 = \cos\left(\frac{2\pi}{3a} \cdot (2m+n)x\right) \cdot \cos\left(\frac{2\pi}{3a} \cdot 3n \cdot \frac{y}{\sqrt{3}}\right)$$
$$\psi_2 = \cos\left(\frac{2\pi}{3a} \cdot (m-n)x\right) \cdot \cos\left(\frac{2\pi}{3a} \cdot 3(m+n) \cdot \frac{y}{\sqrt{3}}\right)$$
$$\psi_3 = \cos\left(\frac{2\pi}{3a} \cdot (m+2n)x\right) \cdot \cos\left(\frac{2\pi}{3a} \cdot (-3m) \cdot \frac{y}{\sqrt{3}}\right)$$



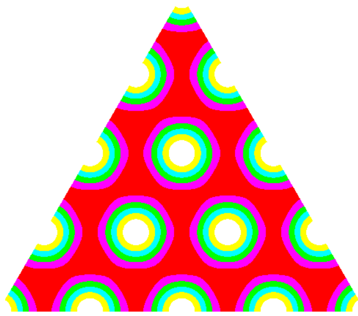
m = 1, n = 1



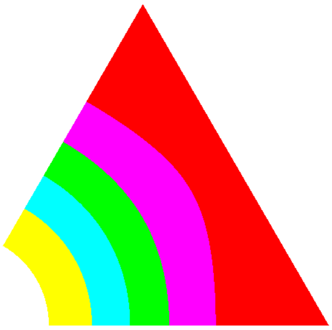
m = 2, n = 2



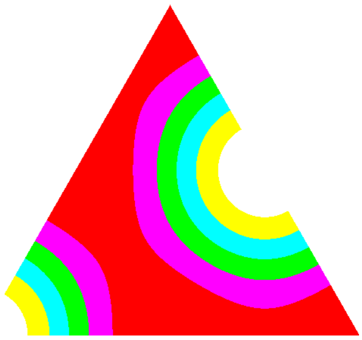
m = 3, n = 3



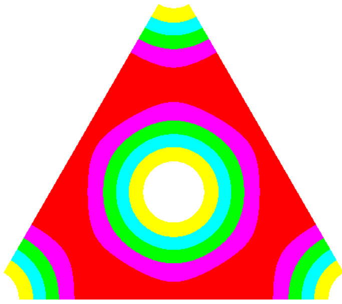
m = 4, n = 4



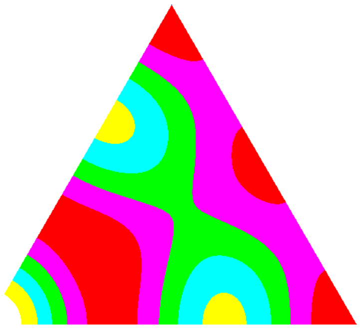
m = 0, n = 1



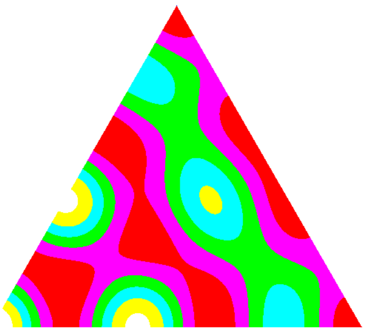
m = 0, n = 2



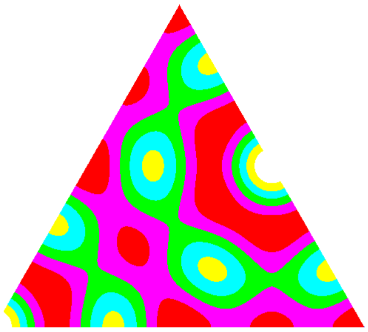
m = 0, n = 3



m = 1, n = 2



m = 2, n = 3



m = 2, n = 4.

