On the GIT stratification in the non-split case

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博　士　論　文

On the GIT stratification in the non-split case

(分裂的でない場合の GIT stratification について)

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On the GIT stratification in the non-split case

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Abstract

In early 80's, the notion of the GIT stratification of reductive group actions was studied by Kirwan and Ness. If the group is split over k, their works tell us that these stratifications are rationally defined over a perfect ground field k. In this thesis, we extend these stratifications for all (not necessarily split) reductive algebraic groups over k.

Notation

The symbols $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $\mathbb{Z}$ denote respectively the set of rational, real and complex numbers and the rational integers. The set of positive real numbers will be denoted by $\mathbb{R}_{>0}$. If V is a vector space defined over a field k, we define $\mathbb{P}(V) = (V \setminus \{0\})/\sim$, where $v \sim \lambda v'$ for all $\lambda \in k \setminus \{0\}$. The map $\pi_V : V \setminus \{0\} \ni v \mapsto \bar{v} \in \mathbb{P}(V)$ is the natural projection.

For any field k, the symbol $\bar{k}$ and k^{sep} denote an algebraic closure of k and a separable closure of k respectively. If K/k is an extension of fields, $\operatorname{Aut}_k K$ denotes the set of all isomorphisms of K as a k-algebra. Then $\operatorname{Gal}(k^{\text{sep}}/k) = \operatorname{Aut}_k k^{\text{sep}}$ is the absolute Galois group of k (endowed with Krull topology).

For natural numbers m, n, $\mathrm{M}_{m,n}(R)$ denotes the set of all $m \times n$ matrices whose entries in a ring R. The unit matrix of the size $n \times n$ is denoted by I_n. For any $n \times n$ matrix A, $\operatorname{tr} A$ and $\det A$ denote the trace of A and the determinant of A respectively.

For any ring R with 1, X_R denotes the set of R-points of a scheme X.

For any natural number n, the symbol GL_n denotes the general linear group. The special linear group will be denoted by SL_n.

For any algebraic group G defined over a field k and $S \subset G$, the symbol $Z_G(S)$ (resp. $N_G(S)$) stands for the centralizer of S in G (resp. the normalizer of S in G). The symbol $[G, G]$ denotes the commutator subgroup of G.

For any algebraic group G defined over a field k, a homomorphism from $\mathbb{G}_m = \mathrm{GL}_1$ to G is called a one parameter subgroup (which will be abbreviated as 1-PS from now on). Let $X^*(G) = \operatorname{Hom}(G, \mathbb{G}_m)$ and $X_*(G) = \operatorname{Hom}(\mathbb{G}_m, G)$ be the group of characters of G and the group of 1-PS's of G (defined over the algebraic closure $\bar{k}$) respectively. Also let $X_k^*(G) = \operatorname{Hom}_k(G, \mathbb{G}_m)$ and $X_{*,k}(G) = \operatorname{Hom}_k(\mathbb{G}_m, G)$ be the group of rational characters of G and the group of rational 1-PS's of G respectively.

Suppose that L/k is a finite separable extension. Then $\mathfrak{R}_{L/k} X$ denotes the restriction of scalar of a L-variety X.

1 Introduction

In this thesis, we discuss a result concerning an extension of the notion of the GIT (geometric invariant theory) stratification for the non-split case which A.Yukie and the author have proved in [24]. In this section, we state the main result of [24] and discuss historical backgrounds.

Let G be a connected reductive algebraic group and V a representation of G both defined over a field k.

The GIT stratification is a stratification of $V \setminus V^{\mathrm{ss}}$ where V^{ss} is the set of semistable points of V which is defined by geometric invariant theory. In the split case, the main result of this thesis was proved in principle by Kirwan [13], Ness [17]. But, the construction of the GIT stratification is complicated, so we are going to explain it by an example at first.

Let $G = \mathrm{SL}_2, V = \mathrm{Sym}^3 \mathrm{Aff}^2$. We regard V as the space of homogeneous polynomials in two variables $v = (v_1, v_2)$ of degree three. The group G acts on V by $gx(v) = x(vg)$ for $g \in G, x \in V$. Then $S = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix} \right) \mid a \in k^{\times} \right\}$ is a maximal k-split torus of G. Put $\mathfrak{s}^*_{\mathbb{R}} = X^*(S) \otimes \mathbb{R}$. We identify $\mathfrak{s}^*_{\mathbb{R}}$ with $\mathbb{R}$. We express elements of V as $x(v) = x_0 v_1^3 + x_1 v_1^2 v_2 + x_2 v_1 v_2^2 + x_3 v_2^3$. We use the coordinate (x_0, x_1, x_2, x_3) on V, by which S acts diagonally. Since

$$\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} (x_0, x_1, x_2, x_3) = (t^{-3} x_0, t^{-1} x_1, t x_2, t^3 x_3),$$

the set weights of coordinates respect to S is identified with $\{-3, -1, 1, 3\} \subset \mathbb{R}$. We only consider positive weights, such as $\beta_1 = 1, \beta_2 = 3$. Note that β_1 (resp. β_2) is the closest point of the convex hull of the set of weights $\{1\}$ (resp. $\{3\}$) to the origin.

We define

$$\begin{aligned} Z_{\beta_1 k} &= \{(0, 0, x_2, 0) \mid x_2 \in k\}, \quad Z_{\beta_2 k} = \{(0, 0, 0, x_3) \mid x_3 \in k\}, \\ W_{\beta_1 k} &= \{(0, 0, 0, x_3) \mid x_3 \in k\}, \quad W_{\beta_2 k} = \{0\}. \end{aligned}$$

$Z_{\beta k}$ is a subspace of V with coordinates corresponding to the weights which are on "the edge" of the convex hull of the weights. $W_{\beta k}$ is a subspace of V with coordinates corresponding to the weights which are on "the outside" of the convex hull of the weights. We also define

$$Z^{\mathrm{ss}}_{\beta_1 k} = \{(0, 0, x_2, 0) \mid x_2 \in k, x_2 \neq 0\}, \quad Z^{\mathrm{ss}}_{\beta_2 k} = \{(0, 0, 0, x_3) \mid x_3 \in k, x_3 \neq 0\}.$$

$Z^{\mathrm{ss}}_{\beta_i k}$ is defined as the set of semistable points of $Z_{\beta_i k}$ with respect to the action of a reductive subgroup $G_{\beta_i k} \subset G_k$ for $i = 1, 2$. In this case, the group $G_{\beta_i k}$ is trivial for

$i = 1, 2$. Define

$$Y^{\mathrm{ss}}_{\beta_1\,k} = \{(0,0,x_2,x_3)\,|\,x_2,x_3 \in k, x_2 \neq 0\}, \quad Y^{\mathrm{ss}}_{\beta_2\,k} = \{(0,0,0,x_3)\,|\,x_3 \in k, x_3 \neq 0\}.$$

Put $S_{\beta_1\,k} = G_k Y^{\mathrm{ss}}_{\beta_1\,k}$ and $S_{\beta_2\,k} = G_k Y^{\mathrm{ss}}_{\beta_2\,k}$. In this case, applying the main result of this thesis, we have

$$V_k \setminus \{0\} = V^{\mathrm{ss}}_k \amalg S_{\beta_1\,k} \amalg S_{\beta_2\,k}.$$

Furthermore, $S_{\beta_1\,k} = G_k \times_{P_{\beta_1\,k}} Y^{\mathrm{ss}}_{\beta_1\,k}$ and $S_{\beta_2\,k} = G_k \times_{P_{\beta_2\,k}} Y^{\mathrm{ss}}_{\beta_2\,k}$ hold, where $P_{\beta_1\,k} = P_{\beta_2\,k} = \{\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) |a,c,d \in k, a,d \neq 0\} \subset G_k$ is the standard Borel subgroup of G_k (for the precise definition of P_β, see (1.1)) .

Now we return to the general settings and give a precise definition of the GIT stratification.

Choose a maximal k-split torus S of G and a maximal torus T of G defined over k containing S ([1, 18.2 Theorem, p.218]). We put

$$\begin{aligned} &\mathfrak{s}_{\mathbb{R}} = X_*(S) \otimes \mathbb{R} = X_{*,k}(S) \otimes \mathbb{R},\ \mathfrak{s}^*_{\mathbb{R}} = X^*(S) \otimes \mathbb{R} = X^*_k(S) \otimes \mathbb{R}, \\ &\mathfrak{t}_{\mathbb{R}} = X_*(T) \otimes \mathbb{R},\ \mathfrak{t}^*_{\mathbb{R}} = X^*(T) \otimes \mathbb{R}. \end{aligned}$$

Since T is defined over k, the Galois group $\mathrm{Aut}_k\,\bar{k}$ $(= \mathrm{Gal}(k^{\mathrm{sep}}/k))$ acts on $\mathfrak{t}_{\mathbb{R}}$ and $\mathfrak{t}^*_{\mathbb{R}}$. The action of the Galois group $\mathrm{Aut}_k\,\bar{k}$ on $\mathfrak{t}^*_{\mathbb{R}}$ is defined by $\chi^\sigma(t) = \sigma(\chi(\sigma^{-1}(t)))$ for $\sigma \in \mathrm{Aut}_k\,\bar{k}$ and $\chi \in \mathfrak{t}^*_{\mathbb{R}}$ (we define the action on $\mathfrak{t}_{\mathbb{R}}$ similarly). We also put $\mathfrak{s}_{\mathbb{Q}} = X_*(S) \otimes \mathbb{Q}$, etc. Let $\mathbb{W} = N_G(T)/T$, ${}_k\mathbb{W} = N_G(S)/Z_G(S)$ be the Weyl group of G and the relative Weyl group of G respectively.

There is a natural pairing $\langle\ ,\ \rangle_T : X^*(T) \times X_*(T) \to \mathbb{Z}$ defined by $t^{\langle \chi, \lambda \rangle} = \chi(\lambda(t))$ for $\chi \in X^*(T), \lambda \in X_*(T)$. This is a perfect paring ([1, pp.113–115]). Similarly, there is a perfect pairing $\langle\ ,\ \rangle_S : X^*(S) \times X_*(S) \to \mathbb{Z}$.

There exists an inner product $(\ ,\ )_{\mathfrak{t}_{\mathbb{R}}}$ on $\mathfrak{t}_{\mathbb{R}}$ which is invariant under the action of $\mathbb{W}$. Since $\mathrm{Aut}_k\,\bar{k}$ leaves $N_G(T)$ invariant, we may assume that this inner product is invariant under the action of $\mathrm{Aut}_k\,\bar{k}$. We may assume that this inner product is rational, i.e., $(\lambda, \nu)_{\mathfrak{t}} \in \mathbb{Q}$ for all $\lambda, \nu \in \mathfrak{t}_{\mathbb{Q}}$. We denote the inner product on $\mathfrak{s}_{\mathbb{R}}$ obtained by restricting $(\ ,\ )_{\mathfrak{t}_{\mathbb{R}}}$ to $\mathfrak{s}_{\mathbb{R}}$ by $(\ ,\ )_{\mathfrak{s}_{\mathbb{R}}}$. It is easy to see that $(\ ,\ )_{\mathfrak{s}_{\mathbb{R}}}$ is also rational. Any element of ${}_k\mathbb{W}$ is represented by an element of $N_G(T)$ (Lemma 8.1). In fact ${}_k\mathbb{W}$ can be regarded as a subgroup of $\mathbb{W}$ ([2, 5.5.Corollaire.]). Therefore, $(\ ,\ )_{\mathfrak{s}_{\mathbb{R}}}$ is invariant by the action of ${}_k\mathbb{W}$. Let $\|\ \|_{\mathfrak{s}_{\mathbb{R}}}$ (resp. $\|\ \|_{\mathfrak{t}_{\mathbb{R}}}$) be the norm on $\mathfrak{s}_{\mathbb{R}}$ (resp. $\mathfrak{t}_{\mathbb{R}}$) defined by $(\ ,\ )_{\mathfrak{s}_{\mathbb{R}}}$ (resp. $(\ ,\ )_{\mathfrak{t}_{\mathbb{R}}}$). We choose a Weyl chamber $\mathfrak{s}_{\mathbb{R},+} \subset \mathfrak{s}_{\mathbb{R}}$ (resp. $\mathfrak{t}_{\mathbb{R},+} \subset \mathfrak{t}_{\mathbb{R}}$) for the action of ${}_k\mathbb{W}$ (resp. $\mathbb{W}$).

For $\lambda \in \mathfrak{s}_{\mathbb{R}}$, let $\beta = \beta(\lambda)$ be the element of $\mathfrak{s}^*_{\mathbb{R}}$ such that $\langle \beta, \nu \rangle_S = (\lambda, \nu)_{\mathfrak{s}_{\mathbb{R}}}$ for all $\nu \in \mathfrak{s}_{\mathbb{R}}$. The map $\lambda \mapsto \beta(\lambda)$ is a bijection and we denote the inverse map by $\lambda = \lambda(\beta)$. We have a similar bijection between $\mathfrak{t}_{\mathbb{R}}$ and $\mathfrak{t}^*_{\mathbb{R}}$. We use the same notation $\beta(\lambda), \lambda(\beta)$ for this bijection. There is a unique positive rational number a such that $a\lambda(\beta) \in X_*(S)$ or $X_*(T)$ and is indivisible. We use the notation λ_β for $a\lambda(\beta)$.

Identifying $\mathfrak{s}_\mathbb{R}$ (resp. $\mathfrak{t}_\mathbb{R}$) with $\mathfrak{s}^*_\mathbb{R}$ (resp. $\mathfrak{t}^*_\mathbb{R}$), we have a ${}_k\mathbb{W}$-invariant (resp. $\mathbb{W}$-invariant) inner product $(\ ,\ )_{\mathfrak{s}^*_\mathbb{R}}$ (resp. $(\ ,\ )_{\mathfrak{t}^*_\mathbb{R}}$) on $\mathfrak{s}^*_\mathbb{R}$ (resp. $\mathfrak{t}^*_\mathbb{R}$), the norm $\|\ \|_{\mathfrak{s}^*_\mathbb{R}}$ (resp. $\|\ \|_{\mathfrak{t}^*_\mathbb{R}}$) determined by $(\ ,\ )_{\mathfrak{s}^*_\mathbb{R}}$ (resp. $(\ ,\ )_{\mathfrak{t}^*_\mathbb{R}}$) and a Weyl chamber $\mathfrak{s}^*_{\mathbb{R},+}$ (resp. $\mathfrak{t}^*_{\mathbb{R},+}$).

Since S is a split torus, its action is diagonalizable over the ground field k. So we choose a coordinate system $v = (v_0, v_1, \ldots, v_N)$ on V by which S acts diagonally. Let $\gamma_i \in \mathfrak{s}^*_\mathbb{R}$ and e_i be the weight and the coordinate vector which corresponds to the i-th coordinate. For a subset $\mathfrak{I} \subset \{\gamma_i \,|\, i = 0, 1, \ldots, N\}$, we denote the convex hull of $\mathfrak{I}$ by $\operatorname{Conv}\mathfrak{I}$. If $v \in V \setminus \{0\}$ and $x = \pi_V(v)$ then we put $\mathfrak{I}_v = \mathfrak{I}_x = \{\gamma_i \,|\, v_i \neq 0\}$.

For $\mathfrak{I} \subset \{\gamma_i \,|\, i = 0, 1, \ldots, N\}$ such that $0 \notin \operatorname{Conv}\mathfrak{I}$, let β be the closest point of $\operatorname{Conv}\mathfrak{I}$ to the origin. Then β lies in $\mathfrak{s}^*_\mathbb{Q}$. Note that $(\xi, \beta)_{\mathfrak{s}^*_\mathbb{R}} \geqq (\beta, \beta)_{\mathfrak{s}^*_\mathbb{R}}$ holds for all $\xi \in \operatorname{Conv}\mathfrak{I}$ since $\operatorname{Conv}\mathfrak{I}$ is convex. Let $\mathfrak{B}$ be the set of all such β which lies in $\mathfrak{s}^*_{\mathbb{R},+}$.

We define

$$\begin{aligned}
&Y_\beta = \operatorname{span}\{e_i \,|\, (\gamma_i, \beta)_{\mathfrak{s}^*_\mathbb{R}} \geqq (\beta, \beta)_{\mathfrak{s}^*_\mathbb{R}}\}, \quad Z_\beta = \operatorname{span}\{e_i \,|\, (\gamma_i, \beta)_{\mathfrak{s}^*_\mathbb{R}} = (\beta, \beta)_{\mathfrak{s}^*_\mathbb{R}}\}, \\
&W_\beta = \operatorname{span}\{e_i \,|\, (\gamma_i, \beta)_{\mathfrak{s}^*_\mathbb{R}} > (\beta, \beta)_{\mathfrak{s}^*_\mathbb{R}}\}.
\end{aligned}$$

Clearly $Y_\beta = Z_\beta \oplus W_\beta$ and $\mathbb{P}(Z_\beta), \mathbb{P}(Y_\beta)$ can be regarded as subspaces of $\mathbb{P}(V)$.

If λ is a non-trivial 1-PS of G, we define

$$\begin{aligned}
&P(\lambda) = \left\{p \in G \,\middle|\, \lim_{t\to 0} \lambda(t)p\lambda(t)^{-1} \text{ exists}\right\},\ M(\lambda) = Z_G(\lambda), \\
&U(\lambda) = \left\{p \in G \,\middle|\, \lim_{t\to 0} \lambda(t)p\lambda(t)^{-1} = 1\right\}.
\end{aligned} \tag{1.1}$$

The group $P(\lambda)$ is a parabolic subgroup of G ([22, p.148]) with Levi part $M(\lambda)$ and unipotent radical $U(\lambda)$. We put $P_\beta = P(\lambda_\beta)$, $M_\beta = Z_G(\lambda_\beta)$ and $U_\beta = U(\lambda_\beta)$.

Let χ_β be the indivisible rational character of M_β such that the restriction of χ^a_β to S coincides with $b\beta$ for some positive integers a, b. We define $G_\beta = \{g \in M_\beta \,|\, \chi_\beta(g) = 1\}^\circ$ (the identity component). Then G_β acts on Z_β. Note that M_β and G_β are defined over k, and since $\langle\chi_\beta, \lambda_\beta\rangle_S$ is a positive multiple of $\|\beta\|_{\mathfrak{s}^*_\mathbb{R}}$, $M_\beta = G_\beta\lambda_\beta$. Moreover, if ν is any rational 1-PS in G_β, then $(\nu, \lambda_\beta)_{\mathfrak{s}_\mathbb{R}} = 0$.

Let $\mathbb{P}(Z_\beta)^{\mathrm{ss}}$ be the set of G_β-semistable points of $\mathbb{P}(Z_\beta)$. We regard $\mathbb{P}(Z_\beta)^{\mathrm{ss}}$ as a subset of $\mathbb{P}(V)$. Put

$$\begin{aligned}
&Z^{\mathrm{ss}}_\beta = \pi_V^{-1}(\mathbb{P}(Z_\beta)^{\mathrm{ss}}),\ Y^{\mathrm{ss}}_\beta = \{(z, w) \,|\, z \in Z^{\mathrm{ss}}_\beta, w \in W_\beta\}, \\
&\mathbb{P}(Y_\beta)^{\mathrm{ss}} = \{\pi_V((z, w)) \,|\, (z, w) \in Y^{\mathrm{ss}}_\beta\}.
\end{aligned}$$

We define $S_\beta = GY^{\mathrm{ss}}_\beta$. Note that S_β can be the empty set. We denote the set of k-rational points of S_β, etc., by $S_{\beta\,k}$, etc.

The following theorem is the main result of this thesis.

Theorem 1.2. *Suppose that k is a perfect field. Then we have*

$$V_k \setminus \{0\} = V_k^{\mathrm{ss}} \amalg \coprod_{\beta \in \mathfrak{B}} S_{\beta\, k}.$$

Moreover, $S_{\beta\, k} \cong G_k \times_{P_{\beta\, k}} Y_{\beta\, k}^{\mathrm{ss}}$.

We remind the reader that $G_k \times_{P_{\beta\, k}} Y_{\beta\, k}^{\mathrm{ss}}$ means $(G_k \times Y_{\beta\, k}^{\mathrm{ss}})/\sim$ where $(g,v),(g'v') \in G_k \times Y_{\beta\, k}^{\mathrm{ss}}$ are equivalent if there exists an element $p \in P_{\beta\, k}$ such that $g' = gp^{-1}$ and $v' = pv$. In Theorem 1.2, the bijection $S_{\beta\, k} \cong G_k \times_{P_{\beta\, k}} Y_{\beta\, k}^{\mathrm{ss}}$ is induced from the canonical map $G_k \times_{P_{\beta\, k}} Y_{\beta\, k}^{\mathrm{ss}} \ni (g,v) \mapsto gv \in S_{\beta\, k}$.

If G is a split reductive group, Theorem 1.2 was proved in principle by Kirwan [13], Ness [17]. The main purpose of [13] was to calculate equivariant cohomology groups by using the equivariant morse stratification. Kirwan used the inductive structure of the strata for that purpose. On the other hand, in [17], Ness studied the stratification of the null cone depending more on geometric invariant theory. She studied the stratification of the null cone by the convexity of the moment map. The rationality of the stratification is basically due to the earlier work of Kempf [12]. In this thesis, we call the stratification of the null cone which was introduced in their works "the GIT stratification of the null cone".

We intend to study zeta functions associated to prehomogeneous vector spaces using the stratification as in [26]. For that purpose, we need a completely algebraic approach and the rationality of the inductive structure. Also in number theoretic situations, there are interesting non-split groups such as orthogonal groups, unitary groups, restrictions of scalars, etc. So even though our proof is fairly easy if we assume the split case, it is probably worth pointing out how the non-split case is reduced to the split case. This is the main purpose of this thesis.

We fix a perfect field k. The representation of $G = \mathrm{GL}_2$ on $V = \mathrm{Sym}^3\mathrm{Aff}^2$ (over k) is an example of what we call a "prehomogeneous vector space", where Aff^2 denotes the two dimensional affine space which is regard as a vector space of dimension two (similarly, we use the notation Aff^n for the n dimensional affine space). For the notion of prehomogeneous vector spaces, we summarized various definitions and properties in section 3 (see also [14] or [21]). In this situation we are interested in G_k-orbits in V_k. However, if we are to use Theorem 1.2 in this situation, we have to consider the action of SL_2 on V instead of GL_2. So we would like to modify Theorem 1.2 so that it is applicable to the action of the groups which correspond to GL_2 in this situation.

Let G be a reductive group and V a representation of G both defined over k. We assume that there is a reductive subgroup G_1 of G, a torus $T_0 \subset Z_G(G)$ (the center of G) with positive split rank and a rational character ψ of T_0 such that $T_0 \cap G_1$ is finite and that $G = T_0 G_1$ as algebraic groups (i.e., $G_{\bar{k}} = T_{0\,\bar{k}} G_{1\,\bar{k}}$). We also assume that the action of $t \in T_0$ on V is given by the scalar multiplication by $\psi(t)$. Let S be a maximal split torus of G_1 (this is the difference from the situation of Theorem 1.2) and we define $\mathfrak{s}^*_{\mathbb{R}}$, $\mathfrak{s}^*_{\mathbb{R},+}$, $\mathfrak{B}$, etc., with respect to the group G_1. For $\beta \in \mathfrak{B}$, we define $Z_\beta, W_\beta, Y_\beta, Y^{\mathrm{ss}}_\beta$, etc., as in Introduction with respect to G_1 also. Then we have the following corollary.

