Girsanov transformation of symmetric Markov processes and its applications

著者	MIURA YUSUKE
学位授与機関	Tohoku University
学位授与番号	11301甲第16718号
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博 士 論 文

Girsanov transformation of symmetric Markov processes and its applications

(対称マルコフ過程のギルサノフ変換とその応用)

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Yusuke Miura

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Contents

Chapter 1

Introduction

In this paper, we study Girsanov transformations of symmetric Markov processes.

Let $\{B_t\}_{t\geq 0}$ be the Brownian motion in $\mathbb{R}^d$. We consider a transformation of $\{B_t\}$ by the multiplicative functional

$$L_t^\rho = \exp\left(\int_0^t \frac{\nabla\rho}{\rho}(B_s)\cdot dB_s - \frac{1}{2}\int_0^t \left|\frac{\nabla\rho}{\rho}\right|^2 (B_s)\, ds\right).$$

Here ρ is a nonnegative function in the 1-order Sobolev space. This transformation is called a *Girsanov transformation*. It is known that the transformed process is a symmetric diffusion process in $\mathbb{R}^d$ with generator, $\frac{1}{2}\Delta + \frac{\nabla\rho}{\rho}\cdot\nabla$. When ρ decreases to 0 near infinity, the drift $\frac{\nabla\rho}{\rho}$ forces the transformed process to move back inward. Thus, it is expected that the new process hardly approaches to the infinity and the zero set of ρ. Indeed, the non-attainability to the set $\{\rho(x) = 0\}$ and the recurrence of the transformed process are shown in [31, 34]. We treat transformations of general symmetric Markov processes by multiplicative functionals of this type and investigate properties of transformed processes.

Let E be a locally compact separable metric space and m a positive Radon measure on E with full topological support. Let $\mathbb{M} = (X_t, \mathbb{P}_x)$ be an m-symmetric Hunt process on E. $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ denotes the regular Dirichlet form on $L^2(E; m)$ generated by $\mathbb{M}$. Let ρ be a nonnegative function belonging to the space $\dot{\mathcal{D}}^\dagger_{\mathrm{loc}}(\mathcal{E})$ (for the definition of $\dot{\mathcal{D}}^\dagger_{\mathrm{loc}}(\mathcal{E})$, see the next chapter). It is shown in [25, 26] that $\rho(X_t) - \rho(X_0)$ has the following Fukushima's decomposition:

$$\rho(X_t) - \rho(X_0) = M_t^{[u]} + N_t^{[u]},$$

where $M^{[u]}$ is a local martingale additive functional locally of finite energy and $N^{[u]}$ is a continuous additive functional locally of zero energy. Let L_t^ρ be the solution to the following stochastic differential equation:

$$L_t^\rho = 1 + \int_0^t L_{s-}^\rho \frac{1}{\rho(X_{s-})} \, dM_s^{[\rho]}.$$

Then L_t^ρ is a positive supermartingale multiplicative functional and defines a family of probability measures $\{\widetilde{\mathbb{P}}_x\}$ by $d\widetilde{\mathbb{P}}_x := L_t^\rho d\mathbb{P}_x$. It is known that under new measures $\{\widetilde{\mathbb{P}}_x\}$, X_t is a symmetric right Markov process on $\{\rho(x) > 0\}$. We denote this transformed process by $\widetilde{\mathbb{M}}^\rho$.

Girsanov transformations of symmetric Markov processes have been considered by many authors (for example, see [6, 8, 13, 18, 22, 31, 34]). It is shown in [18, §6.3] that if $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a strong local Dirichlet form and ρ has a finite energy measure, then the process $\widetilde{\mathbb{M}}^\rho$ is also conservative and never attains to the set $\{\rho(x) = 0 \text{ or } \rho(x) = \infty\}$. We prove that the same result holds without assuming the local property (Theorem 3.12). Note that $\widetilde{\mathbb{M}}^\rho$ is conservative if and only if the exponential martingale L_t^ρ is a martingale. Novikov's condition is well known as a sufficient condition for an exponential martingale to be a martingale. However, we cannot apply Novikov's condition when $\mathbb{M}$ has jumps. We overcome this problem by checking the criterion for uniform integrability of exponential martingales due to Chen [4]. For more general symmetric Markov processes, Chen et al. [6] showed that $\widetilde{\mathbb{M}}^\rho$ is recurrent for all positive $\rho \in \mathcal{D}(\mathcal{E})$. Using ideas from [6], we extend this result to an element ρ of the extended Dirichlet space $\mathcal{D}_e(\mathcal{E})$ (Theorem 3.9).

Let $\mathbb{M}$ be a transient Markov process with strong Feller property. As an application of Girsanov transformation, we consider Hardy's inequality:

$$\int_E u^2 d\mu \leq \mathcal{E}(u, u), \quad \text{for all } u \in \mathcal{D}(\mathcal{E}), \tag{1.1}$$

where μ is a Green-tight measure (see Definition 4.1). Let $\lambda(\mu)$ be the bottom of the spectrum of the time changed process of $\mathbb{M}$ by A_t^μ, a positive continuous additive functional whose Revuz measure is μ:

$$\lambda(\mu) = \inf\left\{ \mathcal{E}(u, u) \,\middle|\, u \in \mathcal{D}(\mathcal{E}), \int_E u^2 d\mu = 1 \right\}.$$

If $\lambda(\mu) > 1$, then the gauge function $\mathbb{E}.\left[\exp(A_\zeta^\mu)\right]$ is bounded ([3, Theorem 5.1]). If $\lambda(\mu) = 1$, then the ground state of the operator $\mathcal{L} + \mu$ exists in the extended Dirichlet space $\mathcal{D}_e(\mathcal{E})$, where $\mathcal{L}$ is the generator of $\mathbb{M}$ ([39]). Assume $\lambda(\mu) > 1$ (resp. $\lambda(\mu) = 1$) and let ρ be the gauge function (resp. ground state). Then ρ is in $\dot{\mathcal{D}}_{\text{loc}}^\dagger(\mathcal{E})$, and thus the Girsanov transformed process $\widetilde{\mathbb{M}}^\rho$ by L_t^ρ can be defined. Then by using Itô's formula, L_t^ρ

can be expressed by

$$L_t^\rho = \frac{\rho(X_t)}{\rho(X_0)} \exp\left(A_t^\mu\right).$$

This expression tells us that the Girsanov transformed process $\widetilde{\mathbb{M}}^\rho$ coincides with the process generated by the composition of Doob's h-transform and the Feynman-Kac multiplicative functional $e^{A_t^\mu}$. As a corollary, we have the identity

$$\mathcal{E}(u,u) - \int_E u^2 d\mu = \widetilde{\mathcal{E}}^\rho\left(\frac{u}{\rho}, \frac{u}{\rho}\right) \qquad \text{for all } u \in \mathcal{D}(\mathcal{E}),$$

where $\widetilde{\mathcal{E}}^\rho$ is the Dirichlet form generated by the process $\widetilde{\mathbb{M}}^\rho$. Applying the results above on the Girsanov transformation, we can precisely express the right-hand side, which implies an improvement of the inequality (1.1). Improvements of Hardy-type inequalities are studied by many authors with analytical methods (for example, see [15, 16, 27]). We think that our probabilistic method gives an interpretation to Hardy's inequalities.

A probability measure μ on E is said to be a *quasi-stationary distribution* of $\mathbb{M}$ if for all $t \geq 0$,

$$\mu(\cdot) = \mathbb{P}_\mu(X_t \in \cdot \,|\, t < \zeta),$$

where $\mathbb{P}_\mu$ denotes the probability of the process with initial distribution μ and ζ is the lifetime of $\mathbb{M}$. In [23], they prove that if a Markov semigroup is intrinsically ultracontractive, then a measure ν on E defined by

$$\nu(B) = \frac{\int_B \rho \, dm}{\int_E \rho \, dm}$$

is a unique quasi-stationary distribution. Here ρ is a ground state of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. We will give another proof of this fact by applying Fukushima's ergodic theorem to the Girsanov transformed process $\widetilde{\mathbb{M}}^\rho$ (Corollary 5.5).

Chapter 2

Preliminaries

Let E be a locally compact separable metric space and m a positive Radon measure with full topological support on E. Let $\mathbb{M} = (\Omega, \mathscr{F}_t, \theta_t, X_t, \mathbb{P}_x)$ be an m-symmetric Hunt process with a state space E. Here $\{\mathscr{F}_t\}_{t\geq 0}$ is the minimal (augmented) admissible filtration and θ_t, $t \geq 0$ is the shift operator satisfying $X_s(\theta_t) = X_{s+t}$ identically for $s, t \geq 0$. Let ∂ be a one point added to E so that $E_\partial := E \cup \{\partial\}$ is the one point compactification of E. The point ∂ also serves as the cemetery point for $\mathbb{M}$, that is, $\zeta := \inf\{t \geq 0 : X_t = \partial\}$ is the lifetime of $\mathbb{M}$. For each measure μ on E, we denote by $\mathbb{P}_\mu$ (resp., $\mathbb{E}_\mu$) the probability (resp., the expectation) of the process with initial distribution μ. For any $x \in E$, we simply write $\mathbb{P}_x$ and $\mathbb{E}_x$ for $\mathbb{P}_{\delta_x}$ and $\mathbb{E}_{\delta_x}$. We define the semigroup $\{P_t\}_{t\geq 0}$ by

$$P_t f(x) = \mathbb{E}_x[f(X_t); t < \zeta], \quad f \in \mathfrak{B}_b(E),$$

where $\mathfrak{B}_b(E)$ is the space of bounded Borel functions on E. By the right continuity of paths of $\mathbb{M}$, $\{P_t\}_{t>0}$ can be extended to an $L^2(E;m)$-strongly continuous semigroup ([18, Lemma 1.4.3]). Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the *Dirichlet form* on $L^2(E;m)$ generated by $\mathbb{M}$:

$$\begin{cases} \mathcal{D}(\mathcal{E}) = \left\{ u \in L^2(E;m) \,\middle|\, \lim_{t\downarrow 0} \dfrac{1}{t}(u - P_t u, u)_m < \infty \right\}, \\ \mathcal{E}(u,v) = \lim_{t\downarrow 0} \dfrac{1}{t}(u - P_t u, v)_m, \quad u, v \in \mathcal{D}(\mathcal{E}), \end{cases}$$

where $(\cdot,\cdot)_m$ denotes the inner product on $L^2(E;m)$. For any $\beta > 0$, set

$$\mathcal{E}_\beta(u,v) := \mathcal{E}(u,v) + \beta(u,v)_m, \quad u, v \in \mathcal{D}(\mathcal{E}).$$

Then $\mathcal{D}(\mathcal{E})$ becomes a Hilbert space with inner product $\mathcal{E}_\beta$ for any $\beta > 0$.

For a closed subset F of E, we define

$$\mathcal{D}(\mathcal{E})_F := \{u \in \mathcal{D}(\mathcal{E}) \mid u = 0 \ \ m\text{-a.e. on } \ E \setminus F\}.$$

An increasing sequence $\{F_n\}_{n\geq 1}$ of closed sets of E is said to be an $\mathcal{E}$*-nest* if $\bigcup_{n\geq 1}\mathcal{D}(\mathcal{E})_{F_n}$ is $\mathcal{E}_1$-dense in $\mathcal{D}(\mathcal{E})$. A subset N of E is said to be $\mathcal{E}$*-exceptional* if there is an $\mathcal{E}$-nest $\{F_n\}_{n\geq 1}$ such that $N\subset\bigcap_{n\geq 1}(E\setminus F_n)$. A statement depending on $x\in E$ is said to hold $\mathcal{E}$*-quasi-everywhere* ($\mathcal{E}$-q.e. in abbreviation) on E if there exists an $\mathcal{E}$-exceptional set N such that the statement is true for every $x\in E\setminus N$. A function u is said to be $\mathcal{E}$*-quasi-continuous* if there exists an $\mathcal{E}$-nest $\{F_n\}_{n\geq 1}$ such that $u|_{F_n}$ is finite and continuous on F_n for each n: we denote this situation briefly by writing $u\in C(\{F_n\})$. When we deal with a fixed Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$, for convenience we drop "$\mathcal{E}$-" from the terminology "$\mathcal{E}$-q.e." and "$\mathcal{E}$-quasi-continuous" and will simply call them q.e. and quasi-continuous, respectively.

Let $\mathcal{D}_e(\mathcal{E})$ be the family of m-measurable functions u on E such that $|u|<\infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of $\mathcal{D}(\mathcal{E})$ such that $\lim_{n\to\infty}u_n=u$ m-a.e. We call $\{u_n\}$ an approximating sequence for $u\in\mathcal{D}_e(\mathcal{E})$. For $u,v\in\mathcal{D}_e(\mathcal{E})$ and approximating sequences $\{u_n\}$, $\{v_n\}$, the limit $\mathcal{E}(u,v)=\lim_{n\to\infty}\mathcal{E}(u_n,v_n)$ exists and does not depend on the choices of the approximating sequences for u,v. We call $\mathcal{D}_e(\mathcal{E})$ the *extended Dirichlet space* of $(\mathcal{E},\mathcal{D}(\mathcal{E}))$. For $u,v\in\mathcal{D}_e(\mathcal{E})$, the following Beurling-Deny formula holds:

$$\begin{aligned}\mathcal{E}(u,v)=\mathcal{E}^{(c)}(u,v)&+\int_{E\times E\setminus d}(\widetilde{u}(x)-\widetilde{u}(y))(\widetilde{v}(x)-\widetilde{v}(y))J(dx,dy)\\&+\int_E\widetilde{u}(x)\widetilde{v}(x)\kappa(dx).\end{aligned}\tag{2.1}$$

Here $\widetilde{u}$ denotes a quasi-continuous m-version of u, that is, $u=\widetilde{u}$ m-a.e. Here $\mathcal{E}^{(c)}$ is a symmetric form possessing the strong local property, i.e., $\mathcal{E}^{(c)}(u,v)=0$ whenever u has a compact support and v is constant on a neighborhood of supp$[u]$. J is a symmetric Radon measure on $E\times E\setminus d$, where d denotes the diagonal set, and κ is a Radon measure on E (see [18, Theorem 4.5.2]). J and κ are called the *jumping measure* and the *killing measure* of $\mathbb{M}$, respectively. We define the family Θ of finely open sets by

$$\Theta=\left\{\{G_n\}\,|\,G_n \text{ is finely open for all } n,\ G_n\subset G_{n+1},\ \bigcup_{n=1}^{\infty}G_n=E\ \text{ q.e.}\right\}$$

(the definition of a finely open set can be found in [18]). A function u on E is said to be *locally in* $\mathcal{D}(\mathcal{E})$ *in the broad sense* ($u\in\dot{\mathcal{D}}_{\rm loc}(\mathcal{E})$ in notation) if there exist $\{G_n\}\in\Theta$ and $\{u_n\}\subset\mathcal{D}(\mathcal{E})$ such that $u=u_n$ m-a.e. on G_n for each $n\in\mathbb{N}$. It is shown in [24, Theorem 4.1] that $\mathcal{D}_e(\mathcal{E})\subset\dot{\mathcal{D}}_{\rm loc}(\mathcal{E})$ and $u\in\dot{\mathcal{D}}_{\rm loc}(\mathcal{E})$ admits a quasi-continuous m-version $\tilde{u}$. In the sequel, we always take a quasi-continuous m-version for every element of $\dot{\mathcal{D}}_{\rm loc}(\mathcal{E})$.

A positive Borel measure μ on E is said to be *smooth* if it satisfies the following two conditions:

(i) μ charges no $\mathcal{E}$-exceptional set,

(ii) there exists an $\mathcal{E}$-nest $\{F_n\}$ such that $\mu(F_n) < \infty$ for each n.

A stochastic process $A = \{A_t\}_{t\ge 0}$ is said to be an *additive functional* (AF in abbreviation) if it satisfies the following conditions:

(i) $A_t(\cdot)$ is $\mathscr{F}_t$-measurable for all $t \ge 0$,

(ii) there exists a set $\Lambda \in \mathscr{F}_\infty = \sigma\left(\bigcup_{t\ge 0}\mathscr{F}_t\right)$ such that $\mathbb{P}_x(\Lambda) = 1$ for q.e. $x \in E$, $\theta_t\Lambda \subset \Lambda$ for all $t > 0$, and for each $\omega \in \Lambda$, $A_\cdot(\omega)$ is a function satisfying: $A_0(\omega) = 0$, $A_t(\omega) < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \ge \zeta(\omega)$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \ge 0$.

An AF A is said to be a *continuous additive functional* (CAF in abbreviation) if $t \mapsto A_t(\omega)$ is continuous on $[0, \infty[$ for each $\omega \in \Lambda$. A $[0, \infty[$-valued CAF is called a *positive continuous additive functional* (PCAF in abbreviation). We call A an AF on $[\![0, \zeta[\![$ if A is $\{\mathscr{F}_t\}$-adapted and satisfies (i) and the property (ii)$'$ in which (ii) is modified so that additivity condition is required only for $t + s < \zeta$. From [5, Remark 2.2], any PCAF A on $[\![0, \zeta[\![$ can be extended to a PCAF by setting

$$A_t(\omega) := \begin{cases} \lim\limits_{s\uparrow\zeta} A_s(\omega), & \text{if } \ t \ge \zeta(\omega) > 0, \\ 0, & \text{if } \ t \ge \zeta(\omega) = 0. \end{cases}$$

The family of all smooth measures and the set of all PCAF's are in one-to-one correspondence as follows: for each smooth measure μ, there exists a unique PCAF $A = \{A_t\}_{t\ge 0}$ such that for any nonnegative Borel function f and γ-excessive function h ($\gamma \ge 0$), that is, $e^{-\gamma t}P_t h \le h$,

$$\lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}_{hm}\left[\int_0^t f(X_s)dA_s\right] = \int_E f(x)h(x)\mu(dx) \tag{2.2}$$

([18, Thorem 5.1.4]). Here $\mathbb{E}_{hm}[\,\cdot\,] = \int_E \mathbb{E}_x[\,\cdot\,]h(x)m(dx)$. We say that a smooth measure μ and an AF A are in the *Revuz correspondence* if they satisfy the relation (2.2). In this case, μ is called the *Revuz measure* of A and denoted by μ_A.

Let $(N, H) = (N(x, dy), H_t)$ be a Lévy system for $\mathbb{M}$; that is, $N(x, dy)$ is a kernel on $(E_\partial, \mathcal{B}(E_\partial))$ with $N(x, \{x\}) = 0$ and H is a PCAF of $\mathbb{M}$ such that for any nonnegative Borel function f on $E_\partial \times E_\partial$ vanishing on the diagonal and for any $x \in E_\partial$,

$$\mathbb{E}_x\left[\sum_{s\le t} f(X_{s-}, X_s)\right] = \mathbb{E}_x\left[\int_0^t \int_{E_\partial} f(X_s, y)N(X_s, dy)dH_s\right].$$

Let μ_H be the Revuz measure of the PCAF H. Then the jumping measure J and the killing measure κ of $\mathbb{M}$ are given by

$$J(dx, dy) = \frac{1}{2}N(x, dy)\mu_H(dx) \quad \text{and} \quad \kappa(dx) = N(x, \{\partial\})\mu_H(dx). \tag{2.3}$$

For an AF A, the *energy* of A is defined by

$$\mathbf{e}(A) := \lim_{t \downarrow 0} \frac{1}{2t}\mathbb{E}_m\left[A_t^2\right]$$

if the limit exists. We then define

$$\mathcal{M} := \left\{ M = \{M_t\}_{t \geq 0} \;\middle|\; \begin{array}{l} M \text{ is a finite AF, } \mathbb{E}_x[M_t^2] < \infty, \, \mathbb{E}_x[M_t] = 0 \\ \text{for q.e. } x \in E \text{ and all } t \geq 0, \end{array} \right\},$$
$$\mathring{\mathcal{M}} := \{M \in \mathcal{M} \mid \mathbf{e}(M) < \infty\},$$
$$\mathcal{N}_c := \left\{ N = \{N_t\}_{t \geq 0} \;\middle|\; \begin{array}{l} N \text{ is a CAF, } \mathbb{E}_x\big[|N_t|\big] < \infty \text{ q.e. } x \in E \\ \text{for each } t \geq 0, \text{ and } \mathbf{e}(N) = 0 \end{array} \right\}.$$

An element of $\mathring{\mathcal{M}}$ is called a *martingale additive functional* (MAF in abbreviation) *of finite energy* and an element of $\mathcal{N}_c$ is called a *continuous additive functional* (CAF in abbreviation) *of zero energy*. For $M \in \mathcal{M}$, there exists a unique PCAF $\langle M \rangle$ such that $M^2 - \langle M \rangle$ is an MAF. $\langle M \rangle$ is called the *sharp bracket* of M. Let M^c be the continuous part of $M \in \mathcal{M}$ and define the *square bracket* $[M]$ by

$$[M]_t := \langle M^c \rangle_t + \sum_{s \leq t} \Delta M_s^2,$$

where $\Delta M_s := M_s - M_{s-}$. Then $[M]^p = \langle M \rangle$. Here for an AF A of integrable variation, A^p denotes *the dual predictable projection* of A so that $A - A^p$ is an MAF (see [18, section A.3.3]). For $L, M \in \mathcal{M}$, we put

$$\langle L, M \rangle := \frac{1}{2}\big(\langle L, M \rangle - \langle L \rangle - \langle M \rangle\big),$$
$$[L, M] := \frac{1}{2}\big([L, M] - [L] - [M]\big).$$

We set

$$\mathring{\mathcal{M}}_{\text{loc}} := \left\{ \{M_t\}_{t \geq 0} \;\middle|\; \begin{array}{l} \text{there exist } \{G_n\} \in \Theta \text{ and } \{M^{(n)}\} \subset \mathring{\mathcal{M}} \text{ such that} \\ M_t = M_t^{(n)} \text{ for all } t < \tau_{G_n} \text{ and } n \in \mathbb{N}, \mathbb{P}_x\text{-a.s. q.e. } x \end{array} \right\}.$$

Here $\tau_{G_n} := \inf\{t > 0 : X_t \notin G_n\}$ and $\lim_{n\to\infty} \tau_{G_n} = \zeta$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$ by [18, Lemma 5.5.2]. The space $\mathcal{N}_{c,\mathrm{loc}}$ is defined similarly. An element of $\mathring{\mathcal{M}}_{\mathrm{loc}}$ is called an *MAF locally of finite energy* and an element of $\mathcal{N}_{c,\mathrm{loc}}$ is called a *CAF locally of zero energy*. For every $M \in \mathring{\mathcal{M}}_{\mathrm{loc}}$, its sharp bracket process $\langle M \rangle$ can be defined to be a PCAF by setting

$$\langle M\rangle_t := \begin{cases} \langle M^{(n)}\rangle_t, & \text{if } t < \tau_{G_n}, \\ \lim_{s\uparrow\zeta}\langle M\rangle_s, & \text{if } t \geq \zeta \end{cases}$$

([5, Proposition 2.8]).

We introduce the subclass $\dot{\mathcal{D}}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$ of $\dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E})$ as follows:

$$\dot{\mathcal{D}}^{\dagger}_{\mathrm{loc}}(\mathcal{E}) := \left\{ u \in \dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E}) \;\middle|\; \int_{y\in E} (u(y)-u(x))^2 J(dx,dy) \text{ is a smooth measure} \right\}.$$

By [5, Remark 3.9], we see $\mathcal{D}_e(\mathcal{E}) \cup \big(\dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E})\big)_b \subset \dot{\mathcal{D}}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$. Here $\big(\dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E})\big)_b := \{u \in \dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E}) \mid u \text{ is bounded}\}$.

Remark 2.1. For any $u \in \dot{\mathcal{D}}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$, there exists an $\mathcal{E}$-nest $\{F_n\}$ of compact sets such that $u \in C(\{F_n\})$ and

$$\int_{F_n\times E} (u(x)-u(y))^2 J(dx,dy) < \infty \tag{2.4}$$

for each n. Then we can define $\mathcal{E}(u,v)$ by

$$\begin{aligned}\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \int_{E\times E\setminus d} (u(x)-u(y))(v(x)-v(y))J(dx,dy) \\ + \int_E \big(u(x)-u(\partial)\big)v(x)\kappa(dx)\end{aligned}$$

for any $v \in \bigcup_{n\geq 1} \mathcal{D}(\mathcal{E})_{F_n}$, To see this, we have only to check the jumping part is finite, that is,

$$\int_{E\times E} (u(x)-u(y))(v(x)-v(y))J(dx,dy) < \infty.$$

For $v \in \mathcal{D}(\mathcal{E})_{F_n}$, the left-hand side is decomposed as

$$\begin{aligned}&\int_{F_n\times E} (u(x)-u(y))(v(x)-v(y))J(dx,dy) \\ &\qquad + \int_{F_n\times F_n^c} (u(x)-u(y))(v(x)-v(y))J(dx,dy).\end{aligned}$$

By Schwarz's inequality and (2.4), the integrals are finite.