8.2 Visualization by Maple

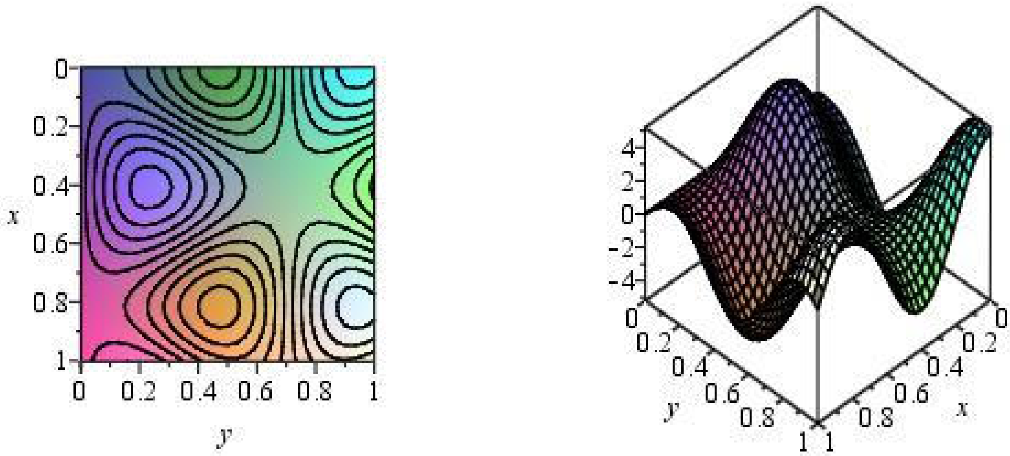
We show the eigenfunctions for both the Dirichlet problems and the Neumann problems on $D(R)$, which are of the type $A_2,\ B_2,\ G_2$.

We figure out the eigenfunctions for the Dirichlet problems by the determinant and the eigenfunctions for the Neumann problems by the permanent.

Type A_2

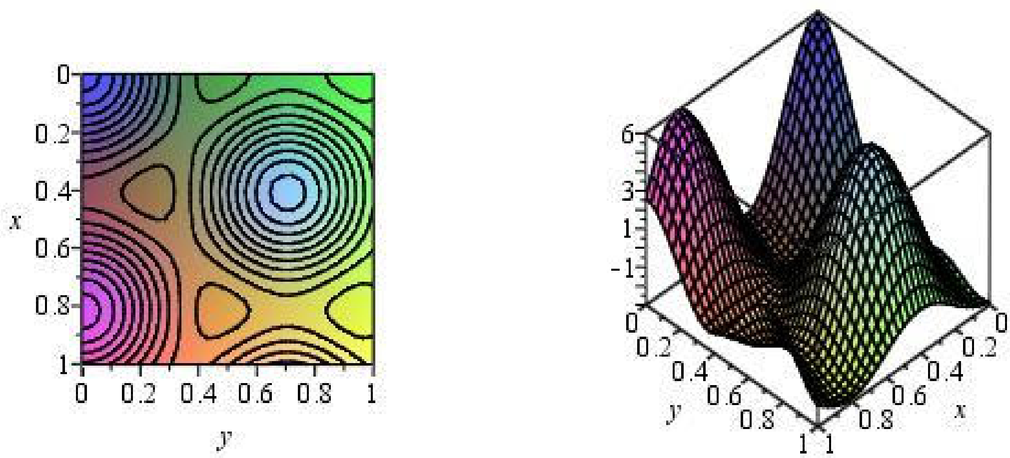
They are the same as the above eigenfunctions of the equilateral triangle.

The Dirichlet eigenvalue problem



m = 1, n = 1.

The Neumann eigenvalue problem

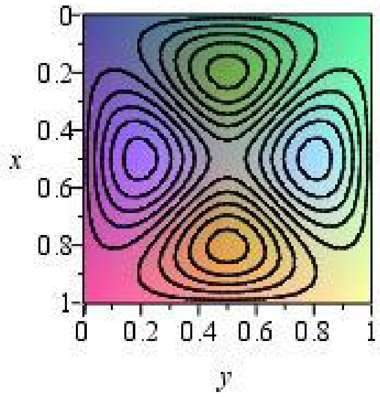


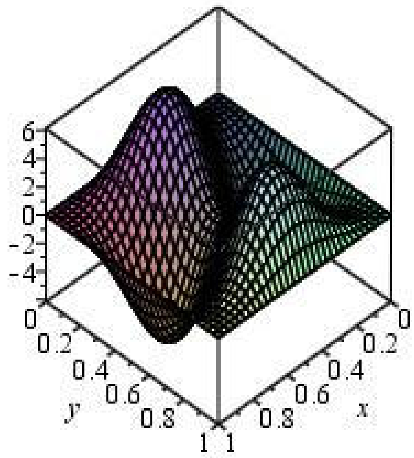
m = 1, n = 1.