Corollary 1.3. *In the above situation, the statement of Theorem 1.2 holds.*

Let (G, V, χ) be a prehomogeneous vector space defined a perfect field k (see section 3). For simplicity, we assume that V is an irreducible representation of G. Then, χ is essentially unique (Proposition 3.6). Thus, we can say that "(G, V) is a prehomogeneous vector space". Let $\Delta \in k[V]$ be a relative invariant polynomial of (G, V, χ), that is $\Delta(gv) = \chi(g)\Delta(v)$ for all $g \in G$ and $v \in V$. Put $G_1 = \ker \chi \subset G$. Then there is a torus $T \subset Z_G(G)$ such that $G_{\bar{k}} = T_{\bar{k}} G_{1\,\bar{k}}$ and $G_1 \cap T$ is finite. By Schur's lemma, the action of $t \in T$ on V is given by the scalar multiplication by $\psi(t)$ for some $\psi \in X^*(Z_G(G)^\circ)$. We put $V' = \{v \in V \mid \Delta(v) \neq 0\}$. The set V' does not depend on the choice of Δ (Proposition 3.3). The purpose of using our formulation is to define the notion of a generic point from the viewpoint of geometric invariant theory (see section 4). In fact, our definition of V' coincides with $V^{\mathrm{ss}} = \pi_V^{-1}(\mathbb{P}(V)^{\mathrm{ss}})$ where $\mathbb{P}(V)^{\mathrm{ss}}$ is the set of semistable point respect to the action of G_1 (not G). In the global theory of prehomogeneous vector spaces, the set $V_k \setminus V_k^{\mathrm{ss}}$ is called the singular set of V. Corollary 1.3 applies to this situation, in particular, we obtain a stratification of the singular set $V_k \setminus V_k^{\mathrm{ss}}$. There are several interesting prehomogeneous vector spaces with non-split groups, and our result provides basic information of the singular orbits for those prehomogeneous vector spaces. This is useful for studying the global zeta function associated with a prehomogeneous vector space.

This thesis is organized as follows.

In section 2, we recall elementary results of algebraic groups. The notion of prehomogeneous vector spaces is discussed in section 3. In section 4, we recall the main theorem of geometric invariant theory which we call "the Hilbert–Mumford criterion of stability". In section 5, we recall Kempf's result which will be used in sections 6, 8. The rationality property of Kempf's result will be used mainly in section 8. In section 6, we give a proof of Theorem 1.2 in the case where the ground field is algebraically closed (and so the group is split). This case is in principle known. However, we are interested in the rationality question and we use a formalization which is a combination of methods in [17], [13] so that it is easier to deduce the rationality result. Therefore, we included the proof of this case (inductive structure of the strata is proved in section 7). We give a proof of Theorem 1.2 and Corollary 1.3 in the non-split case in section 8. Finally, in section 9, we give two series of prehomogeneous vector spaces which have the same sets of weights with respect to the action of maximal split tori. One is similar to the space of pairs of ternary quadratic forms and the other is slightly easier and similar to the space of pairs of binary quadratic forms.

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2 Notion regarding algebraic groups

Let k be an arbitrary field and G an algebraic group defined over k. We say that G is *solvable* (*resp. nilpotent*) if G is solvable (resp. nilpotent) as an abstract group. If G is a solvable (resp. nilpotent, abelian) algebraic group, then its Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ (= the tangent space $T_e(G)$ of G at the identity) is solvable (resp. nilpotent, abelian). For a closed subgroup $H \subset G$, we denote the normalizer and the centralizer of H by $N_G(H)$ and $Z_G(H)$. If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, we define

$$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} \mid [X, Y] \in \mathfrak{h} \text{ for all } Y \in \mathfrak{h}\}$$
$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{h}\}.$$

The subalgebras $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}), \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ are called the normalizer and the centralizer of $\mathfrak{h}$ respectively.

An algebraic group T over k is called a *torus* if T is isomorphic to GL_1^n over $\bar{k}$ for some positive integer n. If $T \cong \mathrm{GL}_1^n$ over K ($\supset k$), we say that T is *split* (over K).

Example 2.1. Consider the torus

$$T = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a^2 + b^2 \neq 0 \right\} \subset \mathrm{GL}_2.$$

The characteristic polynomial of the matrix $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ is $X^2 - 2aX + a^2 + b^2 = (X-a)^2 + b^2$ and so its eigenvalues are $\lambda = a \pm b\sqrt{-1}$. Therefore, T is split over K if and only if $\sqrt{-1} \in K$.

It is known that if T is a torus, $T \times_k \bar{k}$ is split (see 8.11 Proposition [1, p.117]). If T splits over the ground field k, then $X^*(T) \cong \mathbb{Z}^m$ for some m. So any character is of the form $(t_1, \ldots, t_m) \mapsto t_1^{p_1}, \ldots, t_m^{p_m}$. Since this is the case over $\bar{k}$ also, $X^*(T) \cong X^*(T \times_k \bar{k})$.

A torus T is said to be *anisotropic* if $X^*(T) = \{0\}$. If G is an algebraic group and a closed subgroup $T \subset G$ is isomorphic to a torus, it is called a *subtorus*. If T is a torus, then there exist subtori $T_a, T_d \subset T$ such that T_a is anisotropic, T_d is split, $T = T_a \cdot T_d$ and $T_a \cap T_d$ is finite. Moreover, $T_a = \cap_{\chi \in X^*(T)} \ker \chi$. Also T/T_a is a split torus.

The following proposition is proved in 8.2 Proposition [1, pp.111,112].

Proposition 2.2. *(1) If T is a torus and $H \subset T$ is a closed connected subgroup, then H is a torus.*

(2) If $\rho : T \to \mathrm{GL}_n$ is a finite representation, then there exists $g \in \mathrm{GL}_n(k^{\mathrm{sep}})$ such that $g\rho(T)g^{-1}$ consists of diagonal matrices.

Considering the dimension, there always exists a maximal torus. The following theorem is proved in 18.2, 19.2 Theorems [1, pp.218–220,223].

Theorem 2.3. *Let G be connected. Then G contains a maximal torus T such that $T \times_k \bar{k} \subset G \times_k \bar{k}$ is also a maximal torus. Any two of them are conjugate by an element of G_k.*

Suppose G is connected and $S \subset G$ is a subtorus. It is known that the centralizer $Z_G(S)$ is connected (see 11.2 Corollary [1, p.152]). Let $T \subset G$ be a maximal torus as in Theorem 2.3. Then $Z_G(T)$ is called a *Cartan subgroup*. Also by 19.2 Theorem [1, p.223], any two Cartan subgroup are conjugate by an element of G_k.

For an algebraic group G over k, G° denotes the connected component of the identity of G, and we call it the *identity component* of G.

We have a natural pairing $\langle\ ,\ \rangle_G : X^*(G) \times X_*(G) \to \mathbb{Z}$ such that $\chi(\lambda(t)) = t^{\langle \chi, \lambda \rangle}$ for all $\chi \in X^*(G), \lambda \in X_*(G)$. It is known that this pairing is perfect if $G = T$ is a torus ([1, pp.113–115]).

A maximal connected solvable closed subgroup of $G \times_k \bar{k}$ is called a *Borel subgroup*. Note that G may not have a Borel subgroup defined over k and if it does, G is said to be *quasi-split*.

Example 2.4. Let B be the set of upper triangular matrices which are contained in GL_n (resp. SL_n). Then B is a Borel subgroup of GL_n (resp. SL_n).

For a while, we assume k is algebraically closed. The following theorem is proved in 10.4 Theorem [1, p.137].

Theorem 2.5 (Borel's fixed point theorem)**.** *Let G be a connected solvable group acting on a non-empty complete variety V. Then G has a fixed point in V.*

Suppose $G = \mathrm{GL}_n$. Let $B \subset G$ be the subgroup of upper triangular matrices, and B' be another Borel subgroup. Since G/B is complete, B' has a fixed point xB in G/B. This implies that $B'xB = xB$ or equivalently, $B' \subset xBx^{-1}$. By the maximality of Borel subgroups, $B' = xBx^{-1}$. Therefore, all Borel subgroups of GL_n are conjugate.

Let G be an connected algebraic group again. The above argument can be generalized to prove the following theorem. For the proof, see 11.1 Theorem [1, p.147].

Theorem 2.6. *Let $B \subset G$ be a Borel subgroup. Then G/B is a projective variety. Moreover, all Borel subgroup of G are conjugate.*

A closed group $P \subset G$ is called a *parabolic subgroup* if it contains a Borel subgroup.

By Theorem 2.5, it is easy to see that P is parabolic if and only if G/P is complete. The following theorem is proved in [1, pp.154, 155].

Theorem 2.7. *If $P \subset G$ is a parabolic subgroup, it is connected and $N_G(P) = P$.*

Let $\mathscr{B}$ be the set of all Borel subgroups of G. Then, the group $R(G) = (\bigcap_{B \in \mathscr{B}} B)^\circ$ is called the *radical* of G. $R(G)$ is the maximum connected solvable normal subgroup of G, and its unipotent part $R_u(G)$ is called the *unipotent radical* of G.

Definition 2.8. A connected algebraic group G is said to be *semi-simple* (resp. *reductive*) if $R(G) = \{e\}$ (resp. $R_u(G) = \{e\}$).

Suppose G is reductive and that k is algebraic closed. It is known that $R(G) = Z_G(G)^\circ$ is a torus.

Definition 2.9. A *Levi subgroup* of G is a connected subgroup L such that G is the semi-direct product of L and $R_u(G)$.

A Levi subgroup maps isomorphically onto $G/R_u(G)$, hence it is reductive.

If λ is a 1-PS of G, we denote by $P(\lambda)$ the closed subgroup formed by the $g \in G$ such that $\lim_{t \to 0} \lambda(t) g \lambda(t)^{-1}$ exists. The group $P(\lambda)$ was already introduced in Introduction. As was shown in 8.4.5 Proposition [22, p.148], $P(\lambda)$ is a parabolic subgroup of G. We denote by $U(\lambda)$ the normal subgroup of $P(\lambda)$ formed by the $g \in P(\lambda)$ for which the limit equals 1. The centralizer of $\operatorname{im} \lambda$ is denoted by $Z(\lambda)$. It is a closed subgroup of $P(\lambda)$.

The following theorem is proved in 13.4.2 Theorem [22, p.234].

Theorem 2.10. *Assume that k is an arbitrary field. Let $\lambda \in X_{*,k}(G)$. Then $P(\lambda), U(\lambda)$ and $Z(\lambda)$ are connected k-subgroups and $U(\lambda)$ is a unipotent normal subgroup of $P(\lambda)$. Moreover, the product morphism $Z(\lambda) \times U(\lambda) \to P(\lambda)$ is a k-isomorphism of varieties.*

We put $C = Z_G(G)^\circ$. Then $G = C \cdot (G, G)$, (G, G) is semi-simple, and $C \cap (G, G)$ is finite. If T is a maximal torus with Lie algebra $\mathfrak{t}$, then $Z_G(T) = T$ and $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$, where $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t})$ denotes the centralizer of $\mathfrak{t}$.

Definition 2.11. Let T be a maximal torus of G and S a maximal k-split torus of G. We call $\dim T$ (resp. $\dim T \cap [G, G]$) the *rank* (resp. *semi-simple rank*) of G. Also, we call $\dim S$ the *split rank* or *k-rank* of G.

Let $\rho : T \to \mathrm{GL}(V)$ be a finite dimensional representation. Since $\bar{k} = k$, T is split and so this representation is diagonalizable. For $\alpha \in X^*(T)$, we define

$$V_\alpha = \{v \in V \mid \rho(t)v = \alpha(t)v \text{ for all } t \in T\}.$$

If $V_\alpha \neq \{0\}$, α is called a *weight* of T in V. We consider the adjoint action of T on $\mathfrak{g}$. Let Φ be the set of non-zero weights of T in $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right). \tag{2.12}$$

It is known that $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$.

Definition 2.13. The group $\mathbb{W}(T, G) = N_G(T)/T$ is called the *Weyl group* of G.

Example 2.14. Let $G = \mathrm{GL}_n$ and T be the set of diagonal matrices which are contained in G. If $\sigma \in \mathfrak{S}_n$, we denote by E_σ the matrix whose $(i, \sigma(i))$ entry is 1 and other entries are 0. Then such E_σ's form set of representatives for the set $N_G(T)_k/T_k$. Thus, we have $\mathbb{W}(T, G) \cong \mathfrak{S}_n$.

Let $n \in N_G(T)$ and let $\alpha \in X^*(T)$. We define $\alpha^n(t) = \alpha(n^{-1}tn)$. This defines a left action of $\mathbb{W} = \mathbb{W}(T, G)$ on $X^*(T)$, and hence $\mathfrak{t}^*_{\mathbb{R}} = X^*(T) \otimes \mathbb{R}$. If $v \in \mathfrak{g}_\alpha$, then

$$\mathrm{Ad}(ntn^{-1})\mathrm{Ad}(n)v = \alpha(t)\mathrm{Ad}(n)v.$$

Replacing t by $n^{-1}tn$, we have $\mathrm{Ad}(n)\mathfrak{g}_\alpha \subset \mathfrak{g}_{\alpha^n}$. So if $\alpha \in \Phi$, then $\alpha^n \in \Phi$. Therefore, $\mathbb{W}$ leaves Φ stable.

Let $\alpha \in \Phi$. Then there exists a unique closed T-invariant subgroup U_α such that $\mathrm{Lie}\, U_\alpha = \mathfrak{g}_\alpha$. Moreover, $U_\alpha \cong \mathbb{G}_a$. Let $T_\alpha = (\ker \alpha)^\circ, G_\alpha = Z_G(T_\alpha)$. Then G_α is a reductive subgroup of semi-simple rank one. Moreover, G_α is generated by $U_\alpha, U_{-\alpha}$ and T. Since

$$\mathbb{W}(T, G_\alpha) = (N_G(T) \cap Z_G(T_\alpha))/T \subset \mathbb{W}(T, G),$$

we regard $\mathbb{W}(T, G_\alpha)$ as a subgroup of $\mathbb{W}(T, G)$. The order of this group is two and if we denote the generator by r_α, then $r_\alpha(\alpha) = -\alpha$.

If $\alpha, \beta \in \Phi$, it turns out that

$$r_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$$

with $n_{\beta,\alpha} \in \mathbb{Z}$. Moreover, $n_{\alpha,\alpha} = 2$.

Let V be a vector space over $\mathbb{R}$ with inner product $(\ ,\ )$. $r \in \mathrm{GL}(V)$ is called a reflection with respect to $\alpha \in V$ if $r(\alpha) = -\alpha$ and r fixes each point of the hyperplane $\{\beta \in V \mid (\beta, \alpha) = 0\}$.

Definition 2.15. A *root system* is a pair (V, Φ) where V is a vector space over $\mathbb{R}$, and $\Phi \subset V$ is a subset satisfying.

(1) Φ is finite, spans V, and does not contain zero.

(2) For each $\alpha \in \Phi$ there is a reflection r_α with respect to α which leaves Φ stable.

(3) If $\alpha, \beta \in \Phi$ then $r_\alpha(\beta) = \beta - n_{\beta,\alpha}\alpha$ with $n_{\beta,\alpha} \in \mathbb{Z}$. The elements of Φ are called *roots.*

The notion of isomorphism of root system is defined in the obvious manner. We will usually denote the root system by Φ, and say that "Φ is a root system in V". In particular, we have $\mathrm{Aut}\,\Phi \subset \mathrm{GL}(V)$. The subgroup $\mathbb{W}(\Phi)$ of $\mathrm{Aut}\,\Phi$ generated by the r_α ($\alpha \in \Phi$) is called the *Weyl group* of Φ.

Let $\alpha \in \Phi$ be an element such that the only roots, $a\alpha$, proportional to α are such that $|a| \leqq 1$. If $a\alpha$ is such a root, then

$$-a\alpha = r_\alpha(a\alpha) = a\alpha - n_{a\alpha,\alpha}\alpha,$$

which implies that $2a = n_{a\alpha,\alpha} \in \mathbb{Z}$. Thus the roots proportional to α are either $\{-\alpha, \alpha\}$ or $\{-\alpha, -\alpha/2, \alpha/2, \alpha\}$. If the latter case never occurs, then the root system is said to be *reduced.*

A subset Δ of Φ is called a *basis* if the two conditions are hold.

(1) Δ is a basis of V.

(2) Each root β can be written as $\beta = \sum k_\alpha \alpha$ ($\alpha \in \Delta$) with integral coefficients k_α all non negative or all non positive.

The roots in Δ are called *simple.* If all k_α are non negative (resp. all k_α are non positive), we call β *positive* (resp. *negative*) and write $\beta \succ 0$ (resp. $\beta \prec 0$). The collection of positive and negative roots (relative to Δ) will usually just be denoted by Φ^+ and Φ^-.

Clearly $\Phi^- = -\Phi^+$ holds. It is known that any root system has a basis (see 14.7 Theorem [1, p.188]).

The root system (V, Φ) is *irreducible* if one cannot write $V = V_1 \oplus V_2$ as a non-trivial direct sum so that $\Phi = (\Phi \cap V_1) \cup (\Phi \cap V_2)$.

Let $\langle\, , \,\rangle : V \times V^* \to \mathbb{R}$ be the natural pairing between V and V^*. We call

$$WC(\Delta) = \{\lambda \in V^* \,|\, \langle \alpha, \lambda \rangle > 0 \text{ for all } \alpha \in \Delta\}$$

the *Weyl chamber* of Δ or Φ^+. Call $\lambda \in V^*$ *regular* if $\langle \alpha, \lambda \rangle \neq 0$ for all $\alpha \in \Phi$. For example, a Weyl chamber clearly consists of regular elements. If λ is regular, we write

$$\Phi^+(\lambda) = \{\alpha \in \Phi \,|\, \langle \alpha, \lambda \rangle > 0\}$$

and

$$\Delta(\lambda) = \{\alpha \in \Phi^+(\lambda) \,|\, \alpha \text{ is not the sum of two elements of } \Phi^+(\lambda)\}.$$

The following theorem is 14.7 Theorem [1, p.188].

Theorem 2.16. *Let Φ be a root in V.*

(1) If $\lambda \in V^*$, then $\Delta(\lambda)$ is a basis of Φ. It is the unique basis contained in $\Phi^+(\lambda)$. Thus, $\Delta \mapsto WC(\Delta)$ is a bijection from the set of bases to the set of Weyl chambers.

(2) The group $\mathbb{W}(\Phi)$ acts simply transitivity on the set of bases of Φ, and on the set of Weyl chambers.

(3) Reflexions r_α ($\alpha \in \Delta$) generate $\mathbb{W}(\Phi)$.

(4) We have $\Phi = \bigcup_{w \in \mathbb{W}(\Phi)} w\Delta$.

We can identify $V = X^*(T/Z_G(G)^\circ) \otimes \mathbb{R}$ canonically with a subgroup of $\mathfrak{t}^*_{\mathbb{R}} = X^*(T) \otimes \mathbb{R}$. Then $\Phi = \Phi(T, G)$ is a reduced root system in V, with Weyl group $\mathbb{W} = \mathbb{W}(T, G)$ (14.8 Theorem [1, p.189]).

Let $\mathscr{B}^T$ be the set of all Borel subgroups of G containing T. Let $B \in \mathscr{B}^T$ with Lie algebra $\mathfrak{b}$. Then $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$ and the set $\Phi(B)$ of non-zero weights of T in $\mathfrak{b}$ may be identified with a subset of Φ. Let $\Delta = \Delta(B)$ be the set of $\alpha \in \Phi(B)$ which are not sums of two elements in $\Phi(B)$. Then $\Delta(B)$ is a basis of Φ and G_α ($\alpha \in \Delta(B)$) generate G (14.8 Corollary 1 [1, p.189]). We call $\Delta(B)$ the set of simple roots associated with B.

If we identify V^* with $\mathfrak{t}_{\mathbb{R}} = X_*(T) \otimes \mathbb{R}$, then the Weyl chambers in $\mathfrak{t}_{\mathbb{R}}$ of bases of Φ are obtained form Borel subgroups $B \in \mathscr{B}^T$ by Theorem 2.16. The Weyl chamber in $\mathfrak{t}_{\mathbb{R}}$ which corresponds $B \in \mathscr{B}^T$ is $WC(B) = \{\lambda \in \mathfrak{t}_{\mathbb{R}} \,|\, \langle \alpha, \lambda \rangle_T > 0 \text{ for all } \alpha \in \Phi(B)\}$. Moreover, $\mathbb{W}$ acts simply transitively on these Weyl chambers.

Let k be an arbitrary field again. Suppose G is a connected reductive group over k. For the rest of this section, we consider the structure of $X^*(G)$.

If G is semi-simple, then $Z_G(G)^\circ = R(G) = \{e\}$. So $\dim G = \dim(G,G)$. Therefore, $G = (G,G)$. This implies that $X^*(G) = \{0\}$.

In general, we put $G_1 = (G,G)$ and $T = G/G_1$. We denote the natural homomorphism $G \to T$ by π. Let $G_2 = \pi^{-1}(T_a)$. Put $\bar{T} = G/G_2 \cong T/T_a$. Then $\bar{T}$ is a split torus. If $\chi \in X^*(G)$, then it is trivial on G_2 and so it is induced by a character of $\bar{T}$. This implies that $X^*(G) \cong \mathbb{Z}^m$ for some m.

A character χ of an algebraic group is said to be *primitive* (or *indivisible*) if ψ is another character and $\chi = \psi^p$ for an integer p, then $p = \pm 1$. If T is a torus, $\chi \in X^*(T)$ is primitive if and only if it is a primitive vector in $X^*(T) \cong \mathbb{Z}^m$.

Proposition 2.17. *Suppose G is connected reductive. If $\chi \in X^*(G)$ is primitive, then it is primitive in $X^*(G \times_k \bar{k})$.*

Proof. Suppose $\psi \in X^*(G \times_k \bar{k})$ and $\phi = \psi^p$. Since $\chi|_{G_2}$ is trivial, $(\psi|_{G_2 \times_k \bar{k}})^p$ is trivial. Since $X^*(G_2 \times_k \bar{k})$ is torsion free, $\psi|_{G_2 \times_k \bar{k}}$ is trivial. So ψ is induced by a character of $\bar{T}$. Since $\bar{T}$ is split, $X^*(\bar{T}) \cong X^*(\bar{T} \times_k \bar{k})$. Since χ corresponds to a primitive integer point, p must be ± 1. □

3 Notion of prehomogeneous vector spaces

In this section, we summarize the notion of prehomogeneous vector spaces. This mainly due to unpublished note [28].

Let k be an arbitrary field. Let G be a connected reductive group, V a representation of G, and χ a non-trivial rational character of G, all defined over k.

Definition 3.1 (M.Sato)**.** A triple (G, V, χ) is called a *prehomogenous vector space* if the following two conditions are satisfied:

(1) There exists a Zariski open orbit. More precisely, there exist an open set $U \subset V$ and $v \in U_{\bar{k}}$ such that $G_{\bar{k}} v = U_{\bar{k}}$.

(2) There exists a non-constant polynomial $\Delta \in k[V]$ such that $\Delta(gv) = \chi(g)\Delta(v)$.