We see from [18, Theorem 5.2.2] and [26, Theorem 1.2] that for $u \in \mathcal{D}_e(\mathcal{E})$ (resp. $u \in \dot{\mathcal{D}}^{\dagger}_{\rm loc}(\mathcal{E})$), the additive functional $u(X_t) - u(X_0)$ admits the following Fukushima decomposition:

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \quad \text{for } t \in [0,\infty[\text{ (resp. } t \in [0,\zeta[\,), \tag{2.5}$$

where $M^{[u]} \in \overset{\circ}{\mathcal{M}}$ and $N^{[u]} \in \mathcal{N}_c$ (resp. $M^{[u]} \in \overset{\circ}{\mathcal{M}}_{\rm loc}$ and $N^{[u]} \in \mathcal{N}_{c,{\rm loc}}$). Moreover, for $u \in \mathcal{D}_e(\mathcal{E})$ (or $u \in \dot{\mathcal{D}}^{\dagger}_{\rm loc}(\mathcal{E})$), $M^{[u]}$ can be decomposed as

$$M^{[u]} = M^{[u],c} + M^{[u],j} + M^{[u],k},$$

where $M^{[u],c}$, $M^{[u],j}$ and $M^{[u],k}$ are the continuous, jumping and killing parts of martingale $M^{[u]}$. $M^{[u],j}$ and $M^{[u],k}$ are defined by

$$\begin{aligned} M_t^{[u],j} &= \lim_{\varepsilon\downarrow 0}\Bigg\{\sum_{0<s\le t}(u(X_s)-u(X_{s-}))\mathbb{1}_{\{|u(X_s)-u(X_{s-})|>\varepsilon\}}\mathbb{1}_{\{s<\zeta\}} \\ &\qquad - \int_0^t\left(\int_{\{y\in E\,:\,|u(y)-u(X_s)|>\varepsilon\}}(u(y)-u(X_s))N(X_s,dy)\right)dH_s\Bigg\}, \\ M_t^{[u],k} &= \int_0^t u(X_s)N(X_s,\{\partial\})dH_s - u(X_{\zeta-})\mathbb{1}_{\{t\ge\zeta\}}. \end{aligned}$$

Let $\mu_{\langle u\rangle}$, $\mu^c_{\langle u\rangle}$, $\mu^j_{\langle u\rangle}$ and $\mu^k_{\langle u\rangle}$ be the smooth Revuz measures associated with the PCAF's $\langle M^{[u]}\rangle$, $\langle M^{[u],c}\rangle$, $\langle M^{[u],j}\rangle$ and $\langle M^{[u],k}\rangle$, respectively. Then

$$\mu_{\langle u\rangle} = \mu^c_{\langle u\rangle} + \mu^j_{\langle u\rangle} + \mu^k_{\langle u\rangle}$$

and

$$\mu^j_{\langle u\rangle}(dx) = 2\int_{y\in E}(u(x)-u(y))^2 J(dx,dy), \quad \text{and} \quad \mu^k_{\langle u\rangle}(dx) = u(x)^2\kappa(dx). \tag{2.6}$$

For $t > 0$, let r_t denote the time-reversal operator on the path space Ω as follows: for $\omega \in \{t < \zeta\}$,

$$r_t(w)(s) := \begin{cases} \omega\left((t-s)-\right), & \text{if } 0 \le s < t, \\ \omega(0), & \text{if } s \ge t. \end{cases}$$

Here $\omega(r-) := \lim_{s\uparrow r}\omega(s)$ for $r > 0$. The symmetry of $\mathbb{M}$ implies that the restriction of the measure $\mathbb{P}_m$ to $\mathscr{F}_t$ is invariant under r_t on $\Omega \cap \{t < \zeta\}$, that is, for every nonnegative random valuable $\xi \in \mathscr{F}_t$,

$$\mathbb{E}_m[\,\xi\,; t<\zeta] = \mathbb{E}_m[\,\xi\circ r_t\,; t<\zeta]. \tag{2.7}$$

An additive functional A_t is said to be *even* if $A_t \circ r_t = A_t$ $\mathbb{P}_m$-a.s. on $\{t < \zeta\}$ for each $t > 0$. From [12], CAFs of bounded variation (or of zero energy) are even (although it was proved in [12] for symmetric diffusion processes, the proof works for general symmetric Markov processes).

For $u \in \dot{\mathcal{D}}^{\dagger}_{\rm loc}(\mathcal{E})$, $u(X_t) - u(X_0)$ has Fukushima's decomposition:

$$u(X_t) - u(X_0) = M^{[u]}_t + N^{[u]}_t, \quad t \in [0, \zeta[.$$

By the definition of $\dot{\mathcal{D}}_{\rm loc}(\mathcal{E})$, there exist $\{G_n\} \in \Theta$ and $\{u_n\} \subset \mathcal{D}(\mathcal{E})$ such that $u = u_n$ m-a.e. on G_n for each $n \in \mathbb{N}$. Then we have for $t \in [0, \tau_{G_n}[$,

$$u(X_t) - u(X_0) = u_n(X_t) - u_n(X_0) = M^{[u_n]}_t + N^{[u_n]}_t$$

$\mathbb{P}_x$-a.s. for q.e. $x \in E$. By the uniqueness of the decomposition,

$$M^{[u]}_t = M^{[u_n]}_t \ \text{ and } \ N^{[u]}_t = N^{[u_n]}_t, \quad t < \tau_{G_n}, \ \mathbb{P}_x\text{-a.s. for q.e. } x \in E.$$

Hence, by the calculation similar to that in the proof of [18, Theorem 5.7.1], we can show that

Lemma 2.2. *For any* $u \in \dot{\mathcal{D}}^{\dagger}_{\rm loc}(\mathcal{E})$ *and* $T > 0$, $\mathbb{P}_m$-a.s. *on* $\{T < \zeta\}$

$$N^{[u]}_t(r_T) = N^{[u]}_T - N^{[u]}_{T-t} \quad \textit{for} \ \ t \in [0, T].$$

In particular, $N^{[u]}$ *is even.*

2.1 CAF's locally of zero energy

An AF $\{A_t\}_{t\geq 0}$ is said to be *of bounded variation* if A_t can be expressed as a difference of two PCAF's:

$$A_t = A^{(1)}_t - A^{(2)}_t, \quad t < \zeta.$$

A sufficient condition for $N^{[u]}_t$ in (2.5) being of bounded variation is given in [18, §5]. Our first aim in this section is to extend it and this result is used in Chapter 4.

We say that a function u is *locally in* $\mathcal{D}(\mathcal{E})$ ($u \in \mathcal{D}_{\rm loc}(\mathcal{E})$ in notation) if for any relatively compact open set $D \subset E$, there exists a function $v \in \mathcal{D}(\mathcal{E})$ such that $u = v$ m-a.e. on D. For $u \in \mathcal{D}_{\rm loc}(\mathcal{E})$ and a Borel set B, define

$$\mu^j_{\langle u\rangle}(B) := \int_{B\times E} (u(x) - u(y))^2 J(dx, dy).$$

Note that $\mu^j_{\langle u\rangle}$ is not necessarily a Radon measure. We introduce a subclass $\mathcal{D}^\dagger_{\rm loc}(\mathcal{E})$ of $\mathcal{D}_{\rm loc}(\mathcal{E})$:

$$\mathcal{D}^\dagger_{\rm loc}(\mathcal{E}) := \{u \in \mathcal{D}_{\rm loc}(\mathcal{E}) \mid \mu^j_{\langle u\rangle} \text{ is a Radon measure on } E\}.$$

It is noted in [25] that $\mathcal{D}(\mathcal{E}) \cup \big(\mathcal{D}_{\rm loc}(\mathcal{E})\big)_b \subset \mathcal{D}^\dagger_{\rm loc}(\mathcal{E})$, where $\big(\mathcal{D}_{\rm loc}(\mathcal{E})\big)_b = \mathcal{D}_{\rm loc}(\mathcal{E}) \cap \mathfrak{B}_b(E)$. For $u \in \mathcal{D}^\dagger_{\rm loc}(\mathcal{E})$ and $\varphi \in \mathcal{D}(\mathcal{E})$ with compact support ,

$$\mathcal{E}(u,\varphi) = \frac{1}{2}\int_E d\mu^c_{\langle u,\varphi\rangle} + \int_{E\times E} (u(x)-u(y))(\varphi(x)-\varphi(y))J(dx,dy) + \int_E u\varphi\, d\kappa$$

is well-defined ([14, Theorem 3.5]).

Recall that for a closed subset F of E, $\mathcal{D}(\mathcal{E})_F$ is the space defined by

$$\mathcal{D}(\mathcal{E})_F = \{u \in \mathcal{D}(\mathcal{E}) \mid u = 0 \text{ q.e. on } E \setminus F\}.$$

The spaces $\mathcal{D}_e(\mathcal{E})_F$ and $\mathcal{D}_b(\mathcal{E})_F$ are defined similarly, where $\mathcal{D}_b(\mathcal{E})$ is the set of bounded functions in $\mathcal{D}(\mathcal{E})$. For $f \in \mathfrak{B}_b(E)$ and a Borel set $A \subset E$, define

$$H_A f(x) := \mathbb{E}_x[f(X_{\sigma_A})\,;\, \sigma_A < \infty].$$

Lemma 2.3. *Let $u \in \mathcal{D}^\dagger_{\rm loc}(\mathcal{E})$ and F a compact set. It holds that*
(i) $u - H_{F^c}u \in \mathcal{D}_e(\mathcal{E})_F$ *and*

$$\begin{aligned}\mathcal{E}(u - H_{F^c}u, u - H_{F^c}u) \le \frac{1}{2}\mu^c_{\langle u\rangle}(F) &+ \int_{F\times F}(u(x)-u(y))^2 J(dx,dy)\\ &+ 2\int_{F\times F^c}(u(x)-u(y))^2 J(dx,dy) + \int_F u^2\, d\kappa.\end{aligned} \tag{2.8}$$

(ii) $H_{F^c}u \in \dot{\mathcal{D}}^\dagger_{\rm loc}(\mathcal{E})$ *and* $\mathcal{E}(H_{F^c}u, v) = 0$ *for any* $v \in \mathcal{D}_b(\mathcal{E})_F$.

Proof. The proof is similar to that of [7, Lemma 6.2.10]. Note that $H_{F^c}u = u$ q.e. on $E \setminus F$.

First suppose that $u \in \mathcal{D}_e(\mathcal{E})$. Then by [18, Lemma 4.6.5], $H_{F^c}u \in \mathcal{D}_e(\mathcal{E})$ and $\mathcal{E}(H_{F^c}u, v) = 0$ for all $v \in \mathcal{D}_e(\mathcal{E})_F$. Hence,

$$\mathcal{E}(u - H_{F^c}u, u - H_{F^c}u) = \mathcal{E}(u,u) - \mathcal{E}(H_{F^c}u, H_{F^c}u).$$

Since

$$\begin{aligned}\mathcal{E}(H_{F^c}u, H_{F^c}u) &\ge \frac{1}{2}\mu^c_{\langle H_{F^c}u\rangle}(F^c) + \int_{F^c\times F^c}\big(H_{F^c}u(x) - H_{F^c}u(y)\big)^2 J(dx,dy)\\ &\quad + \int_{F^c}\big(H_{F^c}u\big)^2\, d\kappa\\ &= \frac{1}{2}\mu^c_{\langle u\rangle}(F^c) + \int_{F^c\times F^c}(u(x)-u(y))^2 J(dx,dy) + \int_{F^c} u^2\, d\kappa,\end{aligned}$$

we have (2.8).

Suppose next that $u \in \big(\mathcal{D}_{\rm loc}(\mathcal{E})\big)_b = \mathcal{D}_{\rm loc}(\mathcal{E}) \cap \mathfrak{B}_b(E)$. Take an increasing sequence of relatively compact open sets $\{D_k\}$ with $\bigcup_{k=1}^{\infty} D_k = E$ and $F \subset D_k$ for each k. Then there exists $\{g_k\} \in \mathcal{D}(\mathcal{E})$ such that $u = g_k$ q.e. on D_k. We may assume $|g_k(x)| \leq \|u\|_\infty$. By applying (2.8) to g_k, we have

$$\begin{aligned}\mathcal{E}(g_k - H_{F^c}g_k, g_k - H_{F^c}g_k) &\leq \frac{1}{2}\,\mu^c_{\langle g_k\rangle}(F) + \int_{F\times F}(g_k(x)-g_k(y))^2 J(dx,dy) \\ &\quad + 2\int_{F\times F^c}(g_k(x)-g_k(y))^2 J(dx,dy) + \int_F g_k^2\,d\kappa \\ &= \frac{1}{2}\,\mu^c_{\langle u\rangle}(F) + \int_{F\times F}(u(x)-u(y))^2 J(dx,dy) \\ &\quad + 2\int_{F\times(F^c\cap D_1)}(u(x)-u(y))^2 J(dx,dy) \\ &\quad + 2\int_{F\times(F^c\cap D_1^c)}(u(x)-g_k(y))^2 J(dx,dy) + \int_F u^2\,d\kappa.\end{aligned}$$

Since $J(F\times D_1^c) < \infty$,

$$\int_{F\times(F^c\cap D_1^c)}(u(x)-g_k(y))^2 J(dx,dy) \to \int_{F\times(F^c\cap D_1^c)}(u(x)-u(y))^2 J(dx,dy)$$

as $k\to\infty$ by the dominated convergence theorem. Therefore we have

$$\begin{aligned}\limsup_{k\to\infty}\mathcal{E}(g_k - H_{F^c}g_k, g_k - H_{F^c}g_k) &\leq \frac{1}{2}\,\mu^c_{\langle u\rangle}(F) + \int_{F\times F}(u(x)-u(y))^2 J(dx,dy) \\ &\quad + 2\int_{F\times F^c}(u(x)-u(y))^2 J(dx,dy) + \int_F u^2\,d\kappa.\end{aligned}$$

Since the right-hand side is finite, we see from the Banach-Saks theorem ([7, Theorem A.4.1]) that there exists a subsequence $\{g_{k_j}\}_{j\geq 1}$ such that $\psi_j := \frac{1}{j}\sum_{\ell=1}^{j}\big(g_{k_\ell} - H_{F^c}g_{k_\ell}\big)$ is an $\mathcal{E}$-Cauchy sequence. Noting that $\|g_k\|_\infty \leq \|u\|_\infty$ and $g_k \to u$ q.e., we see $\psi_j \to u - H_{F^c}u$ q.e. Hence $u - H_{F^c}u$ belongs to $\mathcal{D}_e(\mathcal{E})_F \cap \mathfrak{B}_b(E) = \mathcal{D}_b(\mathcal{E})_F$ and satisfies the inequality (2.8) because

$$\mathcal{E}(u - H_{F^c}u, u - H_{F^c}u) = \lim_{j\to\infty}\mathcal{E}(\psi_j,\psi_j) \leq \limsup_{k\to\infty}\mathcal{E}(g_k - H_{F^c}g_k, g_k - H_{F^c}g_k).$$

We next show (ii). For the subsequence $\{g_{k_j}\}_{j\geq 1}$ above, we put $\overline{g}_j := \frac{1}{j}\sum_{\ell=1}^{j} g_{k_\ell}$. Then it holds that for $v \in \mathcal{D}_b(\mathcal{E})_F$

$$0 = \mathcal{E}(H_{F^c}\overline{g}_j, v) = \mathcal{E}(\overline{g}_j, v) - \mathcal{E}(\psi_j, v). \tag{2.9}$$

Since $\overline{g}_j = u$ q.e. on $D_1 \supset F$, we have

$$\begin{aligned}\mathcal{E}(\overline{g}_j, v) &= \frac{1}{2}\mu^c_{\langle f,u\rangle}(E) + \int_{F\times F}(u(x)-u(y))(v(x)-v(y))J(dx,dy)\\ &\quad + 2\int_{F\times(F^c\cap D_1)}(u(x)-u(y))(v(x)-v(y))J(dx,dy)\\ &\quad + 2\int_{F\times(F^c\cap D_1^c)}(u(x)-\overline{g}_j(y))(v(x)-v(y))J(dx,dy) + \int_E uv\,d\kappa.\end{aligned}$$

Thus $\lim_{j\to\infty}\mathcal{E}(\overline{g}_j, v) = \mathcal{E}(u,v)$ by the dominated convergence theorem. Therefore, by letting $j\to\infty$ in (2.9), we have

$$0 = \mathcal{E}(u,v) - \mathcal{E}(u - H_{F^c}u, v) = \mathcal{E}(H_{F^c}u, v).$$

We finally treat the general case that u belongs to $\mathcal{D}^\dagger_{\rm loc}(\mathcal{E})$. By considering a decomposition $u = (u\vee 0) - ((-u)\vee 0)$, we may assume that u is nonnegative. Put $u_k := u\wedge k$. Then u_k is a normal contraction of u and $H_{F^c}u_k$ tends to $H_{F^c}u$ as $k\to\infty$ by the monotone convergence theorem. Applying the result in the last paragraph to u_k, we see that $u_k - H_{F^c}u_k \in \mathcal{D}_b(\mathcal{E})_F$ and

$$\begin{aligned}\mathcal{E}(u_k - H_{F^c}u_k, u_k - H_{F^c}u_k) &\le \frac{1}{2}\,\mu^c_{\langle u\rangle}(F) + \int_{F\times F}(u(x)-u(y))^2 J(dx,dy)\\ &\quad + 2\int_{F\times F^c}(u(x)-u(y))^2 J(dx,dy) + \int_F u^2\,d\kappa.\end{aligned}$$

Hence, by repeating the argument above, we can prove the lemma. □

On account of Lemma 2.3, we see that for any $u\in\mathcal{D}^\dagger_{\rm loc}(\mathcal{E})$ and compact set F, $H_{F^c}u(X_t) - H_{F^c}u(X_0)$ has Fukushima's decomposition:

$$H_{F^c}u(X_t) - H_{F^c}u(X_0) = M_t^{[H_{F^c}u]} + N_t^{[H_{F^c}u]}, \quad t<\zeta.$$

Lemma 2.4. *Let F be a compact set. Then for any $u\in\mathcal{D}^\dagger_{\rm loc}(\mathcal{E})$,*

$$\mathbb{P}_x(N_t^{[H_{F^c}u]} = 0,\ t<\tau_F) = 1 \quad \text{q.e. } x\in E.$$

Proof. This lemma can be shown by the argument similar to that in [7, Lemma 5.5.5].

$(F^c)^r$ denotes the set of all regular points of F^c. Since $F^c\setminus(F^c)^r$ is semi-polar by [18, Theorem A.2.6], we can choose a properly exceptional set $N\supset F^c\setminus(F^c)^r$ by [18, Theorem 4.1.3, Theorem 4.1.1]. Then it follows that $X_{\tau_F} \in (F^c)^r \cup \{\partial\}$ and $\tau_F \circ \theta_{\tau_F} = 0$ $\mathbb{P}_x$-a.s. for $x \in E \setminus N$. Hence, by the strong Markov property,

$$\begin{aligned} H_{F^c}u(X_{t\wedge\tau_F}) = \mathbb{E}_{X_{t\wedge\tau_F}}\left[u(X_{\tau_F})\right] &= \mathbb{E}_x\left[u\big(X_{\tau_F(\theta_{t\wedge\tau_F})} \circ \theta_{t\wedge\tau_F}\big) | \mathscr{F}_{t\wedge\tau_F}\right] \\ &= \mathbb{E}_x\left[u(X_{\tau_F}) | \mathscr{F}_{t\wedge\tau_F}\right] \quad \mathbb{P}_x\text{-a.s., } x \in E \setminus N, \end{aligned}$$

namely, $H_{F^c}u(X_{t\wedge\tau_F}) - H_{F^c}u(X_0)$ is a martingale relative to $\{\mathscr{F}_{t\wedge\tau_F}\}_{t\ge0}$ under $\mathbb{P}_x$ for $x \in E \setminus N$.

Let $C_t := H_{F^c}u(X_{t\wedge\tau_F}) - H_{F^c}u(X_0) - M^{[H_{F^c}u]}_{t\wedge\tau_F}$. Then $C_t = N^{[H_{F^c}u]}_{t\wedge\tau_F}$ and $\{C_t\}_{t\ge0}$ is a local martingale relative to $\{\mathscr{F}_{t\wedge\tau_F}\}_{t\ge0}$ under $\mathbb{P}_x$ for q.e. $x \in E$. Since $H_{F^c}u \in \dot{\mathcal{D}}^\dagger_{\mathrm{loc}}(\mathcal{E})$, there exist a sequence $\{G_n\} \in \Theta$ and a sequence $\{v_n\} \subset \mathcal{D}(\mathcal{E})$ such that $H_{F^c}u = v_n$ q.e. on G_n. Then by the uniqueness of decomposition,

$$\mathbb{P}_x(C_t = N^{[v_n]}_t,\ t < \tau_F \wedge \tau_{G_n}) = 1, \quad \text{q.e. } x \in E.$$

Since $N^{[v_n]}$ has zero energy, we have for each fixed $t > 0$,

$$\begin{aligned} \mathbb{E}_{\mathbb{1}_F\cdot m}\left[\langle C\rangle_t\,; t < \tau_F \wedge \tau_{G_n}\right] &= \mathbb{E}_{\mathbb{1}_F\cdot m}\left[\lim_{k\to\infty}\sum_{j=1}^k \left(N^{[v_n]}_{jt/k} - N^{[v_n]}_{(j-1)t/k}\right)^2 ; t < \tau_F \wedge \tau_{G_n}\right] \\ &\le \lim_{k\to\infty} \mathbb{E}_m\left[\sum_{j=1}^k \left(N^{[v_n]}_{jt/k} - N^{[v_n]}_{(j-1)t/k}\right)^2\right] = 0. \end{aligned}$$

Hence, by letting $n \to \infty$, we see that $\langle C\rangle_t = 0$ $\mathbb{P}_{\mathbb{1}_F\cdot m}$-a.e. on $\{t < \tau_F\}$ for every $t > 0$. Thus on $\{t < \tau_F\}$, $C_t = 0$, namely, $N^{[H_{F^c}u]}_t = 0$. □

Theorem 2.5. *Let $\nu = \nu^{(1)} - \nu^{(2)}$ be a difference of positive smooth measures on E. If $u \in \mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E})$ satisfies*

$$\mathcal{E}(u, v) = \int_E v\, d\nu, \quad \textit{for all } v \in \bigcup_{k=1}^\infty \mathcal{D}_b(\mathcal{E})_{F_k} \tag{2.10}$$

for an $\mathcal{E}$-nest $\{F_k\}$ of compact sets associated with ν, then

$$\mathbb{P}_x(N^{[u]} = -A^{(1)} + A^{(2)} \textit{ on } [0,\zeta)) = 1 \quad \text{q.e. } x \in E,$$

where $A^{(i)}$ is a PCAF *with Revuz measure $\nu^{(i)}$, $i = 1, 2$.*

Proof. If $u \in \mathcal{D}^\dagger_{\mathrm{loc}}(\mathcal{E})$ satisfies the equation (2.10), then for each k,

$$\mathcal{E}(u - H_{F_k^c}u, v) = \int_E v\, d\nu, \quad \text{for all } v \in \mathcal{D}_b(\mathcal{E})_{F_k}$$

by Lemma 2.3 (ii). Note that $u - H_{F_k^c} u \in \mathcal{D}_e(\mathcal{E})_{F_k}$ by Lemma 2.3 (i). By applying [18, Lemma 5,4,4] and Lemma 2.4, we have

$$\mathbb{P}_x(N_t^{[u]} = -A_t^{(1)} + A_t^{(2)},\ t < \tau_{F_k}) = 1,\ \text{q.e. } x \in E.$$

Therefore, we have the assertion by letting $k \to \infty$. □

By the same argument as in the proof of [18, Corollary 5.4.1], we have the next corollary.

Corollary 2.6. *Let $\nu = \nu^{(1)} - \nu^{(2)}$ be a difference of positive smooth measures on E. Suppose $u \in \mathcal{D}^{\dagger}_{\mathrm{loc}}(\mathcal{E})$ satisfies*

$$\mathcal{E}(u, v) = \int_E v\, d\nu \quad \textit{for all } v \in \mathcal{D}(\mathcal{E}) \cap C_0(E),$$

where $C_0(E) := \{u \in C(E) \mid \mathrm{supp}[u]$ is compact$\}$. Then

$$\mathbb{P}_x(N^{[u]} = -A^{(1)} + A^{(2)} \textit{ on } [0, \zeta)) = 1 \quad \text{q.e. } x \in E,$$

where $A^{(i)}$ is a PCAF *with Revuz measure $\nu^{(i)}$, $i = 1, 2$.*

Chapter 3

Girsanov transformations

3.1 Girsanov's transformed processes

An increasing sequence $\{F_n\}$ of closed sets of E is said to be a *strict $\mathcal{E}$-nest* if

$$\lim_{n\to\infty} \mathrm{Cap}_{1,G_1\varphi}(E \setminus F_n) = 0,$$

where $\mathrm{Cap}_{1,G_1\varphi}$ is the weighted capacity defined in [28, Chapter V, Definition 2.1] and a family $\{F_n\}$ of closed sets is a strict $\mathcal{E}$-nest if and only if

$$\mathbb{P}_x(\lim_{n\to\infty} \sigma_{E\setminus F_n} < \infty) = 0 \quad \text{q.e. } x \in E$$

in view of [28, Chapter V, Proposition 2.6]. A function u defined on E_∂ is said to be *strictly $\mathcal{E}$-quasi-continuous* if there exists a strict $\mathcal{E}$-nest $\{F_n\}$ such that u is continuous on each $F_n \cup \{\partial\}$. Denote by $QC(E_\partial)$ the totality of strictly $\mathcal{E}$-quasi-continuous functions on E_∂.

Throughout this chapter, we assume that ρ is a nonnegative function in $\dot{\mathcal{D}}^{\dagger}_{\mathrm{loc}}(\mathcal{E}) \cap QC(E_\partial)$ such that $m(\{\rho > 0\}) > 0$ and $0 \le \rho(\partial) < \infty$. Set

$$N := \{x \in E \,|\, \rho(x) = 0 \ \text{ or } \ \rho(x) = \infty\}$$

and define a stopping time σ_N by $\sigma_N := \inf\{t > 0 \,|\, X_t \in N\}$. From Fukushima's decomposition,

$$\rho(X_t) - \rho(X_0) = M^{[\rho]}_t + N^{[\rho]}_t, \quad t \in [0,\zeta), \mathbb{P}_x\text{-a.s. for q.e. } x \in E,$$

where $M^{[\rho]}$ is an MAF locally of finite energy and $N^{[\rho]}$ is a CAF locally of zero energy. Define a local martingale M on the random interval $[\![0, \sigma_N \wedge \zeta[\![$ by

$$M_t := \int_0^t \frac{1}{\rho(X_{s-})} dM^{[\rho]}_s. \tag{3.1}$$

Note that

$$\Delta M_t = \frac{1}{\rho(X_{t-})}(M_t^{[\rho]} - M_{t-}^{[\rho]}) = \frac{1}{\rho(X_{t-})}(\rho(X_t) - \rho(X_{t-}))$$
$$= \frac{\rho(X_t)}{\rho(X_{t-})} - 1. \tag{3.2}$$

Let L_t^ρ be the Doléans-Dade exponential of M_t, that is, the unique solution of

$$L_t^\rho = 1 + \int_0^t L_{s-}^\rho dM_s, \quad \mathbb{P}_x\text{-a.s.}, \ x \in E \setminus N. \tag{3.3}$$

It is known from the Doléans-Dade formula ([20, Theorem 9.39]) that for $t < \sigma_N \wedge \zeta$

$$\begin{aligned} L_t^\rho &= \exp\left(M_t - \frac{1}{2}\langle M^c\rangle_t\right) \prod_{0<s\le t} (1+\Delta M_s)e^{-\Delta M_s} \\ &= \exp\left(M_t - \frac{1}{2}\langle M^c\rangle_t\right) \prod_{0<s\le t} \frac{\rho(X_s)}{\rho(X_{s-})} \exp\left(1 - \frac{\rho(X_s)}{\rho(X_{s-})}\right). \end{aligned} \tag{3.4}$$

Since L_t^ρ is a positive local martingale on the random interval $[\![0, \sigma_N \wedge \zeta[\![$, so is a positive supermartingale. Consequently, the formula

$$d\widetilde{\mathbb{P}}_x = L_t^\rho d\mathbb{P}_x \qquad \text{on } \mathscr{F}_t \cap \{t < \sigma_N \wedge \zeta\} \text{ for } x \in E \setminus N, \tag{3.5}$$

uniquely determines a family of probability measures on $(\Omega, \mathscr{F})$. It follows from [35, (62.19)] that under these new measures, $\{X_t\}$ is a right Markov process on the finely open set $E \setminus N$. We denote by $\widetilde{\mathbb{M}}^\rho := (\Omega, \mathscr{F}_t, \widetilde{X}_t, \widetilde{\mathbb{P}}_x, \widetilde{\zeta})$ the transformed process of $\mathbb{M}$ by L_t^ρ. Here for $\omega \in \Omega$,

$$\widetilde{X}_t(\omega) := \begin{cases} X_t(\omega), & 0 \le t < \sigma_N, \\ \partial, & \sigma_N \le t \le \infty, \end{cases} \qquad \widetilde{\zeta}(\omega) := \sigma_N(\omega) \wedge \zeta(\omega).$$

The semigroup $\{\widetilde{P}_t\}$ of $\widetilde{\mathbb{M}}^\rho$ equals

$$\widetilde{P}_t f(x) = \widetilde{\mathbb{E}}_x\big[f(\widetilde{X}_t) : t < \widetilde{\zeta}\big] = \mathbb{E}_x[L_t^\rho f(X_t)\,;\, t < \sigma_N \wedge \zeta]. \tag{3.6}$$

We introduce the space $\dot{\mathcal{D}}^{++}_{\rm loc}(\mathcal{E})$ defined by

$$\dot{\mathcal{D}}^{++}_{\rm loc}(\mathcal{E}) := \left\{ u \in \dot{\mathcal{D}}_{\rm loc}(\mathcal{E}) \;\middle|\; \begin{array}{l} \text{there exists a constant } a \in (1,\infty) \\ \text{such that } a^{-1} \le u \le a \end{array} \right\}. \tag{3.7}$$

Since each element of $\dot{\mathcal{D}}^{++}_{\rm loc}(\mathcal{E})$ is bounded, we see $\dot{\mathcal{D}}^{++}_{\rm loc}(\mathcal{E}) \subset \dot{\mathcal{D}}^{\dagger}_{\rm loc}(\mathcal{E})$.