Type B_2

They are the same as the above eigenfunctions of the right isosceles triangle.

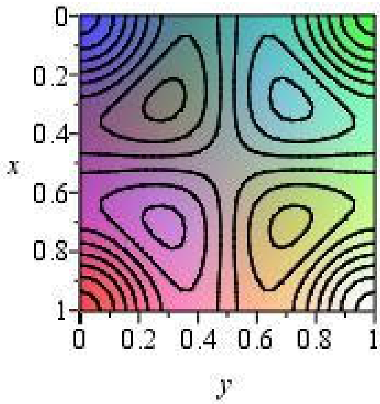
The Dirichlet eigenvalue problem

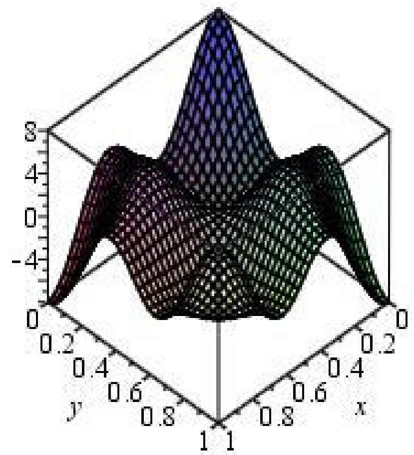




m = 1, n = 1.

The Neumann eigenvalue problem



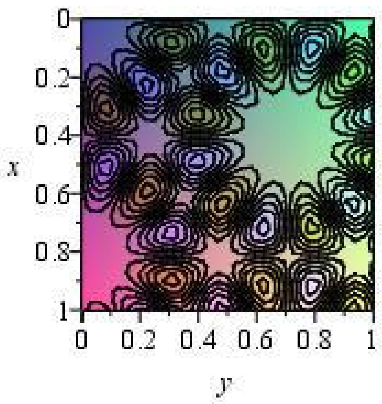


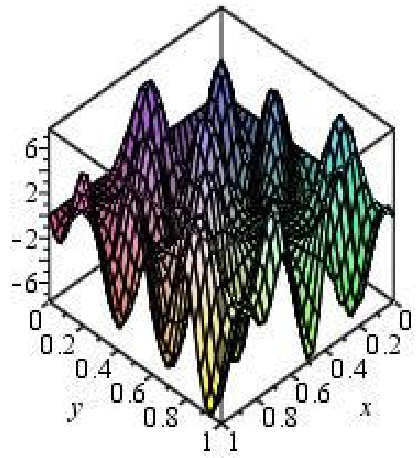
m = 1, n = 1.

Type G_2

They are the same as the above eigenfunctions of the right triangle which has 30° and 60° angles.

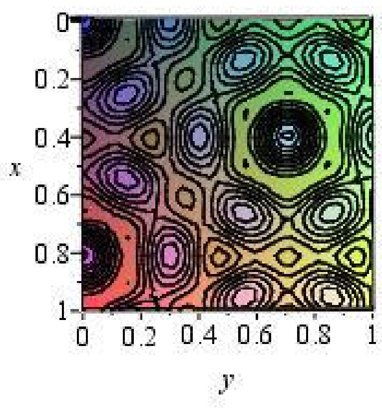
The Dirichlet eigenvalue problem

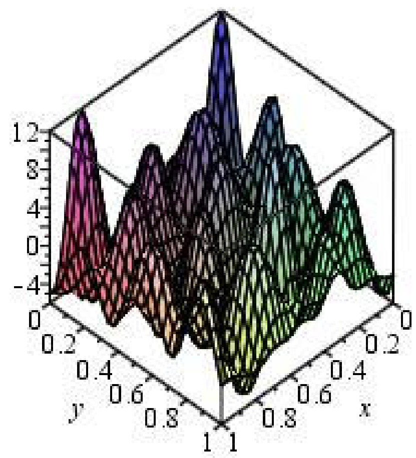




m = 1, n = 1.

The Neumann eigenvalue problem





m = 1, n = 1.