The polynomial $\Delta(v)$ is called the *relative invariant polynomial*.

We are mainly interested in irreducible representations. However, if $\operatorname{char} k = p > 0$, we sometimes have to consider reducible representations which are obtained by reducing irreducible representations modulo p. Those representations can be handled more or less in the same manner as irreducible representations. Therefore, we consider the following condition for that purpose.

Let Z be the identity component of the center of G. Since G is reductive, Z is a torus.

Condition 3.2. There exist $\psi \in X^*(Z)$ such that if $t \in Z$, then $tv = \psi(t)v$ for all $v \in V$.

Note that because of Schur's lemma, Condition 3.2 is satisfied if V is a irreducible representation.

Now we put $V^{\mathrm{ss}} = \{v \in V \mid \Delta(v) \neq 0\}$ and call it *the set of semistable points* of V. We show the set V^{ss} does not depend on the choice of Δ.

Proposition 3.3. *If Δ_1, Δ_2 are relative invariant polynomials, then $\Delta_1(v)/\Delta_2(v)$ is a constant.*

Proof. By definition, we have

$$\Delta_1(gv) = \chi(g)\Delta_1(v), \quad \Delta_2(gv) = \chi(g)\Delta_2(v).$$

Then $\Delta_1(v)/\Delta_2(v)$ is invariant by the action of G. Since V has an open G-orbit, $\Delta_1(v)/\Delta_2(v)$ is a constant. □

Lemma 3.4. *If $\Delta(v)$ is a relative invariant polynomial, it is a homogeneous polynomial.*

Proof. Suppose $\Delta(gv) = \chi(g)\Delta(v)$. Let $t \in \mathrm{GL}_1$. Then $\Delta(tgv) = \Delta(gtv) = \chi(g)\Delta(tv)$. So by the proof of Proposition 3.3, there exists $c(t) \in \mathrm{GL}_1$ such that $\Delta(tv) = c(t)\Delta(v)$. Obviously, $t \mapsto c(t)$ is a character of GL_1 and so there exists an integer N such that $c(t) = t^N$. Since $\Delta(v)$ is a non-constant polynomial, $N > 0$. So $\Delta(v)$ is a homogeneous polynomial. □

Lemma 3.5. *Suppose $F \subset V$ is a G-invariant closed subset and $F = \cup_{i=1}^N F_i$ is the irreducible decomposition. Then each F_i is G-invariant.*

Proof. Without loss of generality, it suffice to prove that F_1 is G-invariant. Let $U_1 = F_1 \setminus (\cup_{i=2}^N F_i)$. Then U_1 is an open set of F_1 and so is irreducible.

Let $\mu : G \times V \to V$ be the group action, $\widetilde{U}_1 = \mu(G \times U_1)$, and $\widetilde{F}_1$ be the closure of $\widetilde{U}_1$. Since G is irreducible also, $\widetilde{U}_1, \widetilde{F}_1$ are irreducible. Since $U_1 \subset \widetilde{U}_1 \subset \widetilde{F}_1 \subset F$, $F_1 \subset \widetilde{F}_1 \subset F$. This implies that $F_1 = \widetilde{F}_1$ and so F_1 is a G-invariant subset. □

Suppose F_i in the above lemma is the zero set of an irreducible polynomial Δ_i (this is the case if the codimension of F_i is one). By Hilbert's Nullstellensatz, there exists a character χ_i of G such that $\Delta_i(gv) = \chi_i(g)\Delta_i(v)$.

If (G, V, χ) satisfies Condition 3.2, we will prove that the choice of χ is essentially unique.

Proposition 3.6. *Suppose that V is a representation of G satisfying Condition 3.2. If χ_1, χ_2 are primitive rational characters of G and $(G, V, \chi_1), (G, V, \chi_2)$ are prehomogeneous vector space, then $\chi_1 = \chi_2$.*

Proof. We may assume that $k = \bar{k}$. Let $G_1 = [G, G]$. By assumption, G_1 is semi-simple and so has no non-trivial character. Also G contains a torus T which is contained in the center of G such that $G = TG_1$.

By Condition 3.2, there exists $\psi \in X^*(T)$ such that $tv = \psi(t)v$ for all $t \in T, v \in V$. Suppose that $\Delta_1(v), \Delta_2(v) \in k[V]$ and

$$\Delta_1(gv) = \chi_1(g)\Delta_1(v), \Delta_2(gv) = \chi_2(g)\Delta_2(v).$$

If $g = tg_1$ with $t \in T, g_1 \in G_1$, then

$$\begin{aligned}\Delta_1(g) &= \chi_1(g)\Delta_1(v) = \chi_1(t)\Delta_1(v) = \Delta_1(tv) \\ &= \Delta_1(\psi(t)v) = \psi(t)^{\deg \Delta_1}\Delta_1(v).\end{aligned}$$

This implies that $\chi_1(tg_1) = \psi(t)^{\deg \Delta_1}$. Similarly, $\chi_1(tg_2) = \psi(t)^{\deg \Delta_2}$. Therefore $\chi_1(tg_1)^{\deg \Delta_2}$ $= \chi_2(tg_2)^{\deg \Delta_1}$. Since both χ_1 and χ_2 are primitive, we have $\chi_1 = \chi_2$. □

By this proposition, if a triple (G, V, χ) satisfies Condition 3.2, we may use the notation (G, V) instead of (G, V, χ).

Corollary 3.7. *Let (G, V) be a prehomogeneous vector space satisfying Condition 3.2, and $\Delta(x)$ a relative invariant polynomial of the lowest degree. Then $\Delta(x)$ is irreducible over $\bar{k}$. Moreover, the zero set of $\Delta(v)$ is the only G-invariant irreducible codimension one closed subset of V.*

Proof. We assume that $k = \bar{k}$ (we omit the proof of the general case). Let

$$\Delta(v) = \Delta_1(v)^{p_1} \cdots \Delta_N(v)^{p_N}$$

be the prime decomposition, and

$$F = \{v \in V \mid \Delta(v) = 0\}, \quad F_i = \{v \in V \mid \Delta_i(v) = 0\}$$

for all i. More precisely, $F = \operatorname{Spec} k[V]/(\Delta(v))$, etc. By Hilbert's Nullstellensatz, $F_i \not\subset F_j$ if $i \neq j$.

By the comment after Lemma 3.5, for each i, there exists a character χ_i of G such that $\Delta_i(gv) = \chi_i(g)\Delta_i(v)$. Suppose $\chi_i = \psi_i^{n_i}$ where ψ_i is primitive and n_i is a positive integer for all i. By Proposition 3.6, $\psi_i = \chi$ for all i. Therefore, $\Delta_i(x)$ is a relative invariant polynomial (corresponding to χ). Since the degree of $\Delta(v)$ is smallest, $N = 1, p_1 = 1$ and so $\Delta(v)$ is irreducible.

If $\overline{F} \subset V$ is any irreducible codimension one closed subset, it is the zero set of an irreducible polynomial $\overline{\Delta}(v)$. Then by Hilbert's Nullstellensatz, there exists a character $\overline{\chi}$ of G such that $\overline{\Delta}(gv) = \overline{\chi}(g)\overline{\Delta}(v)$. Then by Proposition 3.6 again, $\overline{\Delta}(v)$ is a relative invariant polynomial. Since both $\Delta(v)$ and $\overline{\Delta}(x)$ are irreducible, $\overline{\Delta}(v)$ must be a constant multiple of $\Delta(v)$. □

Suppose that (G, V) is a prehomogenous vector space satisfying Condition 3.2. In general, $V^{\mathrm{ss}}_{k^{\mathrm{sep}}}$ is not a single $G_{k^{\mathrm{sep}}}$-orbit. However, if (G, V) satisfies a condition called *regularity*, then $V^{\mathrm{ss}}_{k^{\mathrm{sep}}}$ becomes a single $G_{k^{\mathrm{sep}}}$-orbit.

The notion of regularity was introduced by Sato–Kimura in § 4 in [21]. We use the property of Proposition 25 ([21, p.72]) as the definition of regularity. Note that we only consider prehomogeneous vector spaces which satisfy Condition 3.2 in this thesis.

We recall Sato–Kimura's original definition of the notion of regularity for the convenience of readers. Let (G, V) be a (reductive) prehomogeneous vector space defined over the complex number field $\mathbb{C}$. Then the set $\mathrm{Sing}(V) = V \setminus V^{\mathrm{ss}} = \{v \in V \mid \Delta(v) = 0\}$ is called the set of *unstable points* (or *singular set*) of (G, V).

The next definition is Definition 7 in [21, p.60].

Definition 3.8 (Sato–Kimura)**.** A prehomogeneous vector space (G, V) is called *regular* if there exists a relative invariant polynomial $f(v)$ such that the Hessian $H_f = \det\left(\frac{\partial^2 f}{\partial v_i \partial v_j}(v)\right)_{i,j}$ of $f(x)$ is not identically zero.

In the book by T. Kimura, there is a proof of the following theorem ([14, p.43, Theorem 2.28]).

Theorem 3.9. *Let (G, V) be a (reductive) prehomogeneous vector space defined over $\mathbb{C}$. Then the following conditions are equivalent.*

(1) A pair (G, V) is a regular prehomogeneous vector space.

(2) The singular set $\mathrm{Sing}(V)$ *is a hypersurface.*

(3) The open orbit $Gv = V \setminus \mathrm{Sing}(V)$ is an affine variety.

(4) Each generic isotropy subgroup G_v ($v \in V \setminus \mathrm{Sing}(V)$) is reductive.

(5) Each generic isotropy subalgebra $\mathrm{Lie}(G_v)$ *($v \in V \setminus \mathrm{Sing}(V)$) is reductive in* $\mathrm{Lie}\, G$.

The statement of Proposition 25 ([21, p.72]) is a part of Theorem 3.9 (i.e. (1) follows form (4)).

Now we return to the general situation. Let (G, V) be a (reductive) prehomogeneous vector space defined over an arbitrary field k. Assume that there exists $w \in V_k$ such that $U = Gw$ is Zariski open. In this situation, A.Yukie proved the following theorem ([27]) .

Theorem 3.10. *Suppose that G_w is reductive. Then, the following conditions are hold.*

(1) $V \setminus U$ is a hypersurface.

(2) $U_{k^{\mathrm{sep}}}$ is a single $G_{k^{\mathrm{sep}}}$-orbit set-theoretically.

Corollary 3.11. *If the assumption of Theorem 3.10 is satisfied, there exits a relative invariant polynomial and so (G, V) is a prehomogenous vector space. Moreover, $V^{\mathrm{ss}}_{k^{\mathrm{sep}}} = G_{k^{\mathrm{sep}}}w$.*

Proof. Let $U = Gw$ and $F_1, \dots, F_n$ be the irreducible codimension one components of $V \setminus U$, $F = F_1 \cup \cdots \cup F_n$, and $W = V \setminus F$. Then W is affine, $U \subset W$, and the codimension of $W \setminus U$ in W is greater than one. Suppose $W = \operatorname{Spec} A$ and $U = \operatorname{Spec} B$. Then $A \subset B$ and A is a normal ring because W is smooth over k. Since the codimension of U in W is greater than one, any regular function on U (i.e., an element of B) extends to a regular function on W. Therefore, $A = B$ and so $W = U$.

Clearly, the orbit of the origin 0 consists of 0 itself. Since U is a single G-orbit, $0 \notin U$. This implies that $V \setminus U \neq \emptyset$. We have shown that any irreducible component of $F = V \setminus U$ is of codimension one.

Since F is a G-invariant closed subset of V, by Corollary 3.7, F is irreducible and is the zero set of a relative invariant polynomial of the lowest degree. This shows that $V^{\mathrm{ss}} = U$ and so $V^{\mathrm{ss}}_{k^{\mathrm{sep}}}$ is a single $G_{k^{\mathrm{sep}}}$-orbit by Theorem 3.10. □

In this thesis, we define the notion of regularity as follows.

Definition 3.12. A prehomogeneous vector space (G, V) satisfying Condition 3.2 is said to be *regular* if there exists a point $w \in V_k$ such that $U = Gw$ is Zariski open and G_w is reductive (and so smooth as a group scheme).

We shall prove a proposition which is convenient for verifying the assumption of Theorem 3.10 for actual examples.

Proposition 3.13. *If there exists a point $w \in V$ such that*

$$\dim T_e(G_w) = \dim G - \dim V,$$

then Gw is open in V. Moreover, G_w is smooth over k.

Proof. We may assume that $k = \bar{k}$. Consider the map $G \to V$ defined by $g \mapsto gw$. Then the image Gw is a constructible set. Since G is irreducible, $F = \overline{Gw}$ is irreducible. If $Gw = \cup_{i=1}^{l}(F_i \cap U_i)$ where F_i is closed and U_i is open for all i, then there must exists i such that $F_i = F$. Hence Gw contains an open set of F. Since G acts on Gw transitively, Gw itself is an open set of F. Therefore, Gw is a variety.

Note that for any $y \in Gw$, the fibre over y is isomorphic to G_w. Then,

$$\begin{aligned}\dim G &= \dim G_w + \dim Gw \\ &\leqq \dim T_e(G_w) + \dim Gw \\ &= \dim G - \dim V + \dim Gw.\end{aligned}$$

Therefore, $\dim Gw \geqq \dim V$ and so $Gw \subset V$ is open and $\dim T_e(G_w) = \dim G_w$. This implies that G_w is smooth over k. □

Next we discuss the notion of the castling transform (see [21, pp.37–39]). If $\rho : G \to \mathrm{GL}(V)$ is a representation, we define a representation of G on the dual vector space V^* by $(\rho^*(g)f)(v) = f(\rho(g^{-1})v)$ for $g \in G, f \in V^*$. This is also a left action and is called the *contragredient representation* of V.

To explain the castling transform, we have to discuss the notion of the Grassmann varieties. Let $V = \mathrm{Aff}^n$ be the n-dimensional affine space, regarded as a vector space of dimension n. If $0 \leqq m \leqq n$, there is an algebraic variety (which is not affine) called the Grassmann variety $\mathrm{Grass}_{n,m} = \mathrm{Grass}_m(V)$ of m-planes in V whose points are in one-to-one correspondence with m-dimensional subspaces of V.

We propose to put on the set $\mathrm{Grass}_m(V)$ of m-dimensional subspaces of V the projective variety. Define $f : \mathrm{Grass}_m(V) \to \mathbb{P}(\wedge^m V)$ by sending W to the point in the structure of a projective space corresponding to the line $\wedge^m W \subset \wedge^m V$. It is easily verified that f is injective, so we need only to show that its image is closed. This fact is proved in 10.3 [1, pp.135, 136].

If $W \subset V$ is an m-dimensional subspace,

$$\widehat{W} = \{f \in V^* \mid f(v) = 0 \text{ for all } v \in W\}$$

is an $(m-n)$-dimensional subspace of V^*. Therefore, there is a map $\mathrm{Grass}_m(V) \to \mathrm{Grass}_{n-m}(V^*)$. It is easy to see that this is set-theoretically bijective, and it is known that this map is in fact an isomorphism of algebraic varieties.

Now let $\rho : G \to \mathrm{GL}(V)$ be a representation of a reductive group G, and $\rho^* : g \to \mathrm{GL}(V^*)$ its contragredient representation. Consider the following two representations.

(1) $G \times \mathrm{GL}_m$ acting on $V \otimes \mathrm{Aff}^m$.

(2) $G \times \mathrm{GL}_{n-m}$ acting on $V^* \otimes \mathrm{Aff}^{n-m}$.

Note that we are regarding $\mathrm{Aff}^m, \mathrm{Aff}^{n-m}$ as a set of column vectors and so $\mathrm{GL}_m, \mathrm{GL}_{n-m}$ acts on them respectively.

The following proposition is Proposition 7 [21, p.37].

Proposition 3.14. *The representation (1) has an open orbit if and only if the representation (2) has an open orbit. If so, their generic stabilizer are isomorphic over $\bar{k}$.*

Proof. We may assume that k is algebraically closed. Let

$$U = \{v = (v_1, \dots v_m) \,|\, V \ni v_1, \dots, v_m \text{ are linearly independent}\},$$
$$U^* = \{v = (v_1, \dots v_{n-m}) \,|\, V \ni v_1, \dots, v_{n-m} \text{ are linearly independent}\}.$$

Note that $U \subset V \otimes \mathrm{Aff}^m, U^* \subset V^* \otimes \mathrm{Aff}^{n-m}$ are Zariski open subsets.

If $v = (v_1, \dots, v_m) \in U$, let $\langle v \rangle \in \mathrm{Grass}_m(V)$ be the m-dimensional subspace spanned by $v_1, \dots, v_m$. Then G acts on $\mathrm{Grass}_m(V), \mathrm{Grass}_{n-m}(V^*)$ and the maps

$$\pi : U \ni v \mapsto \langle v \rangle \in \mathrm{Grass}_m(V), \quad \pi^* : U^* \ni v \mapsto \langle v \rangle \in \mathrm{Grass}_{n-m}(V^*)$$

are equivariant with respect to the action of G. Moreover, points in $\mathrm{Grass}_m(V)$ (resp. $\mathrm{Grass}_{n-m}(V^*)$) are in one-to-one correspondence with GL_m-orbits in U (resp. GL_{n-m}-orbits in U^*). So there is an open orbit in U if and only if there is an open orbit in $\mathrm{Grass}_m(V) \cong \mathrm{Grass}_{n-m}(V^*)$. Therefore, this is the case if and only if there is an open orbit in U^*.

Suppose that the orbit of $w \in U$ is open and corresponds to the orbit of $w^* \in U^*$. Note that the action of GL_m on U does not have a fixed point (in other words the action is *free*). Therefore, the stabilizer of w in $G \times \mathrm{GL}_m$ and the stabilizer of $\pi(w)$ in G are isomorphic. Similarly, the stabilizer of w^* in $G \times \mathrm{GL}_{n-m}$ and the stabilizer of $\pi^*(w^*)$ in G are isomorphic. Since the stabilizers of $\pi(w), \pi^*(w^*)$ are isomorphic, this proves the second assertion. □

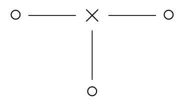
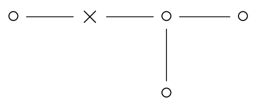
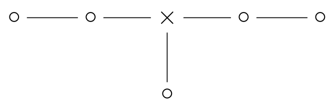
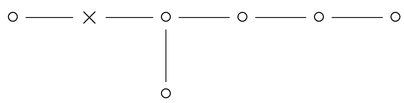
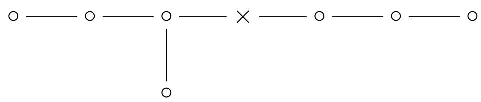
Corollary 3.15. *The representation (1) is a regular prehomogeneous vector space if and only if the representation (2) is a regular prehomogeneous vector space.*

Definition 3.16. Two prehomogeneous vector spaces which are related as the two prehomogeneous vector spaces (1), (2) as above are called the *castling transform* of each other.

If k is an algebraically closed field of characteristic zero, by monumental work, Sato and Kimura classified irreducible reduced regular prehomogeneous vector spaces in [21] into 29 classes.

We review an important method to construct prehomogeneous vector spaces. Let k be a field, and E a simple algebraic group defined over k. We choose a maximal parabolic subgroup P of E. Then we have the Levi decomposition $P = GU$ where G is the Levi part of P and U is the unipotent part of P. Since G acts on U by conjugation, G acts on $U^{\mathrm{ab}} \stackrel{\mathrm{def}}{=} U/[U, U]$. The latter action can be regarded as a representation of G defined over k. Vinberg proved the couple (G, U^{ab}) has a Zariski open orbit (see [19]). Therefore, (G, U^{ab}) is a prehomogeneous vector space. Prehomogeneous vector spaces of such type are calld *prehomogeneous vector space of parabolic type.*

The following table is the list of spaces which was treated in [25].

type of E	$P = GU$	G	$V = U^{\mathrm{ab}}$
C_2		GL_2	$\mathrm{Sym}^2 \mathrm{Aff}^2$
D_4		GL_2^3	$\mathrm{Aff}^2 \otimes \mathrm{Aff}^2 \otimes \mathrm{Aff}^2$
D_5		$\mathrm{GL}_2 \times \mathrm{GL}_4$	$\mathrm{Aff}^2 \otimes \wedge^2 \mathrm{Aff}^4$
G_2		GL_2	$\mathrm{Sym}^3 \mathrm{Aff}^2$
E_6		$\mathrm{GL}_2 \times \mathrm{GL}_3^2$	$\mathrm{Aff}^3 \otimes \mathrm{Aff}^3 \otimes \mathrm{Aff}^3$
E_7		$\mathrm{GL}_6 \times \mathrm{GL}_2$	$\wedge^2 \mathrm{Aff}^6 \otimes \mathrm{Aff}^2$
F_4		$\mathrm{GL}_3 \times \mathrm{GL}_2$	$\mathrm{Sym}^2 \mathrm{Aff}^3 \otimes \mathrm{Aff}^2$
E_8		$\mathrm{GL}_5 \times \mathrm{GL}_4$	$\wedge^2 \mathrm{Aff}^5 \otimes \mathrm{Aff}^4$

Now, we give examples of prehomogeneous vector spaces.

Example 3.17. Let $G = \mathrm{GL}_1 \times \mathrm{GL}_2$ and $V = \mathrm{Sym}^2 \mathrm{Aff}^2$ or $V = \mathrm{Sym}^3 \mathrm{Aff}^2$. We regard V as a space of homogeneous polynomials in two variables $v = (v_1, v_2)$ of degree two or three. We regard v as a row vector. We express elements of V as

$$x = x(v) = x_0 v_1^2 + x_1 v_1 v_2 + x_2 v_2^2 \quad \text{or} \quad x = x(v) = x_0 v_1^3 + x_1 v_1^2 v_2 + x_2 v_1 v_2^2 + x_3 v_2^3.$$

The group G acts on V by $gx(v) = tx(vg_1)$ for $g = (t, g_1) \in G, x(v) \in V$. This is an irreducible representation unless $\mathrm{char}\, k = 2$ in the case $V = \mathrm{Sym}^2 \mathrm{Aff}^2$. Define $\widetilde{T} = \{(t^{-2}, tI_2) \mid t \in \mathrm{GL}_1)\}$ (resp. $\widetilde{T} = \{(t^{-3}, tI_2) \mid t \in \mathrm{GL}_1\}$). Then $\widetilde{T} = \ker(V \to \mathrm{GL}(V)) \cong \mathrm{GL}_1$ in both cases. Put $w = v_1 v_2$ or $w = v_1 v_2 (v_1 - v_2)$.

We show that (G, V) is a regular prehomogenous vector space (Moreover, $G_w^\circ \cong \mathrm{GL}_1^2$ or $G_w \cong \mathrm{GL}_1$ holds). Note that $\mathrm{char}\, k$ can be 2.