Lemma 3.1. *The operator* $\widetilde{P}_t$ *defined by* (3.6) *is symmetric on* $L^2(E \setminus N; \rho^2 m)$.

Proof. For $f, g \in \mathfrak{B}_b^+(E)$, we have by the time reversal property (2.7)

$$\begin{aligned}(\widetilde{P}_t f, g)_{\rho^2 m} &= \mathbb{E}_m[L_t^\rho\, f(X_t)g(X_0)\rho(X_0)^2\,;\, t < \sigma_N \wedge \zeta] \\ &= \mathbb{E}_m[L_t^\rho \circ r_t\, f(X_0)g(X_t)\rho(X_t)^2\,;\, t < \sigma_N \wedge \zeta].\end{aligned}$$

For the proof of symmetry,

$$(\widetilde{P}_t f, g)_{\rho^2 m} = (f, \widetilde{P}_t g)_{\rho^2 m} = \mathbb{E}_m[L_t^\rho\, f(X_0)g(X_t)\rho(X_0)^2\,;\, t < \sigma_N \wedge \zeta],$$

it suffices to prove the following identity:

$$L_t^\rho \circ r_t = L_t^\rho \frac{\rho(X_0)^2}{\rho(X_t)^2}, \quad \mathbb{P}_m\text{-a.s. on } \{t < \sigma_N \wedge \zeta\}. \tag{3.8}$$

We first consider the case $\rho \in \dot{\mathcal{D}}_{\rm loc}^{++}(\mathcal{E})$. Then $\widetilde{\zeta}$ equals ζ. The function $\log \rho$ is bounded and in $\dot{\mathcal{D}}_{\rm loc}(\mathcal{E})$ by [24, Corollary 6.2], and thus $\log \rho$ belongs to $\dot{\mathcal{D}}_{\rm loc}^\dagger(\mathcal{E})$. Hence $\log \rho$ admits the following decomposition:

$$\log \rho(X_t) - \log \rho(X_0) = M_t^{[\log \rho]} + N_t^{[\log \rho]} \quad t < \zeta,\ \mathbb{P}_x\text{-a.s. for q.e. } x \in E.$$

Moreover, $M^{[\log \rho]}$ is decomposed to $M^{[\log \rho]} = M^{[\log \rho],c} + M^{[\log \rho],d}$ ([20, Theorem 8.23]), where $M^{[\log \rho],c}$ (resp. $M^{[\log \rho],d}$) is the continuous (resp. purely discontinuous) part of $M^{[\log \rho]}$. By Itô's formula ([24, Theorem 7.2] and [25, Corollary 4.4]), it holds that for $t \in [0, \zeta[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$

$$\begin{aligned}M_t^{[\log \rho],c} &= \int_0^t \frac{1}{\rho(X_{s-})} dM_s^{[\rho],c} \;=\; M_t^c, \\ M_t^{[\log \rho],d} &= \int_0^t \frac{1}{\rho(X_{s-})} dM_s^{[\rho],d} + \sum_{s \le t} \left(\log \frac{\rho(X_s)}{\rho(X_{s-})} + 1 - \frac{\rho(X_s)}{\rho(X_{s-})}\right) \\ &\qquad - \int_0^t \int_{E_\partial} \left(\log \frac{\rho(y)}{\rho(X_s)} + 1 - \frac{\rho(y)}{\rho(X_s)}\right) N(X_s, dy) dH_s.\end{aligned}$$

Thus we get

$$\begin{aligned}M_t^{[\log \rho]} = M_t &+ \sum_{s \le t} \left(\log \frac{\rho(X_s)}{\rho(X_{s-})} + 1 - \frac{\rho(X_s)}{\rho(X_{s-})}\right) \\ &- \int_0^t \int_{E_\partial} \left(\log \frac{\rho(y)}{\rho(X_s)} + 1 - \frac{\rho(y)}{\rho(X_s)}\right) N(X_s, dy) dH_s.\end{aligned}$$

By this expression and (3.4), we have for $t \in [0, \zeta[$

$$\begin{aligned} L_t^\rho &= \exp\left(M_t - \frac{1}{2}\langle M^{[\log\rho],c}\rangle_t + \sum_{s\le t}\left(\log\frac{\rho(X_s)}{\rho(X_{s-})} + 1 - \frac{\rho(X_s)}{\rho(X_{s-})}\right)\right) \\ &= \exp\left(M_t^{[\log\rho]} + A_t\right), \end{aligned} \tag{3.9}$$

where

$$A_t := \int_0^t \int_{E_\partial}\left(\log\frac{\rho(y)}{\rho(X_s)} + 1 - \frac{\rho(y)}{\rho(X_s)}\right) N(X_s, dy)dH_s - \frac{1}{2}\langle M^{[\log\rho],c}\rangle_t.$$

Hence we have $\mathbb{P}_m$-a.s. on $\{t < \zeta\}$

$$\begin{aligned} L_t^\rho \circ r_t &= \exp\left(M_t^{[\log\rho]} \circ r_t + A_t \circ r_t\right) \\ &= \exp\Big(\log\rho(X_0) - \log\rho(X_t) - N_t^{[\log\rho]} \circ r_t + A_t \circ r_t\Big). \end{aligned}$$

Since A_t is a CAF of bounded variation, A_t is even, $A_t \circ r_t = A_t$. Moreover, $N_t^{[\log\rho]}$ is also even by Lemma 2.2. Thus the right-hand side is equal to

$$\begin{aligned} &\exp\left(\log\rho(X_0) - \log\rho(X_t) - N_t^{[\log\rho]} + A_t\right) \\ &\qquad = \exp\left(2(\log\rho(X_0) - \log\rho(X_t)) + M_t^{[\log\rho]} + A_t\right) \\ &\qquad = L_t^\rho\,\frac{\rho(X_0)^2}{\rho(X_t)^2}. \end{aligned}$$

Therefore (3.8) holds for $\rho \in \dot{\mathcal{D}}_{\mathrm{loc}}^{++}(\mathcal{E})$.

For a general nonnegative $\rho \in \dot{\mathcal{D}}_{\mathrm{loc}}^{\dagger}(\mathcal{E})$, we define $E_n := \{x \in E \mid \frac{1}{n} < \rho(x) < n\}$, $\tau_n := \inf\{t > 0 \mid X_t \notin E_n\}$ and $\rho_n := \left(\frac{1}{n} \vee \rho\right) \wedge n$. Then, on $\{t < \tau_n\}$, $\rho(X_s) = \rho_n(X_s)$ for $s \in [0, t]$ and thus $L_t^\rho = L_t^{\rho_n}$. By applying the result above to $\rho_n \in \dot{\mathcal{D}}_{\mathrm{loc}}^{++}(\mathcal{E})$, we have

$$L_t^\rho \circ r_t = L_t^{\rho_n} \circ r_t = L_t^{\rho_n}\frac{\rho_n(X_0)^2}{\rho_n(X_t)^2} = L_t^\rho\,\frac{\rho(X_0)^2}{\rho(X_t)^2} \quad \mathbb{P}_m\text{-a.s. on } \{t < \tau_n\}.$$

Since $\tau_n \to \sigma_N \wedge \zeta$ as $n \to \infty$, we get (3.8) by letting n to infinity. □

The next theorem is proved in [13, Lemma 4.4] for symmetric diffusion processes. However, its proof works for general symmetric right Markov processes.

Theorem 3.2. *If A is a* PCAF *of $\mathbb{M}$ with Revuz measure μ, then the Revuz measure for A as a* PCAF *of $\widetilde{\mathbb{M}}^\rho$ equals $\rho^2\mu$.*

Lemma 3.3. *For $u \in \mathcal{D}(\mathcal{E})$, the inequality*

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}\widetilde{\mathbb{E}}_{\rho^2 m}\big[\big(u(\widetilde{X}_t)-u(\widetilde{X}_0)\big)^2\,;t<\widetilde{\zeta}\,\big] &\\ \le \int_E \rho(x)^2\mu^c_{\langle u\rangle}(dx) + 2\int_{E\times E}(u(x)-u(y))^2\rho(x)\rho(y)J(dx,dy) &\\ + \rho(\partial)\int_E u(x)^2\rho(x)\kappa(dx). &\end{aligned} \tag{3.10}$$

holds, whenever the integrals on the right-hand side exist.

Proof. Our proof is similar to that of [6, Thorem 2.6]. We give the details here for the reader's convenience.

Take $u \in \mathcal{D}(\mathcal{E})$ such that the right-hand side of (3.10) is finite. Then $u(X_t) - u(X_0)$ can be decomposed as

$$u(X_t) - u(X_0) = M^{[u]}_t + N^{[u]}_t, \quad t>0,\ \mathbb{P}_x\text{-a.s. for q.e. } x \in E,$$

where $M^{[u]} \in \overset{\circ}{\mathcal{M}}$ and $N^{[u]} \in \mathcal{N}_c$. Moreover, the sharp bracket process $\langle M^{[u]}\rangle$ is given by

$$\langle M^{[u]}\rangle_t = \langle M^{[u],c}\rangle_t + \int_0^t\int_{E_\partial}(u(y)-u(X_s))^2N(X_s,dy)dH_s \tag{3.11}$$

for all $t>0$.

By the Girsanov transform,

$$\widetilde{M}^{[u]}_t := M^{[u]}_t - \int_0^t \frac{1}{L^\rho_{s-}}\,d\langle M^{[u]},L^\rho\rangle_s = M^{[u]}_t - \langle M^{[u]}, M\rangle_t, \quad t<\widetilde{\zeta},$$

is a local MAF under $\widetilde{\mathbb{P}}_x$ for $x \in E\setminus N$ and

$$\big[\widetilde{M}^{[u]}\big]_t(\widetilde{\mathbb{P}}) = [M^{[u]}]_t(\mathbb{P}), \quad \widetilde{\mathbb{P}}_m\text{-a.s. on } \{t<\widetilde{\zeta}\} \tag{3.12}$$

(see [20, Chapter 12]). Here $\big[\widetilde{M}^{[u]}\big](\widetilde{\mathbb{P}})$ is the square bracket of the martingale $\widetilde{M}^{[u]}$ under $\widetilde{\mathbb{P}}_x$, and $[M^{[u]}](\mathbb{P})$ is the square bracket of martingale $M^{[u]}$ under $\mathbb{P}_x$. Then $\langle\widetilde{M}^{[u]}\rangle(\widetilde{\mathbb{P}}) = \big[\widetilde{M}^{[u]}\big]^p(\widetilde{\mathbb{P}})$ and $\langle M^{[u]}\rangle(\mathbb{P}) = [M^{[u]}]^p(\mathbb{P})$, that is, $\langle\widetilde{M}^{[u]}\rangle(\widetilde{\mathbb{P}})$ and $\langle M^{[u]}\rangle(\mathbb{P})$ are dual predictable projections of $[\widetilde{M}^{[u]}](\widetilde{\mathbb{P}})$ and $[M^{[u]}](\mathbb{P})$ under $\widetilde{\mathbb{P}}_x$ and $\mathbb{P}_x$, respectively. It follows from (3.12) and [20, Corollary 12.18] that for $t<\widetilde{\zeta}$,

$$\begin{aligned}\langle\widetilde{M}^{[u]}\rangle_t(\widetilde{\mathbb{P}}) = \big[\widetilde{M}^{[u]}\big]^p_t(\widetilde{\mathbb{P}}) &= \langle M^{[u]}\rangle_t(\mathbb{P}) + \int_0^t \frac{1}{L^\rho_{s-}}\,d\langle [M^{[u]}], L^\rho\rangle_s \\ &= \langle M^{[u]}\rangle_t(\mathbb{P}) + \langle [M^{[u]}], M\rangle_t.\end{aligned}$$

Noting that

$$\big[[M^{[u]}],M\big]_t=\sum_{s\le t}\Delta[M^{[u]}]_s\Delta M_s=\sum_{s\le t}(u(X_s)-u(X_{s-}))^2\left(\frac{\rho(X_s)}{\rho(X_{s-})}-1\right)$$

we have by (3.11)

$$\begin{aligned}\langle\widetilde{M}^{[u]}\rangle_t(\widetilde{\mathbb{P}})&=\langle M^{[u]}\rangle_t(\mathbb{P})+\left(\sum_{s\le\cdot}(u(X_s)-u(X_{s-}))^2\left(\frac{\rho(X_s)}{\rho(X_{s-})}-1\right)\right)_t^p(\mathbb{P})\\&=\langle M^{[u]}\rangle_t(\mathbb{P})+\int_0^t\int_{E_\partial}(u(y)-u(X_s))^2\left(\frac{\rho(y)}{\rho(X_s)}-1\right)N(X_s,dy)dH_s\\&=\langle M^{[u],c}\rangle_t(\mathbb{P})+\int_0^t\int_E(u(y)-u(X_s))^2\frac{\rho(y)}{\rho(X_s)}N(X_s,dy)dH_s\\&\qquad\qquad+\rho(\partial)\int_0^t\frac{u(X_s)^2}{\rho(X_s)}N(X_s,\{\partial\})dH_s.\end{aligned}$$

Therefore, the Revuz measure of the PCAF $\langle\widetilde{M}^{[u]}\rangle(\widetilde{\mathbb{P}})$ for $\mathbb{M}$ is

$$\mu^c_{\langle u\rangle}(dx)+2\int_{y\in E}(u(y)-u(x))^2\,\frac{\rho(y)}{\rho(x)}\,J(dx,dy)+\rho(\partial)u(x)^2\rho(x)^{-1}\kappa(dx)$$

by (2.3) and (2.6). We see from Theorem 3.2 that the Revuz measure of the PCAF $\langle\widetilde{M}^{[u]}\rangle(\widetilde{\mathbb{P}})$ for $\widetilde{\mathbb{M}}^\rho$ is

$$\rho(x)^2\mu^c_{\langle u\rangle}(dx)+2\int_{y\in E}(u(x)-u(y))^2\rho(x)\rho(y)J(dx,dy)+\rho(\partial)u(x)^2\rho(x)\kappa(dx).\quad(3.13)$$

Noting that $N^{[u]}$ and $\langle M^{(n)},M\rangle$ are even, we have

$$\begin{aligned}u(\widetilde{X}_t)-u(\widetilde{X}_0)&=\frac{1}{2}(M^{[u]}_t-M^{[u]}_t\circ r_t)\\&=\frac{1}{2}\big(\widetilde{M}^{[u]}_t-\widetilde{M}^{[u]}_t\circ r_t\big)\quad\mathbb{P}_m\text{-a.s. on }\{t<\widetilde{\zeta}\}.\end{aligned}$$

It holds from this equality and the reversibility of the measure $\widetilde{\mathbb{P}}_{\rho^2m}$ that

$$\begin{aligned}&\lim_{t\to0}\frac{1}{t}\widetilde{\mathbb{E}}_{\rho^2m}\big[\big(u(\widetilde{X}_t)-u(\widetilde{X}_0)\big)^2;\,t<\widetilde{\zeta}\big]\\&\quad\le\lim_{t\to0}\frac{1}{2t}\left(\widetilde{\mathbb{E}}_{\rho^2m}\left[\big(\widetilde{M}^{[u]}_t\big)^2;\,t<\widetilde{\zeta}\right]+\widetilde{\mathbb{E}}_{\rho^2m}\left[\big(\widetilde{M}^{[u]}_t\circ r_t\big)^2;\,t<\widetilde{\zeta}\right]\right)\\&\quad=\lim_{t\to0}\frac{1}{t}\widetilde{\mathbb{E}}_{\rho^2m}\left[\big(\widetilde{M}^{[u]}_t\big)^2;\,t<\widetilde{\zeta}\right]\\&\quad=\lim_{t\to0}\frac{1}{t}\widetilde{\mathbb{E}}_{\rho^2m}\left[\langle\widetilde{M}^{[u]}\rangle_t\right].\end{aligned}$$

Since the right-hand side equals (3.13), we have the assertion. □

Recall that the transformed process $\widetilde{\mathbb{M}}^\rho$ by L_t^ρ is a $\rho^2 m$-symmetric right process by Lemma 3.1. We denote by $(\widetilde{\mathcal{E}}^\rho, \mathcal{D}(\widetilde{\mathcal{E}}^\rho))$ the Dirichlet form on $L^2(E \setminus N, \rho^2 m)$ associated with $\widetilde{\mathbb{M}}^\rho$. It is known that $(\widetilde{\mathcal{E}}^\rho, \mathcal{D}(\widetilde{\mathcal{E}}^\rho))$ is *quasi-regular* (see [28]).

Lemma 3.4. *Define* $\widetilde{N}(x, dy) := \frac{\rho(y)}{\rho(x)} \cdot N(x, dy)$*. Then* $(\widetilde{N}(x, dy), H_t)$ *is a Lévy system of* $\widetilde{\mathbb{M}}$*. Consequently, by Theorem 3.2,*

$$\widetilde{J}(dx, dy) := \rho(x)\rho(y)J(dx, dy), \quad \widetilde{\kappa}(dx) := \rho(\partial)\rho(x)\kappa(dx)$$

are the jumping and killing measure of $(\widetilde{\mathcal{E}}^\rho, \mathcal{D}(\widetilde{\mathcal{E}}^\rho))$*, respectively.*

Proof. Let f be a nonnegative bounded function on $E_\partial \times E_\partial$ such that $f(x, x) = 0$ for each $x \in E_\partial$ and put $f_n := f \mathbb{1}_{\{f > 1/n\}}$. Then

$$F_t^n := \sum_{s \le t} f_n(X_{s-}, X_s) - \int_0^t \int_{E_\partial} f_n(X_s, y) N(X_s, dy) dH_s$$

is a $\mathbb{P}_x$-martingale. By the Girsanov theorem,

$$F_t^n - \langle F^n, M \rangle_t = \sum_{s \le t} f_n(X_{s-}, X_s) - \int_0^t \int_{E_\partial} f_n(X_s, y) \frac{\rho(y)}{\rho(x)} N(X_s, dy) dH_s$$

is a $\widetilde{\mathbb{P}}_x$-martingale, and thus

$$\widetilde{\mathbb{E}}_x \left[\sum_{s \le t} f_n(X_{s-}, X_s) \right] = \widetilde{\mathbb{E}}_x \left[\int_0^t \int_{E_\partial} f_n(X_s, y)\, \widetilde{N}(X_s, dy) dH_s \right].$$

We then see by the monotone convergence theorem

$$\widetilde{\mathbb{E}}_x \left[\sum_{s \le t} f(X_{s-}, X_s) \right] = \widetilde{\mathbb{E}}_x \left[\int_0^t \int_{E_\partial} f(X_s, y)\, \widetilde{N}(X_s, dy) dH_s \right].$$

□

For a closed subset F of E, $\mathcal{D}_b(\mathcal{E})_F$ is the space defined by

$$\mathcal{D}_b(\mathcal{E})_F = \{u \in \mathcal{D}_b(\mathcal{E}) \,|\, u = 0 \text{ q.e. on } E \setminus F\},$$

where $\mathcal{D}_b(\mathcal{E})$ is the set of bounded functions in $\mathcal{D}(\mathcal{E})$.

Theorem 3.5. *Suppose that* $\rho > 0$ q.e. *Then there exists an* $\mathcal{E}$*-nest* $\{F_n\}$ *of compact sets such that* $\bigcup_{n\geq 1}\mathcal{D}_b(\mathcal{E})_{F_n} \subset \mathcal{D}(\widetilde{\mathcal{E}}^\rho)$ *and for* $u \in \bigcup_{n\geq 1}\mathcal{D}_b(\mathcal{E})_{F_n}$,

$$\begin{aligned}\widetilde{\mathcal{E}}^\rho(u,u) = \frac{1}{2}\int_E \rho(x)^2\mu^c_{\langle u\rangle}(dx) + \int_{E\times E}(u(x)-u(y))^2\rho(x)\rho(y)J(dx,dy)\\ + \rho(\partial)\int_E u(x)^2\rho(x)\kappa(dx).\end{aligned} \tag{3.14}$$

Proof. There exist $\{G_n\} \in \Theta$ and $\{\rho_n\} \subset \mathcal{D}(\mathcal{E})$ such that $\rho = \rho_n$ m-a.e. on G_n for each n. Take $f \in L^2(E;m)$ with $0 < f \leq 1$ on E and set

$$R_1^{G_n}f(x) := \mathbb{E}_x\left[\int_0^{\tau_{G_n}} e^{-s}f(X_s)ds\right].$$

Then $R_1^{G_n}f(x) > 0$ on G_n and $R_1^{G_n}f$ is $\mathcal{E}$-quasi-continuous for each n. Take a common $\mathcal{E}$-nest $\{K_m\}$ such that all $R_1^{G_n}f$, $n \geq 1$ are continuous on each K_m. We set $F_n^{(1)} := \{x \in K_n : R_1^{G_n}f(x) \geq 1/n\}$. Then since $A_n := \{R_1^{G_n}f \geq 1/n\}$ is increasing and $E \setminus \bigcup_{n\geq 1} A_n$ is $\mathcal{E}$-exceptional, $\{F_n^{(1)}\}$ is an $\mathcal{E}$-nest by [24, Lemma 3.3]. For each n, $(E \setminus G_n)^r \subset E \setminus F_n^{(1)}$, where $(E\setminus G_n)^r = \{x \in E : R_1^{G_n}f(x) = 0\}$ is the set of regular points for $E \setminus G_n$. Therefore we have

$$F_n^{(1)} \setminus G_n \subset F_n^{(1)} \cap \big((E\setminus G_n)\setminus (E\setminus G_n)^r\big).$$

Since $\big((E\setminus G_n)\setminus(E\setminus G_n)^r\big)$ is $\mathcal{E}$-exceptional, $F_n^{(1)} \subset G_n$ q.e. and thus $\rho = \rho_n$ m-a.e. on $F_n^{(1)}$.

By the quasi-regularity of $(\widetilde{\mathcal{E}}^\rho, \mathcal{D}(\widetilde{\mathcal{E}}^\rho))$, we can choose an $\widetilde{\mathcal{E}}^\rho$-nest $\{F_n^{(2)}\}$ of compact sets and a sequence $\{g_n\} \subset \mathcal{D}(\widetilde{\mathcal{E}}^\rho)$ such that $g_n = 1$ on $F_n^{(2)}$ (see [28]). Note that $\sigma_N = \infty$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$ because $\rho > 0$ q.e. Hence, by using probabilistic characterization of $\widetilde{\mathcal{E}}^\rho$-exceptional set and $\widetilde{\mathcal{E}}^\rho$-nest, we see that $\{F_n^{(2)}\}$ is an $\mathcal{E}$-nest.

Since ρ is an element of $\dot{\mathcal{D}}^\dagger_{\mathrm{loc}}(\mathcal{E})$, there exists an $\mathcal{E}$-nest $\{F_n^{(3)}\}$ of compact sets such that $\rho \in C(\{F_n^{(3)}\})$ and

$$\int_{F_n^{(3)}\times E}(\rho(x)-\rho(y))^2 J(dx,dy) < \infty$$

for each n. We put $F_n := \bigcap_{k=1}^3 F_n^{(k)}$. Then $\{F_n\}$ is also an $\mathcal{E}$-nest. We first claim that for any $u \in \bigcup_{n\geq 1}\mathcal{D}_b(\mathcal{E})_{F_n}$,

$$\int_E \rho^2 d\mu^c_{\langle u\rangle} + \int_{E\times E}(u(x)-u(y))^2\rho(x)\rho(y)J(dx,dy) + \rho(\partial)\int_E u^2\rho\, d\kappa < \infty. \tag{3.15}$$

Take $u \in \mathcal{D}_b(\mathcal{E})_{F_n}$. Define $C_n = \sup_{x\in F_n} |\rho(x)|$ and $\rho^{(n)} = ((-C_n) \vee \rho_n) \wedge C_n$. We then see that $\rho u = \rho^{(n)} u$ m-a.e. Thus ρu is in $\mathcal{D}(\mathcal{E})$ and by the derivation property of μ^c,

$$\mathcal{E}(\rho u, \rho u) = \frac{1}{2}\int_E u^2 d\mu^c_{\langle\rho\rangle} + \int_E \rho u \, d\mu^c_{\langle\rho,u\rangle} + \frac{1}{2}\int_E \rho^2 d\mu^c_{\langle u\rangle} + \mathcal{E}^j(\rho u, \rho u) + \int_E (\rho u)^2 d\kappa,$$

where

$$\mathcal{E}^j(f,g) := \int_{E\times E} (f(x)-f(y))(g(x)-g(y))J(dx,dy).$$

Note that the value of $\mathcal{E}(\rho, \rho u^2)$ is finite and equal to

$$\frac{1}{2}\int_E u^2 d\mu^c_{\langle\rho\rangle} + \int_E \rho u \, d\mu^c_{\langle\rho,u\rangle} + \mathcal{E}^j(\rho, \rho u^2) + \int_E \big(\rho(x) - \rho(\partial)\big)\rho(x) u(x)^2 \kappa(dx)$$

by Remark 2.1 and the derivation property. Since

$$\mathcal{E}^j(\rho u, \rho u) - \mathcal{E}^j(\rho, \rho u^2) = \int_{E\times E} (u(x)-u(y))^2 \rho(x)\rho(y) J(dx,dy),$$

it holds that

$$\frac{1}{2}\int_E \rho^2 d\mu^c_{\langle u\rangle} + \int_{E\times E} (u(x)-u(y))^2\rho(x)\rho(y)J(dx,dy) + +\rho(\partial)\int_E u^2\rho \, d\kappa$$
$$= \mathcal{E}(\rho u, \rho u) - \mathcal{E}(\rho, \rho u^2) < \infty.$$

Therefore (3.15) holds.