9 Concluding Remarks

In Section 5, we figured out the alcove $D(R)$ and the crystallographic groups, and we showed the explicit form for the eigenvalues and eigenfunctions.

In Section 6, by the formulas in Section 5, we showed the explicit form for the eigenvalues and eigenfunctions in 2 and 3 dimensions. By the root systems and affine Weyl group theory, we showed the coordinate expression of the alcove $D(R)$ (type $A_2, B_2, G_2, A_3, B_3, C_3$), which are the fundamental domains in affine Weyl groups. Furthermore, we figured out the corresponding elements of transformations for each $D(R)$ in crystallographic groups as matrices, and showed the eigenvalues and eigenfunctions.

We showed the eigenfunctions for the Dirichlet boundary eigenvalue problems as the determinant.

We showed all the eigenfunctions for the Neumann boundary eigenvalue problems in terms of the permanent.

In Section 7, we obtained Poisson's summation formula of the heat kernel on $D(R)$. This formula is the first result, and a very interesting result, for the Dirichlet and Neumann heat kernel on $D(R)$.

In Section 8, we visualized some eigenfunctions.

We showed the eigenfunctions for the Dirichlet and the Neumann boundary eigenvalue problems on the square and right isosceles triangle (of type B_2) and equilateral triangle (of type A_2) and right triangle which has 30° and 60° angles (of type G_2).

We hope to visualize the types $A_3,\ B_3,\ C_3$ of 3 dimensions.

Mathematical science is applied to many other fields (physics, engineering, etc.). We hope the boundary value problems for affine Weyl groups will be developed much more, and applied in many fields, more and more. We hope that our research can be applied to other mathematical sciences in the future.

Appendices

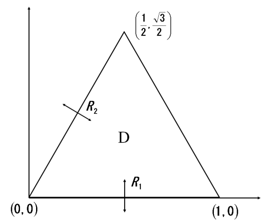
A The relation between our work and Section 3 of Pinsky's work

A.1 Figuration of elements in crystallographic groups section 3

Concerning the symbols, E means "identity transformation", RE means "reflection with respect to", RO means "rotation", I means "inversion".

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (RE \quad y=0),$$
$$R_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad (RE \quad y=\sqrt{3}x).$$

When we do symmetric operation with the z axis (in the planar case, the origin) as the center, symmetric operations are figured by 6 elements as follows.



(1) $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (E)$

(2) $R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (RE \quad y=0)$

(3) $R_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad (RE \quad y=\sqrt{3}x)$

(4) $R_2R_1R_2 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad (RE \quad y=-\sqrt{3}x)$

(5) $R_1R_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad (RO \quad -120^\circ)$

(6) $R_2R_1 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad (RO \quad 120^\circ)$

A.2 The relation between Section 3 (type A_2) and Section 4 (crystallographic groups)

We show the relation between the figure of elements in crystallographic groups which Pinsky used and the one which Bérard used. Pinsky had the idea of symmetric operation around the z axis, and on the other hand, Bérard had the idea symmetric operation around the direction $(1, 1, 1)$ in Bourbaki's book. Then, we consider that transformation from the symmetric operation around the z axis to the one around the x axis, and from the one around the x axis to the one around the direction $(1, 1, 1)$.

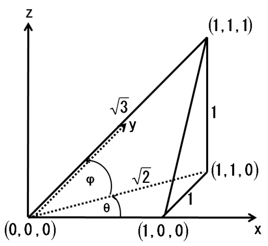
Transformation from symmetric operation around the z axis to the one around the x axis.

The matrix for this operation is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

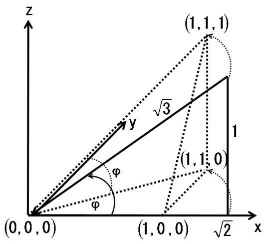
Transformation from symmetric operation around the x axis to the one around the direction $(1, 1, 1)$.

Next, we consider the matrix for this operation.

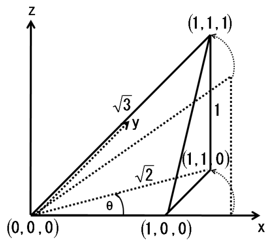


This transformation consists of 2 rotations as follows.