We first determine $T_e(G_w)$. Consider the action of

$$\left(1 + \varepsilon t, I_2 + \varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

where $k[\varepsilon]/(\varepsilon^2)$ is the ring of dual numbers. We write w as $w(v)$. Since v is replaced by $(av_1 + cv_2, bv_1 + dv_2)$, $w(v)$ is replaced by

$$w(v) \to \varepsilon t w(v) + w(v_1 + \varepsilon(av_1 + cv_2), v_2 + \varepsilon(bv_1 + dv_2)) \tag{3.18}$$

We first consider in the case $V = \mathrm{Sym}^2 \mathrm{Aff}^2$. Then (3.18) equals

$$w(v) + \varepsilon(bv_1^2 + (t + a + d)v_1v_2 + cv_2^2).$$

So if this is $w(v)$, then $b = c = 0, t = -(a + d)$. This implies that $T_e(G_w)$ consists of elements of the form

$$\left(-(a + d), \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right).$$

Therefore, $\dim T_e(G_w) = 2 = \dim G - \dim V$. So (G, V) is a regular prehomogenous vector space defined over k.

Let

$$H = \left\{ \left((t_1t_2)^{-1}, \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right) \,\middle|\, t_1, t_2 \in \mathrm{GL}_1 \right\}.$$

Then obviously $H \cong \mathrm{GL}_1^2$ is contained in G_w. Since we have already shown that $\dim G_w = 2$, $G_w^\circ = H$. So G_w is reductive.

We next consider in the case $V = \mathrm{Sym}^3 \mathrm{Aff}^2$. Then (3.18) equals

$$w(v) + \varepsilon(bv_1^3 + (t + 2a - 2b + d)v_1^2v_2 - (t + a - 2c + 2d)v_1v_2^2 - cv_2^3).$$

So if this is $w(v)$, $b = c = 0, a = d, t = -3a$. This implies that $T_e(G_w)$ consists of elements of the form

$$(-3a, aI_2).$$

Therefore, $\dim T_e(G_w) = 1 = \dim G - \dim V$. So (G, V) is a regular prehomogenous vector space defined over k. It is obvious that $\widetilde{T} \subset G_w$. Since $\dim G_w = 1$, $G_w^\circ = \widetilde{T} \cong \mathrm{GL}_1$.

Example 3.19. Suppose that $\mathrm{char}\, k \neq 2, G = \mathrm{GL}_1 \times \mathrm{GL}_3, V = \mathrm{Sym}^2 \mathrm{Aff}^3$. We regard V as a space of ternary quadratic forms. The group G acts on V by $gx(v) = tx(vg_1)$ for $g = (t, g_1) \in G, x(v) \in V$.

Now we consider the action ρ_2 of GL_2 on $\mathrm{Sym}^2 \mathrm{Aff}^2$. We express $x, y \in \mathrm{Sym}^2 \mathrm{Aff}^2$ as

$$x(v) = x_0v_1v_2 + x_1v_1v_2 + x_2v_2^2$$
$$y(v) = y_0v_1v_2 + y_1v_1v_2 + y_2v_2^2$$

where v_1, v_2 are the variables. Note that regarding x as a polynomial $x(v)$ of $v = (v_1\, v_2)$, $g \in \mathrm{GL}_2$ acts on $\mathrm{Sym}^2 \mathrm{Aff}^2$ by $\rho_2(g)x(v) = x(vg)$.

Define

$$\langle x, y\rangle = x_0y_2 - \binom{2}{1}^{-1} x_1y_1 + \binom{2}{2}^{-1} x_2y_0 = x_0y_2 - \frac{1}{2}x_1y_1 + x_2y_0.$$

It is easy to show that $\langle \rho_2(g)x, \rho_2(g)y\rangle = (\det g)^2\langle x, y\rangle$ for all $g \in \mathrm{GL}_2, x, y \in \mathrm{Sym}^2 \mathrm{Aff}^2$. We choose $\{v_1^2, v_1v_2, v_2^2\}$ as a basis for $\mathrm{Sym}^2 \mathrm{Aff}^2$. Then with respect to this basis, the matrix of the bilinear form $\langle\, ,\, \rangle$ is

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1/2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Regard w as an element of $\mathrm{Sym}^2 \mathrm{Aff}^3$.

Consider the representation $\bar\rho$ defined by $\bar\rho(g) = (\det g)^{-1}\rho_2(g)$. Then $\langle \bar\rho(g)x, \bar\rho(g)y\rangle = \langle x, y\rangle$ for all $x, y \in \mathrm{Sym}^2 \mathrm{Aff}^2$. We regard x, y as column vectors and $\bar\rho(g)$ as a 3×3 matrix. Since the matrix of $\langle\, ,\, \rangle$ is w, ${}^tx{}^t\bar\rho(g)w\bar\rho(g)y = {}^txwy$ for all x, y. Therefore, ${}^t\bar\rho(g)w\bar\rho(g) = w$.

Let

$$L_1 = \left\{ \left(0, \begin{pmatrix} a & 2b & 0 \\ c & 0 & b \\ 0 & 2c & -2a \end{pmatrix} \right) \,\middle|\, a, b, c \in k \right\}, \quad L_2 = \{(-2t, tI_3) \mid t \in k\}.$$

Let $A = (a_{ij}) \in \mathrm{GL}_3(k)$. Then $A \in T_e(G_w)$ if and only if $Aw + w{}^tA = 0$. Writing down this condition explicitly,

$$\begin{pmatrix} a_{13} & -\frac{1}{2}a_{12} & a_{11} \\ a_{23} & -\frac{1}{2}a_{22} & a_{21} \\ a_{33} & -\frac{1}{2}a_{32} & a_{31} \end{pmatrix} + \begin{pmatrix} a_{13} & a_{23} & a_{33} \\ -\frac{1}{2}a_{12} & -\frac{1}{2}a_{22} & -\frac{1}{2}a_{32} \\ a_{11} & a_{21} & a_{31} \end{pmatrix} = 0.$$

So

$$2a_{13}, \quad 2a_{31}, \quad -a_{22}, \quad a_{11} + a_{33}, \quad -\frac{1}{2}a_{12} + a_{23}, \quad -\frac{1}{2}a_{32} + a_{21}$$

are all zero. Since $\mathrm{char}\, k \neq 2$, this is equivalent to the condition $A \in L_1 \oplus L_2$. This follows that $T_e(G_w) = L_1 \oplus L_2$. Since $\dim T_e(G_w) = 4 = \dim G - \dim V$, Gw is open and G_w is smooth over k. So, (G, V) is a regular prehomogeneous vector space.

Example 3.20. Let $G_1 = \mathrm{GL}_3, G_2 = \mathrm{GL}_2, G = G_1 \times G_2$ and $V = \mathrm{Sym}^2 \mathrm{Aff}^3 \otimes \mathrm{Aff}^2$. We regard elements of V either as pairs $x = (x_1, x_2)$ of ternary quadratic forms or as forms $x(u, v)$ in two sets of variables which are quadratic in $v = (v_1, v_2, v_3)$ and linear in $u = (u_1, u_2)$ where u, v are regarded as row vectors. The action of $g = (g_1, g_2) \in G$ where $g_2 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is given either as

$$(3.21) \qquad (x_1(v), x_2(v)) \mapsto (ax_1(vg_1) + bx_2(vg_1), cx_1(vg_1) + dx_2(vg_1))$$

or by $x(u,v) \mapsto x(vg_1, ug_2)$. This action is compatible with the action of GL_3 on $\mathrm{Sym}^2 \mathrm{Aff}^3$ which is defined in Example 3.19. This is an irreducible representation unless $\mathrm{char}\, k = 2$, and even so, it is obvious that Condition 3.2 is satisfied.

We express a ternary quadratic form as

$$x(v) = x_{11}v_1^2 + x_{12}v_1v_2 + x_{13}v_1v_3 + x_{22}v_2^2 + x_{23}v_2v_3 + x_{33}v_3^2. \tag{3.22}$$

If $\mathrm{char}\, k \neq 2$, we identify $\mathrm{Sym}^2 \mathrm{Aff}^3$ with the space of symmetric 3×3 matrices by the map

$$x(v) \mapsto S_x = \begin{pmatrix} 2x_{11} & x_{12} & x_{13} \\ x_{12} & 2x_{22} & x_{23} \\ x_{13} & x_{23} & 2x_{33} \end{pmatrix}. \tag{3.23}$$

Then the action of $g \in \mathrm{GL}_3$ is given by $M \mapsto gM{}^tg$.

For $x(v)$, we define

$$P(x) = 4x_{11}x_{22}x_{33} + x_{12}x_{13}x_{23} - x_{11}x_{23}^2 - x_{12}^2x_{33} - x_{13}^2x_{22}. \tag{3.24}$$

Easy computation shows that $P(x) = (1/2)\det S_x$. Therefore,

$$P(gx) = (1/2)\det S_{gx} = (1/2)(\det g)^2 \det S_x = (\det g)^2 P(x).$$

For $x \in V$, we define a binary cubic form $F_x(u)$ by the map

$$x \mapsto F_x(u) = -P(u_1x_1 + u_2x_2).$$

It is easy to see that if $g = (g_1, g_2)$

$$F_{gx}(u) = (\det g_1)^2 F_x(ug_2).$$

Let $\Delta(x)$ be the discriminant of F_x. Then $\Delta(x)$ is a polynomial with degree 12 and it satisfies a relation $\Delta(gx) = \chi(g)\Delta(x)$ where $\chi(g) = (\det g_1)^8(\det g_2)^6$.

We define $w = (w_1, w_2)$ where

$$w_1 = v_2v_3 - v_1v_3, \quad w_2 = v_1v_2 - v_2v_3.$$

By the definition of $F_w(u)$,

$$F_w(u) = u_1u_2(u_1 - u_2), \quad \Delta(w) = 1 \neq 0.$$

Let $\widetilde{T} = \ker(G \to \mathrm{GL}(V))$. Then it is easy to see that

$$\widetilde{T} = \{(tI_3, t^{-2}I_2) \mid t \in \mathrm{GL}_1\}.$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Suppose $g(A, B) = (I_3 + \varepsilon A, I_2 + \varepsilon B)$ fixes w where $k[\varepsilon]/(\varepsilon^2)$ is the ring of dual numbers. Then

$$F_{g(A,B)w} = (\det(I_3 + \varepsilon A))^2 F_w(u(I_2 + \varepsilon B)) = (1 + \varepsilon \cdot \operatorname{tr} A) F_w(u(I_2 + \varepsilon B)).$$

Note that $F_w = u_1 u_2 (u_- u_2)$ and we have already determined the stabilizer of this element in $\mathrm{GL}_1 \times \mathrm{GL}_2$ in Example 3.17. So B must be a scalar matrix of the form bI_2 and $2(a_{11} + a_{22} + a_{33}) = -3b$. Writing down the action explicitly,

$$g(A, bI_2)(w_1, w_2) = ((1 + \varepsilon b) w_1(v(I_3 + \varepsilon A)), (1 + \varepsilon b) w_2(v(I_3 + \varepsilon A))).$$

If this is w,

$$(1 + \varepsilon b) w_i(v(I_3 + \varepsilon A)) = w_i$$

for $i = 1, 2$. We simplify the above equation for $i = 1$ as follows

$$\begin{aligned}
(1 + \varepsilon b) w_1(v(I_3 + \varepsilon A)) &= w_1 + \varepsilon b(v_2 v_3 - v_1 v_3) \\
&\quad + \varepsilon(v_3(a_{12} v_1 + a_{22} v_2 + a_{32} v_3)) \\
&\quad + \varepsilon(v_2(a_{13} v_1 + a_{23} v_2 + a_{33} v_3)) \\
&\quad - \varepsilon(v_3(a_{11} v_1 + a_{21} v_2 + a_{31} v_3)) \\
&\quad - \varepsilon(v_1(a_{13} v_1 + a_{23} v_2 + a_{33} v_3)) \\
&= w_1 + \varepsilon(a_{13} v_1^2 + (a_{13} - a_{23}) v_1 v_2)) \\
&\quad + \varepsilon((-a_{11} + a_{12} - a_{33} - b) v_1 v_3 + a_{23} v_2^2) \\
&\quad + \varepsilon((-a_{21} + a_{22} + a_{33} + b) v_2 v_3 + (-a_{31} + a_{32}) v_3^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(1 + \varepsilon b) w_2(v(I_3 + \varepsilon A)) &= w_2 + \varepsilon b(v_1 v_2 - v_2 v_3) \\
&\quad - \varepsilon(v_3(a_{12} v_1 + a_{22} v_2 + a_{32} v_3)) \\
&\quad - \varepsilon(v_2(a_{13} v_1 + a_{23} v_2 + a_{33} v_3)) \\
&\quad + \varepsilon(v_2(a_{11} v_1 + a_{21} v_2 + a_{31} v_3)) \\
&\quad + \varepsilon(v_1(a_{12} v_1 + a_{22} v_2 + a_{32} v_3)) \\
&= w_2 + \varepsilon(a_{12} v_1^2 + (a_{11} - a_{13} + a_{22} + b) v_1 v_2)) \\
&\quad + \varepsilon((-a_{12} + a_{32}) v_1 v_3 + (a_{21} - a_{23}) v_2^2) \\
&\quad + \varepsilon((a_{22} - a_{31} + a_{33} + b) v_2 v_3 + a_{32} v_3^2).
\end{aligned}$$

So we get the following system of linear equations

$$a_{13} = a_{13} - a_{23} = a_{11} - a_{12} + a_{33} + b = 0,$$
$$a_{23} = -a_{21} + a_{22} + a_{33} + b = a_{31} - a_{32} = 0,$$
$$a_{12} = a_{11} - a_{13} + a_{22} + b = a_{12} - a_{32} = 0,$$
$$a_{21} - a_{23} = a_{22} - a_{31} + a_{33} + b = a_{32} = 0.$$

Solving these equations, $a_{ij} = 0$ if $i \neq j$, $a_{11} = a_{22} = a_{33}$, and $b = -2a_{11}$. Therefore,

$$T_e(G_w) = \{(tI_3, -2tI_2) \mid t \in k\}.$$

Since $\widetilde{T} \subset G_w^\circ$ and $\dim G_w \leqq \dim T_e(G_w) = 1$, G_w must be smooth over k and $G_w^\circ = \widetilde{T} \cong \mathrm{GL}_1$. Since $\dim T_e(G_w) = 1 = \dim G - \dim V$, (G, V) is a regular prehomogeneous vector space.

4 Review on Geometric Invariant Theory

We recall the definition of stability over $\bar{k}$. Let $\pi_V : V \setminus \{0\} \to \mathbb{P}(V)$ be the natural projection map. Let $\bar{k}[V]^{G_{\bar{k}}}$ be the ring of polynomials invariant under the action of $G_{\bar{k}}$. Suppose that $P \in \bar{k}[V]^{G_{\bar{k}}} \setminus \bar{k}$ is a homogeneous polynomial. We define $\mathbb{P}(V)_P = \{\pi(v) \mid P(v) \neq 0\}$.

Definition 4.1. Let $x \in \mathbb{P}(V)_{\bar{k}}$.

(1) x is *semistable* if there exists a homogeneous polynomial $P \in \bar{k}[V]^{G_{\bar{k}}} \setminus \bar{k}$ such that $x \in \mathbb{P}(V)_P$.

(2) x is *properly stable* if exists a homogeneous polynomial $P \in \bar{k}[V]^{G_{\bar{k}}} \setminus \bar{k}$ such that $x \in \mathbb{P}(V)_P$, all the orbits in $\mathbb{P}(V)_P$ are closed, and the stabilizer of x in $G_{\bar{k}}$ is finite.

(3) x is *unstable* if it is not semistable.

We use the notation $\mathbb{P}(V)^{\mathrm{ss}}_{\bar{k}}$ and $\mathbb{P}(V)^{\mathrm{s}}_{(0)\,\bar{k}}$ for the set of semistable points and properly stable points respectively These are $\mathrm{Aut}_k\,\bar{k}$-invariant open subsets in $\mathbb{P}(V)_{\bar{k}}$. Also if $x = \pi_V(v) \in \mathbb{P}(V)_k \cap \mathbb{P}(V)^{\mathrm{ss}}_{\bar{k}}$ then there exists $P \in \bar{k}[V]^{G_{\bar{k}}} \cap (k[V] \setminus k)$ such that $P(v) \neq 0$. Therefore, the notion of semistability is rational over the ground field.

Let λ be a non-trivial 1-PS of G over $\bar{k}$. Suppose that $v \in V \setminus \{0\}$, $\pi_V(v) = x$ and $v = \sum_{i=1}^n v_i$ is the eigen decomposition with respect to λ, i.e., $\lambda(t)v = \sum_{i=1}^n t^{r_i} v_i$, $v_i \neq 0$ for all i, and $r_i \neq r_j$ if $i \neq j$. Then we define a *numerical function* by $\mu(x, \lambda) \overset{\mathrm{def}}{=} \min\limits_{1 \leq i \leq n} r_i$. For later purposes, we would like to define $\mu(x, \lambda)$ for $\lambda \in \mathfrak{t}_{\mathbb{Q}} \setminus \{0\}$. For that, if $m > 0$ is a positive integer and $\nu = m\lambda$ (written additively) is an element of $X_*(T)$, then we define $\mu(x, \lambda) = (1/m)\mu(x, \nu)$. This definition is apparently well-defined.

Theorem 4.2 (Hilbert–Mumford criterion of stability [16])**.** *Let $x \in \mathbb{P}(V)_{\overline{k}}$, then x is semistable if and only if $\mu(x, \lambda) \leqq 0$ for all non-trivial* 1*-PS's λ.*

Note that the above statement is equivalent to the statement that x is unstable if and only if $\mu(x, \lambda) > 0$ for a non-trivial 1-PS λ.

The notion of "properly stable" will not be need in this thesis. However, we point out without proof how the notions of stability and quotients are related.

Definition 4.3. An algebraic group G is *geometrically reductive* (resp. *linearly reductive*) if for any rational representation $\rho : G \to \mathrm{GL}(V)$ and any non zero invariant vector v there exists a G-invariant homogeneous polynomial f on V with $\deg f \geqq 1$ (resp. $\deg f = 1$) such that $f(v) \neq 0$.

Nagata and Miyata (1963) proved that all geometrically reductive groups are reductive. Furthermore, Weyl's unitary trick shows that all reductive groups are linearly reductive if $\mathrm{char}\, k = 0$. But, this is not true for reductive groups defined over a field of characteristic $p > 0$. However, it turns out that "all reductive groups are geometrically reductive". This was known as the *Mumford conjecture.* The Mumford conjecture was proved by Habush in 1975 ([5]).

Theorem 4.4 (Nagata)**.** *Let G be a geometrically reductive group which acts rationally on an affine variety* $\mathrm{Spec}\, A$*. Then A^G is a finitely generated k-algebra.*

For the proof of Theorem 4.4, see [18, p.43–50].

Let k be an algebraic closed. Let G an algebraic group and X a variety both defined over k. Assume G acts on X rationally. The action of G on X is determined by a k-morphism $\mu : G \times_k X \to X$.

Let Y be a k-scheme. A k-morphism $f : X \to Y$ is *G-invariant* if the diagram

$$\begin{array}{ccc} G \times_k X & \xrightarrow{\mu} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes.

Definition 4.5. A scheme Y of finite type over k and a k-morphism $f : X \to Y$ is called a *categorical quotient* if f is G-invariant and any G-invariant morphism $X \to Z$ factors through Y uniquely.

By universal property, categorical quotients are uniquely determined up to isomorphisms.

Definition 4.6. A scheme Y of finite type over k and a k-morphism $f : X \to Y$ is called a *geometric quotient* if the following condition (1)–(4) are hold.

(1) f is G-invariant.

(2) For any $y \in Y$, the geometric fibre $f^{-1}(y) \times_{k(y)} \overline{k(y)}$ consists of a single orbit set-theoretically (this implies that f is surjective).

(3) A subset $U \subset Y$ is open if and only if $f^{-1}(U)$ is open (i.e. f is submersive).

(4) The structure sheaf $\mathscr{O}_Y$ is the subsheaf of $\mathscr{O}_X$ consisting of invariant functions.

The following proposition is Proposition 0.1 in [16, p.4].

Proposition 4.7. *Suppose $f : X \to Y$ is a geometric quotient. Then it is a categorical quotient.*

The following theorem is Theorem 1.10 in [16, p.38].

Theorem 4.8. *Let X be an algebraic variety over k, and G a reductive algebraic group acting on X. Then categorical quotient (Y, f) of X^{ss} by G exists. Moreover, the following conditions hold.*

(1) f is affine and submersive.

(2) There is an open subset $Y' \subset Y$ such that $f^{-1}(Y') = X^{\mathrm{s}}_{(0)}$ and such that $(Y', f|_{X_{(0)^{\mathrm{s}}}})$ is a geometric quotient.

5 Kempf's result

Let S be a maximal k-split torus of G and T a maximal torus of G with $S \subset T$. In this section, we assume k is a perfect field.

First we assume that $k = \bar{k}$ and so $S = T$. Let K be a field and X a variety over K and $f : \mathbb{G}_m \to X$ a K-morphism . We embed $\mathbb{G}_m$ to the one dimensional affine space Aff^1 in the usual manner. We say that $\lim_{t \to 0} f(t) = y$ if there exists a K-morphism $g : \mathrm{Aff}^1 \to X$ such that $g|_{\mathbb{G}_m} = f$ and $g(0) = y$.

Let $v \in V \setminus \{0\}$ and $x = \pi_V(v) \in \mathbb{P}(V)$. We define

$$|V, v| = \left\{ \lambda \in X_*(G) \;\middle|\; \lim_{t \to 0} \lambda(t) v \text{ exists in } V \right\},$$
$$|V, v|_{\{0\}} = \left\{ \lambda \in X_*(G) \;\middle|\; \lim_{t \to 0} \lambda(t) v = 0 \right\}.$$

If λ is any 1-PS of G then it is conjugate to an element γ of $X_*(T)$. If γ_1 is another such element of $X_*(T)$ then γ, γ_1 are conjugate by an element of $\mathbb{W}$. Since the norm is invariant by the action of $\mathbb{W}$, $\|\gamma\|_{\mathfrak{t}_\mathbb{R}}$ depends only on λ. So we define $\|\lambda\| = \|\gamma\|_{\mathfrak{t}_\mathbb{R}}$.

The following theorem is Theorem 3.4 in Kempf [12].

Theorem 5.1 (Kempf [12]). *(1) The function $\mu(x,\lambda)/|\|\lambda\|$ has a maximal value (which we denote by $M(x)$) on the set $|V,v|$ if it is not empty.*

(2) The condition $\overline{Gv} \ni 0$ is equivalent to the condition that $|V,v|_{\{0\}}$ is not empty.

(3) Suppose that $|V,v|_{\{0\}}$ is not empty, and let Λ_x be the set of indivisible 1-PS λ's such that $\mu(x,\lambda) = M(x)\|\lambda\|$.

(a) The set Λ_x is non empty, and there exists a parabolic subgroup P_x of G such that $P_x = P(\lambda)$ for all $\lambda \in \Lambda_x$.

(b) The set Λ_x is a principal homogeneous space under the action of the unipotent radical of P_x.

(c) Any maximal torus of P_x contains a unique element of Λ_x.

Remark 5.2. The condition $\overline{Gv} \ni 0$ holds if and only if x is unstable.

Next we assume that k is an arbitrary perfect field. Now we recall the rationality property of Kempf's result. This property of Kempf's result will be used mainly in section 8. The next theorem is Theorem 4.2 (and Corollary 4.4) in Kempf [12].