Let $u \in \mathcal{D}_b(\mathcal{E})_{F_n}$. Noting that $u = 0$ m-a.e. on $E \setminus F_n$ and $g_n \in \mathcal{D}(\widetilde{\mathcal{E}}^\rho)$ with $g_n = 1$ on F_n, we have $u = u \cdot g_n$ m-a.e. Thus it follows from [7, Theorem 4.2.1 (ii)] that

$$\lim_{t\to 0}\frac{1}{t}(1-\widetilde{P}_t 1,\, u^2)_{\rho^2 m} \le \|u\|_\infty^2 \lim_{t\to 0}\frac{1}{t}(1-\widetilde{P}_t 1,\, g_n^2)_{\rho^2 m} \le \|u\|_\infty^2 \int_E g_n(x)^2\,\widetilde{\kappa}(dx).$$

Hence we have by Lemma 3.3

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}(u - \widetilde{P}_t u, u)_{\rho^2 m} \\
&\quad = \lim_{t\to 0}\frac{1}{2t}\left(\widetilde{\mathbb{E}}_{\rho^2 m}\big[\big(u(\widetilde{X}_t) - u(\widetilde{X}_0)\big)^2; t < \widetilde{\zeta}\,\big] + (1-\widetilde{P}_t 1, u^2)_{\rho^2 m}\right) \\
&\quad \le \frac{1}{2}\int_E \rho(x)^2 \mu^c_{\langle u\rangle}(dx) + \int_{E\times E}(u(x)-u(y))^2\rho(x)\rho(y)J(dx,dy) \\
&\qquad + \frac{\rho(\partial)}{2}\int_E u(x)^2\rho(x)\kappa(dx) + \frac{\|u\|_\infty^2}{2}\int_E g_n(x)^2\,\widetilde{\kappa}(dx) \\
&\quad < \infty.
\end{aligned}$$

Therefore u belongs to $\mathcal{D}(\widetilde{\mathcal{E}}^\rho)$ and u admits Fukushima's decomposition under $\widetilde{\mathbb{P}}_x$:

$$u(\widetilde{X}_t) - u(\widetilde{X}_0) = M_t^* + N_t^*,$$

where M^* is a $\widetilde{\mathbb{P}}_x$-square integrable MAF of finite energy for $\widetilde{\mathbb{M}}^\rho$ and N^* is a CAF of zero energy for $\widetilde{\mathbb{M}}^\rho$.

Recall that by the Girsanov theorem,

$$\widetilde{M}_t^{[u]} = M_t^{[u]} - \langle M^{[u]}, M \rangle_t$$

is an MAF under $\widetilde{\mathbb{P}}_x$. Hence, Fukushima's decomposition $u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}$ under $\mathbb{P}_x$ leads us to the following decomposition:

$$u(X_t) - u(X_0) = \widetilde{M}_t^{[u]} + \big(N_t^{[u]} + \langle M^{[u]}, M \rangle_t\big).$$

Since $N^{[u]} + \langle M^{[u]}, M \rangle$ is a CAF of zero energy for $\widetilde{\mathbb{M}}^\rho$, we have by the uniqueness of Fukushima's decomposition

$$M_t^* = \widetilde{M}_t^{[u]}.$$

Now we have

$$\begin{aligned}
\widetilde{\mathcal{E}}^\rho(u,u) &= \lim_{t\to 0} \frac{1}{2t} \widetilde{\mathbb{E}}_{\rho^2 m}\big[\big(u(\widetilde{X}_t) - u(\widetilde{X}_0)\big)^2\big] + \frac{1}{2}\int_E u^2 d\widetilde{\kappa} \\
&= \lim_{t\to 0} \frac{1}{2t} \widetilde{\mathbb{E}}_{\rho^2 m}\left[\big(M_t^*\big)^2\right] + \frac{1}{2}\int_E u^2 d\widetilde{\kappa} \\
&= \lim_{t\to 0} \frac{1}{2t} \widetilde{\mathbb{E}}_{\rho^2 m}\left[\langle \widetilde{M}^{[u]} \rangle_t\right] + \frac{1}{2}\int_E u^2 d\widetilde{\kappa}.
\end{aligned}$$

We see from Lemma 3.4 and (3.13) in the proof of Lemma 3.3 that the right-hand side equals

$$\frac{1}{2}\int_E \rho^2 d\mu_{\langle u \rangle}^c + \int_{E\times E} (u(x)-u(y))^2 \rho(x)\rho(y) J(dx,dy) + \rho(\partial)\int_E u(x)^2 \rho(x)\kappa(dx).$$

Therefore (3.14) holds for $u \in \bigcup_{n\geq 1} \mathcal{D}_b(\mathcal{E})_{F_n}$. □

Suppose that ρ is bounded. Then we obtain by Theorem 3.5 the following inequality:

$$\widetilde{\mathcal{E}}_1^\rho(u,u) \leq \big(\|\rho\|_\infty \vee \rho(\partial)\big)^2 \cdot \mathcal{E}_1(u,u), \qquad u \in \bigcup_{n\geq 1} \mathcal{D}_b(\mathcal{E})_{F_n}, \tag{3.16}$$

where $\widetilde{\mathcal{E}}_1^\rho = \widetilde{\mathcal{E}}^\rho + (\cdot,\cdot)_{\rho^2 m}$. Since $\bigcup_{n\geq 1} \mathcal{D}_b(\mathcal{E})_{F_n}$ is dense in $\mathcal{D}(\mathcal{E})$ with respect to the norm $\sqrt{\mathcal{E}_1(\cdot,\cdot)}$, the inequality (3.16) tells us that $\mathcal{D}(\mathcal{E})$ is contained in $\mathcal{D}(\widetilde{\mathcal{E}}^\rho)$. By repeating the computation above, we can extend (3.14) to $u \in \mathcal{D}(\mathcal{E})$.

Theorem 3.6. (a) *If ρ is bounded, then $\mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\widetilde{\mathcal{E}}^{\rho})$ and the formula* (3.14) *holds for all $u \in \mathcal{D}(\mathcal{E})$.*
(b) *If $\rho \in \dot{\mathcal{D}}^{++}_{\mathrm{loc}}(\mathcal{E})$, that is, there exists a constant $c > 1$ such that $c^{-1} < \rho < c$, then $\mathcal{D}(\widetilde{\mathcal{E}}^{\rho}) = \mathcal{D}(\mathcal{E})$.*

Proof. (a) is already shown above.

Suppose $\rho \in \dot{\mathcal{D}}^{++}_{\mathrm{loc}}(\mathcal{E})$. Then $1/\rho$ and $\log \rho$ are in $\big(\dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E})\big)_b$, and so in $\big(\dot{\mathcal{D}}_{\mathrm{loc}}(\widetilde{\mathcal{E}}^{\rho})\big)_b$ by (a). Hence, we have

$$\begin{aligned} \frac{1}{\rho}(X_t) - \frac{1}{\rho}(X_0) &= \widetilde{M}^{[1/\rho]}_t + \widetilde{N}^{[1/\rho]}_t, \\ \log \rho(X_t) - \log \rho(X_0) &= \widetilde{M}^{[\log \rho]}_t + \widetilde{N}^{[\log \rho]}_t, \quad \widetilde{\mathbb{P}}_x\text{-a.s.} \end{aligned}$$

Let $\widetilde{L}^{[1/\rho]}_t$ be the solution of

$$\widetilde{L}^{[1/\rho]}_t = 1 + \int_0^t \widetilde{L}^{[1/\rho]}_{s-}\, \rho(X_{s-})\, d\widetilde{M}^{[1/\rho]}_s$$

and $\mathbb{M}^* = \big(\Omega, \mathbb{P}^*_x, X_t\big)$ the transformed process of $\widetilde{\mathbb{M}}^{\rho}$ by $\widetilde{L}^{[1/\rho]}$, $d\mathbb{P}^*_x := \widetilde{L}^{[1/\rho]}_t d\widetilde{\mathbb{P}}_x$. Denote by $\big(\mathcal{E}^*, \mathcal{D}(\mathcal{E}^*)\big)$ the Dirichlet form generated by $\mathbb{M}^*$. Since $1/\rho$ is bounded, we see $\mathcal{D}\big(\widetilde{\mathcal{E}}^{\rho}\big) \subset \mathcal{D}(\mathcal{E}^*)$ by (a). Hence, it is enough to prove $\mathcal{D}(\mathcal{E}^*) = \mathcal{D}(\mathcal{E})$. Owing to (3.9) and Lemma 3.4, $\widetilde{L}^{[1/\rho]}_t$ is expressed by

$$\widetilde{L}^{[1/\rho]}_t = \exp\big(-\widetilde{M}^{[\log \rho]}_t + \widetilde{A}_t\big),$$

where

$$\widetilde{A}_t := \int_0^t \int_{E_\partial} \left(\log \frac{\rho(X_s)}{\rho(y)} + 1 - \frac{\rho(X_s)}{\rho(y)} \right) \frac{\rho(y)}{\rho(X_s)} N(X_s, dy) dH_s + \frac{1}{2} \langle \widetilde{M}^{[\log \rho],c} \rangle_t.$$

Noting that

$$\begin{aligned} \widetilde{M}^{[\log \rho]}_t &= M^{[\log \rho]}_t - \langle M^{[\log \rho]}, M \rangle_t \\ &= M^{[\log \rho]}_t - \int_0^t \int_{E_\partial} \log \frac{\rho(y)}{\rho(X_s)} \left(\frac{\rho(y)}{\rho(X_s)} - 1 \right) N(X_s, dy) dH_s \end{aligned}$$

and $\langle \widetilde{M}^{[\log \rho],c} \rangle_t = \langle M^{[\log \rho],c} \rangle_t$, we see $\widetilde{L}^{[1/\rho]}_t = 1/L^{[\rho]}_t$ by (3.9). This implies $\mathbb{M}^* = \mathbb{M}$, and thus $\mathcal{D}(\mathcal{E}^*) = \mathcal{D}(\mathcal{E})$. □

Let us recall the definitions of transience and recurrence of Dirichlet forms.

Definition 3.7. (1) A Dirichlet space $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ on $L^2(E;m)$ is said to be *transient* if the extended Dirichlet space $\mathcal{D}_e(\mathcal{E})$ is a Hilbert space with inner product $\mathcal{E}$.
(2) $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is said to be *recurrent* if the constant function 1 belongs to $\mathcal{D}_e(\mathcal{E})$ and $\mathcal{E}(1,1)=0$. Namely, there exists a sequence $\{u_n\}\subset\mathcal{D}(\mathcal{E})$ such that $\lim_{n\to\infty}u_n=1$ m-a.e. and $\lim_{n,m\to\infty}\mathcal{E}(u_n-u_m,u_n-u_m)=0$.

Corollary 3.8. *Suppose $\rho\in\dot{\mathcal{D}}^{++}_{\mathrm{loc}}(\mathcal{E})$. If $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is a transient (or recurrent) regular Dirichlet form, then so is $\big(\widetilde{\mathcal{E}}^\rho,\mathcal{D}(\widetilde{\mathcal{E}}^\rho)\big)$.*

Proof. If $c^{-1}<\rho<c$, then it follows from Theorem 3.6 that $\mathcal{D}(\widetilde{\mathcal{E}}^\rho)=\mathcal{D}(\mathcal{E})$ and

$$c^{-2}\mathcal{E}(u,u)\le\widetilde{\mathcal{E}}^\rho(u,u)\le c^2\mathcal{E}(u,u),\quad u\in\mathcal{D}(\widetilde{\mathcal{E}}^\rho).$$

Hence, if $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is transient (or recurrent), then so is $\big(\widetilde{\mathcal{E}}^\rho,\mathcal{D}(\widetilde{\mathcal{E}}^\rho)\big)$.

Moreover, since $\mathcal{E}_1$-norm and $\widetilde{\mathcal{E}}^\rho_1$-norm are equivalent by the inequality above, it holds that

$$\overline{\mathcal{D}(\widetilde{\mathcal{E}}^\rho)\cap C_0(E)}^{\widetilde{\mathcal{E}}^\rho_1}=\overline{\mathcal{D}(\mathcal{E})\cap C_0(E)}^{\mathcal{E}_1}=\mathcal{D}(\mathcal{E})=\mathcal{D}(\widetilde{\mathcal{E}}^\rho).$$

Here $\overline{\mathcal{D}(\widetilde{\mathcal{E}}^\rho)\cap C_0(E)}^{\widetilde{\mathcal{E}}^\rho_1}$ and $\overline{\mathcal{D}(\mathcal{E})\cap C_0(E)}^{\mathcal{E}_1}$ denote closures of sets $\mathcal{D}(\widetilde{\mathcal{E}}^\rho)\cap C_0(E)$ and $\mathcal{D}(\mathcal{E})\cap C_0(E)$ with respect to $\widetilde{\mathcal{E}}^\rho_1$- and $\mathcal{E}_1$-norm, respectively. Clearly, $\mathcal{D}(\widetilde{\mathcal{E}}^\rho)\cap C_0(E)$ is dense in $C_0(E)$ with respect to the uniform norm. Therefore $\big(\widetilde{\mathcal{E}}^\rho,\mathcal{D}(\widetilde{\mathcal{E}}^\rho)\big)$ is regular. □

We now obtain an extension of [6, Thorem 2.6].

Theorem 3.9. *Let $\rho\in\mathcal{D}_e(\mathcal{E})$ with $\rho>0$* q.e. *Then the Dirichlet form $(\widetilde{\mathcal{E}}^\rho,\mathcal{D}(\widetilde{\mathcal{E}}^\rho))$ is recurrent.*

Proof. We see from [18, Lemma 1.6.7] that there exists a strictly positive bounded function g in $L^1(E;m)$ such that $\rho\in\mathcal{D}_e(\mathcal{E}^g)$, where $\mathcal{E}^g$ is a perturbed form on $L^2(E;m)$ defined by

$$\mathcal{E}^g(u,v)=\mathcal{E}(u,v)+(u,v)_{g\cdot m},\quad u,v\in\mathcal{D}(\mathcal{E}).$$

Then $(\mathcal{E}^g,\mathcal{D}(\mathcal{E}))$ is a transient Dirichlet form and thus its extended Dirichlet space $\mathcal{D}_e(\mathcal{E}^g)$ is a Hilbert space with inner product $\mathcal{E}^g$ ([18, Theorem 1.6.2]). By Theorem 3.5, we can take an $\mathcal{E}$-nest $\{F_n\}$ of compact sets such that $\rho\in C(\{F_n\})$ and $\bigcup_{n\ge1}\mathcal{D}_b(\mathcal{E})_{F_n}\subset\mathcal{D}(\widetilde{\mathcal{E}}^\rho)$. Let $K_n:=\{x\in F_n\,|\,\rho(x)\ge1/n\}$. Then $\{K_n\}$ is an $\mathcal{E}$-nest because $E\setminus\bigcup_{n\ge1}\{\rho\ge1/n\}$ is $\mathcal{E}$-exceptional. Since the norm $\sqrt{\mathcal{E}^g_1(\cdot,\cdot)}$ is equivalent to $\sqrt{\mathcal{E}_1(\cdot,\cdot)}$, $\{K_n\}$ is an $\mathcal{E}^g$-nest as well. We set $\mathcal{D}_e(\mathcal{E}^g)_{K_n}:=\{u\in\mathcal{D}_e(\mathcal{E}^g)\,|\,u=0$ m-a.e. on $E\setminus K_n\}$. Then $\mathcal{D}_e(\mathcal{E}^g)_{K_n}$ is a closed subspace of the Hilbert space $(\mathcal{D}_e(\mathcal{E}^g),\mathcal{E}^g)$ and by [7, Corollary

3.4.4], $\bigcup_{n\geq 1}\mathcal{D}_e(\mathcal{E})_{K_n}$ is dense in $\mathcal{D}_e(\mathcal{E})$. Let ρ_{K_n} be the $\mathcal{E}^g$-orthogonal projection of ρ onto $\mathcal{D}_e(\mathcal{E}^g)_{K_n}$. Then ρ_{K_n} converges to ρ in $(\mathcal{D}_e(\mathcal{E}^g),\mathcal{E}^g)$. Let $\rho_n := (0\vee\rho_{K_n})\wedge\rho$. Then we easily see that $\rho_n\in\mathcal{D}_b(\mathcal{E})_{K_n}$ for each n and $\rho_n\to\rho$ m-a.e. as $n\to\infty$. Noting that $\rho-\rho_n=(\rho-\rho_{K_n})^+$, we have by the contraction property

$$\begin{aligned}\mathcal{E}(\rho-\rho_n,\rho-\rho_n) &\leq \mathcal{E}^g(\rho-\rho_n,\rho-\rho_n)\\ &\leq \mathcal{E}^g(\rho-\rho_{K_n},\rho-\rho_{K_n})\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

By taking subsequence if necessary, we may assume ρ_n converges to ρ $\mathcal{E}$-q.e. on E (cf. [7, Theorem 2.3.4]). For $n\geq 1$, define a function h_n by

$$h_n(x) := \begin{cases}\rho_n(x)/\rho(x) & \text{if } \rho(x)>0,\\ 0 & \text{if } \rho(x)=0.\end{cases}$$

Then $0\leq h_n\leq 1$ and $h_n\to 1$ $\mathcal{E}$-q.e. on E as $n\to\infty$. Moreover, for $(x,y)\in K_n\times K_n$,

$$\begin{aligned}&|h_n(x)|\leq n|\rho_n(x)|,\\ &|h_n(x)-h_n(y)|\leq \frac{|\rho_n(x)-\rho_n(y)|}{\rho(x)}+\frac{|\rho_n(x)-\rho_n(y)|}{\rho(y)}+\frac{|\rho(x)\rho_n(x)-\rho(y)\rho_n(y)|}{\rho(x)\rho(y)}\\ &\qquad\qquad\qquad\leq 2n|\rho_n(x)-\rho_n(y)|+n^2|\rho(x)\rho_n(x)-\rho(y)\rho_n(y)|.\end{aligned}$$

By noting that ρ_n and $\rho\cdot\rho_n$ belong to $\mathcal{D}_b(\mathcal{E})_{K_n}$, this inequality and [18, Theorem 1.5.2 (ii)] tell us that h_n is also in $\mathcal{D}_b(\mathcal{E})_{K_n}$. Hence, since $\rho\in\mathcal{D}_e(\mathcal{E})\cap QC(E_\partial)$ and thus $\rho(\partial)=0$, it follows from Theorem 3.5 that $h_n\in\mathcal{D}(\widetilde{\mathcal{E}}^\rho)$ and

$$\widetilde{\mathcal{E}}^\rho(h_n,h_n)=\frac{1}{2}\int_E \rho(x)^2\mu^c_{\langle h_n\rangle}(dx)+\int_{E\times E}(h_n(x)-h_n(y))^2\rho(x)\rho(y)J(dx,dy).$$

By a calculation found in the proof of [18, Theorem 6.3.2], we can show that the right-hand side of the equality above tends to 0 as $n\to\infty$. Therefore $h_n\to 1$ q.e. and $\widetilde{\mathcal{E}}^\rho(h_n,h_n)\to 0$ as $n\to\infty$, which implies that the constant function 1 belongs to $\mathcal{D}_e(\widetilde{\mathcal{E}}^\rho)$ and $\widetilde{\mathcal{E}}^\rho(1,1)=0$. Hence, $(\widetilde{\mathcal{E}}^\rho,\mathcal{D}(\widetilde{\mathcal{E}}^\rho))$ is recurrent. □

Theorem 3.9 is interesting in the sense that for $\rho\in\mathcal{D}_e(\mathcal{E})$, the transformed process $\widetilde{\mathbb{M}}^\rho$ always becomes recurrent (in particular, conservative) even if $\mathbb{M}$ is transient.

3.2 Non-attainability to zero sets

In this section, we assume that $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is conservative, $\mathbb{P}_m(\zeta<\infty)=0$, and that ρ is a nonnegative function in $\dot{\mathcal{D}}^\dagger_{\mathrm{loc}}(\mathcal{E})$ with finite energy measure, $\mu_{\langle\rho\rangle}(E)<\infty$. It is shown in

[18, §6.3] that under assumption of the strong local property, the transformed process $\widetilde{\mathbb{M}}^\rho$ never approaches in finite time to the set $\{x \in E \,|\, \rho(x) = 0 \text{ or } \rho(x) = \infty\}$. The objective of this section is to obtain the non-attainability without assuming the local property. We use ideas from [18, §6.3] but modifications are needed because $\mathbb{M}$ is allowed to have jumps.

Lemma 3.10. *Let $\rho \in \dot{\mathcal{D}}^{++}_{\mathrm{loc}}(\mathcal{E})$ with $\mu_{\langle\rho\rangle}(E) < \infty$, where $\dot{\mathcal{D}}^{++}_{\mathrm{loc}}(\mathcal{E})$ is the space defined in* (3.7). *Then the transformed process $\widetilde{\mathbb{M}}^\rho$ is conservative in the sense that*

$$\widetilde{\mathbb{P}}_{\rho^2 m}(\zeta < \infty) = 0. \tag{3.17}$$

Proof. Let M be a local martingale defined by (3.1). Let $\{T_n\}$ be a sequence of $\{\mathscr{F}_t\}$-stopping times defined by

$$T_n := \inf\{t > 0 \,|\, \langle M\rangle_t \geq n\}.$$

Since the Revuz measure of the PCAF $\langle M\rangle$ for $\mathbb{M}$ is $(1/\rho)^2\mu_{\langle\rho\rangle}$, that for $\widetilde{\mathbb{M}}^\rho$ is $\mu_{\langle\rho\rangle}$ by Theorem 3.2. Hence, we get

$$\widetilde{\mathbb{P}}_{\rho^2 m}(T_n \leq t) \leq \frac{1}{n}\widetilde{\mathbb{E}}_{\rho^2 m}[\langle M\rangle_t] \leq \frac{t}{n}\mu_{\langle\rho\rangle}(E).$$

By letting n to infinity, we obtain $\lim_{n\to\infty} T_n = \infty$ $\widetilde{\mathbb{P}}_{\rho^2 m}$-a.s.

Put $M^{T_n}_t := M_{t\wedge T_n}$ and $L^{(n)}_t := L^\rho_{t\wedge T_n}$. Then for each n, $L^{(n)}$ is a solution to the following SDE:

$$L^{(n)}_t = 1 + \int_0^t L^{(n)}_{s-}\, dM^{T_n}_s, \quad t > 0.$$

From the definition of $\dot{\mathcal{D}}^{++}_{\mathrm{loc}}(\mathcal{E})$, there exists a constant $a > 1$ such that $a^{-1} \leq \rho \leq a$. Hence we have by (3.2)

$$\Delta M^{T_n}_t = \frac{\rho(X_{t\wedge T_n})}{\rho(X_{(t\wedge T_n)-})} - 1 \geq \frac{1}{a^2} - 1, \quad t \geq 0.$$

Moreover, it holds that

$$\mathbb{E}_x\big[[M^{T_n}]_\infty\big] = \mathbb{E}_x\big[\langle M\rangle_{T_n}\big] \leq n.$$

By the same argument as that in the proof of [4, Theorem 4.3.2], we can show that there exists a constant $C > 0$ such that $L^{(n)}_t \leq C\,\mathbb{E}_x\big[L^{(n)}_\infty \,|\, \mathscr{F}_t\big]$ for every $x \in E$ and $t > 0$. Therefore $L^{(n)}$ is of class (D), that is, $\{L^{(n)}_\tau \,|\, \tau \text{ is a stopping time}\}$ is a uniformly integrable family. Thus $L^{(n)}$ is a uniformly integrable martingale by [20, Theorem 7.12]. Hence we have by (3.5)

$$\widetilde{\mathbb{P}}_x(t \wedge T_n < \zeta) = \mathbb{E}_x\big[L^{(n)}_t\big] = 1, \quad t > 0$$

for each n. Therefore we have for all $t > 0$

$$\widetilde{\mathbb{P}}_x(t < \zeta) = \lim_{n\to\infty} \widetilde{\mathbb{P}}_x(t \wedge T_n < \zeta) = 1 \quad \rho^2 m\text{-a.e.},$$

which leads to (3.17). □

Let $\rho \in \dot{\mathcal{D}}^{++}_{\mathrm{loc}}(\mathcal{E})$ with $\mu_{\langle\rho\rangle}(E) < \infty$. Then there exist $\{G_n\} \in \Theta$ and $\{\rho_n\} \subset \mathcal{D}(\mathcal{E})$ such that $\rho = \rho_n$ m-a.e. on G_n. Then by LeJan's formula ([18, Thorem 3.2.2]), we see that

$$\int_E \rho^2 \, d\mu^c_{\langle \log \rho\rangle} = \lim_{n\to\infty} \int_{G_n} \rho_n^2 \, d\mu^c_{\langle \log \rho_n\rangle} = \lim_{n\to\infty} \int_{G_n} d\mu^c_{\langle \rho_n\rangle} = \mu^c_{\langle\rho\rangle}(E).$$

On the other hand, substituting $\rho(y)/\rho(x)$ for the inequality

$$t(\log t)^2 \le (1-t)^2, \quad t \in (0,\infty),$$

we have

$$\begin{aligned} 2\int_{E\times E} (\log\rho(x) - \log\rho(y))^2 \rho(x)\rho(y) J(dx,dy) &\le 2\int_{E\times E} (\rho(x)-\rho(y))^2 J(dx,dy) \\ &= \mu^j_{\langle\rho\rangle}(E). \end{aligned}$$

Since $\log\rho \in \big(\dot{\mathcal{D}}_{\mathrm{loc}}(\mathcal{E})\big)_b$, it is in $\big(\dot{\mathcal{D}}_{\mathrm{loc}}(\widetilde{\mathcal{E}}^\rho)\big)_b$ as well by Theorem 3.6. Thus $\log\rho(X_t) - \log\rho(X_0)$ admits Fukushima's decompositions under $\mathbb{P}_x$ and $\widetilde{\mathbb{P}}_x$, respectively:

$$\begin{aligned} \log\rho(X_t) - \log\rho(X_0) &= M^{[\log\rho]}_t + N^{[\log\rho]}_t, \quad \mathbb{P}_x\text{-a.s.} \\ \log\rho(X_t) - \log\rho(X_0) &= \widetilde{M}^{[\log\rho]}_t + \widetilde{N}^{[\log\rho]}_t, \quad \widetilde{\mathbb{P}}_x\text{-a.s.} \end{aligned}$$

By the same argument as in the proof of Theorem 3.5, $\widetilde{M}^{[\log\rho]}$ and $\widetilde{N}^{[\log\rho]}$ are expressed as

$$\widetilde{M}^{[\log\rho]}_t = M^{[\log\rho]}_t - \langle M^{[\log\rho]}, M\rangle_t, \qquad \widetilde{N}^{[\log\rho]}_t = N^{[\log\rho]}_t + \langle M^{[\log\rho]}, M\rangle_t.$$

Moreover, in a similar way to the proof of Lemma 3.3, we can show that

$$\begin{aligned} &\lim_{t\downarrow 0} \frac{1}{t} \widetilde{\mathbb{E}}_{\rho^2 m}\big[\langle \widetilde{M}^{[\log\rho]}\rangle_t\big] \\ &\qquad \le \int_E \rho^2 \, d\mu^c_{\langle\log\rho\rangle} + 2\int_E (\log\rho(x) - \log\rho(y))^2 \rho(x)\rho(y) J(dx,dy). \end{aligned}$$

Noting that $\mu_{\langle\rho\rangle} = \mu^c_{\langle\rho\rangle} + \mu^j_{\langle\rho\rangle}$ because of the conservativeness of $\mathbb{M}$, we get

$$\widetilde{\mathbb{E}}_{\rho^2 m}\big[\langle \widetilde{M}^{[\log\rho]}\rangle_t\big] \le t\, \mu_{\langle\rho\rangle}(E), \quad t > 0. \tag{3.18}$$

Since $\mu_{\langle\rho\rangle}(E) < \infty$, this inequality implies that $\widetilde{M}^{[\log\rho]}$ is a $\widetilde{\mathbb{P}}_x$-square integrable martingale for $\rho^2 m$-a.e. x.