1.Rotation(ϕ) in the zx-plane.



2.Rotation(θ) in the xy-plane.



1.Rotation(ϕ) in the zx-plane.

If we put R_ϕ for the operation of 1, then the matrix is given as follows.

Because of $\cos\phi = \sqrt{\frac{2}{3}}$, $\sin\phi = \frac{1}{\sqrt{3}}$, R_ϕ is showed by

$$R_\phi = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & -1 \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}.$$

2.Rotation(θ) in the xy-plane.

If we put R_θ for the operation here, the matrix is given as follows.

Because of $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$, R_θ is shown by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The transform operation from the symmetric operation around the x axis to the one around the direction (1,1,1) means operation 1 as first and operation 2 as second. Since the matrix is shown by

$$\begin{aligned} R_\theta R_\phi &= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & -1 \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}, \end{aligned}$$

the inverse matrix of this matrix is the transposed matrix of the elements

$$(R_\theta R_\phi)^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{pmatrix}.$$

Transformation from the symmetric operation around the z axis to the one around the direction (1,1,1)

This operation means the transformation which is a combination from the symmetric operation around the z axis to the one around the x axis firstly and from the symmetric operation around the x axis to the one around the direction $(1,1,1)$ secondly. We put A_Z for this operation. The matrices $A_Z,\ A_Z^{-1}$ are given by

$$A_Z = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad A_Z^{-1} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Transformation from the symmetric operation around the *z* axis to the one around the direction (1,1,1)(calculation)

We calculate the transformation of symmetric operation which is around the direction $(1,1,1)$ by the symmetric operation around the z axis, by using elements $E,\ R_1,\ R_2,\ R_2R_1R_2,\ R_1R_2,\ R_2R_1$ in Pinsky's symmetric operation.

If we put R for symmetric operation around the z axis by Pinsky generally, around the direction $(1,1,1)$, the symmetric operation $R_{1,1,1}$ is shown by

$$R_{1,1,1} = A_Z\ R\ A_Z^{-1}.$$

When we put

$$R = E,\ R_1,\ R_2,\ R_2R_1R_2,\ R_1R_2,\ R_2R_1,$$

we calculate $R_{1,1,1}$, then they are in accord with 6 elements in symmetric operation for A_2. The calculations are as follows.

In the case of $\boldsymbol{R} = \boldsymbol{E}$,

$$R_{1,1,1} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (E)$$

In the case of $\boldsymbol{R} = \boldsymbol{R_1}$,

$$R_{1,1,1} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (RE \quad y = x)$$

In the case of $\boldsymbol{R} = \boldsymbol{R_2}$,

$$R_{1,1,1} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (RE \quad x = z)$$

In the case of $\boldsymbol{R} = \boldsymbol{R_1 R_2}$,

$$R_{1,1,1} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (RO \quad -120^\circ)$$

In the case of $\boldsymbol{R} = \boldsymbol{R_2 R_1}$,

$$R_{1,1,1} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (RO \quad 120^\circ)$$

In the case of $\boldsymbol{R} = \boldsymbol{R_2 R_1 R_2}$.

$$R_{1,1,1} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (RE \quad z = y)$$

B The definition of the permanent

For n-degree square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots a_{nn} \end{pmatrix},$$

we consider the sum

$$\sum_{\sigma} a_{\sigma(1)1} a_{\sigma(2)} \cdots a_{\sigma(n)n}.$$

Here, $\sum_{\sigma}$ runs over all the $n!$ terms in all the permutations in the set $\{1, 2, \cdots, n\}$.

We call this the permanent of A and write it as

$$\mathrm{Perm}(A) = \mathrm{Perm} \begin{pmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots a_{nn} \end{pmatrix}.$$

Also, we call this the n-degree permanent of an $n \times n$ square matrix. This is with changed sign $\varepsilon(\sigma)$ of the determinant

$$|A| = \sum_{\sigma} \varepsilon(\sigma) a_{\sigma(1)1} a_{\sigma(2)} \cdots a_{\sigma(n)n} = \begin{vmatrix} a_{11} & a_{12} & \cdots a_{1n} \\ a_{21} & a_{22} & \cdots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots a_{nn} \end{vmatrix}$$

into $+1$.

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