Theorem 5.3 (Kempf [12]). *Suppose that $v \in V_k$, and that $|V,v|_{\{0\}\,\bar{k}}$ is not empty. Put Let Λ_x be the set of indivisible 1-PS λ's of $X_*(G \times_k \bar{k})$ which are defined over k such that $\mu(x,\lambda) = M(x)\|\lambda\|$, where $x = \pi_V(v)$.*

Then all elements in Λ_x are defined over k, P_x is rationally conjugate to a standard parabolic subgroup, and Λ_x is a principal homogeneous space under the action of the k-points of the unipotent radical of P_x.

This theorem implies that if $v \in V_k$ and $|V,v|_{\{0\}\,\bar{k}}$ is not empty, v is conjugate to $v' \in V_k$ by an element of G_k and Λ_y contains a split 1-PS of T, where $y = \pi_V(v')$.

Finally, we introduce the terminology of "adapted 1-PS" which we will be use in section 6, 8.

Definition 5.4. We say that a 1-PS λ is *G-adapted (or adapted, simply)* for x if $M(x)\|\lambda\| = \mu(x,\lambda)$.

Now we prove Kempf's theorem. We first introduce the concepts of a state and its associated numerical function. A *state* Ξ is the assignment of a non-empty subset $\Xi(T) \subset X^*(T)$ of characters to every torus T of G such that, if $T_1 \subset T_2$ are two tori of G, the image of $\Xi(T_2)$ under the restriction $X^*(T_2) \to X^*(T_1)$ is $\Xi(T_1)$.

Let $g \in G_{\bar{k}}$ and T be a torus of G. We have an isomorphism $g_! : X^*(g^{-1}Tg) \xrightarrow{\sim} X^*(T)$, where $g_!\chi(r) = \chi(g^{-1}rg)$ for each character of $g^{-1}Tg$. If Ξ is any state, we may define a conjugate state $g * \Xi$ by the equation $g * \Xi(T) = g_!\Xi(g^{-1}Tg)$. It is trivial to verify the restriction property for $g * \Xi$. Hence, $g * \Xi$ is a state.

A state Ξ is called *bounded* if, for any torus T of G, the union $\bigcup_{g\in G_{\bar{k}}} g * \Xi(T)$ is a finite subset of $X^*(T)$. The *numerical function* $\mu(\Xi, *)$ of a state Ξ is the function on $X_*(G)$ with values in $\mathbb{Z} \cup \{-\infty\}$ given by

$$\mu(\Xi, \lambda) = \min_{\chi \in \Xi(\operatorname{im} \lambda)} \langle \chi, \lambda \rangle$$

for any $\lambda \in X_*(G)$.

A state Ξ is called *admissible* if its numerical function has the property,

$$\mu(\Xi, p * \lambda) = \mu(\Xi, \lambda) \text{ for all } \lambda \in X_*(G) \text{ and all } p \in P(\lambda)_{\bar{k}},$$

where $p * \lambda = p^{-1}\lambda p$.

Theorem 5.5. *Let Ξ and Υ are two bounded states.*

(a) The function $\mu(\Upsilon, \lambda)/\|\lambda\|$ has a maximum value $M(\Xi, \Upsilon)$ for λ in

$$\{\lambda \in X_*(G) \mid \lambda \text{ is non trivial, } \mu(\Xi, \Upsilon) \geqq 0\},$$

if this set is not empty.

(b) If $M(\Xi, \Upsilon)$ is defined and positive, and if Ξ and Υ are admissible, then the set

$$\Lambda(\Xi, \Upsilon) = \left\{ \lambda \in X_*(G) \;\middle|\; \begin{array}{c} \lambda \text{ is non trivial and indivisible,} \\ \mu(\Xi, \lambda) \geqq 0, \mu(\Upsilon, \lambda) = M(\Xi, \Upsilon)\|\lambda\| \end{array} \right\}$$

has the following properties.

(1) $\Lambda(\Xi, \Upsilon)$ is not empty.

(2) There is a parabolic subgroup $P(\Xi, \Upsilon)$ of G such that $P(\Xi, \Upsilon) = P(\lambda)$ for any $\lambda \in \Lambda(\Xi, \Upsilon)$.

(3) $\Lambda(\Xi, \Upsilon)$ is a principal homogeneous space under conjugation by the $\bar{k}$-points of the unipotent radical of $P(\Xi, \Upsilon)$.

(4) Any maximal torus of $P(\Xi, \Upsilon)$ contains a unique element of $\Lambda(\Xi, \Upsilon)$.

To prove Theorem 5.5, we need following lemma.

Lemma 5.6. *Let V be a finite dimensional real vector space with an inner product $(\ ,\ )$. Let F and G be two non-empty finite set consisting of real-valued linear functions on V. Set $f(v) = \min_{\alpha \in F} \alpha(v)$ and $g(v) = \min_{\alpha \in G} \alpha(v)$. Assume that $\mathscr{S} = \{v \neq 0 \,|\, f(v) \geqq 0\}$ is a non-empty subset of V. Then, we have the following (a), (b).*

(a) The function $g(s)/\|s\|$ on $\mathscr{S}$ has a maximum value M.

(b) If $M > 0$, then $\mathscr{R} = \{s \in \mathscr{S} \,|\, g(s) = M\|s\|\}$ is an open ray.

Furthermore, if the above inner product and functions in F and G are integral valued on a lattice L of V, then the following (c), (d) holds.

(c) $L \cap \mathscr{R}$ is not empty.

(d) If $M > 0$, $L \cap \mathscr{R}$ consists of all positive integral multiples of its unique element.

Proof of Lemma 5.6. The set $\mathscr{S}$ is a semi-cone, i.e. a union of open rays. Since the elements of G are linear, the function $g(s)/\|s\|$ is constant on open rays in $\mathscr{S}$. Any continuous function on a non-empty compact space is continuous. In particular, any continuous function on the intersection of $\mathscr{S}$ and the unit sphere of V, must obtain a maximum value. So (a) follows from these facts.

To prove (b), we consider the set $\mathscr{T} = \{s \in \mathscr{S} \,|\, g(s) \geqq 1\}$. Then, $\mathscr{T}$ is a closed convex subset of V which does not contain the origin. When $M > 0$, $\mathscr{T}$ is not empty. Thus, there is a unique point t of $\mathscr{T}$ which is closest to the origin. Since there is $t_0 \in \mathscr{T}$ such that $g(t_0) = 1$ and t is the closest point of $\mathscr{T}$ to the origin, $g(t)$ must be 1. Therefore, $g(s)/\|s\|$ reaches its maximum value, $1/\|t\|$, only on the ray through t. Thus, $\mathscr{R} = \mathbb{R}_{>0}t$ and (b) holds.

The proof of (c) must be done by an honest calculation. Let s be a point of $\mathscr{R}$. If we replace (L, V) by $(L \cap V', V')$, where

$$V' = \left\{ v \in V \;\middle|\; \begin{array}{c} \alpha(v) = 0 \text{ for all } \alpha \text{ in } F \text{ such that } \alpha(s) = 0 \text{ and} \\ \beta(v) = \gamma(v) \text{ for all } \beta \text{ and } \gamma \text{ in } G \text{ such that } \beta(s) = M\|s\| = \gamma(s) \end{array} \right\},$$

then we may assume that $\mathscr{S}$ contains a neighborhood of s in V on which $g(v)$ is the restriction of a linear function $h(v)$ on V, where h has integral values on L.

There are three cases. When $M < 0$, then V is a one-dimensional subspace spanned by s. When $M = 0$, $g(v)/\|v\|$ has constant value 0 in a neighborhood of s. When $M > 0$, s spans the line N orthogonal to the hyperplane $h(v) = 0$ on V. Here, $N \cap L$ is an infinite cyclic group as $(\ ,\ )$ and h are both integral on L. The proof of (c) and (d) follows directly from the above facts. □

Proof of Theorem 5.5. We first prove the case where G is a torus T. The only relevant statements are (a), (b) (1) and (b) (2). Lemma 5.6 is exactly what we need. To apply Lemma 5.6, we put $V = X_*(G) \otimes \mathbb{R}, L = X_*(G)$, where V is given the extended inner product from the length on $X_*(G)$. Set $F = \Xi(T)$ and $G = \Upsilon(T)$ equal to the subsets linear functions on V, obtained by extending $\Xi(T)$ and $\Upsilon(T)$ in $X^*(T) \cong \mathrm{Hom}(X_*(T), \mathbb{Z})$. Both F and G are finite because $\Xi(T)$ and $\Upsilon(T)$ are, by the boundedness assumption. Further, $\mu(\Xi, \lambda) = f(\lambda)$ and $\mu(\Upsilon, \lambda) = g(\lambda)$ for λ in $X_*(T)$ in the notation of Lemma 5.6. Therefore, the proof for this case follows from Lemma 5.6.

In the general case, we need to find a maximum value for the function $\mu(\Upsilon, s)/\|s\|$ on all of $\mathscr{S} = \{\lambda \in X_*(G) \mid \lambda \text{ is non trivial}, \mu(\Xi, \Upsilon) \geqq 0\}$. We know that it has a maximum value on any non-empty subset of the form $\mathscr{S} \cap X_*(g^{-1}Tg)$, where T is a maximal torus of G and g is a $\bar{k}$-point of G. As these subsets cover $\mathscr{S}$, it will be enough to see that there are only finite numbers of maximum values on such subset. The boundedness of Ξ and Υ implies that there are only finitely many possibilities for $g * \Xi(T)$ and $g * \Upsilon(T)$. This implies (a).

For (b), let T be any torus of G containing a non-trivial subgroup λ such that $\mu(\Xi, \lambda) \geqq 0$ and $\mu(\Upsilon, \lambda) = M(\Xi, \Upsilon)\|\lambda\|$. By the case $G = T$, we know that there is a unique λ_T in T such that $\lambda_T \in \Lambda(\Xi, \Upsilon)$. Hence, (1) holds because $M(\Xi, \Upsilon)$ exists. Let λ be any element of $\Lambda(\Xi, \Upsilon)$. For any $p \in P(\lambda)_{\bar{k}}$, $p * \lambda$ is also contained in $\Lambda(\Xi, \Upsilon)$ as Ξ and Υ are admissible. As any maximal torus of $P(\lambda)$ contains some $p * \lambda$, it possesses a unique such member of $\Lambda(\Xi, \Upsilon)$ by the above argument.

Let λ_1 and λ_2 be two members of $\Lambda(\Xi, \Upsilon)$. The intersection $P(\lambda_1) \cap P(\lambda_2)$ of these parabolic subgroups contains a maximal torus T of G. This is the crucial point of the proof. Then $p_1 * \lambda_1 = \lambda_T = p_2 * \lambda_2$ for some $p_i \in P(\lambda_i)$ $(i = 1, 2)$. Therefore,

$$P(\lambda_1) = P(p_1 * \lambda_1) = P(\lambda_T) = P(p_2 * \lambda_2) = P(\lambda_2),$$

and λ_1, λ_2 are conjugate in this parabolic subgroup. Thus we have proved that $P(\lambda)$ has a fixed value on $\Lambda(\Xi, \Upsilon)$, i.e., (2) holds. We have already noted that (4) must hold. (3) follows from the description of $P(\lambda) * \lambda$. □

Next we recall the eigen decomposition of V as a representation of T, where T is a torus of G. Explicitly, $V = \bigoplus V_\chi$, where V_χ is the non-zero subspace of V on which T acts by multiplication via the character χ of T. The finite set of characters χ which occur in this decomposition is called the set of *T-weights* of V.

The state $\Xi_{v,V}$ assigns to each torus T of G the set of T-weights χ of V such that v has non zero projection on the weight space V_χ. As v is not zero, $\Xi_{v,V}(T)$ is a non empty subset of $X^*(T)$. The restriction property is trivial. Therefore, $\Xi_{v,V}$ is a state.

Lemma 5.7. *Let $\mu(\Xi_{v,V},\lambda)$ be the numerical function of the state $\Xi_{v,V}$ of the vector $v\in V\setminus\{0\}$. Then, the following conditions are hold.*

(a) $|V,v|=\{\lambda\in X_*(G)\,|\,\mu(v,\lambda)\geqq 0\}$.

(b) $|V,v|_{\{0\}}=\{\lambda\in X_*(G)\,|\,\mu(v,\lambda)>0\}$.

(c) For any $g\in G_{\bar{k}}$, $g*(\Xi_{v,V})=\Xi_{g\cdot v,V}$.

(d) $\Xi_{v,V}$ *is an admissible bounded state.*

(e) $\mu(v,\lambda)=\mu(\Xi_{v,V},\lambda)$ *for all* $\lambda\in|V,v|$.

Proof. Let $v=\sum_i v_i$ be an eigen decomposition respect to λ, i.e., $\lambda(t)v=\sum_{i\in I}t^iv_i, v_i\neq 0$, where $I\subset\mathbb{Z}$ is a finite and non-empty subset. If λ is a 1-PS of T, then $I=\{\langle\lambda,\chi\rangle\,|\,\chi\in\Xi_{v,V}(T)\}$. Since $\mu(v,\lambda)=\min I$, (a), (b) hold.

Let $v=\sum_\chi v_\chi$ be an eigen decomposition respect to T, where χ runs through all elements of $\Xi_{v,V}(T)$. Then $gv=\sum gv_\chi$. Put $T'=gTg^{-1}$. Since

$$\begin{aligned} t(gv_\chi)&=g(g^{-1}tg)v_\chi=g\cdot\chi(g^{-1}tg)v_\chi\\ &=\chi(g^{-1}tg)g\cdot v_\chi, \end{aligned}$$

we have

$$\Xi_{g\cdot v,V}(T')=g_!(\Xi_{v,V}(g^{-1}T'g))=g*\Xi_{v,V}(T').$$

This shows (c).

By definition, the set $\Xi_{v,V}$ is a subset of the set of T-weights. Let $\Phi(T)$ be the set of T-weights. By (c), we have

$$g*\Xi_{v,V}(T)=\Xi_{g\cdot v,V}(T)\subset\Phi(T).$$

Since $\Phi(T)$ is a finite set, $g*\Xi_{v,V}(T)$ is also finite. Therefore, $\Xi_{v,V}$ is a bounded state. We also have to check the admissibility. Let λ be a 1-PS, and p a k-point of $P(\lambda)$. As $P(\lambda)=P(p*\lambda)$, it will suffice to show that $\mu(v,\lambda)\leqq\lambda(v,p*\lambda)$. By (a), $\mu(v,\lambda)$ is the largest integer m such that $\lim_{t\to 0}t^{-m}\lambda(t)v$ exists in V. Therefore, we need to show the following claim.

Claim 1. If $\lim_{t\to 0}\lambda(t)\cdot p^{-1}\cdot\lambda(t)^{-1}=p_0^{-1}$ exists in G and $\lim_{t\to 0}t^{-m}\lambda(t)\cdot v=v_m$ exists in V, then $\lim_{t\to 0}t^{-m}p*\lambda(t)v$ exists and equals $pp_0^{-1}\cdot v_m$.

In fact,

$$t^{-m}p*\lambda(t)v=t^{-m}p\lambda(t)p^{-1}v=p(\lambda(t)p^{-1}\lambda(t)^{-1})(t^{-m}\lambda(t)v).$$

Passing to the limit, we have $\lim_{t\to 0} t^{-m} p * \lambda(t)v = p \cdot p_0^{-1} \cdot v_m$. This proves the fact, and therefore (d) is true.

Finally, we have to prove (e). But this follows from (a) and (d). □

Proof of Theorem 5.1. By Lemma 5.7, (1) and (3) follow from Theorem 5.5. (2) follows from Lemma 5.7 (b) and (e). □

Since k is perfect, $k^{\mathrm{sep}} = \bar{k}$ holds. We denote the Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k) = \mathrm{Aut}_k \bar{k}$ by Γ.

Let V be a vector space with k-structure V_k. Then Γ operates on $V_{\bar{k}} = V_k \otimes_k \bar{k}$ through the second factor, and it is clear that V_k is the set $V_{\bar{k}}^{\Gamma}$ of fixed point under the action of Γ. If W is another vector space with k-structure, then Γ operates on $\mathrm{Hom}(V,W)_{\bar{k}} = \mathrm{Hom}_{\bar{k}}(V_{\bar{k}}, W_{\bar{k}})$ by $({}^{\sigma}f)(v) = \sigma(f(\sigma^{-1}v))$. Here $\sigma \in \Gamma, f : V \to W$ is defined over $\bar{k}$ and $v \in V_{\bar{k}}$. Then the following conditions on such f are equivalent.

(i) f is defined over k.

(ii) $f : V_{\bar{k}} \to W_{\bar{k}}$ is Γ-equivariant.

(iii) $f \in \mathrm{Hom}(V,W)_{\bar{k}}^{\Gamma}$.

Note that $\| \ \|$ is Γ-invariant i.e. $\|{}^{\sigma}\lambda\| = \|\lambda\|$ for all $\lambda \in X_*(G \times_k \bar{k})$ and $\sigma \in \Gamma$.

Lemma 5.8. *Suppose that $v \in V_k$.*

(1) $|V_{\bar{k}}, v|$ and $|V_{\bar{k}}, v|_{\{0\}}$ are Γ-invariant subsets of $X_*(G \times_k \bar{k})$.

(2) For λ in $|V_{\bar{k}}, v|$, $\mu(x,\lambda) = \mu(x, {}^{\sigma}\lambda)$ for all $\sigma \in \Gamma$, where $x = \pi_V(v)$.

Proof of Lemma 5.8. Let λ be in $X_*(G \times_k \bar{k})$. As the action of G on V and v are rationally defined over k, we have ${}^{\sigma}\lambda(t) \cdot v = \sigma(\lambda(\sigma^{-1}(t))) \cdot v = \sigma(\lambda(\sigma^{-1}(t)) \cdot v)$ for any $\sigma \in \Gamma$. Therefore, for any $\sigma \in \Gamma$, $\lim_{t\to 0} {}^{\sigma}\lambda(t) \cdot v$ exists if and only if $\lim_{t\to 0} \lambda(t) \cdot v$ exists. This shows (1). Furthermore, as in the proof of Lemma 5.7, we have $\mu(x,\lambda) = \mu(x, {}^{\sigma}\lambda)$ for all $\sigma \in \Gamma$. Therefore, (2) is true. □

Proof of Theorem 5.3. The function $\mu(x,\lambda)/\|\lambda\|$ has a positive maximum value $M(x)$ on $|V_{\bar{k}}, v|$ by Theorem 5.1 (2). To show that P_x is defined over k, we must see that it is invariant under Γ. By Theorem 5.1 (3), it will suffice to note that Λ_x is invariant under Γ. The invariance of Λ_x follows from Lemma 5.8 because the norm $\| \ \|$ is Galois invariant. Therefore, any $\lambda \in \Lambda_x$ is defined over k.

For $g \in G_k$, we have $|V, gv|_{\{0\}} = g * |V, v|_{\{0\}}$ and $\mu(gx, g * \lambda) = \mu(x, \lambda)$ for $\lambda \in |V, v|$. As the length satisfy $\|\lambda\| = \|g * \lambda\|$, $g * \Lambda_x = \Lambda_x$. Hence, $gP_xg^{-1} = P_{gx}$ i.e. $P(g * \lambda) = gP(\lambda)g^{-1}$. This follows P_x is rationally conjugate to a standard parabolic subgroup.

The rest follows formally from the analogue results in Theorem 5.1. □

6 Proof of the main theorem in the split case

We first assume that $k = \bar{k}$ (so $S = T$) and so we shall not use the subscript $_k$ until we consider the rationality questions.

The next lemma is proved in [16, p.57, Proposition 2.7].

Lemma 6.1. *Let ξ be a 1-PS of G. Then $\mu(qx, \xi) = \mu(x, \xi)$ holds for all $q \in P(\xi)$.*

Proof. Let $x = \pi_V(v)$. We choose a coordinate system $v = (v'_0, v'_1, \ldots, v'_N)$ so that $\xi(t)v = (t^{a_i} v'_i)$ and $a_0 \leqq \cdots \leqq a_N$. Let e'_i be the i-th coordinate vector. We put

$$Z = \bigoplus_{i:a_i = \mu(x,\xi)} k e'_i, \; W = \bigoplus_{i:a_i > \mu(x,\xi)} k e'_i. \tag{6.2}$$

Then $Z \oplus W$ is a subspace of V, and by the definition of $\mu(x, \xi)$, $v \in Z \oplus W$. So we express v in the form $v = (z, w)$ where $z \in Z, w \in W$ according to the decomposition (6.2).

We remind the reader that $P(\xi) = M(\xi)U(\xi)$ is the Levi decomposition of $P(\xi)$. We write $q = mu$ for $m \in M(\xi)$ and $u \in U(\xi)$. Then qx is in the form (mz, w') where $w' \in W$. Since $z \neq 0$, we have $mz \neq 0$. Since the weights of the non-zero coordinates of w' are strictly greater than $\mu(x, \xi)$ and $\xi(t)mz = t^{\mu(x,\xi)} mz$, this proves that $\mu(qx, \xi) = \mu(x, \xi)$. □

The next lemma is proved in [17, Lemma 9.2]. However, since there are minor inaccuracies in the proof, we give a full proof here.

Lemma 6.3. *Let $x = \pi_V((z, w)) \in \mathbb{P}(Y_\beta)$ where $z \in Z_\beta \setminus \{0\}$ and $w \in W_\beta$. We put $M = \mu(x, \lambda_\beta)/\|\lambda_\beta\| > 0$. Then $\pi_V(z)$ is G_β-unstable if and only if $\mu(x, \xi)/\|\xi\| > M$ for some $\xi \in X_*(P_\beta)$.*

Proof. We first show that if $\pi_V(z)$ is G_β-semistable then $\mu(x, \xi)/\|\xi\| \leq M$ for all $\xi \in X_*(P_\beta)$.

Since ξ is conjugate to a 1-PS of M_β, there exists $p \in P_\beta$ such that $\xi_1 \stackrel{\text{def}}{=} p\xi p^{-1} \in X_*(M_\beta)$. If we write $p = mu_1$ where $m \in M_\beta$ and $u_1 \in U_\beta$ then

$$\xi = p^{-1}\xi_1 p = u_1^{-1} m^{-1} \xi_1 m u_1 = m^{-1}\xi_1 m u$$

where $u = (m^{-1}\xi_1 m)^{-1} u_1^{-1} (m^{-1}\xi_1 m) u_1 \in U_\beta$. If we put $\eta = m^{-1}\xi_1 m$, then $\eta \in X_*(M_\beta)$ is conjugate to ξ by an element of P_β. Therefore, $\|\xi\| = \|\eta\|$.