Lemma 3.11. *It holds that for $\lambda > 0$ and $\rho \in \dot{\mathcal{D}}^{++}_{\rm loc}(\mathcal{E})$ with $\mu_{\langle\rho\rangle}(E) < \infty$,*

$$\widetilde{\mathbb{P}}_{\rho^2 m}\left(\sup_{0\le s\le t}\left(\frac{\rho(X_s)}{\rho(X_0)} \vee \frac{\rho(X_0)}{\rho(X_s)}\right) \ge e^{\lambda}\right) \le \frac{8t}{\lambda^2}\,\mu_{\langle\rho\rangle}(E). \tag{3.19}$$

Proof. By Lemma 2.2, it holds that

$$\widetilde{N}^{[\log\rho]}_s \circ r_t = \widetilde{N}^{[\log\rho]}_t - \widetilde{N}^{[\log\rho]}_{t-s}, \quad 0 \le s \le t, \quad \widetilde{\mathbb{P}}_{\rho^2 m}\text{-a.s.}$$

Moreover, we can show in the same way as in the proof of [18, Thorem 5.7.1] that

$$\log\rho(X_s) - \log\rho(X_0) = \frac{1}{2}\widetilde{M}^{[\log\rho]}_s + \frac{1}{2}\left(\widetilde{M}^{[\log\rho]}_{t-s}\circ r_t - \widetilde{M}^{[\log\rho]}_t \circ r_t\right),$$
$$0 \le s \le t, \quad \widetilde{\mathbb{P}}_{\rho^2 m}\text{-a.s.}$$

Hence, the left-hand side of (3.19) is equal to

$$\begin{aligned}
&\widetilde{\mathbb{P}}_{\rho^2 m}\left(\sup_{0\le s\le t}|\log\rho(X_s) - \log\rho(X_0)| \ge \lambda\right)\\
&\qquad = \widetilde{\mathbb{P}}_{\rho^2 m}\left(\sup_{0\le s\le t}\left|\widetilde{M}^{[\log\rho]}_s + \widetilde{M}^{[\log\rho]}_{t-s}\circ r_t - \widetilde{M}^{[\log\rho]}_t\circ r_t\right| \ge 2\lambda\right)
\end{aligned}$$

and the right-hand side is dominated by

$$\begin{aligned}
&\widetilde{\mathbb{P}}_{\rho^2 m}\left(\sup_{0\le s\le t}\left|\widetilde{M}^{[\log\rho]}_s\right| \ge \lambda\right) + \widetilde{\mathbb{P}}_{\rho^2 m}\left(\sup_{0\le s\le t}\left|\widetilde{M}^{[\log\rho]}_{t-s}\circ r_t - \widetilde{M}^{[\log\rho]}_t\circ r_t\right| \ge \lambda\right)\\
&\le \widetilde{\mathbb{P}}_{\rho^2 m}\left(\sup_{0\le s\le t}\left|\widetilde{M}^{[\log\rho]}_s\right| \ge \frac{\lambda}{2}\right) + \widetilde{\mathbb{P}}_{\rho^2 m}\left(\sup_{0\le s\le t}\left|\widetilde{M}^{[\log\rho]}_s\circ r_t\right| \ge \frac{\lambda}{2}\right)\\
&= 2\,\widetilde{\mathbb{P}}_{\rho^2 m}\left(\sup_{0\le s\le t}\left|\widetilde{M}^{[\log\rho]}_s\right| \ge \frac{\lambda}{2}\right).
\end{aligned}$$

Here the last equality is derived from the reversibility of the measure $\widetilde{\mathbb{P}}_{\rho^2 m}$. We have by Doob's inequality and (3.18)

$$\widetilde{\mathbb{P}}_{\rho^2 m}\left(\sup_{0\le s\le t}\left|\widetilde{M}^{[\log\rho]}_s\right| \ge \frac{\lambda}{2}\right) \le \frac{4}{\lambda^2}\,\widetilde{\mathbb{E}}_{\rho^2 m}\left[\langle\widetilde{M}^{[\log\rho]}\rangle_t\right] \le \frac{4t}{\lambda^2}\,\mu_{\langle\rho\rangle}(E).$$

□

Theorem 3.12. *Let $\rho \in \dot{\mathcal{D}}_{\rm loc}(\mathcal{E})$ such that $\rho \ge 0$ m-a.e., $m(\{\rho(x) > 0\}) > 0$ and $\mu_{\langle\rho\rangle}(E) < \infty$. Then the transformed process $\widetilde{\mathbb{M}}^{\rho}$ is conservative in the sense of* (3.17) *and it never attains to the set $N = \{x \in E \mid \rho(x) = 0$ or $\rho(x) = \infty\}$ in the following sense:*

$$\widetilde{\mathbb{P}}_{\rho^2 m}(\sigma_N < \infty) = 0, \tag{3.20}$$

where $\sigma_N = \inf\{t > 0 : X_t \in N\}$.

Proof. Our proof is quite similar to that of [18, Thorem 6.3.4]. For the reader's convenience, we spell out the details. The assertion holds for $\rho \in \mathcal{D}^{++}_{\mathrm{loc}}(\mathcal{E})$ because of Lemma 3.10. We assume that $E \setminus E_n \neq \emptyset$ for any $n \geq 1$, where $E_n := \{x \in E \mid \frac{1}{n} \leq \rho(x) \leq n\}$. We set $\rho_n := \left(\frac{1}{n} \vee \rho\right) \wedge n$ and define stopping times τ_n by $\tau_n := \inf\{t > 0 \mid X_t \notin E_n\}$. Then $\rho_n \in \dot{\mathcal{D}}^{++}_{\mathrm{loc}}(\mathcal{E})$ and $\rho = \rho_n$ on E_n. Moreover, it holds that $\mu_{\langle \rho_n \rangle}(E) \leq \mu_{\langle \rho \rangle}(E)$ for each n because of the following inequality:

$$|\rho_n(x) - \rho_n(y)| \leq |\rho(x) - \rho(y)| \quad \text{for all } x, y \in E.$$

Let us denote by $\widetilde{\mathbb{M}}^{(n)} := (\Omega, \mathscr{F}, X_t, \widetilde{\mathbb{P}}^{(n)}_x, \{\widetilde{P}^{(n)}_t\})$ the transformed process by $L^{\rho_n}_t$. Then we see from Lemma 3.10 that $\widetilde{\mathbb{M}}^{(n)}$ is conservative, $\widetilde{P}^{(n)}_t 1 = 1$, $\rho^2 m$-a.e.

Note that $L^{\rho}_t = L^{\rho_n}_t$ on $\{t < \tau_n\}$, and thus

$$\widetilde{\mathbb{P}}^{(n)}_x(t < \tau_n) = \mathbb{E}_x\big[L^{\rho_n}_t\,; t < \tau_n\big] = \mathbb{E}_x\big[L^{\rho}_t\,; t < \tau_n\big] = \widetilde{\mathbb{P}}_x(t < \tau_n). \tag{3.21}$$

Hence, for any $1 < \ell < n$ and $t > 0$, we have

$$\begin{aligned}\widetilde{\mathbb{P}}_{\rho^2 m}\Big(\frac{1}{\ell} \leq \rho(X_0) \leq \ell,\ \tau_n \leq t\Big) &= \int_{\left\{\frac{1}{\ell} \leq \rho \leq \ell\right\}} \widetilde{\mathbb{P}}_x\left(\tau_n \leq t\right) \rho(x)^2 m(dx) \\ &= \int_{\left\{\frac{1}{\ell} \leq \rho \leq \ell\right\}} \widetilde{\mathbb{P}}^{(n)}_x\left(\tau_n \leq t\right) \rho(x)^2 m(dx).\end{aligned}$$

Since $\left\{\frac{1}{\ell} \leq \rho \leq \ell\right\} \subset \{\rho = \rho_n\}$, the right-hand side is equal to

$$\int_{\left\{\frac{1}{\ell} \leq \rho_n \leq \ell\right\}} \widetilde{\mathbb{P}}^{(n)}_x\left(\tau_n \leq t\right) \rho_n(x)^2 m(dx) = \widetilde{\mathbb{P}}^{(n)}_{\rho_n^2 m}\left(\frac{1}{\ell} \leq \rho_n(X_0) \leq \ell,\ \tau_n \leq t\right). \tag{3.22}$$

Since $\widetilde{\mathbb{M}}^{(n)}$ is conservative, we see that $X_{\tau_n} \in E \setminus E_n$ $\widetilde{\mathbb{P}}^{(n)}_{\rho_n^2 m}$-a.s. on $\{\tau_n \leq t\}$ and thus the value of $\rho_n(X_{\tau_n})$ is either n or $1/n$. Therefore (3.22) is dominated by

$$\widetilde{\mathbb{P}}^{(n)}_{\rho_n^2 m}\left(\sup_{0 \leq s \leq t}\left(\frac{\rho_n(X_s)}{\rho_n(X_0)} \vee \frac{\rho_n(X_0)}{\rho_n(X_s)}\right) \geq \frac{n}{\ell}\right).$$

By applying Lemma 3.11, this is dominated by

$$8t\left(\log\frac{n}{\ell}\right)^{-2} \mu_{\langle \rho_n \rangle}(E) \leq 8t\left(\log\frac{n}{\ell}\right)^{-2} \mu_{\langle \rho \rangle}(E).$$

Consequently, we have by letting n to infinity

$$\widetilde{\mathbb{P}}_{\rho^2 m}\left(\frac{1}{\ell} \leq \rho(X_0) \leq \ell,\ \sigma_N \leq t\right) = 0.$$

Since the left-hand side tends to $\widetilde{\mathbb{P}}_{\rho^2 m}(\sigma_N \le t)$ as $\ell \to \infty$, we attain (3.20). Now we have for any $t > 0$ and $f \in \mathfrak{B}_b(E)$,

$$\lim_{n\to\infty} \widetilde{P}_t^{(n)} f = \widetilde{P}_t f \quad \rho^2 m\text{-a.e.} \tag{3.23}$$

Indeed, we see from (3.21) and (3.20) that for $\rho^2 m$-a.e. x,

$$\begin{aligned} \widetilde{\mathbb{E}}_x^{(n)}\big[f(X_t)\,;\tau_n \le t\big] &\le \|f\|_\infty \widetilde{\mathbb{P}}_x^{(n)}(\tau_n \le t) \\ &= \|f\|_\infty \widetilde{\mathbb{P}}_x(\tau_n \le t) \quad \to 0 \quad \text{as } \ n \to \infty. \end{aligned}$$

Hence, noting that $L_t^\rho = L_t^{\rho_n}$ on $\{t < \tau_n\}$, we get

$$\begin{aligned} \lim_{n\to\infty} \widetilde{\mathbb{E}}_x^{(n)}\big[f(X_t)\big] &= \lim_{n\to\infty} \widetilde{\mathbb{E}}_x^{(n)}\big[f(X_t)\,;t < \tau_n\big] \\ &= \lim_{n\to\infty} \mathbb{E}_x\big[L_t^\rho f(X_t)\,;t < \tau_n\big] \\ &= \mathbb{E}_x\big[L_t^\rho f(X_t)\,;t < \sigma_N\big], \end{aligned}$$

which implies (3.23). Since $\widetilde{\mathbb{M}}^{(n)}$ is conservative for each n, so is $\widetilde{\mathbb{M}}^\rho$ by (3.23). $\square$

Chapter 4

Hardy-type inequalities

4.1 Schrödinger forms

From this section, we impose the next assumptions on $\mathbb{M}$:

Irreducibility: If a Borel set A is P_t-invariant, i.e., $P_t(\mathbb{1}_A f) = \mathbb{1}_A P_t f$ m-a.e. for any $t > 0$ and any $f \in L^2(E; m) \cap \mathfrak{B}_b(E)$, then the set A satisfies either $m(A) = 0$ or $m(E \setminus A) = 0$.

Strong Feller Property (SF): For each t, $P_t(\mathfrak{B}_b(E)) \subset C_b(E)$, where $C_b(E)$ is the space of bounded continuous functions on E.

We remark that **(SF)** implies

Absolute Continuity Condition (AC): The transition probability of $\mathbb{M}$ is absolutely continuous with respect to m, $p_t(x, dy) = p_t(x, y)m(dy)$ for each $t > 0$ and $x \in E$.

For $\beta > 0$, we define the β-order resolvent kernel by

$$R_\beta(x, y) = \int_0^\infty e^{-\beta t} p_t(x, y)dt, \quad x, y \in E.$$

If $\mathbb{M}$ is transient, we can define the 0-order resolvent kernel $R(x, y) := R_0(x, y) < \infty$ for $x, y \in E$ with $x \neq y$. $R(x, y)$ is called the *Green function* of $\mathbb{M}$. For a measure μ, we define the *β-potential* of μ by

$$R_\beta \mu(x) := \int_E R_\beta(x, y)\mu(dy).$$

We introduce two classes of measures.

Definition 4.1. Suppose that μ is a positive smooth Radon measure on E.
(i) A measure μ is said to be in the *Kato class* ($\mu \in \mathcal{K}$ in abbreviation), if

$$\lim_{\alpha \to \infty} \|R_\alpha \mu\|_\infty = 0.$$

(ii) Suppose that $\mathbb{M}$ is transient. A measure $\mu \in \mathcal{K}$ is said to be *Green-tight* ($\mu \in \mathcal{K}_\infty$ in abbreviation), if for any $\varepsilon > 0$ there exists a compact set $K = K(\varepsilon)$ such that

$$\sup_{x \in E} \int_{K^c} R(x,y)\mu(dy) < \varepsilon.$$

By [1, Theorem 3.9], $\mu \in \mathcal{K}$ if and only if

$$\lim_{t \downarrow 0} \sup_{x \in E} \mathbb{E}_x[A_t^\mu] = \lim_{t \downarrow 0} \sup_{x \in E} \int_0^t \int_E p_s(x,y)\mu(dy)ds = 0. \tag{4.1}$$

We see from [22, Lemma 4.1] that the class $\mathcal{K}_\infty$ is the same as that defined in [3, Definition 2.2 (1)] under **(SF)**. We denote the Green-tight class by $\mathcal{K}_\infty(R)$ if we would like to emphasize the dependence of the Green kernel. We see from the Stollmann-Voigt inequality (4.11) below that for $\alpha \geq 0$ and $\mu \in \mathcal{K}$

$$\int_E u^2 d\mu \leq \|R_\alpha \mu\|_\infty \cdot \mathcal{E}_\alpha(u,u), \quad u \in \mathcal{D}(\mathcal{E}).$$

Let $\mu \in \mathcal{K}$. We define the *Schrödinger form* by

$$\begin{cases} \mathcal{D}(\mathcal{E}^\mu) = \mathcal{D}(\mathcal{E}), \\ \mathcal{E}^\mu(u,v) = \mathcal{E}(u,v) - \displaystyle\int_E uv\, d\mu. \end{cases}$$

Denoting by $\mathcal{L}^\mu = \mathcal{L} + \mu$ the self-adjoint operator generated by the closed symmetric form $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$, $(-\mathcal{L}^\mu u, v)_m = \mathcal{E}^\mu(u,v)$. Let $\{P_t^\mu\}$ be the semigroup generated by $\mathcal{L}^\mu$, $P_t^\mu = e^{t\mathcal{L}^\mu}$. By the Feynman-Kac formula, P_t^μ is expressed by

$$P_t^\mu f(x) = \mathbb{E}_x[\exp(A_t^\mu) f(X_t)\,; t < \zeta].$$

It is known from [1] that $\{P_t^\mu\}$ has the strong Feller property.

For $\mu \in \mathcal{K}$, we set a function space:

$$\mathcal{H}^+(\mu) := \{h \in \mathcal{D}_{\rm loc}^\dagger(\mathcal{E}) \cap C(E_\partial) \mid h > 0 \text{ and } P_t^\mu h \leq h\}. \tag{4.2}$$

Suppose $\mathcal{H}^+(\mu) \neq \emptyset$. For $h \in \mathcal{H}^+(\mu)$, we define the bilinear form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ by

$$\begin{cases} \mathcal{D}(\mathcal{E}^{\mu,h}) = \{u \in L^2(E; h^2m) \,|\, hu \in \mathcal{D}(\mathcal{E}^{\mu})\}, \\ \mathcal{E}^{\mu,h}(u,v) = \mathcal{E}^{\mu}(hu, hv), \quad u, v \in \mathcal{D}(\mathcal{E}^{\mu,h}). \end{cases}$$

The closedness of $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ follows from that of $(\mathcal{E}^{\mu}, \mathcal{D}(\mathcal{E}))$. Then the semigroup $\{P_t^{\mu,h}\}$ generated by $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is h^2m-symmetric and expressed as

$$\begin{aligned} P_t^{\mu,h} f(x) &= \frac{1}{h(x)} P_t^{\mu}(hf)(x) \\ &= \frac{1}{h(x)} \mathbb{E}_x[\exp(A_t^{\mu}) h(X_t) f(X_t)\,;\, t < \zeta], \quad f \in \mathfrak{B}_b(E). \end{aligned} \tag{4.3}$$

Moreover, by the definition of $\mathcal{H}^+(\mu)$, $\{P_t^{\mu,h}\}$ is a Markovian semigroup and this implies that $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is a Dirichlet form on $L^2(E; h^2m)$.

Lemma 4.2. *For $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$, the function φ/h belongs to $\mathcal{D}(\mathcal{E}) \cap C_0(E)$.*

Proof. Let K be the support of φ and put $c = (\inf_{x \in K} h(x))^{-1}$. Then, for $(x, y) \in K \times K$

$$\begin{aligned} \left| \frac{\varphi}{h}(x) \right| &\leq c|\varphi(x)|, \\ \left| \frac{\varphi}{h}(x) - \frac{\varphi}{h}(y) \right| &\leq \frac{|\varphi(x) - \varphi(y)|}{h(x)} + \frac{|\varphi(x) - \varphi(y)|}{h(y)} + \frac{|h(x)\varphi(x) - h(y)\varphi(y)|}{h(x)h(y)} \\ &\leq 2c|\varphi(x) - \varphi(y)| + c^2|h(x)\varphi(x) - h(y)\varphi(y)|. \end{aligned}$$

Since φ and $h\varphi$ belong to $\mathcal{D}(\mathcal{E})$, the function φ/h also belongs to $\mathcal{D}(\mathcal{E})$ by [18, Theorem 1.5.2 (ii)]. □

Lemma 4.3. $\mathcal{D}(\mathcal{E}^{\mu,h}) \cap C_0(E) = \mathcal{D}(\mathcal{E}) \cap C_0(E)$.

Proof. By the definition of $\mathcal{D}(\mathcal{E}^{\mu,h})$, $u \in \mathcal{D}(\mathcal{E}^{\mu,h}) \cap C_0(E)$ if and only if $hu \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$. On the other hand, it follows from Lemma 4.2 that $hu \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$ if and only if $u \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$. □

Lemma 4.4. *The Dirichlet form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is regular.*

Proof. We see from Lemma 4.3 and the regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ that $\mathcal{D}(\mathcal{E}^{\mu,h}) \cap C_0(E)$ is dense in $C_0(E)$ with respect to the uniform norm.

Suppose $u \in \mathcal{D}(\mathcal{E}^{\mu,h})$. Then by the definition of $\mathcal{D}(\mathcal{E}^{\mu,h})$, $hu \in \mathcal{D}(\mathcal{E})$ and by the regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and (4.11), there exists a sequence $\{\varphi_n\} \subset \mathcal{D}(\mathcal{E}) \cap C_0(E)$ such that $\mathcal{E}^\mu(hu-\varphi_n, hu-\varphi_n)$ converges to 0 as $n\to\infty$. Then the function φ_n/h is in $\mathcal{D}(\mathcal{E})\cap C_0(E)$ by Lemma 4.2, and

$$\mathcal{E}^{\mu,h}\left(u-\frac{\varphi_n}{h}, u-\frac{\varphi_n}{h}\right)=\mathcal{E}^\mu(hu-\varphi_n, hu-\varphi_n)\to 0 \quad \text{as } n\to\infty,$$

which implies the the regularity of $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$. □

Let us denote by $\mathbb{M}^{\mu,h}$ the Hunt process generated by the regular Dirichlet form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$. Then by (4.3), the irreducibility of $\mathbb{M}^{\mu,h}$ follows from that of $\mathbb{M}$ because $\exp(A_t^\mu)h(X_t)>0$ for $t<\zeta$ $\mathbb{P}_x$-a.s.

Remark 4.5. The process $\mathbb{M}^{\mu,h}$ possesses the following property:

(LSC): For $\gamma>0$, every γ-excessive function is lower-semi-continuous.

Indeed, let $R_\gamma^{\mu,h}$ be the γ-resolvent of $\mathbb{M}^{\mu,h}$. Then for $g\in\mathfrak{B}_b(E)$,

$$\frac{1}{h(x)}R_\gamma^\mu(g\,(h\wedge n))(x)\uparrow R_\gamma^{\mu,h}g(x), \quad \text{as } n\to\infty.$$

The function $R_\gamma^\mu(g\,(h\wedge n))$ is continuous on E by the strong Feller property of P_t^μ, and thus $R_\gamma^{\mu,h}g$ is lower-semi-continuous. By [18, Lemma A.2.8], for any γ-excessive function u, there exists a sequence $\{g_n\}$ of bounded nonnegative Borel functions such that $R_\gamma^{\mu,h}g_n(x)\uparrow u(x)$ as $n\to\infty$. Hence **(LSC)** holds.

$\mathcal{D}_e(\mathcal{E}^\mu)$ denotes the family of functions u on E such that $|u|<\infty$ m-a.e. and there exists an $\mathcal{E}^\mu$-Cauchy sequence $\{u_n\}$ of $\mathcal{D}(\mathcal{E}^\mu)$ such that $\lim_{n\to\infty}u_n=u$ m-a.e. For $u\in\mathcal{D}_e(\mathcal{E}^\mu)$ and the sequence $\{u_n\}$, define

$$\mathcal{E}^\mu(u,u):=\lim_{n\to\infty}\mathcal{E}^\mu(u_n,u_n).$$

Lemma 4.6. *Let $\mathcal{D}_e(\mathcal{E}^{\mu,h})$ be the extended Dirichlet space of $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$. Then*

$$\begin{cases}\mathcal{D}_e(\mathcal{E}^{\mu,h})=\{u\,|\,hu\in\mathcal{D}_e(\mathcal{E}^\mu)\},\\ \mathcal{E}^{\mu,h}(u,u)=\mathcal{E}^\mu(hu,hu), \quad u\in\mathcal{D}_e(\mathcal{E}^{\mu,h}).\end{cases}$$

Proof. Suppose that $hu\in\mathcal{D}_e(\mathcal{E}^\mu)$. Then there exists an $\mathcal{E}^\mu$-Cauchy sequence $\{\varphi_n\}\subset\mathcal{D}(\mathcal{E}^\mu)$ such that $\lim_{n\to\infty}\varphi_n=hu$ m-a.e. Hence, the sequence $\{\varphi_n/h\}\subset\mathcal{D}(\mathcal{E}^{\mu,h})$ satisfies the following condition: $\lim_{n\to\infty}\varphi_n/h=u$ h^2m-a.e. and

$$\mathcal{E}^{\mu,h}\left(\frac{\varphi_n}{h}-\frac{\varphi_m}{h}, \frac{\varphi_n}{h}-\frac{\varphi_m}{h}\right)=\mathcal{E}^\mu(\varphi_n-\varphi_m, \varphi_n-\varphi_m)\to 0$$

as $m, n \to \infty$, which implies $u \in \mathcal{D}_e(\mathcal{E}^{\mu,h})$.

For any $u \in \mathcal{D}_e(\mathcal{E}^{\mu,h})$, there exists an $\mathcal{E}^{\mu,h}$-Cauchy sequence $\{u_n\} \subset \mathcal{D}(\mathcal{E}^{\mu,h})$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. Then we have

$$\mathcal{E}^{\mu}(hu_n - hu_m, hu_n - hu_m) = \mathcal{E}^{\mu,h}(u_n - u_m, u_n - u_m) \to 0$$

as $m, n \to \infty$. Therefore $hu \in \mathcal{D}_e(\mathcal{E}^{\mu})$. Moreover, it holds that

$$\mathcal{E}^{\mu,h}(u,u) = \lim_{n\to\infty} \mathcal{E}^{\mu,h}(u_n, u_n) = \lim_{n\to\infty} \mathcal{E}^{\mu}(hu_n, hu_n) = \mathcal{E}^{\mu}(hu, hu).$$

□

4.2 Hardy-type inequalities

We next consider the following Hardy-type inequality:

$$\int_E u^2\, d\mu \le \mathcal{E}(u,u), \quad u \in \mathcal{D}(\mathcal{E}), \tag{4.4}$$

where μ is a positive smooth measure. We set a function space:

$$\widetilde{\mathcal{H}}^+(\mu) := \left\{ h \;\middle|\; \begin{array}{l} h \in \mathcal{D}^{\dagger}_{\mathrm{loc}}(\mathcal{E}) \cap C(E_\partial) \text{ is strictly positive and} \\ \mathcal{E}^{\mu}(h,\varphi) \ge 0 \text{ for all } \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0^+(E) \end{array} \right\}.$$

As an application of Girsanov's transformations, we shall show that the inequality (4.4) holds whenever $\widetilde{\mathcal{H}}^+(\mu) \neq \emptyset$.

Lemma 4.7. *For $h \in \widetilde{\mathcal{H}}^+(\mu)$, there exists a positive smooth measure ν such that*

$$N_t^{[h]} = -\int_0^t h(X_s)\, dA_s^{\mu} - A_t^{\nu}, \quad t < \zeta,\ \mathbb{P}_x\text{-a.s. q.e. } x \in E.$$

Proof. Let $\mathcal{L} := \mathcal{D}(\mathcal{E}) \cap C_0(E)$. Then $\mathcal{L}$ is a *Stone vector lattice*, i.e., $f \wedge g \in \mathcal{L}$, $f \wedge 1 \in \mathcal{L}$ for any $f, g \in \mathcal{L}$. For $h \in \widetilde{\mathcal{H}}^+(\mu)$, define the functional I by

$$I(\varphi) = \mathcal{E}(h,\varphi) - \int_E h\varphi\, d\mu, \quad \varphi \in \mathcal{L}.$$

Then $I(\varphi)$ is pre-integral, that is, $I(\varphi_n) \downarrow 0$ whenever $\varphi_n \in \mathcal{L}$ and $\varphi_n(x) \downarrow 0$ for all $x \in E$. Indeed, let $\psi \in \mathcal{D}(\mathcal{E}) \cap C_0^+(E)$ such that $\psi = 1$ on supp$[\varphi_1]$. Then $\varphi_n \le \|\varphi_n\|_\infty \psi$ and

$$I(\varphi_n) \le \|\varphi_n\|_\infty \cdot I(\psi) \downarrow 0 \quad \text{as } n \to \infty.$$

Noting that the smallest σ-field generated by $\mathcal{L}$ is identical with the Borel σ-field, we see from [10, Theorem 4.5.2] that there exists a Borel measure ν such that

$$I(\varphi) = \int_E \varphi \, d\nu, \quad \varphi \in \mathcal{L}. \tag{4.5}$$

We shall prove that the measure ν is smooth. Let K be a compact set of zero capacity and take a relatively compact open set D such that $K \subset D$. Then there exists a sequence $\{\varphi_n\} \subset \mathcal{D}(\mathcal{E}) \cap C_0^+(D)$ such that $\varphi_n \geq 1$ on K and $\mathcal{E}_1(\varphi_n, \varphi_n) \to 0$ as $n \to \infty$ ([18, Lemma 2.2.7]). Take $\psi \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$ with $\psi = 1$ on D and $0 \leq \psi \leq 1$ on E. Noting that $h\psi = h$ on D and $h\psi \leq h$, we have

$$\begin{aligned}
\mathcal{E}(h\psi, \varphi_n) = {} & \frac{1}{2} \int_E d\mu^c_{\langle h, \varphi_n \rangle} + \int_{D \times D} (h(x) - h(y))(\varphi_n(x) - \varphi_n(y)) J(dx, dy) \\
& + 2 \int_{D \times D^c} (h(x) - h\psi(y)) \varphi_n(x) J(dx, dy) + \int_E h\varphi_n \, d\kappa \\
\geq {} & \frac{1}{2} \int_E d\mu^c_{\langle h, \varphi_n \rangle} + \int_{D \times D} (h(x) - h(y))(\varphi_n(x) - \varphi_n(y)) J(dx, dy) \\
& + 2 \int_{D \times D^c} (h(x) - h(y)) \varphi_n(x) J(dx, dy) + \int_E h\varphi_n \, d\kappa \\
= {} & \mathcal{E}(h, \varphi_n).
\end{aligned}$$

Consequently,

$$\nu(K) \leq \int_E \varphi_n \, d\nu = \mathcal{E}(h, \varphi_n) - \int_E h\varphi_n \, d\mu \leq \mathcal{E}(h\psi, \varphi_n)$$

and the right-hand side is dominated by

$$\mathcal{E}(h\psi, h\psi)^{1/2} \cdot \mathcal{E}(\varphi_n, \varphi_n)^{1/2} \longrightarrow 0 \ \text{ as } n \to \infty.$$

Therefore ν is smooth.

The equation (4.5) is equivalent to

$$\mathcal{E}(h, \varphi) = \int_E \varphi h \, d\mu + \int_E \varphi \, d\nu = \int_E \varphi \, (h \, d\mu + d\nu).$$

Therefore, we have the lemma by Corollary 2.6. □

Suppose $\widetilde{\mathcal{H}}^+(\mu) \neq \emptyset$ and let $h \in \widetilde{\mathcal{H}}^+(\mu)$. Define a local martingale on the random interval $[\![0, \zeta[\![$ by $M_t = \int_0^t (h(X_{s-}))^{-1} dM_s^{[h]}$ and let L_t^h be the solution to the following stochastic differential equation:

$$L_t^h = 1 + \int_0^t L_{s-}^h \, dM_s, \quad t < \zeta.$$

Define

$$d\widetilde{\mathbb{P}}_x = L_t^h d\mathbb{P}_x \quad \text{on } \mathscr{F}_t \cap \{t < \zeta\} \text{ for } x \in E.$$

As we have shown in §3, $\widetilde{\mathbb{M}}^h := (\Omega, \mathscr{F}_t, X_t, \widetilde{\mathbb{P}}_x)$ is an h^2m-symmetric right process on E.

On the other hand, on account of Lemma 4.7, there exists a positive smooth measure ν such that

$$h(X_t) = h(X_0) + M_t^{[h]} - \int_0^t h(X_s) dA_s^\mu - A_t^\nu, \quad t < \zeta, \ \mathbb{P}_x\text{-a.s. q.e. } x \in E. \tag{4.6}$$

By Itô's formula applied to the semimartingale $h(X_t)$ with the function $\log x$, we have

$$\begin{aligned}\log h(X_t) = \log h(X_0) + M_t + \int_0^t \frac{1}{h(X_{s-})} dN_s^{[h]} - \frac{1}{2}\langle M^c \rangle_t \\ + \sum_{0<s\le t} \left(\log \frac{h(X_s)}{h(X_{s-})} + 1 - \frac{h(X_s)}{h(X_{s-})} \right) \quad \mathbb{P}_x\text{-a.s.}\end{aligned}$$

for q.e. $x \in E$, which leads to

$$\begin{aligned}&\frac{h(X_t)}{h(X_0)} \exp\left(-\int_0^t \frac{1}{h(X_{s-})} dN_s^{[h]} \right) \\ &\quad = \exp\left(M_t - \frac{1}{2}\langle M^c \rangle_t + \sum_{0<s\le t} \left(\log \frac{h(X_s)}{h(X_{s-})} + 1 - \frac{h(X_s)}{h(X_{s-})} \right) \right) \\ &\quad = L_t^h\end{aligned}$$

$\mathbb{P}_x$-a.s. for q.e. $x \in E$, and thus for all $x \in E$. Therefore we see from (4.6) that L_t^h has the following expression:

$$L_t^h = \frac{h(X_t)}{h(X_0)} \exp\left(A_t^\mu + \int_0^t \frac{1}{h(X_s)} dA_s^\nu \right).$$

Hence, a transition semigroup $\{\widetilde{P}_t^h\}$ of $\widetilde{\mathbb{M}}^h$ is expressed as

$$\widetilde{P}_t^h u(x) = \mathbb{E}_x\left[L_t^h u(X_t) \, ; t < \zeta \right] = \frac{1}{h(x)} P_t^\eta (hu)(x), \quad u \in \mathfrak{B}_b(E), \tag{4.7}$$

where $\eta := \mu + \frac{1}{h}\nu$ and $P_t^\eta f(x) = \mathbb{E}_x\left[\exp(A_t^\eta) f(X_t); t < \zeta\right]$. The identity (4.7) implies that

$$\frac{1}{t}(u - P_t^\eta u, u)_m = \frac{1}{t}\left(\frac{u}{h} - \widetilde{P}_t^h\left(\frac{u}{h}\right), \frac{u}{h} \right)_{h^2 m}. \tag{4.8}$$

Let $(\widetilde{\mathcal{E}}^h, \mathcal{D}(\widetilde{\mathcal{E}}^h))$ be the Dirichlet form on $L^2(E; h^2 m)$ generated by $\widetilde{\mathbb{M}}^h$. On account of Theorem 3.5, there exists an $\mathcal{E}$-nest $\{F_k\}$ such that $\bigcup_{k\geq 1} \mathcal{D}_b(\mathcal{E})_{F_k} \subset \mathcal{D}\big(\widetilde{\mathcal{E}}^h\big)$ and

$$\begin{aligned}\widetilde{\mathcal{E}}^h(u,u) = \frac{1}{2}\int_E h^2\, d\mu^c_{\langle u\rangle} + \int_{E\times E} (u(x)-u(y))^2 h(x)h(y)J(dx,dy) \\ + h(\partial)\int_E u(x)^2 h(x)\kappa(dx)\end{aligned}$$

for $u \in \bigcup_{k\geq 1} \mathcal{D}_b(\mathcal{E})_{F_k}$. If u is in $\bigcup_{k\geq 1} \mathcal{D}_b(\mathcal{E})_{F_k}$, then so is u/h by the same argument as in the proof of Lemma 4.2. Thus, we see from (4.8) that the identity

$$\begin{aligned}&\mathcal{E}(u,u) - \int_E u^2\, d\mu - \int_E \frac{u^2}{h}\, d\nu \\ &\quad= \frac{1}{2}\int_E h^2\, d\mu^c_{\langle u/h\rangle} + \int_{E\times E}\Big(\frac{u}{h}(x) - \frac{u}{h}(y)\Big)^2 h(x)h(y)J(dx,dy) \\ &\quad+ h(\partial)\int_E \frac{u^2}{h}(x)\,\kappa(dx)\end{aligned} \tag{4.9}$$

holds for $u \in \bigcup_{k\geq 1} \mathcal{D}_b(\mathcal{E})_{F_k}$. Now we obtain the next theorem.

Theorem 4.8. *The identity* (4.9) *holds for all* $u \in \mathcal{D}(\mathcal{E})$.

Proof. For $u \in \mathcal{D}(\mathcal{E})$, there exists a sequence $\{u_n\} \subset \bigcup_{k\geq 1} \mathcal{D}_b(\mathcal{E})_{F_k}$ such that $u_n \to u$ q.e. and $\mathcal{E}(u_n,u_n) \to \mathcal{E}(u,u)$ as $n \to \infty$ ([18, Theorem 2.1.4]). Then we have by Fatou's lemma

$$\int_E u^2\left(d\mu + \frac{1}{h}d\nu\right) \leq \liminf_{n\to\infty}\int_E u_n^2\left(d\mu + \frac{1}{h}d\nu\right).$$

From (4.9), the right-hand side is bounded by

$$\liminf_{n\to\infty}\mathcal{E}(u_n,u_n) = \mathcal{E}(u,u) < \infty,$$

and thus $u \in L^2(E; \mu + \frac{1}{h}\nu)$.

On the other hand, by using (4.9) again, we have

$$\begin{aligned}\widetilde{\mathcal{E}}^h\left(\frac{u_n}{h}, \frac{u_n}{h}\right) &= \mathcal{E}(u_n,u_n) - \int_E u_n^2\, d\mu - \int_E \frac{u_n^2}{h}\, d\nu \\ &\leq \sup_n \mathcal{E}(u_n,u_n) \ < \infty.\end{aligned}$$

Since $u_n/h \to u/h$ q.e., u/h belongs to $\mathcal{D}_e(\widetilde{\mathcal{E}}^h) \cap L^2(E; h^2 m) = \mathcal{D}(\widetilde{\mathcal{E}}^h)$ by [36, Definition 1.6] and [18, Theorem 1.5.2]. Therefore, on account of the relation (4.8), we see that the equation (4.9) holds for all $u \in \mathcal{D}(\mathcal{E})$. □

Theorem 4.8 tells us that if $\widetilde{\mathcal{H}}^+(\mu) \neq \emptyset$, then Hardy's inequality (4.4) holds and the remainder term is given by

$$\widetilde{\mathcal{E}}^h\left(\frac{u}{h}, \frac{u}{h}\right) + \int_E \frac{u^2}{h}\, d\nu.$$

Example 4.9. Denote by S_{00} the family of finite energy measures of finite energy integral with bounded potentials. For $\mu \in S_{00}$ and $\alpha > 0$, the α-potential $R_\alpha\mu$ is in $\mathcal{D}(\mathcal{E})$ and

$$\mathcal{E}_\alpha(R_\alpha\mu, \varphi) - \int_E \varphi\, d\mu = 0 \quad \text{for all } \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(E).$$

Since $\int_E \varphi\, d\mu = \int_E R_\alpha\mu \cdot \varphi \frac{1}{R_\alpha\mu} d\mu$, the potential $R_\alpha\mu$ is in the space $\widetilde{\mathcal{H}}^+\left(\frac{1}{R_\alpha\mu} \cdot \mu\right)$ associated with $(\mathcal{E}_\alpha, \mathcal{D}(\mathcal{E}))$. Thus, we see from Theorem 4.8 that

$$\mathcal{E}_\alpha(u,u) \geq \int_E \frac{u^2}{R_\alpha\mu} d\mu \geq \frac{1}{\|R_\alpha\mu\|_\infty} \int_E u^2 d\mu \tag{4.10}$$

for all $u \in \mathcal{D}(\mathcal{E})$.

Let μ be a smooth measure. Then by [18, Theorem 2.2.4], there exists a compact $\mathcal{E}$-nest $\{F_n\}$ such that $\mu_n := \mathbb{1}_{F_n} \cdot \mu \in S_{00}$ for each n. By the inequality (4.10), we have

$$\int_E u^2 d\mu_n \leq \|R_\alpha\mu_n\|_\infty \cdot \mathcal{E}_\alpha(u,u).$$

Hence, by letting $n \to \infty$, we obtain

$$\int_E u^2 d\mu \leq \|R_\alpha\mu\|_\infty \cdot \mathcal{E}_\alpha(u,u) \quad \text{for all } u \in \mathcal{D}(\mathcal{E}). \tag{4.11}$$

This inequality is well-known as the Stollmann-Voigt inequality ([37]).

Recall that $\mathcal{H}^+(\mu)$ is the space of P_t^μ-excessive functions defined by (4.2). We next show that the space $\mathcal{H}^+(\mu)$ coincides with $\widetilde{\mathcal{H}}^+(\mu)$ under the condition $\kappa = 0$. Here κ is the killing measure of $\mathbb{M}$.

Lemma 4.10. *$\widetilde{\mathcal{H}}^+(\mu)$ is contained in $\mathcal{H}^+(\mu)$. If $\kappa = 0$, then the opposite inclusion holds.*

Proof. Take $h \in \widetilde{\mathcal{H}}^+(\mu)$ and let $\{\widetilde{P}_t^h\}_{t\geq 0}$ be the transition semigroup of the Girsanov transformed process $\widetilde{\mathbb{M}}^h$ defined in pp. 40. We see from the identity (4.7) that

$$P_t^\mu h(x) \leq h(x) \cdot \widetilde{P}_t^h 1(x) \leq h(x)$$

and thus h is in $\mathcal{H}^+(\mu)$.

We next suppose $\kappa = 0$. Take $h \in \mathcal{H}^+(\mu)$ and $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0^+(E)$. Let $K := \operatorname{supp}[\varphi]$. Then for any $u \in \mathcal{D}(\mathcal{E}) \cap C_0(E)$ with $u = 1$ on K and $0 \leq u \leq 1$ on E, it holds that

$$\mathcal{E}(hu, \varphi) - \int_E hu\varphi \, d\mu \geq 0.$$

Indeed, the left-hand side is equal to

$$\lim_{t \downarrow 0} \frac{1}{t} \big(hu - P_t^\mu(hu), \varphi\big)_m = \lim_{t \downarrow 0} \frac{1}{t} \Big(\big(h, \varphi\big)_m - \big(P_t^\mu(hu), \varphi\big)_m \Big).$$

Since $P_t^\mu(hu) \leq P_t^\mu h \leq h$, the right-hand side is nonnegative. Take a sequence of relatively compact open sets $\{D_n\}$ such that $D_n \uparrow E$ and $K \subset D_n$ for each n. Then there exists a sequence $\{u_n\} \subset \mathcal{D}(\mathcal{E}) \cap C_0(E)$ such that $u_n = 1$ on D_n and $0 \leq u_n \leq 1$ on E. Since $u_n = 1$ on $K = \operatorname{supp}[\varphi]$, we have

$$\begin{aligned}
&\mathcal{E}(hu_n, \varphi) \\
&= \frac{1}{2} \int_E d\mu^c_{\langle hu_n, \varphi \rangle} + \int_{E \times E} (hu_n(x) - hu_n(y))(\varphi(x) - \varphi(y)) J(dx, dy) \\
&\quad + \int_E hu_n \varphi \, d\kappa \\
&= \frac{1}{2} \int_E d\mu^c_{\langle h, \varphi \rangle} + \int_{K \times K} (h(x) - h(y))(\varphi(x) - \varphi(y)) J(dx, dy) \\
&\quad + 2 \int_{K \times (D_1 \cap K^c)} (h(x) - h(y))(\varphi(x) - \varphi(y)) J(dx, dy) \\
&\quad + 2 \int_{K \times (D_1^c \cap K^c)} (h(x) - hu_n(y))(\varphi(x) - \varphi(y)) J(dx, dy).
\end{aligned}$$

Since $|h(y)u_n(y)\varphi(x)| \leq h(y)\varphi(x)$ and $\int_{K \times (D_1^c \cap K^c)} h(y)\varphi(x) J(dx, dy) < \infty$, the fourth term on the right-hand side tends to

$$2 \int_{K \times (D_1^c \cap K^c)} (h(x) - h(y))(\varphi(x) - \varphi(y)) J(dx, dy)$$

as $n \to \infty$ by the Lebesgue convergence theorem. Consequently, we have

$$\mathcal{E}(h, \varphi) - \int_E h\varphi \, d\mu = \lim_{n \to \infty} \left(\mathcal{E}(hu_n, \varphi) - \int_E hu_n \varphi \, d\mu \right) \geq 0.$$

□

4.3 Existence of excessive functions

Let $\mu \in \mathcal{K}_\infty$, the set of Green-tight measures. In this section, we consider the existence of a function in $\mathcal{H}^+(\mu)$. Define

$$\lambda(\mu) := \inf\left\{\mathcal{E}(u,u) \,\middle|\, u \in \mathcal{D}(\mathcal{E}^\mu), \int_E u^2 d\mu = 1\right\}. \tag{4.12}$$

Note that the condition $\lambda(\mu) \geq 1$ is equivalent to Hardy's inequality (4.4). Hence, we see from Theorem 4.8 that the next result holds.

Lemma 4.11. *If $\lambda(\mu) < 1$, then $\mathcal{H}^+(\mu) = \emptyset$.*

4.3.1 The case $\lambda(\mu) > 1$

In this subsection we treat the case that $\lambda(\mu) > 1$. For a smooth measure μ, let g_μ be a so-called *gauge function* defined by

$$g_\mu(x) := \mathbb{E}_x\left[\exp(A_\zeta^\mu)\right].$$

It is known in [3, Thoerem 5.1] that g_μ is a bounded function if and only if $\lambda(\mu) > 1$.

Lemma 4.12. *Assume that $\lambda(\mu) > 1$. Then the gauge function g_μ is excessive with respect to $\{P_t^\mu\}$, $P_t^\mu g_\mu(x) \uparrow g_\mu(x)$ as $t \downarrow 0$.*

Proof. Noting that $\mathbb{E}_x\left[e^{A_\zeta^\mu(\theta_t)} \middle| \mathscr{F}_t\right] = \mathbb{E}_{X_t}\left[e^{A_\zeta^\mu}\right]$ by the Markov property,

$$\begin{aligned} P_t^\mu g_\mu(x) = \mathbb{E}_x\left[e^{A_t^\mu} g_\mu(X_t);\, t < \zeta\right] &= \mathbb{E}_x\left[e^{A_t^\mu}\mathbb{E}_{X_t}\left[e^{A_\zeta^\mu}\right];\, t < \zeta\right] \\ &= \mathbb{E}_x\left[\mathbb{E}_x\left[e^{A_t^\mu + A_\zeta^\mu(\theta_t)}\mathbb{1}_{\{t<\zeta\}} \middle| \mathscr{F}_t\right]\right]. \end{aligned}$$

Since $A_t^\mu + A_\zeta^\mu(\theta_t) = A_\zeta^\mu$ on $\{t < \zeta\}$, the right-hand side equals $\mathbb{E}_x\left[e^{A_\zeta^\mu}; t < \zeta\right]$. Therefore

$$P_t^\mu g_\mu(x) = \mathbb{E}_x\left[e^{A_\zeta^\mu}; t < \zeta\right] \uparrow \mathbb{E}_x\left[e^{A_\zeta^\mu}\right] = g_\mu(x) \quad \text{as} \quad t \downarrow 0.$$

□

Lemma 4.13. *It holds that*

$$g_\mu(x) = 1 + R(g_\mu \cdot \mu)(x).$$

Proof. Fix $x \in E$ and define a uniformly integrable martingale $\{M_t\}$ by

$$M_t = \mathbb{E}_x\big[\exp(A^\mu_\zeta)|\mathscr{F}_t\big].$$

Since $A^\mu_t + A^\mu_\zeta(\theta_t) = A^\mu_\zeta$ on $\{t < \zeta\}$, we have

$$\begin{aligned} e^{-A^\mu_t} M_t \mathbb{1}_{\{t<\zeta\}} = e^{-A^\mu_t}\mathbb{E}_x\Big[e^{A^\mu_\zeta}\mathbb{1}_{\{t<\zeta\}}\big|\mathscr{F}_t\Big] &= e^{-A^\mu_t}\mathbb{E}_x\Big[e^{A^\mu_t + A^\mu_\zeta(\theta_t)}\mathbb{1}_{\{t<\zeta\}}\big|\mathscr{F}_t\Big] \\ &= \mathbb{E}_x\Big[e^{A^\mu_\zeta(\theta_t)}\big|\mathscr{F}_t\Big]\mathbb{1}_{\{t<\zeta\}}. \end{aligned}$$

By the Markov property, the right-hand side equals

$$\mathbb{E}_{X_t}\big[e^{A^\mu_\zeta}\big]\mathbb{1}_{\{t<\zeta\}} = g_\mu(X_t)\mathbb{1}_{\{t<\zeta\}},$$

and thus

$$\int_0^t g_\mu(X_s)dA^\mu_s = \int_0^t e^{-A^\mu_s}M_s dA^\mu_s.$$

Hence by Itô's formula,

$$\begin{aligned} e^{-A^\mu_t}M_t &= M_0 + \int_0^t e^{-A^\mu_s}dM_s - \int_0^t e^{-A^\mu_s}M_s dA^\mu_s \\ &= M_0 + \int_0^t e^{-A^\mu_s}dM_s - \int_0^t g_\mu(X_s)dA^\mu_s. \end{aligned}$$

Since $\int_0^t e^{-A^\mu_s}dM_s$ is a $\mathbb{P}_x$-martingale, $\mathbb{E}_x\left[\int_0^t e^{-A^\mu_s}dM_s\right] = 0$ and thus

$$\mathbb{E}_x[M_0] = \mathbb{E}_x\left[e^{-A^\mu_\zeta}M_\zeta\right] + \mathbb{E}_x\left[\int_0^\zeta g_\mu(X_s)dA^\mu_s\right].$$

Noting that $\mathbb{E}_x[M_0] = g_\mu(x)$, $e^{-A^\mu_\zeta}M_\zeta = e^{-A^\mu_\zeta}e^{A^\mu_\zeta} = 1$ and

$$\mathbb{E}_x\left[\int_0^\zeta g_\mu(X_s)dA^\mu_s\right] = R(g_\mu \cdot \mu)(x),$$

we have the lemma. □

Theorem 4.14. *The gauge function g_μ belongs to $\mathcal{H}^+(\mu) \cap C_b(E_\partial)$.*

Proof. First note that $g_\mu \cdot \mu \in \mathcal{K}_\infty$. Hence, on account of Lemma 4.12 and 4.13, we have only to prove that $R\nu$ is in $C_\infty(E) \cap \mathcal{D}_{\rm loc}(\mathcal{E})$ for any $\nu \in \mathcal{K}_\infty$. Here $C_\infty(E)$ is the set of continuous functions vanishing at infinity. Since $R\nu \in \mathfrak{B}_b(E)$ by [3, Proposition 2.2], $P_t\big(R\nu\big) \in C_b(E)$ by the strong Feller property. We have by the Markov property

$$\begin{aligned} \|R\nu - P_t\big(R\nu\big)\|_\infty &= \sup_{x\in E}\big(\mathbb{E}_x\big[A_\zeta^\nu\big] - \mathbb{E}_x\big[A_\zeta^\nu(\theta_t)\big]\big) \\ &= \sup_{x\in E}\mathbb{E}_x[A_t^\nu]. \end{aligned}$$

Since the right-hand side tends to 0 as $t \downarrow 0$ by (4.1), $R\nu$ belongs to $C_b(E)$.