Since $u(t)((z, w)) = (z, w')$, where $w' = w'(t) \in W_\beta$, $\xi((z, w)) = (\eta z, \eta w')$. There exist $m, n \in \mathbb{Z}$ and $\nu \in X_*(G_\beta)$ such that $m > 0$ and $\eta^m = \nu \lambda_\beta^n$. Now $\lambda_\beta(t)$ acts on Z_β by scalar multiplication, say $\lambda_\beta(t)z = t^a z$ where $a > 0$. We choose a coordinate system $z = (z_i)$ of Z_β so that $\nu(t)z = (t^{b_i} z_i)$. Since $\pi_V(z)$ is G_β-semistable, there is i such that $z_i \neq 0$ and $b_i \leqq 0$. Then the i-th component of $\eta^m(t)z$ is $t^{na+b_i} z_i$. Since λ_β and ν are orthogonal,

$$\|\eta^m\| = \sqrt{\|\nu\|^2 + |n|^2\|\lambda_\beta\|^2} > |n|\|\lambda_\beta\|.$$

So,

$$\frac{\mu(x,\xi)}{\|\xi\|} = \frac{\mu(x,\eta^m)}{\|\eta^m\|} \leqq \frac{na + b_i}{\|\eta^m\|} \leqq \frac{na}{|n|\|\lambda_\beta\|} \leqq \frac{a}{\|\lambda_\beta\|} = M.$$

Conversely we assume that $\mu(x,\nu) > 0$. As above we assume $\nu(t)z = (t^{b_i} z_i)$. We choose a coordinate system $w = (w_j)$ so that $\lambda_\beta(t)w = (t^{c_j} w_j)$ and $\nu(t)w = (t^{d_j} w_j)$. Then $c_j > a$ and

$$(\nu\lambda_\beta^n)(t)w = (t^{nc_j + d_j} w_j),\ (\nu\lambda_\beta^n)(t)z = (t^{na+b_i} z_i).$$

Since there are finitely many possibilities for i, j, if n is sufficiently large, then $nc_j + d_j > na + b_i$ for all i, j. Therefore, $\mu(x, \nu\lambda_\beta^n) = \mu(x,\nu) + na = \mu(x,\nu) + n\mu(x,\lambda_\beta)$. So,

$$\frac{\mu(x,\nu\lambda_\beta^n)}{\|\nu\lambda_\beta^n\|} = \frac{\mu(x,\nu) + n\mu(x,\lambda_\beta)}{\sqrt{\|\nu\|^2 + |n|^2\|\lambda_\beta\|^2}}.$$

Put $b = \mu(x,\nu) > 0$. Then, we would like to prove

$$\frac{b + na}{\sqrt{\|\nu\|^2 + |n|^2\|\lambda_\beta\|^2}} > \frac{a}{\|\lambda_\beta\|} = M \tag{6.4}$$

for sufficiently large $n > 0$. Clearly the inequality

$$\begin{aligned} &\|\lambda_\beta\|^2(b^2 + 2abn + n^2a^2) - a^2(\|\nu\| + n^2\|\lambda_\beta\|^2) \\ &= \|\lambda_\beta\|^2(b^2 + 2abn) - a^2\|\nu\|^2 > 0 \end{aligned}$$

is valid for sufficiently large $n > 0$, and so

$$\frac{b^2 + 2nab + n^2a^2}{\|\nu\|^2 + n^2\|\lambda_\beta\|^2} > \frac{a^2}{\|\lambda_\beta\|^2}. \tag{6.5}$$

Therefore, (6.4) follows. □

Proposition 6.6. *Assume that $x \in \mathbb{P}(Y_\beta)$. Then $x \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ if and only if λ_β is adapted for x and $\mu(x,\lambda_\beta)/\|\lambda_\beta\| = \|\beta\|_{\mathfrak{t}^*_\mathbb{R}}$.*

Proof. We show the "if" part first. Suppose that $x = \pi_V(v)$ and $v = (v_0, \ldots, v_N) = (z, w)$ where $z \in Z_\beta$ and $w \in W_\beta$. Since $\lambda(\beta) = a\lambda_\beta$ for a positive rational number a,

$$\frac{\mu(x,\lambda_\beta)}{\|\lambda_\beta\|} = \frac{\min_{v_i\neq 0}\langle\gamma_i, \lambda_\beta\rangle_T}{\|\lambda_\beta\|} = \frac{\min_{v_i\neq 0}\langle\gamma_i, \lambda(\beta)\rangle_T}{\|\lambda(\beta)\|} = \frac{\min_{v_i\neq 0}(\gamma_i, \beta)_{\mathfrak{t}^*_\mathbb{R}}}{\|\beta\|_{\mathfrak{t}^*_\mathbb{R}}}. \tag{6.7}$$

By assumption, $\min_{v_i \neq 0}(\gamma_i, \beta)_{\mathfrak{t}^*_\mathbb{R}} = \|\beta\|^2_{\mathfrak{t}^*_\mathbb{R}}$. So $z \neq 0$. Since λ_β is adapted for x, $x \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ by Lemma 6.3.

We next show the "only if" part. Suppose that $x = \pi((z, w)) \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ where $z \in Z_\beta$, $w \in W_\beta$. Since $z \neq 0$, by (6.7), $M \overset{\mathrm{def}}{=} \mu(x, \lambda_\beta)/\|\lambda_\beta\| = \|\beta\|_{\mathfrak{t}^*_\mathbb{R}}$. By Lemma 6.3, $\mu(x, \xi)/\|\xi\| \leqq M$ holds for all $\xi \in X_*(P_\beta)$.

We want to prove that λ_β is adapted for x. So we must show that

$$M = \frac{\mu(x, \lambda_\beta)}{\|\lambda_\beta\|} = \sup_{\xi \in X_*(G)} \frac{\mu(x, \xi)}{\|\xi\|}.$$

Since x is unstable, there exists an adapted 1-PS ξ for x by Theorem 5.1 (1). Since the intersection of any two parabolic subgroups contains a maximal torus, there is a maximal torus $T' \subset P_\beta \cap P(\xi)$. Since any torus in either P_β or $P(\xi)$ is conjugate to a subtorus of T', there exist $p \in P_\beta$ and $q \in P(\xi)$ such that $p\lambda_\beta p^{-1} = \lambda'_\beta$ and $q\xi q^{-1} = \xi'$ are both in $X_*(T')$.

Since $q^{-1} \in P(\xi) = P(\xi')$, by Lemma 6.1,

$$\mu(x, \xi) = \mu(qx, q\xi q^{-1}) = \mu(q^{-1}qx, q\xi q^{-1}) = \mu(x, \xi').$$

Therefore, ξ' is also adapted for x, and so $M(x) = \mu(x, \xi')/\|\xi'\|$. Also ξ' is a 1-PS in $P(\xi')$, and so $M(x) = \mu(x, \xi')/\|\xi'\| \leqq M$ by Lemma 6.3. Clearly $M(x) \geqq M$ holds, and so $M(x) = M$. Therefore, λ_β is adapted for x. □

Proof of Theorem 1.2 (split case). Let $v \in V \setminus \{0\}$ and $x = \pi_V(v)$ be unstable. We would like to show that there is $\beta \in \mathfrak{B}$ such that $x \in G\mathbb{P}(Y_\beta)^{\mathrm{ss}}$. There is a 1-PS λ for which x is adapted. We choose $g \in G$ so that $\nu = g\lambda(t)g^{-1}$ is a 1-PS in T and that $\nu \in \mathfrak{t}_{\mathbb{R},+}$. Then gx is ν-adapted. So we may assume that λ is a 1-PS in T such that $\lambda \in \mathfrak{t}_{\mathbb{R},+}$ to begin with.

Let $a = M(x)/\|\lambda\|$. Then

$$\frac{\mu(x, a\lambda)}{a^2\|\lambda\|^2} = \frac{\mu(x, \lambda)}{M(x)\|\lambda\|} = 1.$$

So by replacing λ by $a\lambda$, we may assume that $\mu(x, \lambda) = \|\lambda\|^2$. This λ may no longer be a 1-PS, but is an element of $\mathfrak{t}_\mathbb{Q}$. Let $\beta = \beta(\lambda) \in \mathfrak{t}^*_\mathbb{Q}$. Then $M(x) = \|\lambda\| = \|\beta\|_{\mathfrak{t}^*_\mathbb{R}}$ and $\mu(x, \lambda) = \|\beta\|^2_{\mathfrak{t}^*_\mathbb{R}}$.

As before, let $v = (v_i)$ be the coordinate of v and γ_i the weight of the i-th coordinate. Since $\mu(x, \lambda) = \|\beta\|^2_{\mathfrak{t}^*_\mathbb{R}}$,

$$\langle \gamma_i, \lambda \rangle_T = (\gamma_i, \beta)_{\mathfrak{t}^*_\mathbb{R}} \geqq \|\beta\|^2_{\mathfrak{t}^*_\mathbb{R}}$$

for all i and there exists i such that $(\gamma_i, \beta)_{\mathfrak{t}^*_\mathbb{R}} = \|\beta\|^2_{\mathfrak{t}^*_\mathbb{R}}$. So $v \in Y_\beta$ and if we write $v = (z, w)$ where $z \in Z_\beta$ and $w \in W_\beta$, then $z \neq 0$. Since $M(x) = \mu(x, \lambda)/\|\lambda\|$, $\pi_V(z)$ is G_β-semistable by Lemma 6.3. This implies that $x \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}$. Also $\mu(x, \lambda_\beta)/\|\lambda_\beta\| = \mu(x, \lambda)/\|\lambda\| = \|\beta\|_{\mathfrak{t}^*_\mathbb{R}}$.

We show that this β belongs to $\mathfrak{B}$. Put $T_\beta = T \cap G_\beta$. Define $\mathfrak{t}_\beta^* = X^*(T_\beta) \otimes \mathbb{R}$. Let $\beta^\perp = \{\gamma \in \mathfrak{t}_\mathbb{R}^* \mid (\gamma, \beta)_{\mathfrak{t}_\mathbb{R}^*} = 0\}$. Then $\beta^\perp \cong \mathfrak{t}_\beta^*$ by the natural homomorphism $\mathfrak{t}_\mathbb{R}^* \to \mathfrak{t}_\beta^*$ and $\mathfrak{t}_\mathbb{R}^* = \beta^\perp \oplus \mathbb{R}\beta$.

Let $z = (z_j)$ be a coordinate system of Z_β such that the action of T is diagonalized. Let δ_j be the weight of the j-th coordinate of z. Since $(\delta_j, \beta)_{\mathfrak{t}_\mathbb{R}^*} = (\beta, \beta)_{\mathfrak{t}_\mathbb{R}^*}$, $\delta_j' \stackrel{\text{def}}{=} \delta_j - \beta \in \beta^\perp$ for all j. Also, let $w = (w_l)$ be a coordinate system of W_β such that the action of T is diagonalized. Let ε_l be the weight of the l-th coordinate of w.

Let $v = (z, w)$, $I_z = \{j \mid z_j \neq 0\}$ and $I_w = \{l \mid w_l \neq 0\}$. Since $\pi_V(z)$ is G_β-semistable and $\beta^\perp \cong \mathfrak{t}_\beta^*$, the convex hull of $\{\delta_j' \mid j \in I_z\}$ contains the origin. This means that there is $a_j \in \mathbb{R}$ for all $j \in I_z$ such that $0 \leqq a_j \leqq 1$, $\sum_{j \in I_z} a_j = 1$ and $\sum_{j \in I_z} a_j \delta_j' = 0$. Therefore,

$$\sum_{j \in I_z} a_j \delta_j = \beta.$$

So β belongs to the convex hull of $\{\delta_j \mid j \in I_z\}$. Obviously, β belongs to the convex hull of $\{\delta_j \mid j \in I_z\} \cup \{\varepsilon_l \mid l \in I_w\}$.

Suppose that $b_j \in \mathbb{R}$ $(j \in I_z)$, $c_l \in \mathbb{R}$ $(l \in I_w)$, $0 \leqq b_j \leqq 1$, $0 \leqq c_l \leqq 1$, $\sum_{j \in I_z} b_j + \sum_{l \in I_w} c_j = 1$ and

$$\alpha = \sum_j b_j \delta_j + \sum_l c_l \varepsilon_l.$$

Since $(\varepsilon_l, \beta)_{\mathfrak{t}_\mathbb{R}^*} > (\beta, \beta)_{\mathfrak{t}_\mathbb{R}^*}$, $(\varepsilon_l, \beta)_{\mathfrak{t}_\mathbb{R}^*} = d_l (\beta, \beta)_{\mathfrak{t}_\mathbb{R}^*}$ where $d_l > 1$. We put $\varepsilon_l' = \varepsilon_l - d_l \beta$. Then $\varepsilon_l' \in \beta^\perp$. We put

$$\alpha' = \sum_j b_j \delta_j' + \sum_l c_l \varepsilon_l'.$$

Then $\alpha' \in \beta^\perp$ and

$$\alpha = \left(\sum_j b_j + \sum_l c_l d_l \right) \beta + \alpha'.$$

We put $C = \sum_j b_j + \sum_l c_l d_l \geqq \sum_j b_j + \sum_l c_l = 1$. Then

$$(\alpha, \alpha)_{\mathfrak{t}_\mathbb{R}^*} = C^2 (\beta, \beta)_{\mathfrak{t}_\mathbb{R}^*} + (\alpha', \alpha')_{\mathfrak{t}_\mathbb{R}^*} \geqq (\beta, \beta)_{\mathfrak{t}_\mathbb{R}^*} + (\alpha', \alpha')_{\mathfrak{t}_\mathbb{R}^*} \geqq (\beta, \beta)_{\mathfrak{t}_\mathbb{R}^*}.$$

Therefore, β is the closest point to the origin of the convex hull of weights of non-zero coordinates of $v = (z, w)$. Since $\beta \in \mathfrak{t}_{\mathbb{R},+}^*$, $\beta \in \mathfrak{B}$. This proves that

$$V \setminus \{0\} = V^{\mathrm{ss}} \cup \bigcup_{\beta \in \mathfrak{B}} S_\beta. \tag{6.8}$$

Suppose that $g_1, g_2 \in G$, $\beta_1, \beta_2 \in \mathfrak{t}_{\mathbb{R},+}^*$ and that $g_i x$ is λ_{β_i}-adapted for $i = 1, 2$. Then x is adapted for $g_i^{-1} \lambda_{\beta_i} g_i$ for $i = 1, 2$. By Theorem 5.1 (3) (b), $\lambda_{\beta_1}, \lambda_{\beta_2}$ are conjugate. Since $\lambda_{\beta_1}, \lambda_{\beta_2} \in \mathfrak{t}_{\mathbb{R},+}$, $\lambda_{\beta_1} = \lambda_{\beta_2}$. So $\beta_2 = a\beta_1$ for a positive rational number a. Since $\mu(x, \lambda_{\beta_i}) = \|\beta_i\|_{\mathfrak{t}_\mathbb{R}^*}^2$ for $i = 1, 2$, $\beta_1 = \beta_2$. Therefore, the union in (6.8) is disjoint.

Suppose that $g_1, g_2 \in G, v_1, v_2 \in Y_\beta^{\mathrm{ss}}$ and that $g_1 v_1 = g_2 v_2$. Then $g_1 \pi_V(v_1) = g_2 \pi_V(v_2)$. Kirwan [13, 13.5.Theorem.] proved the following theorem.

Theorem 6.9 (Kirwan). $\pi_V(S_\beta) \cong G \times_{P_\beta} \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ *holds.*

Therefore, there exists $p \in P_\beta$ such that $g_1 = g_2 p, \pi_V(v_2) = \pi_V(pv_1)$. So, there exists $c \in \bar{k}^\times$ such that $v_2 = cpv_1$. Since $g_1 v_1 = g_2 v_2$, $g_2 p v_1 = g_2 c p v_1 = c g_2 p v_1$. Since $v_1 \neq 0$, $g_2 p v_1 \neq 0$. Therefore, $c = 1$. This implies that $S_\beta \cong G \times_{P_\beta} Y_\beta^{\mathrm{ss}}$. □

7 Proof of Theorem 6.9

We fix a coordinate system $x = (x_0 : x_2 : \cdots : x_N)$ on $\mathbb{P}(V)$ (so $x = \pi_V(v)$ holds) by which G acts diagonally. Define a morphism $p_\beta : \mathbb{P}(Y_\beta)^{\mathrm{ss}} \to \mathbb{P}(Z_\beta)^{\mathrm{ss}}$ by $p_\beta(x_0 : x_2 : \cdots : x_N) = (x'_0 : x'_2 : \cdots : x'_N)$ where

$$x'_j = \begin{cases} x_j & if\ (\gamma_j, \beta)_{\mathfrak{t}^*_{\mathbb{R}}} = (\beta, \beta)_{\mathfrak{t}^*_{\mathbb{R}}}, \\ 0 & otherwise. \end{cases}$$

Lemma 7.1. $\mathbb{P}(Y_\beta)^{\mathrm{ss}}$ *is invariant under the action of* P_β*.*

Proof. If $\lambda : \mathbb{G}_m \to T$ is any 1-PS which is a positive scalar multiple of $\lambda(\beta)$ in $\mathfrak{t}_{\mathbb{Q}}$ then $\lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$ exists in P_β, and if $y \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ then $p_\beta(y) = \lim_{t\to 0} \lambda(t) y$ for any such λ. The result follows from these facts. □

Note that we have already proved that $\pi_V(S_\beta) \cong G\mathbb{P}(Y_\beta)^{\mathrm{ss}}$ holds. By Lemma 7.1, there is a morphism $\sigma : G \times_{P_\beta} \mathbb{P}(Y_\beta)^{\mathrm{ss}} \to \mathbb{P}(V)$ whose image is $\pi_V(S_\beta)$. In the following, we want to prove that this σ is injective.

Suppose that $\mathbb{G}_m$ acts linearly on $\mathbb{P}(V)$. Then the set of fixed points is a finite disjoint union of closed connected nonsingular subvarieties of $\mathbb{P}(V)$. Let Z be one of these. For every $x \in \mathbb{P}(V)$, the morphism $\mathbb{G}_m \to \mathbb{P}(V)$ given by $t \mapsto tx$ extends uniquely to a morphism $k \to \mathbb{P}(V)$. The image of 0 will be denoted by $\lim_{t\to 0} tx$. Let Y consist of all $x \in \mathbb{P}(V)$ such that $\lim_{t\to 0} tx$ lies in Z. Then Y is a connected locally-closed nonsingular subvariety of $\mathbb{P}(V)$ and the map $p : Y \to Z$ defined by $p(x) = \lim_{t\to 0} tx$ is an locally trivial fibration with fibre some affine space over k.

Proposition 7.2. *For each* $\beta \in \mathfrak{B}$*, the subvarieties* $\mathbb{P}(Y_\beta)^{\mathrm{ss}}$ *and* $\mathbb{P}(Z_\beta)^{\mathrm{ss}}$ *of* $\mathbb{P}(V)$ *are non-singular. The morphism* $p_\beta : \mathbb{P}(Y_\beta)^{\mathrm{ss}} \to \mathbb{P}(Z_\beta)^{\mathrm{ss}}$ *is an locally trivial fibration whose fibre at any point is an affine space.*

Proof. Let $\beta \in \mathfrak{B}$. The definition of $\mathbb{P}(Z_\beta)$ and $\mathbb{P}(Y_\beta)$ shows that the subvarieties $\mathbb{P}(Y_\beta)^{\mathrm{ss}}$ and $\mathbb{P}(Z_\beta)^{\mathrm{ss}}$ of $\mathbb{P}(V)$ are non-singular.

Let $z \in Z_\beta$. By definition, the fibre of p_β at $\pi_V((0,z,0))$ is $\{\pi_V((0,z,w)) \mid w \in W_\beta\}$ and therefore is isomorphic to W_β. $\square$

Now we want to prove Theorem 6.9. For simplicity, we shall assume that the homomorphism $\phi : G \to \mathrm{GL}_{N+1}$ which defines the action of G on $\mathbb{P}(V)$ is faithful. Note that such ϕ always exists by Ado–Iwasawa's theorem. The general result follows immediately from this except that P_β must be replaced by $\phi^{-1}(\phi(P_\beta))$, which is also a parabolic subgroup of G.

Lemma 7.3. *Suppose that G is a subgroup of GL_{N+1}. If $x \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ then*

$$\{g \in G \mid gx \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}\} = P_\beta. \tag{7.4}$$

Proof. Since $\mathbb{P}(Y_\beta)^{\mathrm{ss}}$ is invariant under P_β, we have $P_\beta \subset \{g \in G \mid gx \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}\}$. Now we want to prove that

$$P_\beta \supset \{g \in G \mid gx \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}\}. \tag{7.5}$$

Bruhat's lemma (see [22, 8.3.8 Theorem]) tells us that $\mathbb{W}$ is a system of representatives of the set of double cosets $B\backslash G/P_\beta$ of G, where B denotes the Borel subgroup of G. This implies that an element $g \in G$ can be written as $g = b\nu p$, where $b \in B, \nu \in N_G(T)$ and $p \in P_\beta$. Since $\mathbb{P}(Y_\beta)^{\mathrm{ss}}$ is invariant under the action of B and P_β, it suffices to show that if $\nu x \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ then $\nu \in P_\beta$. We fix $v \in V \setminus \{0\}$ such that $x = \pi_V(v)$. We write $v = (0, z, w)$, where $z \in Z_\beta$ and $w \in W_\beta$.

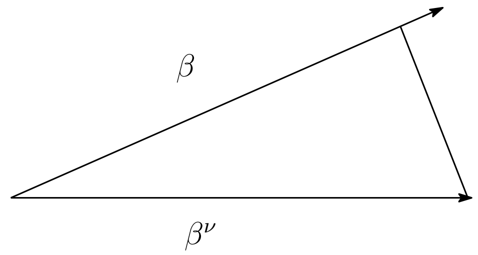
Let $v = (v_i)$ be a coordinate system of V such that the action of T is diagonalized. Let e_i be the coordinate vector which corresponds to v_i. Let γ_i be the weight of v_i. Since γ_i is the weight of v_i, we have $te_i = \gamma_i(t)e_i$ for all $t \in T$. Removing the duplication, we put $\{\gamma_0, \dots, \gamma_N\} = \{\varepsilon_1, \dots, \varepsilon_n\}$ where $\varepsilon_i \neq \varepsilon_j$ if $i \neq j$. Let E_i be the eigen space of ε_i. If $v \in E_i$, then $tv = \varepsilon_i(t)v$ for all t. We define $E_{\rho(\nu)(i)} = \nu E_i$. Since

$$t\nu v = \nu t^\nu v = \nu \varepsilon_i^\nu(t) v = \varepsilon_i^\nu(t)\nu v,$$

the map $\rho : E_i \to E_{\rho(\nu)(i)}$ is bijective. Without loss of generality, we can assume that $Z_\beta \subset \bigoplus_{i=1}^m E_i, W_\beta \subset \bigoplus_{i=m+1}^n E_i$ for some m. We write $v = \sum_{j=1}^m z_j + \sum_{j=m+1}^n w_j$ where $z_j \in E_{m_j}, w_j \in E_{m'_j}$. Then we have $\nu v = \sum_{j=1}^m \nu z_j + \sum_{j=m+1}^n \nu w_j$ where $z_j \in E_{\rho(\nu)(m_j)}, w_j \in E_{\rho(\nu)(m'_j)}$. Since $m_j \neq m'_j$, $\rho(\nu)(m_j) \neq \rho(\nu)(m'_j)$. Since $E_j \neq \{0\}$ for all $j \in I_z$, we have $E_{\rho(\nu)(j)} \neq \{0\}$. Therefore, $\nu z_i \neq 0$ for $j \in I_z$. Also if $j \in I_z$ then for $j' \neq j$, $\varepsilon_{j'} \neq \varepsilon_j$. So $\nu z_{j'}$ $(1 \leqq j' \leqq m, j' \neq j)$, $\nu w_{j'}$ $(m+1 \leqq j' \leqq n)$ do not cancel out with νz_j and so the ε_j component of νv is not zero.