We take an increasing sequence of compact sets $\{K_n\}$ such that $K_n \uparrow E$ and

$$\|R(\mathbb{1}_{K_n^c}\nu)\|_\infty \downarrow 0 \quad \text{as } n \to \infty.$$

The existence of such $\{K_n\}$ follows from the definition of a Green-tight measure. Note that for each n, a measure $\nu_n := \mathbb{1}_{K_n}\nu$ is also Green-tight, and thus $R\nu_n \in C_b(E)$ by the argument above. Since

$$\int_E\int_E R(x,y)\nu_n(dx)\nu_n(dy) < \infty,$$

it follows from [38, Lemma 3.1] that $R\nu_n \in \mathcal{D}_e(\mathcal{E})$, and thus $R\nu_n(x) \to 0$ as $x \to \partial$. Thus $R\nu_n$ belongs to $C_\infty(E)$ and

$$\sup_{x\in E}|R\nu(x) - R\nu_n(x)| = \sup_{x\in E}|R(\mathbb{1}_{K_n^c}\nu)(x)| \downarrow 0 \quad \text{as } n \to \infty.$$

Therefore $R\nu$ is in $C_\infty(E)$.

The function $R\nu$ is an element of $\mathcal{D}_{\mathrm{loc}}(\mathcal{E})$ because a bounded excessive function with respect to $\{P_t\}$ belongs to $\mathcal{D}_{\mathrm{loc}}(\mathcal{E})$. Indeed, take a bounded excessive function u and set $u_n := u \wedge \|u\|_\infty(nR_1f \wedge 1)$ for a strictly positive bounded function $f \in L^2(E;m)$. We further set $E_n := \{x \in E : R_1f(x) > 1/n\}$. Then E_n is an open set by the strong Feller property and $\bigcup_{n\in\mathbb{N}} E_n = E$. Since $u_n \leq \|u\|_\infty(nR_1f \wedge 1)$, $u_n \in \mathcal{D}(\mathcal{E})$ by [18, Lemma 2.3.2] and $u = u_n$ on E_n. Therefore u is in $\mathcal{D}_{\mathrm{loc}}(\mathcal{E})$. □

On account of Theorem 4.14, we can define the Dirichlet form $(\mathcal{E}^{\mu,g_\mu}, \mathcal{D}(\mathcal{E}^{\mu,g_\mu}))$ by

$$\begin{cases} \mathcal{D}(\mathcal{E}^{\mu,g_\mu}) = \{u \in L^2(E; g_\mu^2 m) \mid g_\mu u \in \mathcal{D}(\mathcal{E}^\mu)\}, \\ \mathcal{E}^{\mu,g_\mu}(u,v) = \mathcal{E}^\mu(g_\mu u, g_\mu v), \quad u,v \in \mathcal{D}(\mathcal{E}^{\mu,g_\mu}). \end{cases}$$

Lemma 4.15. *The Dirichlet form $(\mathcal{E}^{\mu,g_\mu}, \mathcal{D}(\mathcal{E}^{\mu,g_\mu}))$ is transient.*

Proof. From the definition of $\lambda(\mu)$,

$$\mathcal{E}(u,u) \geq \lambda(\mu)\int_E u^2 d\mu, \quad u \in \mathcal{D}(\mathcal{E}^\mu),$$

and thus

$$\mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) - \int_E u^2 d\mu \geq \left(\frac{\lambda(\mu)-1}{\lambda(\mu)}\right)\cdot \mathcal{E}(u,u), \quad u \in \mathcal{D}(\mathcal{E}^\mu). \tag{4.13}$$

Take $v \in \mathcal{D}_e(\mathcal{E}^{\mu,g_\mu})$ with $\mathcal{E}^{\mu,g_\mu}(v,v) = 0$, where $\mathcal{D}_e(\mathcal{E}^{\mu,g_\mu})$ denotes the extended Dirichlet space of $(\mathcal{E}^{\mu,g_\mu}, \mathcal{D}(\mathcal{E}^{\mu,g_\mu}))$. Then there exists a sequence $\{v_n\} \subset \mathcal{D}(\mathcal{E}^{\mu,g_\mu})$ such that $v_n \to v$ m-a.e. and $\mathcal{E}^{\mu,g_\mu}(v_n,v_n) \to 0$ as $n \to \infty$. We have by (4.13)

$$\begin{aligned}\mathcal{E}(g_\mu v_n, g_\mu v_n) &\leq \left(\frac{\lambda(\mu)}{\lambda(\mu)-1}\right)\cdot \mathcal{E}^\mu(g_\mu v_n, g_\mu v_n)\\ &= \left(\frac{\lambda(\mu)}{\lambda(\mu)-1}\right)\cdot \mathcal{E}^{\mu,g_\mu}(v_n,v_n) \ \to 0 \quad \text{as} \ \ n\to\infty.\end{aligned}$$

Therefore $g_\mu v \in \mathcal{D}_e(\mathcal{E})$ and $\mathcal{E}(g_\mu v, g_\mu v) = 0$, which implies that $g_\mu v = 0$ m-a.e. because of the transience of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and [18, Theorem 1.6.2]. Since the function g_μ is strictly positive, $v = 0$ m-a.e. and thus $(\mathcal{E}^{\mu,g_\mu}, \mathcal{D}(\mathcal{E}^{\mu,g_\mu}))$ is transient. □

Let R^μ_β be the β-resolvent of $\mathcal{L}^\mu$,

$$R^\mu_\beta f(x) = \int_0^\infty e^{-\beta t}P^\mu_t f(x)dt = \mathbb{E}_x\left[\int_0^\zeta e^{-\beta t + A^\mu_t} f(X_t)dt\right], \quad f \in \mathfrak{B}_b(E).$$

We write R^μ for R^μ_0 simply. Denote by $\mathfrak{B}^+_{b,0}(E)$ the set of nonnegative bounded functions on E with compact support. Next lemmas are used to show the existence of an excessive function when $\lambda(\mu) = 1$.

Lemma 4.16. *Let $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) > 1$. Then for $\varphi \in \mathfrak{B}^+_{b,0}(E)$, $R^\mu\varphi$ is bounded.*

Proof. Put $K := \text{supp}[\varphi]$. Note that P^{μ,g_μ}_t is a transient semigroup with **(LSC)** by Lemma4.15 and Remark 4.5. Hence, we see from [19, Corollary 2.3] that $R^{\mu,g_\mu}\mathbb{1}_K$ is a bounded function. Here R^{μ,g_μ} is the Green operator of $(\mathcal{E}^{\mu,g_\mu}, \mathcal{D}(\mathcal{E}^{\mu,g_\mu}))$:

$$R^{\mu,g_\mu} f = \frac{1}{g_\mu} R^\mu(g_\mu \cdot f).$$

Noting that $\varphi \leq \|\varphi\|_\infty \mathbb{1}_K g_\mu$, we have

$$R^\mu\varphi(x) \leq \|\varphi\|_\infty R^\mu(1_K g_\mu)(x) = \|\varphi\|_\infty\, g_\mu(x)\cdot R^{\mu,g_\mu}\mathbb{1}_K(x).$$

Since g_μ and $R^{\mu,g_\mu}\mathbb{1}_K$ are bounded, the lemma holds. □

Lemma 4.17. *Let $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) > 1$. Then for $\varphi \in \mathfrak{B}^+_{b,0}(E)$, $R^\mu\varphi \in \mathcal{D}_e(\mathcal{E}^\mu)$ and*

$$\mathcal{E}^\mu(R^\mu\varphi, u) = \int_E \varphi u \, dm, \quad u \in \mathcal{D}_e(\mathcal{E}^\mu).$$

Proof. Put $K := \mathrm{supp}[\varphi]$. Then we have by Lemma 4.16

$$\int_E \frac{\varphi}{g_\mu} \cdot R^{\mu, g_\mu}\left(\frac{\varphi}{g_\mu}\right) g_\mu^2 \, dm = \int_E \varphi \cdot R^\mu\varphi \, dm \le m(K) \, \|\varphi\|_\infty \cdot \|R^\mu\varphi\|_\infty < \infty.$$

Thus [18, Theorem 1.5.4] and Lemma 4.6 tell us that $R^\mu\varphi = g_\mu \, R^{\mu, g_\mu}(\varphi/g_\mu) \in \mathcal{D}_e(\mathcal{E}^\mu)$ and for any $u \in \mathcal{D}_e(\mathcal{E}^\mu)$,

$$\mathcal{E}^{\mu, g_\mu}\left(R^{\mu, g_\mu}\left(\frac{\varphi}{g_\mu}\right), \frac{u}{g_\mu}\right) = \left(\frac{\varphi}{g_\mu}, \frac{u}{g_\mu}\right)_{g_\mu^2 m} = (\varphi, u)_m.$$

Noting that the left-hand side above equals

$$\mathcal{E}^{\mu, g_\mu}\left(\frac{R^\mu\varphi}{g_\mu}, \frac{u}{g_\mu}\right) = \mathcal{E}^\mu(R^\mu\varphi, u),$$

we have the lemma. □

4.3.2 The case $\lambda(\mu) = 1$

In this subsection, we treat the case that $\mu \in \mathcal{K}_\infty$ and $\lambda(\mu) = 1$. We see from [40, Theorem 2.1] that there exists a minimizer $\psi \in \mathcal{D}_e(\mathcal{E})$ in (4.12):

$$\psi > 0, \quad \mathcal{E}(\psi, \psi) = 1 \quad \text{and} \quad \int_E \psi^2 d\mu = 1. \tag{4.14}$$

Lemma 4.18. *The measure $\psi \cdot \mu$ is of 0-order finite energy integral with respect to $\mathcal{E}$. Consequently, by* [18, Theorem 2.2.5], *$R(\psi\mu) \in \mathcal{D}_e(\mathcal{E})$ and*

$$\mathcal{E}\big(R(\psi\mu), u\big) = \int_E u\psi \, d\mu, \quad u \in \mathcal{D}_e(\mathcal{E}).$$

Proof. Since $\lambda(\mu) = 1$, it holds that for $u \in \mathcal{D}_e(\mathcal{E})$,

$$\int_E u^2 d\mu \le \mathcal{E}(u, u).$$

Then by Schwarz's inequality and (4.14), we obtain

$$\int_E u\psi \, d\mu \le \left(\int_E \psi^2 d\mu\right)^{\frac{1}{2}} \left(\int_E u^2 d\mu\right)^{\frac{1}{2}} \le \mathcal{E}(u, u)^{\frac{1}{2}}.$$

□

The function ψ is also characterized by

$$0 = \mathcal{E}(\psi, u) - \int_E \psi u \, d\mu, \quad u \in \mathcal{D}_e(\mathcal{E}). \tag{4.15}$$

Hence we see from Lemma 4.18 that

$$\mathcal{E}(\psi, u) = \mathcal{E}\big(R(\psi\mu), u\big), \quad u \in \mathcal{D}_e(\mathcal{E}),$$

and thus

$$\psi(x) = R(\psi\mu)(x) = \mathbb{E}_x\left[\int_0^\zeta \psi(X_t) dA_t^\mu\right], \quad m\text{-a.e.} \tag{4.16}$$

Now we define

$$h(x) := \mathbb{E}_x\left[\int_0^\zeta \psi(X_t) dA_t^\mu\right]. \tag{4.17}$$

By the arguments in [42] and [39], we will show that the function h is in $\mathcal{H}^+(\mu)$ and P_t^μ-invariant, that is, $P_t^\mu h = h$.

Lemma 4.19. *The function h is finely continuous.*

Proof. By the Markov property,

$$\begin{aligned} h(X_s) = \mathbb{E}_{X_s}\left[\int_0^\zeta \psi(X_t) dA_t^\mu\right] &= \mathbb{E}_x\left[\int_0^\zeta \psi(X_{t+s}) dA_t^\mu(\theta_s) \middle| \mathscr{F}_s\right] \\ &= \mathbb{E}_x\left[\int_0^\zeta \psi(X_t) dA_t^\mu \middle| \mathscr{F}_s\right] - \int_0^s \psi(X_t) dA_t^\mu. \end{aligned}$$

Since the first term of the right-hand side is right continuous in s because of the right continuity of $\mathscr{F}_s$, h is finely continuous by [18, Theorem A.2.7]. □

Note that $h = \psi$ q.e. by (4.16) and [18, Lemma 4.1.5]. Hence by [18, Theorem 4.1.2], there exists a nearly Borel set $B \supset \{x \in E : h(x) \neq \psi(x)\}$ such that $\mathbb{P}_x(\sigma_B < \infty) = 0$ for every $x \in E$, where σ_B is the hitting time of B. Therefore, the next lemma follows from (4.17).

Lemma 4.20. *The function h is strictly positive and satisfies*

$$h(x) = \mathbb{E}_x\left[\int_0^\zeta h(X_t) dA_t^\mu\right] \quad \textit{for all } x \in E.$$

Lemma 4.21. *For $w \in \mathfrak{B}_{b,0}^+(E)$ with $\int_E w \, dm > 0$, let $\nu = \mu - w \cdot m$. Then*

$$\lambda(\nu) := \inf\left\{\mathcal{E}(u, u) + \int_E u^2 w \, dm \,\middle|\, u \in \mathcal{D}(\mathcal{E}), \int_E u^2 d\mu = 1\right\} > 1.$$

Proof. It is clear that $\lambda(\nu) \geq \lambda(\mu) = 1$. Suppose $\lambda(\nu) = 1$. Then by the argument above, there exists a strictly positive function $h_0 \in \mathcal{D}_e(\mathcal{E})$ such that

$$\mathcal{E}\big(h_0, h_0\big) + \int_E h_0^2 \cdot w\, dm = 1 \quad \text{and} \quad \int_E h_0^2\, d\mu = 1.$$

Thus we have

$$\begin{aligned}\mathcal{E}(h_0, h_0) &= \mathcal{E}(h_0, h_0) + \int_E h_0^2\, w\, dm - \int_E h_0^2\, w\, dm \\ &= 1 - \int_E h_0^2\, w\, dm < 1.\end{aligned}$$

This implies $\lambda(\mu) < 1$, which is contradictory. □

Lemma 4.22. *The function h is bounded.*

Proof. Since h is quasi-continuous, there exists a compact set K_0 with $m(K_0) > 0$ on which h is continuous. Put $\nu = \mu - \mathbb{1}_{K_0} \cdot m$. Then $\lambda(\nu) > 1$ by Lemma 4.21. Recall that for $\varphi \in \mathfrak{B}^+_{b,0}(E)$ and $\beta > 0$, $R^\nu_\beta \varphi$ and $R^\nu \varphi$ are functions defined by

$$R^\nu_\beta \varphi(x) = \mathbb{E}_x \left[\int_0^\zeta e^{-\beta t + A^\nu_t} \varphi(X_t) dt \right] \quad \text{and} \quad R^\nu \varphi(x) = R^\nu_0 \varphi(x).$$

The function $R^\nu_\beta \varphi$ belongs to $\mathcal{D}(\mathcal{E}^\nu)$ and $R^\nu_\beta \varphi \uparrow R^\nu \varphi$ as $\beta \downarrow 0$. On account of Lemma 4.17, $R^\nu \varphi \in \mathcal{D}_e(\mathcal{E}^\nu)$ and

$$\mathcal{E}^\nu(R^\nu \varphi, u) = \int_E \varphi u\, dm, \quad u \in \mathcal{D}_e(\mathcal{E}^\nu). \tag{4.18}$$

Noting that $\mathcal{E}^\mu(h, R^\nu_\beta \varphi) = 0$ by (4.15), we have

$$\mathcal{E}^\nu(h, R^\nu_\beta \varphi) = \mathcal{E}^\mu(h, R^\nu_\beta \varphi) + \int_{K_0} h \cdot R^\nu_\beta \varphi\, dm = \int_{K_0} h \cdot R^\nu_\beta \varphi\, dm.$$

By letting $\beta \downarrow 0$, we get

$$\mathcal{E}^\nu(h, R^\nu \varphi) = \int_{K_0} h \cdot R^\nu \varphi\, dm = \int_E R^\nu(\mathbb{1}_{K_0} h) \cdot \varphi\, dm.$$

Since the left-hand side above equals $(h, \varphi)_m$ by (4.18), it holds that

$$h = R^\nu(\mathbb{1}_{K_0} h) \quad m\text{-a.e. } x \in E.$$

In the equality above we can replace "m-a.e. x" by "all x" by the same argument as after the proof of Lemma 4.19. Since $R^\nu(\mathbb{1}_{K_0} h)$ is bounded by Lemma 4.21 and 4.16, we have the lemma. □

Lemma 4.23. *The function h satisfies $P_t^\mu h = h$.*

Proof. By (4.15), h satisfies

$$\mathcal{E}(h,v) = \int_E v\,(h\cdot d\mu) \quad \text{for any } \; v\in\mathcal{D}(\mathcal{E})\cap C_0(E).$$

Thus, it follows from [18, Theorem 5.4.2] that

$$h(X_t) = h(X_0) + M_t^{[h]} - \int_0^t h(X_s)dA_s^\mu, \quad \mathbb{P}_x\text{-a.s. for q.e. } x\in E, \tag{4.19}$$

where $M^{[h]}$ is the martingale part of Fukushima's decomposition. Hence we have by Itô's formula

$$\begin{aligned} e^{A_t^\mu}h(X_t) &= h(X_0) + \int_0^t e^{A_s^\mu}dM_s^{[h]} - \int_0^t h(X_s)e^{A_s^\mu}dA_s^\mu + \int_0^t h(X_s)e^{A_s^\mu}dA_s^\mu \\ &= h(X_0) + \int_0^t e^{A_s^\mu}dM_s^{[h]}, \end{aligned}$$

$\mathbb{P}_x$-a.s. q.e. $x\in E$. Let $\tau_n := \inf\{t>0\,;A_t^\mu > n\}$. Then since $\int_0^{\tau_n\wedge t} e^{A_s^\mu}dM_s^{[h]}$ is a martingale, we have

$$h(x) = \mathbb{E}_x\left[e^{A^\mu_{\tau_n\wedge t}}h(X_{\tau_n\wedge t})\right] - \mathbb{E}_x\left[\int_0^{\tau_n\wedge t} e^{A_s^\mu}dM_s^{[h]}\right] = \mathbb{E}_x\left[e^{A^\mu_{\tau_n\wedge t}}h(X_{\tau_n\wedge t})\right].$$

Note that by Lemma 4.22 and the strong Feller property of P_t^μ,

$$e^{A^\mu_{\tau_n\wedge t}}h(X_{\tau_n\wedge t}) \le \|h\|_\infty\cdot e^n \in L^1(\mathbb{P}_x)$$

and that $\tau_n\to\infty$ as $n\to\infty$ $\mathbb{P}_x$-a.s. We then see that by the dominated convergence theorem

$$h(x) = \lim_{n\to\infty}\mathbb{E}_x\left[e^{A^\mu_{\tau_n\wedge t}}h(X_{\tau_n\wedge t})\right] = \mathbb{E}_x\left[e^{A_t^\mu}h(X_t)\right] = P_t^\mu h(x) \quad \text{for q.e. } x\in E,$$

and thus for all $x\in E$. □

Theorem 4.24. *The function h is in $\mathcal{H}^+(\mu)$.*

Proof. Note that $h\in C_b(E)$ by Lemma 4.22, Lemma 4.23 and the strong Feller property of P_t^μ. Hence, the function h is an element of $\mathcal{H}^+(\mu)$ because a bounded function u in $\mathcal{D}_e(\mathcal{E})$ belongs to $\mathcal{D}_{\text{loc}}(\mathcal{E})$. Indeed, let $\{u_n\}\subset\mathcal{D}(\mathcal{E})$ be an approximating sequence for $u\in\mathcal{D}_e(\mathcal{E})\cap\mathfrak{B}_b(E)$, that is, $\lim_{n\to\infty}u_n = u$ m-a.e. and $\sup_n\mathcal{E}(u_n,u_n)<\infty$. We may assume that $|u_n(x)| \leq \|u\|_\infty$ for all n and x. Let G be a relatively compact open set and take a function φ in $\mathcal{D}(\mathcal{E}) \cap C_0(E)$ such that $\varphi = 1$ on G. Then $u_n\varphi \to u\varphi$ m-a.e. and

$$\sup_n \mathcal{E}(u_n\varphi, u_n\varphi)^{1/2} \leq \|u\|_\infty \cdot \mathcal{E}(\varphi,\varphi)^{1/2} + \|\varphi\|_\infty \cdot \mathcal{E}(u_n, u_n)^{1/2} < \infty.$$

Hence, $u\varphi$ belongs to $\mathcal{D}_e(\mathcal{E}) \cap L^2(E;m)$ and so to $\mathcal{D}(\mathcal{E})$ by [18, Theorem 1.5.2 (iii)]. Since $u = u\varphi$ on G, u belongs to $\mathcal{D}_{\rm loc}(\mathcal{E})$. □

Define the Dirichlet form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ by

$$\begin{cases} \mathcal{D}(\mathcal{E}^{\mu,h}) = \{u \in L^2(E; h^2 m) \,|\, hu \in \mathcal{D}(\mathcal{E})\}, \\ \mathcal{E}^{\mu,h}(u,v) = \mathcal{E}^\mu(hu, hv), \quad u, v \in \mathcal{D}(\mathcal{E}^{\mu,h}). \end{cases}$$

Recall that $\{u \,|\, hu \in \mathcal{D}_e(\mathcal{E}^\mu)\} = \mathcal{D}_e(\mathcal{E}^{\mu,h})$ by Lemma 4.6. Since the function h is in $\mathcal{D}_e(\mathcal{E}^\mu)$ and $\mathcal{E}^\mu(h,h) = 0$, the constant function $1 = h/h$ belongs to $\mathcal{D}_e(\mathcal{E}^{\mu,h})$ and $\mathcal{E}^{\mu,h}(1,1) = 0$; this implies that the Dirichlet form $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ is recurrent. Therefore we have the next result.

Lemma 4.25. *The Dirichlet form* $(\mathcal{E}^{\mu,h}, \mathcal{D}(\mathcal{E}^{\mu,h}))$ *is recurrent.*

4.4 Hardy's inequalities for Green-tight measures

We discuss the relation between Schrödinger forms and Girsanov transformed Dirichlet forms treated in Chapter 3.

4.4.1 The case $\lambda(\mu) = 1$

Suppose that $\mu \in \mathcal{K}_\infty$ and $\lambda(\mu) = 1$. Then we see from arguments in the previous subsection that there exists a strictly positive function $h \in \mathcal{D}_e(\mathcal{E}) \cap C_b(E)$ such that

$$\mathcal{E}(h,h) = 1, \quad \int_E h^2 d\mu = 1 \quad \text{and} \quad P_t^\mu h = h.$$

Let $h(X_t) - h(X_0) = M_t^{[h]} + N_t^{[h]}$ be Fukushima's decomposition. Then we see from (4.19) that

$$N_t^{[h]} = -\int_0^t h(X_s) dA_s^\mu, \quad \mathbb{P}_x\text{-a.s. for q.e. } x \in E. \tag{4.20}$$

Let L_t^h be the unique solution of

$$L_t^h = 1 + \int_0^t L_{s-}^h \frac{1}{h(X_{s-})} dM_s^{[h]}$$

and $\widetilde{\mathbb{M}}^h = (\Omega, X_t, \mathbb{P}_x^h)$ the transformed process by multiplicative functional L_t^h, i.e., $d\mathbb{P}_x^h := L_t^h \cdot d\mathbb{P}_x$ on $\mathscr{F}_t \cap \{t < \zeta\}$. Let $\big(\widetilde{\mathcal{E}}^h, \mathcal{D}(\widetilde{\mathcal{E}}^h)\big)$ be the Dirichlet form on $L^2(E; h^2 m)$ generated by $\widetilde{\mathbb{M}}^h$. Since h is bounded, we see from Theorem 3.6 that $\mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\widetilde{\mathcal{E}}^h)$. By the computation similar to that in pp. 40-41, we can show that

$$L_t^h = \frac{h(X_t)}{h(X_0)} \exp\left(A_t^\mu\right)$$

and

$$\begin{aligned}\mathcal{E}^{\mu,h}(u,u) &= \widetilde{\mathcal{E}}^h(u,u) \\ &= \frac{1}{2}\int_E h(x)^2 \mu_{\langle u\rangle}^c(dx) + \int_{E\times E} (u(x)-u(y))^2 h(x)h(y) J(dx,dy)\end{aligned}$$

for $u \in \mathcal{D}(\mathcal{E})$. Consequently, we get the following representation.

Theorem 4.26. *Let $\mu \in \mathcal{K}_\infty$ with $\lambda(\mu) = 1$. Then $\mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\mathcal{E}^{\mu,h})$ and*

$$\mathcal{E}^{\mu,h}(u,u) = \frac{1}{2}\int_E h(x)^2 \mu_{\langle u\rangle}^c(dx) + \int_{E\times E} (u(x)-u(y))^2 h(x)h(y) J(dx,dy)$$

for $u \in \mathcal{D}(\mathcal{E})$.

4.4.2 The case $\lambda(\mu) > 1$

Suppose that $\mu \in \mathcal{K}_\infty(R)$ and $\lambda(\mu) > 1$. Then we see from the argument in §4.3.1 that the gauge function $g_\mu(x) = \mathbb{E}_x\big[\exp\big(A_\zeta^\mu\big)\big]$ is in $\mathcal{H}^+(\mu) \cap C_b(E)$. Note that $g_\mu(\partial) = 1$ because $\mathbb{P}_\partial(A_t \equiv 0) = 1$.

Lemma 4.27. *Define*

$$M_t^{\mu,g_\mu} := e^{A_t^\mu} g_\mu(X_t) - g_\mu(X_0).$$

Then M^{μ,g_μ} is a martingale with respect to $\mathbb{P}_x$.

Proof. From the proof of Lemma 4.13,

$$e^{A_t^\mu} g_\mu(X_t) \mathbb{1}_{\{t<\zeta\}} = \mathbb{E}_x\Big[e^{A_\zeta^\mu} \mathbb{1}_{\{t<\zeta\}} | \mathscr{F}_t\Big]$$

and

$$P_t^\mu g_\mu(x) = \mathbb{E}_x\big[e^{A_\zeta^\mu}\,; t<\zeta\big]. \tag{4.21}$$

Noting that $g_\mu(\partial)=1$, we have

$$\begin{aligned}\mathbb{E}_x\big[e^{A_t^\mu}g_\mu(X_t)\big] &= \mathbb{E}_x\big[e^{A_t^\mu}g_\mu(X_t)\,; t<\zeta\big] + \mathbb{E}_x\big[e^{A_\zeta^\mu}\,; t\geq\zeta\big] \\ &= P_t^\mu g_\mu(x) + \mathbb{E}_x\big[e^{A_\zeta^\mu}\,; t\geq\zeta\big].\end{aligned}$$

The right-hand side equals $g_\mu(x)$ by (4.21), and thus $\mathbb{E}_x\big[M_t^{\mu,g_\mu}\big]=0$. Since

$$\begin{aligned}M_{s+t}^{\mu,g_\mu} &= e^{A_{s+t}^\mu}g_\mu(X_{s+t}) - g_\mu(X_0) \\ &= e^{A_s^\mu}\big(e^{A_t^\mu(\theta_s)}g_\mu(X_{s+t}) - g_\mu(X_s)\big) + e^{A_s^\mu}g_\mu(X_s) - g_\mu(X_0) \\ &= e^{A_s^\mu}M_t^{\mu,g_\mu}(\theta_s) + M_s^{\mu,g_\mu},\end{aligned}$$

we have by the Markov property

$$\mathbb{E}_x\big[M_{s+t}^{\mu,g_\mu}|\mathscr{F}_s\big] = e^{A_s^\mu}\mathbb{E}_{X_s}[M_t^{\mu,g_\mu}] + M_s^{\mu,g_\mu} = M_s^{\mu,g_\mu}.$$

□

Since the gauge function g_μ is in $\mathcal{D}_{\rm loc}(\mathcal{E})\cap C_b(E_\partial)$ by Theorem 4.14, $g_\mu(X_t)-g_\mu(X_0)$ has Fukushima's decomposition:

$$g_\mu(X_t)-g_\mu(X_0) = M_t^{[g_\mu]} + N_t^{[g_\mu]}, \quad t\in[0,\zeta[,\ \mathbb{P}_x\text{-a.s. for q.e. } x\in E.$$

Then by Itô's formula, we have

$$\begin{aligned}g_\mu(X_t) &= e^{-A_t^\mu}\left(g_\mu(X_0)+M_t^{\mu,g_\mu}\right) \\ &= g_\mu(X_0) + \int_0^t e^{-A_s^\mu}dM_s^{\mu,g_\mu} + \int_0^t e^{A_s^\mu}g_\mu(X_s)e^{-A_s^\mu}\left(-dA_s^\mu\right) \\ &= g_\mu(X_0) + \int_0^t e^{-A_s^\mu}dM_s^{\mu,g_\mu} - \int_0^t g_\mu(X_s)dA_s^\mu.\end{aligned}$$

Thus we get

$$M_t^{[g_\mu]} = \int_0^t e^{-A_s^\mu}dM_s^{\mu,g_\mu}, \quad N_t^{[g_\mu]} = -\int_0^t g_\mu(X_s)dA_s^\mu.$$

Define a local martingale by $M_t = \int_0^t (g_\mu(X_{s-}))^{-1}dM_s^{[g_\mu]}$ and let $L_t^{g_\mu}$ be the unique solution of $L_t^{g_\mu} = 1+\int_0^t L_{s-}^{g_\mu}dM_s$. $(\widetilde{\mathcal{E}}^{g_\mu},\mathcal{D}(\widetilde{\mathcal{E}}^{g_\mu}))$ denotes the Girsanov transformed Dirichlet form by $L_t^{g_\mu}$. Then by the same argument as that in §4.4.1,

$$\begin{cases}\mathcal{D}(\widetilde{\mathcal{E}}^{g_\mu}) = \mathcal{D}(\mathcal{E}^{\mu,g_\mu}) = \{u\in L^2(E;g_\mu^2\cdot m) : g_\mu u\in\mathcal{D}(\mathcal{E})\}, \\ \widetilde{\mathcal{E}}^{g_\mu}(u,u) = \mathcal{E}^{\mu,g_\mu}(u,u), \quad u\in\mathcal{D}(\widetilde{\mathcal{E}}^{g_\mu}).\end{cases}$$

Moreover, since $1 \leq g_\mu \leq \|g_\mu\|_\infty$, we see from Theorem 3.6 that $\mathcal{D}(\widetilde{\mathcal{E}}^{g_\mu}) = \mathcal{D}(\mathcal{E})$ and

$$\begin{aligned}\widetilde{\mathcal{E}}^{g_\mu}(u,u) = \frac{1}{2}\int_E g_\mu(x)^2\,\mu^c_{\langle u\rangle}(dx) + \int_{E\times E}(u(x)-u(y))^2 g_\mu(x)g_\mu(y)J(dx,dy)\\ + \int_E u(x)^2 g_u(x)\kappa(dx), \qquad u \in \mathcal{D}(\mathcal{E}).\end{aligned}$$

Therefore we obtain the next conclusion.

Theorem 4.28. *Suppose that* $\mu \in \mathcal{K}_\infty$ *and* $\lambda(\mu) > 1$. *Then* $\mathcal{D}(\mathcal{E}^{\mu,g_\mu}) = \mathcal{D}(\mathcal{E})$ *and*

$$\begin{aligned}\mathcal{E}^{\mu,g_\mu}(u,u) = \frac{1}{2}\int_E g_\mu^2\,d\mu^c_{\langle u\rangle} + \int_{E\times E}(u(x)-u(y))^2 g_\mu(x)g_\mu(y)J(dx,dy)\\ + \int_E u(x)^2 g_u(x)\kappa(dx)\end{aligned}$$

for $u \in \mathcal{D}(\mathcal{E})$.

Chapter 5

Quasi-stationary distributions

5.1 Quasi-stationary distributions

A probability measure ν on E is said to be a *quasi-stationary distribution* (*QSD* in abbreviation) of $\mathbb{M}$ if for all $t \geq 0$ and any Borel set B,

$$\nu(B) = \frac{\mathbb{P}_\nu(X_t \in B, t < \zeta)}{\mathbb{P}_\nu(t < \zeta)}.$$

QSDs capture the long-time behavior of the process that will be surely killed when this process is conditioned to survive (for more informations on QSDs, we refer the recent survey [29]). In this section, we consider the existence of QSDs. The next limiting conditional distribution, so-called *Yaglom limit* is useful to find QSDs.

Definition 5.1. A probability measure ν on E is said to be a *Yaglom limit* of $\mathbb{M}$ if for any $x \in E$ and any Borel set B,

$$\nu(B) = \lim_{t \to \infty} \frac{\mathbb{P}_x(X_t \in B, t < \zeta)}{\mathbb{P}_x(t < \zeta)}. \tag{5.1}$$

We can easily show that Yaglom limit is always a QSD. However, it is known that the existence of a Yaglom limit does not always guarantee the uniqueness of QSDs. In [23], Knobloch and Partzsch proved that for a (not necessary symmetric) Markov process, the *intrinsic ultracontractivity* (see Definition 5.4 below) is a sufficient condition for the uniqueness of QSDs. We will give another proof of this fact for symmetric Markov processes.

Let λ_0 be the bottom of the spectrum:

$$\lambda_0 := \inf\left\{\mathcal{E}(u,u) \;:\; u \in \mathcal{D}(\mathcal{E}),\ \int_E u^2\,dm = 1\right\}.$$

A function ϕ_0 on E is called a *ground state* of the L^2-generator of $\mathcal{E}$ if

$$\phi_0 \in \mathcal{D}(\mathcal{E}), \int_E \phi_0^2 \, dm = 1, \text{ and } \mathcal{E}(\phi_0, \phi_0) = \lambda_0.$$

Suppose that there exists a strictly positive ground state ϕ_0. Then since

$$\mathcal{E}(\phi_0, u) = \lambda_0(\phi_0, u)_m \text{ for any } u \in \mathcal{D}(\mathcal{E}) \cap C_0(E),$$

it follows from [18, Theorem 5.4.2] that $\phi_0(X_t) - \phi_0(X_0)$ is decomposed as

$$\phi_0(X_t) - \phi_0(X_0) = M_t^{[\phi_0]} - \lambda_0 \int_0^t \phi_0(X_s) ds, \quad \mathbb{P}_x\text{-a.s.}$$

Here $M^{[\phi_0]}$ is the martingale part in Fukushima's decomposition. By the calculation similar to that in §4.4.1, we can show that

$$L_t^{\phi_0} = e^{\lambda_0 t} \frac{\phi_0(X_t)}{\phi_0(X_0)}, \quad t < \zeta,$$

where $L_t^{\phi_0}$ be a multiplicative functional defined by (3.3) with $\rho = \phi_0$. Denote by $\widetilde{\mathbb{M}}^{\phi_0} = (\Omega, X_t, \widetilde{\mathbb{P}}_x)$ the Girsanov transformed process by $L_t^{\phi_0}$, i.e., $d\widetilde{\mathbb{P}}_x := L_t^{\phi_0} \, d\mathbb{P}_x$. Its transition semigroup $\{\widetilde{P}_t^{\phi_0}\}$ on $L^2(E; \phi_0^2 m)$ equals

$$\widetilde{P}_t^{\phi_0} f(x) = e^{\lambda_0 t} \frac{1}{\phi_0(x)} \mathbb{E}_x\big[\phi_0(X_t) f(X_t) \, ; t < \zeta\big]. \tag{5.2}$$

The process $\widetilde{\mathbb{M}}^{\phi_0}$ is conservative, $\widetilde{P}_t^{\phi_0} 1 = 1$. Now, we obtain the result on the existence of QSDs. The next theorem due to Fukushima [17] plays a key role for the proof.

Theorem 5.2. *Assume that* $m(E) < \infty$ *and* $\mathbb{M}$ *is conservative,* $P_t 1 = 1$. *Then for* $f \in L^1(E; m)$,

$$\lim_{t \to \infty} P_t f(x) = \frac{1}{m(E)} \int_E f \, dm, \quad m\text{-a.e. and in } L^1(E; m).$$

Note that the process $\widetilde{\mathbb{M}}^{\phi_0}$ satisfies the assumption in Theorem 5.2.

Theorem 5.3. *Assume that there exists a ground state* ϕ_0 *of* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *belonging to* $L^1(E; m) \cap \mathfrak{B}_b(E)$. *Then a measure* ν *on* E *defined by*

$$\nu(B) := \frac{\int_B \phi_0 \, dm}{\int_E \phi_0 \, dm} \tag{5.3}$$

is a unique QSD of $\mathbb{M}$.

Proof. The proof is based on an idea in [41]. Note that $\mathbb{1}_B/\phi_0$ belongs to $L^1(E;\phi_0^2 m)$ for any Borel set B. By applying Theorem 5.2 to $\widetilde{\mathbb{M}}^{\phi_0}$, we have

$$\lim_{t\to\infty} \widetilde{P}_t^{\phi_0}\left(\frac{\mathbb{1}_B}{\phi_0}\right)(x) = \int_B \phi_0\, dm, \quad \phi_0^2 m\text{-a.e.} \tag{5.4}$$

Hence it follows from (5.2) and (5.4) that

$$\lim_{t\to\infty} \frac{\mathbb{P}_x(X_t \in B, t < \zeta)}{\mathbb{P}_x(t<\zeta)} = \lim_{t\to\infty} \frac{\widetilde{P}_t^{\phi_0}\left(\frac{\mathbb{1}_B}{\phi_0}\right)(x)}{\widetilde{P}_t^{\phi_0}\left(\frac{1}{\phi_0}\right)(x)} = \nu(B).$$

Therefore ν is a Yaglom limit, and thus, a QSD.

Secondly, we prove the uniqueness. Let μ be a QSD of $\mathbb{M}$. By the definition of QSD, we have for $t > 0$ and any Borel set B

$$\mu(B) = \frac{\int_E P_t \mathbb{1}_B\, d\mu}{\int_E P_t 1\, d\mu} = \frac{\int_E \phi_0\, \widetilde{P}_t^{\phi_0}\left(\frac{\mathbb{1}_B}{\phi_0}\right) d\mu}{\int_E \phi_0\, \widetilde{P}_t^{\phi_0}\left(\frac{1}{\phi_0}\right) d\mu}.$$

By using (5.4) again, we see that the right-hand side tends to

$$\frac{\int_E \phi_0\, d\mu \int_B \phi_0\, dm}{\int_E \phi_0\, d\mu \int_E \phi_0\, dm} = \nu(B) \quad \text{as } t\to\infty,$$

which implies the uniqueness of a QSD. □

Theorem 5.3 requires that ϕ_0 belongs to $L^1(E;m)$. If m is a finite measure, this is always satisfied. However when $m(E) = \infty$, ϕ_0 does not always belong to $L^1(E;m)$. We now give sufficient conditions for ϕ_0 being in $L^1(E;m)$.

Definition 5.4. Assume that there exists a ground state ϕ_0. We say that a Markov semigroup $\{P_t\}_{t\geq 0}$ has the *intrinsic ultracontractivity* (*IU* in abbreviation) if for any $t > 0$, there exist positive constants α_t, β_t such that

$$\alpha_t \phi_0(x)\phi_0(y) \leq p_t(x,y) \leq \beta_t \phi_0(x)\phi_0(y) \quad \text{for all } x,y \in E. \tag{5.5}$$

The notion of IU was introduced by Davies and Simon [9], and investigated extensively because of its important consequences (see [23, 30, 43] and references therein). Note that the IU implies that ϕ_0 belongs to $L^1(E;m)\cap \mathfrak{B}_b(E)$. Indeed, by integrating the left-hand inequality of (5.5) with respect to y over E, we have

$$\alpha_t \phi_0(x) \int_E \phi_0(y)\, m(dy) \leq \int_E p_t(x,y)\, m(dy) \leq 1.$$

Hence, the next result follows from Theorem 5.3.

Corollary 5.5. *Assume that* $\{P_t\}$ *has the* IU. *Then a measure* ν *defined by* (5.3) *is a unique* QSD *of* $\mathbb{M}$.

5.2 QSD's of one-dimensional diffusion processes

By applying the previous result, we give an example of one-dimensional diffusion processes that has a quasi-stationary distribution.

We consider the stochastic differential equation:

$$dZ_t = \sqrt{Z_t}\,dB_t + (Z_t - Z_t^2)dt, \quad Z_0 > 0,$$

where $\{B_t\}_{t\geq 0}$ is a standard one-dimensional Brownian motion. The solution $Z = \{Z_t\}$ is a diffusion process on $I = (0, \infty)$ with lifetime $\zeta = \inf\{t > 0 : Z_t = 0 \text{ or } \infty\}$. The process Z is called a *logistic Feller diffusion process*, which is derived from biological models. It is proved in [2] that a unique QSD of the process Z exists. We would like to give another proof of this fact.

We firstly make a change of variable and introduce the process $Y = \{Y_t\}$ defined by $Y_t = 2\sqrt{Z_t}$. Y is still absorbed at 0 and a QSD of Z is easily deduced from a QSD of Y. From now on, we focus on the process Y and prove that it has the IU. By Itô's formula,

$$\begin{aligned} dY_t &= \frac{1}{\sqrt{Z_t}}\,dZ_t - \frac{1}{4\sqrt{Z_t^3}}\,Z_t\,dt \\ &= \frac{1}{\sqrt{Z_t}}\Big(\sqrt{Z_t}\,dB_t + (Z_t - Z_t^2)dt\Big) - \frac{1}{4\sqrt{Z_t}}\,dt. \end{aligned}$$

Hence, Y is a solution of the following stochastic differential equation:

$$dY_t = dB_t - q(Y_t)dt, \quad q(u) := \frac{1}{2u} - \frac{u}{2} + \frac{u^3}{8}.$$

We define

$$\begin{aligned} Q(x) &:= 2\int_1^x q(u)du \\ &= \log x - \frac{x^2}{2} + \frac{x^4}{16} + \frac{7}{16} \end{aligned}$$

Since the constant term does not affect further arguments, we may replace $Q(x) := Q(x) - 7/16$. We define functions on I by

$$m(x) := \int_1^x e^{-Q(u)}du, \quad s(x) := \int_1^x e^{Q(u)}du.$$

Then m and s are the *speed measure* and the *scale function* of Y respectively. Note that m is a symmetrization measure of the process Y and $m(I) = \infty$.

Generally, a one-dimensional diffusion process on an open interval (ℓ, r) has the irreducibility and the strong Feller property, and its boundary points ℓ and r are classified into four classes: *regular boundary, exit boundary, entrance boundary* and *natural boundary* (see [11] or [21]).

Lemma 5.6. *For the process Y, its boundary point 0 is exit and ∞ is entrance.*

Proof. We define

$$I(x) := \int_1^x ds(y) \int_1^y dm(z), \quad J(x) := \int_1^x dm(y) \int_1^y ds(z) \quad \text{for } x \in [0, \infty].$$

We first prove that the point 0 is an exit boundary, which is equivalent to $I(0) < \infty$ and $J(0) = \infty$. By the definition,

$$I(0) = \int_0^1 e^{Q(y)} \left(\int_y^1 e^{-Q(z)} dz \right) dy.$$

Since $\int_y^1 e^{-Q(z)} dz$ and $e^{-Q(y)}$ tend to ∞ as $y \to 0$, we have by l'Hôpital's rule

$$\lim_{y \downarrow 0} \frac{\int_y^1 e^{-Q(z)} dz}{e^{-Q(y)}} = \lim_{y \downarrow 0} \frac{1}{Q'(y)} = 0.$$

This yields that $e^{Q(y)} \int_y^1 e^{-Q(z)} dz$ is bounded in $[0, 1]$, which implies $I(0) < \infty$. On the other hand,

$$J(0) = \int_0^1 e^{-Q(y)} \left(\int_y^1 e^{Q(z)} dz \right) dy.$$

Since $e^{-Q(y)} = O(y^{-1})$ and $e^{Q(y)}$ tend to 0 as $y \to 0$, we see that $J(0) = \infty$. Thus 0 is an exit boundary.

We next prove that ∞ is an entrance boundary, which is equivalent to $I(\infty) = \infty$ and $J(\infty) < \infty$. Since

$$\begin{aligned} I(\infty) &\geq \int_2^\infty e^{Q(y)} \left(\int_1^y e^{-Q(z)} dz \right) dy \\ &\geq \int_1^2 e^{-Q(z)} dz \int_2^\infty e^{Q(y)} dy \end{aligned}$$

and $e^{Q(y)}$ tends to ∞ as $y \to \infty$, we get $I(\infty) = \infty$. Finally, we compute the value of $J(\infty)$. We have by l'Hôpital's rule

$$\lim_{y\to\infty} \frac{\int_1^y e^{Q(z)}dz}{y^{-3}e^{Q(y)}} = 4. \tag{5.6}$$

This implies that there exists a constant $C > 0$ such that for sufficiently large y,

$$e^{-Q(y)} \int_1^y e^{Q(z)}dz < \frac{C}{y^3}.$$

Therefore, taking $M > 0$ large enough, we get

$$\begin{aligned} J(\infty) &= \int_1^\infty e^{-Q(y)} \left(\int_1^y e^{Q(z)}dz \right) dy \\ &< \int_1^M e^{-Q(y)} \left(\int_1^y e^{Q(z)}dz \right) dy + \int_M^\infty \frac{C}{y^3}dy \\ &< \infty. \end{aligned}$$

Hence ∞ is an entrance boundary. $\square$

Remark 5.7. Let $\mathbb{M}$ be a general one-dimensional diffusion process on $I = (\ell, r)$. It is shown in Itô [21] that

(a) If r is a regular or exit boundary, then $\lim_{x\to r} R_1 1(x) = 0$.

(b) If r is an entrance boundary, then $\lim_{s\to r} \sup_{x\in(\ell,r)} R_1 1_{(s,r)}(x) = 0$.

(c) If r is a natural boundary, then for $s \in (\ell, r)$, $\lim_{x\to r} R_1 1_{(s,r)}(x) = 1$ and thus $\sup_{x\in(\ell,r)} R_1 1_{(s,r)}(x) = 1$.

Hence, neither boundary is natural if and only if $\mathbb{M}$ has the *tightness* property, that is, for any $\varepsilon > 0$, there exists a compact set K of I such that $\sup_{x\in I} R_1 \mathbb{1}_{K^c}(x) \le \varepsilon$. Thus it follows from [18, Lemma 6.4.5] that there exists a ground state ϕ_0 if no natural boundaries are present.

For diffusion processes with no natural boundaries, a sufficient condition for the IU was given in [43]. We present this condition in case when ℓ is an exit boundary and r an entrance one.

Theorem 5.8 ([43, Theorem 2.11])**.** *Let* $\mathbb{M}$ *be a one-dimensional diffusion process on* $I = (\ell, r)$ *with speed measure* m *and scale function* s*. Assume that* ℓ *is an exit boundary and* r *an entrance one, and there exist points* $c_i \in I, i = 1, 2,$ *such that* $m(c_1) < 0 < m(c_2)$ *and* $s(c_1) < 0 < s(c_2)$*. Further assume that*

$$\int_{\ell}^{c_1} |m(x)| ds(x) < \infty \quad \text{and} \quad \int_{\ell}^{c_1} \frac{\mu(x)}{|m(x)|} dm(x) < \infty, \tag{A1}$$

$$\int_{c_2}^{r} m(x) ds(x) = \infty \quad \text{and} \quad \int_{c_2}^{r} \frac{\nu(x)}{s(x)} ds(x) < \infty, \tag{A2}$$

where

$$\mu(x) := \sup_{\ell < y \le x} |m(y)| \big(s(y) - s(\ell)\big) \quad \text{and} \quad \nu(x) := \sup_{x \le y < r} s(y)\big(m(r) - m(y)\big).$$

Then $\mathbb{M}$ *has the* IU.

By checking this condition, we shall show the next result.

Theorem 5.9. *The process* Y *has the* IU*. Consequently, a unique* QSD *of* Y *exists by Corollary 5.5.*

Proof. We only need to show that **(A1)** and **(A2)** in Theorem 5.8 are satisfied.

The former inequality in (A1): We choose c_1 so that $0 < c_1 < e^{-\frac{1}{2}}$. This gives $m(c_1) < 0$ and $s(c_1) < 0$. We set

$$M_1 := \max_{0 \le u \le 1} \left(-\frac{u^2}{2} + \frac{u^4}{16} \right), \quad M_2 := \min_{0 \le u \le 1} \left(-\frac{u^2}{2} + \frac{u^4}{16} \right).$$

Since $Q(u) = \log u - \frac{u^2}{2} + \frac{u^4}{16}$, it follows that for all $u \in (0, 1)$,

$$e^{Q(u)} \le e^{M_1} u, \quad \text{and} \quad \frac{e^{-M_1}}{u} \le e^{-Q(u)} \le \frac{e^{-M_2}}{u}. \tag{5.7}$$

As a result, we have

$$\begin{aligned}
\int_0^{c_1} |m(x)| ds(x) &= \int_0^{c_1} \left(\int_x^1 e^{-Q(y)} dy \right) e^{Q(x)} dx \\
&\le \int_0^{c_1} \left(\int_x^1 \frac{e^{-M_2}}{y} dy \right) e^{M_1} x \, dx \\
&= e^{M_1 - M_2} \int_0^{c_1} (-x \log x) \, dx \\
&< \infty.
\end{aligned}$$

The latter inequality in (A1): Noting that $s(y) - s(0) = \int_0^y e^{Q(u)}du$, we have

$$\int_0^{c_1} \frac{\mu(x)}{|m(x)|}\, dm(x)$$
$$= \int_0^{c_1} \left(\int_x^1 e^{-Q(z)}dz \right)^{-1} e^{-Q(x)} \sup_{0<y\le x} \left(\int_y^1 e^{-Q(z)}dz \int_0^y e^{Q(u)}du \right) dx.$$

By the estimate (5.7), the right-hand side is dominated by

$$\int_0^{c_1} \left(\int_x^1 \frac{e^{-M_1}}{z} dz \right)^{-1} \frac{e^{-M_2}}{x} \sup_{0<y\le x} \left(\int_y^1 \frac{e^{-M_2}}{z} dz \int_0^y e^{M_1} u\, du \right) dx$$
$$= \frac{e^{2M_1-2M_2}}{2} \int_0^{c_1} \frac{1}{-x\log x} \sup_{0<y\le x} (-y^2 \log y) dx.$$

Since $-y^2 \log y$ is increasing on $(0, c_1)$, the right-hand side is less than

$$\frac{e^{2M_1-2M_2}}{2} \int_0^{c_1} x\, dx < \infty,$$

and thus **(A1)** holds.

The former inequality in (A2): We choose c_2 so that $1 < c_2 < \infty$. This gives $m(c_2) > 0$ and $s(c_2) > 0$. Then

$$\int_{c_2}^{\infty} m(x) ds(x)$$
$$= \int_{c_2}^{c_2+1} \left(\int_1^x e^{-Q(y)}dy \right) e^{Q(x)}dx + \int_{c_2+1}^{\infty} \left(\int_1^x e^{-Q(y)}dy \right) e^{Q(x)}dx$$
$$\ge \int_{c_2+1}^{\infty} \left(\int_1^{c_2+1} e^{-Q(y)}dy \right) e^{Q(x)}dx.$$

A simple calculation shows that the right-hand side is equal to ∞.

The latter inequality in (A2): Noting that $m(\infty) - m(y) = \int_y^\infty e^{-Q(u)}du$, we have

$$\int_{c_2}^{\infty} \frac{\nu(x)}{s(x)}\, ds(x)$$
$$= \int_{c_2}^{\infty} \left(\int_1^x e^{Q(z)}dz \right)^{-1} e^{Q(x)} \sup_{x\le y<\infty} \left(\int_1^y e^{Q(z)}dz \int_y^\infty e^{-Q(u)}du \right) dx. \tag{5.8}$$

By l'Hôpital's rule, it holds that

$$\lim_{y\to\infty} \frac{\int_1^y e^{Q(z)}dz}{\left(\int_y^\infty e^{-Q(u)}du \right)^{-1}} = \lim_{y\to\infty} \left(\frac{\int_y^\infty e^{-Q(u)}du}{e^{-Q(y)}} \right)^2 = \lim_{y\to\infty} \left(\frac{1}{\frac{1}{y} - y + \frac{y^3}{4}} \right)^2,$$

and thus

$$\lim_{y\to\infty} y^6 \int_1^y e^{Q(z)}dz \int_y^\infty e^{-Q(u)}du = \lim_{y\to\infty} y^6 \left(\frac{1}{\frac{1}{y} - y + \frac{y^3}{4}}\right)^2 = 16.$$

By this and (5.6), there exist positive constants C, C' such that for sufficiently large y,

$$\left(\int_1^y e^{Q(z)}dz\right)^{-1} e^{Q(y)} < Cy^3, \qquad \int_1^y e^{Q(z)}dz \int_y^\infty e^{-Q(u)}du < \frac{C'}{y^6}.$$

Thus by taking sufficiently large $K > 0$, the right-hand side of (5.8) is dominated by

$$\int_{c_2}^K \left(\int_1^x e^{Q(z)}dz\right)^{-1} e^{Q(x)} \sup_{x\le y<\infty} \left(\int_1^y e^{Q(z)}dz \int_y^\infty e^{-Q(u)}du\right) dx + \int_K^\infty \frac{CC'}{x^3}dx.$$

Since the integrals above are finite, the condition **(A2)** is satisfied. □

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