Let $z = (z_j)$ be a coordinate system of Z_β such that the action of T is diagonalized. Let δ_j be the weight of the j-th coordinate of z. Put $I_z = \{j \mid z_j \neq 0\}$. Then there is $a_j \in \mathbb{R}$ for all $j \in I_z$ such that $0 \leqq a_j \leqq 1, \sum_{j \in I_z} a_j = 1$ and $\beta = \sum_{j \in I_z} a_j \delta_j$.

Now we assume that $\nu \notin P_\beta$. Then $\beta^\nu \neq \beta$. Since $t\nu z = \nu\nu^{-1}t\nu z = \nu t^\nu z$, δ_j^ν is the weight of νz_j. By definition, $\beta^\nu = \sum_{j \in I_z} a_j \delta_j^\nu$. Note that $\|\beta^\nu\|_{\mathfrak{t}_\mathbb{R}^*} = \|\beta\|_{\mathfrak{t}_\mathbb{R}^*}$. If we consider the orthogonal projection to β of β^ν, we have $(\beta^\nu, \beta)_{\mathfrak{t}_\mathbb{R}^*} < (\beta, \beta)_{\mathfrak{t}_\mathbb{R}^*}$.



Therefore, $(\delta_j^\nu, \beta)_{\mathfrak{t}_\mathbb{R}^*} < (\beta, \beta)_{\mathfrak{t}_\mathbb{R}^*}$ must hold for some $j \in I_z$. This means that $\nu z \notin Z_\beta \oplus W_\beta$ and so $\nu x \notin \mathbb{P}(Y_\beta)^{\mathrm{ss}}$. Thus, we have (7.5). □

We already seen that the canonical map $G \times \mathbb{P}(Y_\beta)^{\mathrm{ss}} \to G\mathbb{P}(Y_\beta)^{\mathrm{ss}} = \pi_V(S_\beta)$ is surjective. Suppose $g_1, g_2 \in G, x_1 x_2 \in \pi_V(S_\beta)$ such that $g_1 x_1 = g_2 x_2$. Since $x_1 = g_1^{-1} g_2 x_2$, there is $p \in P_\beta$ such that $g_1^{-1} g_2 = p$ by (7.4). Then $g_1 x_1 = g_2 x_2$ and $g_1 = g_2 p^{-1}$. This means that $\pi_V(S_\beta) \cong G \times_{P_\beta} \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ set theoretically.

In the following, we want to show that the isomorphism of Theorem 6.9 holds as a scheme.

We recall the definition of the Zariski tangent space of the variety for convenience for the readers. Let X be a variety defined over k and $\mathscr{O}_{X,x}$ a local ring of X at $x \in X$. We denote the maximal ideal of $\mathscr{O}_{X,x}$ by $\mathfrak{m}_{X,x}$. Then $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ is a vector space defined over the residue field $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_{X,x}$ and whose dual $T_x X \stackrel{\mathrm{def}}{=} (\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2)^*$ is called the *Zariski tangent space* of X at x.

Let $\mathfrak{g}$ be the Lie algebra of G and for each $\beta \in \mathfrak{B}$ let $\mathfrak{p}_\beta$ be the Lie algebra of the parabolic subgroup P_β of G. As a k-vector space $\mathfrak{g}$ is just the tangent space to the group G at the origin. The action of G on $\mathbb{P}(V)$ induces a k-linear map $\xi \mapsto \xi_x$ from $\mathfrak{g}$ to the Zariski tangent space $T_x\mathbb{P}(V)$ for each $x \in \mathbb{P}(V)$.

Lemma 7.6. *Suppose that G is a subgroup of GL_{N+1}. If $x \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ then*

$$\{\xi \in \mathfrak{g} \,|\, \xi_x \in T_x(\mathbb{P}(Y_\beta)^{\mathrm{ss}})\} = \mathfrak{p}_\beta.$$

Proof. Since $\mathbb{P}(Y_\beta)^{\mathrm{ss}}$ is invariant under P_β, we have $\mathfrak{p}_\beta \subset \{\xi \in \mathfrak{g} \,|\, \xi_x \in T_x\mathbb{P}(Y_\beta)^{\mathrm{ss}}\}$. It remain to show that $\{\xi \in \mathfrak{g} \,|\, \xi_x \in T_x\mathbb{P}(Y_\beta)^{\mathrm{ss}}\} \subset \mathfrak{p}_\beta$. As in the proof of Lemma 7.1, λ_β acts on V as $t \mapsto \mathrm{diag}(t^{r(\gamma_0,\beta)_{\mathfrak{t}_\mathbb{R}^*}}, \ldots, t^{r(\gamma_N,\beta)_{\mathfrak{t}_\mathbb{R}^*}})$ for some rational number $r > 0$. By definition, the subgroup P_β consists of all $g \in G$ such that $\lim_{t\to 0}(r\lambda(\beta)(t))g(r\lambda(\beta)(t))^{-1}$ exists. Hence, an element $g \in G$ lies in P_β if and only if it is of the form $g = (g_{i,j})$ with $g_{i,j} = 0$ when $(\gamma_i, \beta)_{\mathfrak{t}_\mathbb{R}^*} < (\gamma_j, \beta)_{\mathfrak{t}_\mathbb{R}^*}$.

Let

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha} \mathfrak{g}_\alpha \right)$$

be the root space decomposition of $\mathfrak{g}$ with respect to the Lie algebra $\mathfrak{t}$ of the maximal torus T. If $\xi \in \mathfrak{g}_\alpha \subset \mathfrak{gl}_{N+1}$ has a non zero (i,j)-entry as an element of GL_{N+1} then as $[\eta, \xi] = \alpha(\eta)\xi$ for all $\eta \in \mathfrak{t}$ it follows that $\alpha = \gamma_i - \gamma_j$. So $\mathfrak{g}_\alpha \subset \mathfrak{p}_\beta$ whenever $(\alpha, \beta)_{\mathfrak{t}^*_{\mathbb{R}}} \geqq 0$. Hence it suffice to show that if $\xi \in \bigoplus_{(\alpha,\beta)_{\mathfrak{t}^*_{\mathbb{R}}}<0} \mathfrak{g}_\alpha$ and $\xi_x \in T_x\mathbb{P}(Y_\beta)^{\mathrm{ss}}$ then $\xi \in \mathfrak{p}_\beta$.

Let V_+ (respectively V_0, V_-) be the sum of all subspaces of V on which T acts as multiplication by some character γ_j with $(\gamma_j, \beta)_{\mathfrak{t}^*_{\mathbb{R}}} > (\beta, \beta)_{\mathfrak{t}^*_{\mathbb{R}}}$ (respectively $(\gamma_j, \beta)_{\mathfrak{t}^*_{\mathbb{R}}} = (\beta, \beta)_{\mathfrak{t}^*_{\mathbb{R}}}, (\gamma_j, \beta)_{\mathfrak{t}^*_{\mathbb{R}}} < (\beta, \beta)_{\mathfrak{t}^*_{\mathbb{R}}}$). Then any element of $\bigoplus_{(\alpha,\beta)_{\mathfrak{t}^*_{\mathbb{R}}}<0} \mathfrak{g}_\alpha$ is of block form $\begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & d & e \end{pmatrix}$ with respect to the decomposition of V as $V_+ \oplus V_0 \oplus V_-$. This follows form the fact that $\mathfrak{g}_\alpha V_\gamma \subset V_{\gamma+\alpha}$ holds for all $\alpha, \lambda \in \mathfrak{t}^*_{\mathbb{R}}$. To prove this fact, for $x \in \mathfrak{g}_\alpha, v \in V_\lambda, t \in \mathfrak{t}$, we have

$$\begin{aligned} txv &= xtv + [t,x]v \\ &= \lambda(t)xv + \alpha(t)xv \\ &= (\lambda + \alpha)(t)xv. \end{aligned}$$

If $x \in \mathbb{P}(Z_\beta)^{\mathrm{ss}}$ then x is represented by a vector of the form $(0, v, 0)$ in $V = V_+ \oplus V_0 \oplus V_-$. We have

$$\begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & d & e \end{pmatrix} \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ dv \end{pmatrix}$$

and so by the definition of $\mathbb{P}(Y_\beta)^{\mathrm{ss}}$ if $\xi_x \in T_x\mathbb{P}(Y_\beta)^{\mathrm{ss}}$ then $dv = 0$ and hence $\xi_x = 0$. But this means that ξ is contained in the Lie algebra of the stabilizer of x in G, and by (7.4), the stabilizer of x is contained in P_β. Therefore $\xi \in \mathfrak{p}_\beta$ as required. □

Consider the morphisms

$$G \times \mathbb{P}(Y_\beta) \xrightarrow{\gamma} G \times \mathbb{P}(V) \xrightarrow{\delta} (G/P_\beta) \times \mathbb{P}(V)$$

given by

$$\gamma(g, x) \stackrel{\mathrm{def}}{=} (g, gx), \quad \delta(g, x) \stackrel{\mathrm{def}}{=} (gP_\beta, x).$$

We define

$$M = \delta(\gamma(G \times \mathbb{P}(Y_\beta))), \quad M' = \delta(\gamma(G \times \mathbb{P}(Y_\beta)^{\mathrm{ss}})).$$

Since $\mathbb{P}(Y_\beta)$ is invariant under P_β we have $\delta^{-1}(M) = \{(g, y) \mid g^{-1}y \in \mathbb{P}(Y_\beta)\}$ which is closed in $G \times \mathbb{P}(V)$ and is isomorphic to $G \times \mathbb{P}(Y_\beta)$ via γ. As δ is quotient morphism, M is therefore closed in $(G/P_\beta) \times \mathbb{P}(V)$.

Now $G\mathbb{P}(Y_\beta)$ is the image of M under the projection $\mathrm{pr}_V : (G/P_\beta) \times \mathbb{P}(V) \to \mathbb{P}(V)$. Since G/P_β is complete, this shows $G\mathbb{P}(Y_\beta)$ is closed. Furthermore, it follows that $M' = M \cap \mathrm{pr}_V^{-1}(\pi_V(S_\beta))$ and hence is an open subset of M.

We have $M' = \{(gP_\beta, y) \mid g^{-1}y \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}\}$ which is isomorphic to $G \times_{P_\beta} \mathbb{P}(Y_\beta)^{\mathrm{ss}}$. Since $G \times \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ is non-singular and the action of P_β is free, the quotient group $G \times_{P_\beta} \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ is also non-singular. Moreover, by Lemma 7.6 the restriction of pr_V to M' is a homeomorphism onto $\pi_V(S_\beta)$. Indeed since G/P_β is complete $\mathrm{pr}_V : (G/P_\beta) \times \mathbb{P}(V) \to \mathbb{P}(V)$ is a closed map, so that $\mathrm{pr}_V : M' \to \pi_V(S_\beta)$ is a homeomorphism because M' is locally closed in $(G/P_\beta) \times \mathbb{P}(V)$. To show that $\mathrm{pr}_V : M' \to \pi_V(S_\beta)$ is an isomorphism, we need the following fact.

Let X, Y be varieties defined over k. Fix $x \in X$ and a morphism $f : X \to Y$. By definition, f induces a homomorphism of local rings

$$f_x^\sharp : \mathscr{O}_{f(x),Y} = \varinjlim_{x \in V} \mathscr{O}_{f(x),Y}(V) \to \varinjlim_{x \in V} \mathscr{O}_{x,X}(f^{-1}(V)) = \mathscr{O}_{x,X}.$$

Furthermore, f is an isomorphism if and only if f is a homeomorphism and the induce map $f_x^\sharp$ on local rings is an isomorphism for all $x \in X$.

Lemma 7.7. *Let $\varphi : A \to B$ be a local homomorphism of Noetherian rings (i.e. $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$). Assume that the following conditions hold.*

(1) $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is an isomorphism.

(2) $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective.

(3) B is a finitely generated A-module.

Then φ is surjective.

Proof. We follow the description of [3, Chap. II Lem.7.4, p.153]. Consider the ideal $\mathfrak{a} = \varphi(\mathfrak{m}_A)B$ of B. We have $\mathfrak{a} \subset \mathfrak{m}_B$, and by (2), $\mathfrak{a}$ contains a set of generators for $\mathfrak{m}_B/\mathfrak{m}_B^2$. Hence by Nakayama's lemma for the local ring B and the B-module $\mathfrak{m}_B$, we conclude that $\mathfrak{a} = \mathfrak{m}_B$. Now apply Nakayama's lemma to the A-module B. By (3), B is finitely generated A-module. The element $1 \in B$ gives a generator for $B/\mathfrak{m}_A B = B/\mathfrak{m}_B = A/\mathfrak{m}_A$ by (1), so we conclude that 1 also generates B as an A-module, i.e., f is surjective. □

Clearly, a morphism f induces the $\kappa(x)$-homomorphism $f_{*,x} : T_x X \to T_{f(x)} Y$.

Suppose that f is a homeomorphism and $f_{*,x}$ is injective for all $x \in X$. Then the dual map $\mathfrak{m}_{Y,f(x)}/\mathfrak{m}_{Y,f(x)}^2 \to \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ is surjective. Furthermore, $\kappa(x) = k = \kappa(f(x))$ and $\mathscr{O}_{X,x}$ is a finitely generated $\mathscr{O}_{Y,f(x)}$-module. Therefore, by Lemma 7.7, $f_x^\sharp$ is a surjective homomorphism for all $x \in X$. This implies that f is injective and so f is an isomorphism.

Now we want to show that $\mathrm{pr}_V : M' \to \pi_V(S_\beta)$ is an isomorphism. We are already seen that pr_V is a surjective homomorphism. Therefore, we suffice to check that the induce maps of Zariski tangent spaces $(\mathrm{pr}_V)_{*,m} : T_m M' \to T_{\mathrm{pr}_V(m)}\pi_V(S_\beta)$ are all injective.

It is necessary to consider the case $m = (P_\beta, y)$ for some $y \in \mathbb{P}(Y_\beta)^{\mathrm{ss}}$. Then an element of $T_m M'$ is of the form $(a+\mathfrak{p}_\beta, \xi)$ where $a+\mathfrak{p}_\beta \in \mathfrak{g}/\mathfrak{p}_\beta, \xi \in T_y\mathbb{P}(V)$ and $-a_y+\xi \in T_y\mathbb{P}(Y_\beta)^{\mathrm{ss}}$. So if $0 = (\mathrm{pr}_V)_{*,m}(a+\mathfrak{p}_\beta, \xi) = \xi$ then $a_y \in T_y\mathbb{P}(Y_\beta)^{\mathrm{ss}}$, and hence by Lemma 7.6 $a \in \mathfrak{p}_\beta$ so that $(a+\mathfrak{p}_\beta, \xi)$ is the zero element of $T_m M'$. It follows that $(\mathrm{pr}_V)_{*,m}$ is injective everywhere on M' and hence that $\mathrm{pr}_V : M' \to \pi_V(S_\beta)$ is an isomorphism. Therefore, we conclude that for each $\beta \in \mathfrak{B}$ the stratum $\pi_V(S_\beta)$ is non-singular and isomorphic to $G \times_{P_\beta} \mathbb{P}(Y_\beta)^{\mathrm{ss}}$ and so the proof of Theorem 6.9 is complete.

8 Proof of the main theorem in the non-split case

Assume that k is an arbitrary perfect field from now on. Therefore, $\bar{k} = k^{\mathrm{sep}}$. In this section $S \subset G$ is a maximal split torus and $S \subset T \subset G$ is a maximal torus defined over k.

We include the following two lemmas which are basically well-known for the sake of the reader. As we stated in Introduction, ${}_k\mathbb{W}$ can be regarded as a subgroup of $\mathbb{W}$ ([2, 5.5.Corollaire.]).

Lemma 8.1. *Elements of $N_G(S)/Z_G(S)$ are represented by elements of $N_G(T)$.*

Proof. Let $n \in N_G(S)_{\bar{k}}$ (in fact, it is possible to choose a representative from $N_G(S)_k$). Then $n^{-1}Tn, T$ are maximal tori containing S. So they are contained in $Z_G(S)$. Therefore, there exists an element $g \in Z_G(S)_{\bar{k}}$ such that $g^{-1}n^{-1}Tng = T$. This implies that $h \stackrel{\mathrm{def}}{=} ng \in N_G(T)_{\bar{k}}$. Then $n = hg^{-1} \in hZ_G(S)_{\bar{k}}$. So elements of $N_G(S)/Z_G(S)$ are represented by elements of $N_G(T)$. □

Lemma 8.2. *Suppose that λ_1, λ_2 are 1-PS's in S and that λ_1, λ_2 are conjugate in G_k, i.e., there exists $h \in G_k$ such that $h\lambda_1(t)h^{-1} = \lambda_2(t)$ ($t \in \bar{k}^\times$). Then λ_1, λ_2 are conjugate by an element of ${}_k\mathbb{W} = N_G(S)/Z_G(S)$.*

Proof. Let S_1, S_2 be the images of λ_1, λ_2 respectively. Then $hS_1h^{-1} = S_2$. Since hSh^{-1}, S are both maximal split torus in $Z_G(S_2)$, there is $h_1 \in Z_G(S_2)$ such that $h_1hSh^{-1}h_1^{-1} = S$. So $h_1h \in N_G(S)$. Since $h\lambda_1(t)h^{-1} = \lambda_2(t)$ and $h_1 \in Z_G(S_2)$, $h_1h\lambda_1(t)h^{-1}h_1^{-1} = \lambda_2(t)$. Therefore, λ_1, λ_2 are conjugate by an element of the relative Weyl group ${}_k\mathbb{W}$. □

Proof of Theorem 1.2 in the non-split case. We now prove Theorem 1.2 in the non-split case.

We choose a coordinate system $v' = (v'_0, \ldots, v'_N)$ over $\bar{k}$ so that the action of T is diagonalized. Then the action of S is diagonalized also. However, since we chose a coordinate system $v = (v_0, \dots, v_N)$ so that the action of S is diagonalized over k in Theorem 1.2, we consider the relation between the two coordinates v' and v.

The inclusion map $\mathfrak{s}_{\mathbb{R}} \to \mathfrak{t}_{\mathbb{R}}$ induces a map $\mathfrak{s}_{\mathbb{R}}^* \to \mathfrak{t}_{\mathbb{R}}^*$, which enables us to identify $\mathfrak{s}_{\mathbb{R}}^*$ with a subspace of $\mathfrak{t}_{\mathbb{R}}^*$. The restriction to $\mathfrak{s}_{\mathbb{R}}^*$ induces a map $\mathfrak{t}_{\mathbb{R}}^* \to \mathfrak{s}_{\mathbb{R}}^*$. We show that this is the orthogonal projection (see also [22, p.259]).

If $\alpha \in \mathfrak{s}_{\mathbb{R}}^*$ then $(\lambda(\alpha), \nu)_{\mathfrak{s}_{\mathbb{R}}} = \langle \alpha, \nu \rangle_S$ for all $\nu \in \mathfrak{s}_{\mathbb{R}}$. Regarding $\lambda(\alpha)$ as an element of $\mathfrak{t}_{\mathbb{R}}$, the corresponding element of $\mathfrak{t}_{\mathbb{R}}^*$ is the function g on $\mathfrak{t}_{\mathbb{R}}$ such that $(\lambda(\alpha), \nu)_{\mathfrak{t}_{\mathbb{R}}} = \langle g, \nu \rangle_T$ for all $\nu \in \mathfrak{t}_{\mathbb{R}}$. If we restrict g to $\mathfrak{s}_{\mathbb{R}}$, we obtain the function $\mathfrak{s}_{\mathbb{R}} \ni \nu \mapsto (\lambda(\alpha), \nu)_{\mathfrak{t}_{\mathbb{R}}} = \langle \alpha, \nu \rangle_S$. So the composition $\mathfrak{s}_{\mathbb{R}}^* \to \mathfrak{t}_{\mathbb{R}}^* \to \mathfrak{s}_{\mathbb{R}}^*$ is the identity map. Therefore, if we denote the kernel of $\mathfrak{t}_{\mathbb{R}}^* \to \mathfrak{s}_{\mathbb{R}}^*$ by U then $\mathfrak{t}_{\mathbb{R}}^* = \mathfrak{s}_{\mathbb{R}}^* \oplus U$.

We show that U is orthogonal to $\mathfrak{s}_{\mathbb{R}}^*$. If $v \in U$ then $\langle v, \lambda \rangle_T = 0$ for all $\lambda \in \mathfrak{s}_{\mathbb{R}}$. If $\lambda \in \mathfrak{s}_{\mathbb{R}}$ corresponds to $\beta(\lambda) \in \mathfrak{s}_{\mathbb{R}}^*$ then $0 = \langle v, \lambda \rangle_T = (v, \beta(\lambda))_{\mathfrak{t}_{\mathbb{R}}^*}$. Therefore, v is orthogonal to $\mathfrak{s}_{\mathbb{R}}^*$. This implies that $\mathfrak{t}_{\mathbb{R}}^* \to \mathfrak{s}_{\mathbb{R}}^*$ is the orthogonal projection.

Let $\gamma_i \in \mathfrak{s}_{\mathbb{R}}^*$ (resp. $\eta_i' \in \mathfrak{t}_{\mathbb{R}}^*$) be the weight of the i-th coordinate v_i (resp. v_i'). Note that since S is split, any character of S is defined over k. Let $\eta_i \in \mathfrak{s}_{\mathbb{R}}^*$ be the restriction of η_i' to $\mathfrak{s}_{\mathbb{R}}^*$. Removing the duplication, we put $\{\eta_0, \dots, \eta_N\} = \{\delta_1, \dots, \delta_m\}$ where $\delta_i \neq \delta_j$ if $i \neq j$. Let $A_i = \{j \mid \eta_j = \delta_i\}$. Then A_i is invariant under the action of $\mathrm{Gal}(\bar{k}/k)$. Let e_j (resp. e_j') be the coordinate vector corresponding to the j-th coordinate v_j (resp. v_j'). Let $E_i' \subset V \otimes_k \bar{k}$ be the subspace spanned by $\{e_j' \mid j \in A_i\}$. Then E_i' is invariant under the action of $\mathrm{Gal}(\bar{k}/k)$. Since k is a perfect field, there exists a subspace $E_i \subset V$ such that $E_i' = E_i \otimes_k \bar{k}$. Since $V \otimes_k \bar{k} = \bigoplus_{i=1}^m E_i'$, we have $V = \bigoplus_{i=1}^m E_i$. This implies that E_i is the weight space of δ_i. Since the set of weights of V with respect to S does not depend on the choice of the coordinate, δ_i must coincides with γ_j for some j. So if we put $B_i = \{0 \leqq j \leqq N \mid \gamma_j = \delta_i\}$ then E_i is spanned by $\{e_j \mid j \in B_i\}$. Therefore, we can conclude that if η_i' is the weight of a non-zero coordinate of v' then η_i is the weight of a non-zero coordinate of v where v and v' are related with the change of coordinate.

Suppose that $x \in \mathbb{P}(V)_k \setminus \mathbb{P}(V)_k^{\mathrm{ss}}$ and that x is λ-adapted. Then λ is a 1-PS defined over k (by Theorem 5.3). So there exists $g \in G_k$ such that $g\lambda g^{-1}$ is a 1-PS in $\mathfrak{s}_{\mathbb{R},+}$. As in the split case, there is a positive rational number a such that if $\beta = a\beta(\lambda)$ then $gx \in Y_{\beta\, k}^{\mathrm{ss}}$. Here we have to make sure that the definition of $Y_{\beta\, k}^{\mathrm{ss}}$ is the same whether or not we regard $\beta \in \mathfrak{s}_{\mathbb{R}}^*$ or $\beta \in \mathfrak{t}_{\mathbb{R}}^*$.

To distinguish the difference between $k, \bar{k}$, let $Z_\beta' \subset V \otimes_k \bar{k}$ be the subspace spanned by e_j' such that $(\eta_j', \beta)_{\mathfrak{t}_{\mathbb{R}}^*} = \|\beta\|_{\mathfrak{t}_{\mathbb{R}}^*}^2$. We define W_β' similarly. Let E_i, E_i' be as above. We regard $\beta \in \mathfrak{t}_{\mathbb{R}}^*$. Since $\mathfrak{t}_{\mathbb{R}}^* \to \mathfrak{s}_{\mathbb{R}}^*$ is the orthogonal projection, $(\eta_j', \beta)_{\mathfrak{t}_{\mathbb{R}}^*} = (\eta_j, \beta)_{\mathfrak{s}_{\mathbb{R}}^*}$. So Z_β' is spanned by E_i' such that $(\delta_i, \beta)_{\mathfrak{s}_{\mathbb{R}}^*} = \|\beta\|_{\mathfrak{t}_{\mathbb{R}}^*}^2 = \|\beta\|_{\mathfrak{s}_{\mathbb{R}}^*}^2$. Also Z_β is spanned by E_i such that $(\delta_i, \beta)_{\mathfrak{s}_{\mathbb{R}}^*} = \|\beta\|_{\mathfrak{s}_{\mathbb{R}}^*}^2$. Therefore, $Z_\beta' = Z_\beta \otimes_k \bar{k}$. Similarly we have $W_\beta' = W_\beta \otimes_k \bar{k}$. We pointed out earlier that the notion of semistability does not depend on the ground field.

Therefore, the set $Y^{ss}_{\beta\, k}$ can be regarded as the set of k-rational points of the set Y^{ss}_{β} defined regarding $\beta \in \mathfrak{t}^*_{\mathbb{R}}$.

By these considerations, we can conclude that $x \in Y^{ss}_{\beta\, k}$ where the definition of $Y^{ss}_{\beta\, k}$ is as in Introduction. We have to verify that this β belongs to $\mathfrak{B}$.

Put $x_1 = gx$. Suppose that $x_1 = \pi_V(v)$ where $v = (v_0, \ldots, v_N)$ (this is the coordinate for which the action of S is diagonalized rationally). Let $v' = (v'_0, \ldots, v'_N)$ be the coordinate of x_1 for which the action of T is diagonalized. We claim that this β is the closest point to the origin of the convex hull of $\mathfrak{I}_{x_1} = \{\gamma_j \mid v_j \neq 0\}$. We have already proved this claim in the split case. So β is the closest point to the origin of the convex hull of $\{\eta'_j \mid v'_j \neq 0\}$.

We have $\eta'_j = \eta_j + \varepsilon_j$ where ε_j is orthogonal to $\mathfrak{s}^*_{\mathbb{R}}$. Let $\mathrm{pr} : \mathfrak{t}^*_{\mathbb{R}} \to \mathfrak{s}^*_{\mathbb{R}}$ be the natural map. Since $\beta \in \mathfrak{s}^*_{\mathbb{R}}$, $\mathrm{pr}(\beta) = \beta$. Also since pr is a linear map and β is in the convex hull of $\{\eta'_j \mid v'_j \neq 0\}$, $\beta = \mathrm{pr}(\beta)$ is in the convex hull of $\{\mathrm{pr}(\eta'_j) \mid v'_j \neq 0\} = \{\eta_j \mid v_j \neq 0\}$. Since $\beta \in \mathfrak{s}^*_{\mathbb{R}}$,

$$(\beta, \eta_i)_{\mathfrak{s}^*_{\mathbb{R}}} = (\beta, \eta_i)_{\mathfrak{t}^*_{\mathbb{R}}} = (\beta, \eta'_i)_{\mathfrak{t}^*_{\mathbb{R}}} \geqq \|\beta\|^2_{\mathfrak{t}^*_{\mathbb{R}}} = \|\beta\|^2_{\mathfrak{s}^*_{\mathbb{R}}}.$$

So β is the closest point to the origin of the convex hull of $\{\eta_j \mid v_i \neq 0\}$ in $\mathfrak{s}^*_{\mathbb{R}}$.

We pointed out earlier that if η'_j is the weight of a non-zero coordinate (for which the action of T is diagonalized over $\bar{k}$) of x_1 then η_j coincides with the weight of a non-zero coordinate (for which the action of S is diagonalized over k) of x_1. Since $\beta \in \mathfrak{s}^*_{\mathbb{R},+}$, $\beta \in \mathfrak{B}$.

Finally, we have to prove that $S_{\beta\, k} \cong G_k \times_{P_{\beta\, k}} Y^{ss}_{\beta\, k}$. Let $v \in S_{\beta\, k}$ and $x = \pi_V(v)$. Then any $\lambda \in \Lambda_x$ is split. So there exists $g \in G_k$ such that $gv \in Y^{ss}_{\beta} \cap V_k = Y^{ss}_{\beta\, k}$. Therefore, $G_k \times Y^{ss}_{\beta\, k} \to S_{\beta\, k}$ is surjective. If $g_1, g_2 \in G_k, y_1, y_2 \in Y^{ss}_{\beta\, k}$ and $v = g_1 y_1 = g_2 y_2$ then there exists $h \in P_{\beta\, \bar{k}}$ such that $g_1 = g_2 h$ by the split case. But then $h = g_1 g_2^{-1} \in G_k$ and so $h \in P_{\beta\, k}$. Therefore, $S_{\beta\, k} \cong G_k \times_{P_{\beta\, k}} Y^{ss}_{\beta\, k}$. □

Proof of Corollary 1.3. Let G, G_1, etc., be as in the situation of Corollary 1.3. We remind the reader that we are considering the stability with respect to the group G_1 (not G).

Let $\beta \in \mathfrak{B}$. We put $P_{1\beta} = P_\beta \cap G_1$. We have proved that $S_{\beta\, k} \cong G_{1\, k} \times_{P_{1\beta\, k}} Y^{ss}_{\beta\, k}$. So the map $G_k \times_{P_{\beta\, k}} Y^{ss}_{\beta\, k} \to S_{\beta\, k}$ is surjective. Suppose that $g_1, g_2 \in G_k, v_1, v_2 \in Y^{ss}_{\beta\, k}$ and that $g_1 v_1 = g_2 v_2$. This implies that $g_2^{-1} g_1 v_1 = v_2$. Since $G_{\bar{k}} = T_{0\, \bar{k}} G_{1\, \bar{k}}$, there exist $t \in T_{0\, \bar{k}}, h \in G_{1\, \bar{k}}$ such that $g_2^{-1} g_1 = th$. Since $t \in T_{0\, \bar{k}} \subset Z(G)_{\bar{k}}$, $thv_1 = htv_1 = h\chi(t)v_1 = v_2$. Since Y^{ss}_{β} is invariant under scalar multiplications, we have $\chi(t)v_1 \in Y^{ss}_{\beta\, \bar{k}}$. This implies that $h \in P_{1\beta\, \bar{k}} \subset P_{\beta\, \bar{k}}$. Since $T_0 \subset Z(G)$, $T_0 \subset P_\beta$. So $g_2^{-1} g_1 = th \in P_{\beta\, \bar{k}}$. Since $g_1, g_2 \in G_k$, we have $h \in P_{\beta\, k}$. Therefore, $S_{\beta\, k} \cong G_k \times_{P_{\beta\, k}} Y^{ss}_{\beta\, k}$.

Other statements follow trivially from Theorem 1.2. □

9 Examples of GIT stratifications

We briefly recall the *Cayley–Dickson process* in the following (for more detail, see [4, pp.101–110]). We assume that $\operatorname{char} k \neq 2, 3$. Note that even though it is assumed $k = \mathbb{R}$ in [4], the argument for the Cayley–Dickson process works as long as $\operatorname{char} k \neq 2, 3$.

Definition 9.1. A *normed k-algebra* is a not necessarily associative finite dimensional k-algebra A with multiplicative unit 1, equipped with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ such that the associated square norm $\|x\| \stackrel{\text{def}}{=} \langle x, x\rangle$ satisfies the multiplicative property

$$\|xy\| = \|x\|\|y\|.$$

If A is a normed algebra, then we denote the span of 1 by $\Re A$ and its orthogonal complement $\{x \in A \mid \langle 1, x\rangle = 0\}$ by $\Im A$. Any $x \in A$ has a unique decomposition $x = x_1 + x_2$ with $x_1 \in \Re A, x_2 \in \Im A$. Then we write $\Re x = x_1, \Im x = x_2$. Also we define the conjugation by $\bar{x} = x_1 - x_2$. So we have $\Re x = \frac{1}{2}(x + \bar{x}), \Im x = \frac{1}{2}(\bar{x} - x)$.

Given a normed k-algebra, we make two normed k-algebra $A(\pm)$ as follows. As a vector space, we define $A(\pm) = A \oplus A$. We define the multiplication and the norm by

$$(a, b)(c, d) = (ac \pm \bar{d}b, da + b\bar{c}), \quad \|(a, b)\| = \|a\| + \|b\|.$$

Then we define

$$\langle (a, b), (c, d)\rangle = \frac{1}{2}(\|(a, b) + (c, d)\| - \|(a, b)\| - \|(c, d)\|).$$

We use the notation $a + b\varepsilon$ for (a, b). Note that if k contains $\sqrt{-1}$, $\varepsilon \mapsto \sqrt{-1}\varepsilon$ induces an isomorphism $A(+) \xrightarrow{\sim} A(-)$.

For a normed k-algebra A, we define $[x, y, z] = (xy)z - x(yz)$ for $x, y, z \in A$. This is called the *associator*. If the associator is alternative, A is called an *alternative algebra*. It is known that if A is commutative, $A(\pm)$ is associative, and if A is associative, $A(\pm)$ is alternative. Furthermore, it is known that the norm of $A(\pm)$ is compatible with the product if and only if A is associative. The above process is called the *Cayley–Dickson process*. It easy to see that

$$\Im(a + b\varepsilon) = \Im a + b\varepsilon, \quad \overline{a + b\varepsilon} = \bar{a} - b\varepsilon.$$

The following lemma is proved in [4, Lemma 6.10, p.104].

Lemma 9.2. *(1)* $\overline{xy} = \bar{y}\bar{x}$.

(2) $\langle x, y\rangle = \Re(x\bar{y})$.

(3) $\|x\| = x\bar{x}$.

If A, B are normed k-algebras, a *homomorphism* $\phi : A \to B$ is a k-linear map such that $\phi(1) = 1, \phi(xy) = \phi(x)\phi(y)$ and $\|\phi(x)\| = \|x\|$. The third condition implies $\langle \phi(x), \phi(y) \rangle = \langle x, y \rangle$. So $\phi(\Im A) \subset \Im B$. Suppose $x, y \in \Im A$. Then $\langle x, y \rangle = \Re(x\bar{y}) = -\Re(xy)$. So

$$\begin{aligned} -\Re(\phi(x)\phi(y)) &= \langle \phi(x), \phi(y) \rangle = \langle x, y \rangle = -\Re(xy), \\ \phi(\Im(xy)) &= \phi(xy - \Re(xy)) = \phi(x)\phi(y) - \Re(xy) \\ &= \phi(x)\phi(y) - \Re(\phi(x)\phi(y)) = \Im(\phi(x)\phi(y)). \end{aligned}$$

Conversely, ϕ is a homomorphism if the above conditions are satisfied. So we have proved the following proposition.

Proposition 9.3. *A k-linear map $\phi : A \to B$ is a homomorphism if and only if*

$$\phi(1) = 1, \quad \phi(\Im(xy)) = \Im(\phi(x)\phi(y)), \quad \langle \phi(x), \phi(y) \rangle = \langle x, y \rangle$$

for all $x, y \in \Im A$.

It is easy to see that

$$k(+) \cong \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \,\middle|\, a, b \in k \right\}, \quad k(+)(-) \cong \mathrm{M}_{2,2}(k).$$

For $k(+), \varepsilon = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and the conjugation is

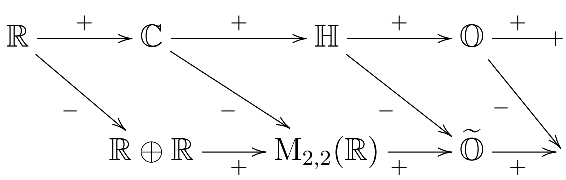
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

For $k(+)(-), \varepsilon = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and the conjugation is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Therefore, $\Re x = \frac{1}{2}\mathrm{tr}(x)$ in both cases and the norm is the determinant.

We define $\mathbb{H} = k(+)(+), \mathbb{O} = \mathbb{H}(+)$ and $\widetilde{\mathbb{O}} = \mathrm{M}_{2,2}(k)(+)$. The normed k-algebra $\mathbb{O}$ is called the *non-split octonion algebra* (if k does not contain $\sqrt{-1}$), and $\widetilde{\mathbb{O}}$ is called the *split octonion algebra*.



Let k be a perfect field, k_1/k a quadratic extension, D a division quaternion algebra over k, $\mathrm{H}_3(k_1)$ the space of 3×3 Hermitian matrices with entries in k_1 (with respect to the action of $\mathrm{Gal}(k_1/k)$), and $\mathrm{H}_3(D)$ the space of 3×3 Hermitian matrices with entries in D (with respect to the canonical involution of D). Explicitly,

$$\mathrm{H}_3(k_1) = \left\{ \begin{pmatrix} h_1 & a & c^\sigma \\ a^\sigma & h_2 & b \\ c & b^\sigma & h_3 \end{pmatrix} \middle| \, h_i \in k\,(i = 1, 2, 3), a, b, c \in k_1 \right\},$$

$$\mathrm{H}_3(D) = \left\{ \begin{pmatrix} h_1 & a & \bar{c} \\ \bar{a} & h_2 & b \\ c & \bar{b} & h_3 \end{pmatrix} \middle| \, h_i \in k\,(i = 1, 2, 3), a, b, c \in D \right\}$$

where σ denotes the generator of $\mathrm{Gal}(k_1/k)$.

Let $\mathbb{O}$ be a non-split octonion. For example, if k is a number field with a real place v and D is a division quaternion algebra such that $D \otimes k_v$ is isomorphic to the Hamiltonian quaternions $\mathbb{H}$, then the algebra $D(+)$ obtained from the Cayley–Dickson process is a non-split octonion. Let J be the exceptional Jordan algebra of 3×3 Hermitian matrices with entries in $\mathbb{O}$. Explicitly,

$$J = \mathrm{H}_3(\mathbb{O}) = \left\{ \begin{pmatrix} h_1 & a & \bar{c} \\ \bar{a} & h_2 & b \\ c & \bar{b} & h_3 \end{pmatrix} \middle| \, h_i \in k\,(i = 1, 2, 3), a, b, c \in \mathbb{O} \right\}.$$

In particular, $\dim_k J = 27$.

Let

$$E_6 = \{g \in \mathrm{GL}_k(J) \mid \det(gx) = \det(x) \text{ for all } x \in J\}.$$

This is a simple group of type E_6 with split rank 2 (see [7]).

We first consider the following four prehomogeneous vector spaces.

(a) $G = \mathrm{GL}_3 \times \mathrm{GL}_2$, $V = \mathrm{Sym}^2\mathrm{Aff}^3 \otimes \mathrm{Aff}^2$.

(b) $G = \mathfrak{R}_{k_1/k}\mathrm{GL}_3 \times \mathrm{GL}_2$, $V = \mathrm{H}_3(k_1) \otimes \mathrm{Aff}^2$.

(c) $G = \mathrm{GL}_3(D) \times \mathrm{GL}_2$, $V = \mathrm{H}_3(D) \otimes \mathrm{Aff}^2$.

(d) $G = E_6 \times \mathrm{GL}_2$, $V = \mathrm{H}_3(\mathbb{O}) \otimes \mathrm{Aff}^2$.

These four representations are prehomogeneous vector spaces of parabolic type coming from simple groups of types F_4, E_6, E_7, E_8 respectively.

Rational orbits for the cases (a)–(c) have interesting arithmetic interpretations. Such interpretations were discussed in [25], [8], [23] for the cases (a)–(c) respectively. However, the interpretation for the case (d) is unknown.

They have exactly the same set of weights as follows.

Let G_1 be $\mathrm{SL}_3\times\mathrm{SL}_2$, $\mathfrak{R}_{k_1/k}\mathrm{SL}_3\times\mathrm{SL}_2$ and $E_6\times\mathrm{SL}_2$ respectively for the cases (a), (b), (d). For the case (c), $\mathrm{GL}_3(D)$ can be identified with a subgroup of GL_{12}. Let $d:\mathrm{GL}_{12}\to\mathrm{GL}_1$ be the determinant and $G_1=\mathrm{Ker}(d)^\circ\times\mathrm{GL}_2$. We use G_1 for the G_1 in Corollary 1.3. In all four cases, let S_1 be the set of diagonal matrices of $\mathrm{SL}_3, \mathfrak{R}_{k_1/k}\mathrm{SL}_3$, $\mathrm{SL}_3(D)$, E_6 with entries in $k^\times$ respectively and S_2 the set of diagonal matrices in SL_2. Then $S=S_1\times S_2$ is a maximal split torus of G_1. Let $T_0=\mathrm{GL}_1\times\mathrm{GL}_1$, $\mathfrak{R}_{k_1/k}\,\mathrm{GL}_1\times\mathrm{GL}_1$, $\mathrm{GL}_1\times\mathrm{GL}_1$ and GL_1 respectively for the cases (a)–(d). In the cases (a), (d), two factors of GL_1 are subgroups of diagonal matrices with entries in $k^\times$. The case (b) is similar. In the case (d), E_6 is simple and GL_1 is the subgroup of diagonal matrices in GL_2. Then we are in the situation of Corollary 1.3.

Let $a_n(t_1,\ldots,t_n)$ be the diagonal matrix with diagonal entries $t_1,\ldots,t_n\in k^\times$. We write in the form $t=(t_1,t_2)\in S$ with

$$t_1=a_3(t_{11},t_{12},t_{13}),\ t_2=a_3(t_{21},t_{22}),\ t_{11}t_{12}t_{13}=t_{21}t_{22}=1$$

where $t_{11},t_{12},t_{13}\in k^\times$ and $t_{21},t_{22}\in k^\times$. We identify $\mathfrak{s}^*_{\mathbb{R}}$ with

$$\{z=(z_{11},z_{12},z_{13};z_{21},z_{22})\in\mathbb{R}^5 \mid z_{11}+z_{12}+z_{13}=0,\ z_{21}+z_{22}=0\}.$$

We use the notation $z_1=(z_{11},z_{12},z_{13}), z_2=(z_{21},z_{22}),\ z=(z_1,z_2)$. For

$$z=(z_1,z_2), z'=(z'_1,z'_2)\quad (z_1=(z_{11},z_{12},z_{13}), z_2=(z_{21},z_{22})\ \text{ and similarly for } z')$$

we define

$$(z,z')_{\mathfrak{s}^*_{\mathbb{R}}}=z_{11}z'_{11}+z_{12}z'_{12}+z_{13}z'_{13}+z_{21}z'_{21}+z_{22}z'_{22}.$$

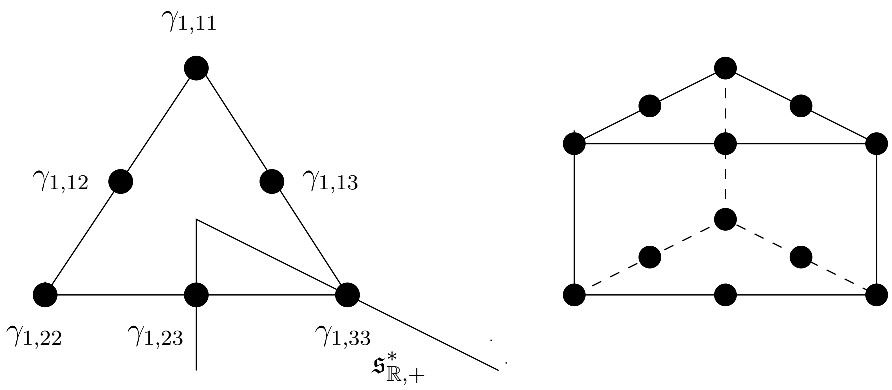
This inner product is Weyl group invariant. Let $\|\ \|_{\mathfrak{s}^*_{\mathbb{R}}}$ be the metric defined by this bilinear form. We choose

$$\mathfrak{s}^*_{\mathbb{R},+}=\{(z_{11},z_{12},z_{13};z_{21},z_{22})\mid z_{11}\leqq z_{12}\leqq z_{13}, z_{21}\leqq z_{22}\}$$

as the Weyl chamber.

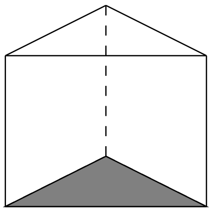
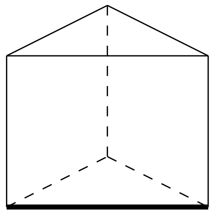
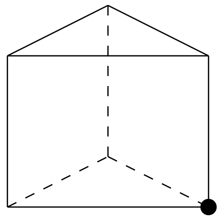
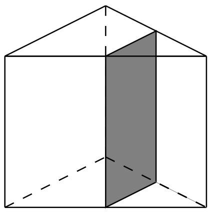
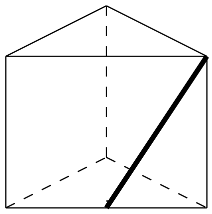
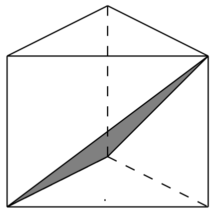
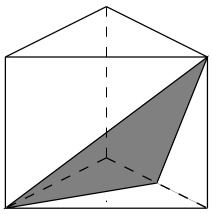
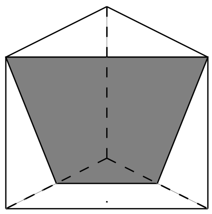
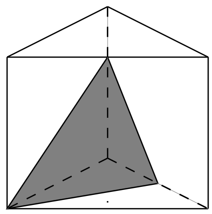
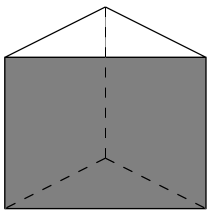
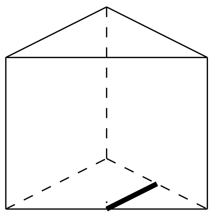
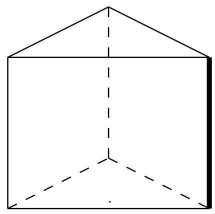
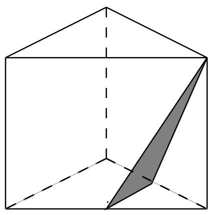
We define $\gamma_{i,jk}$ as follows.

$\gamma_{1,11}$	$(\frac43,-\frac23,-\frac23;\frac12,-\frac12)$	$\gamma_{2,11}$	$(\frac43,-\frac23,-\frac23;-\frac12,\frac12)$
$\gamma_{1,12}$	$(\frac13,\frac13,-\frac23;\frac12,-\frac12)$	$\gamma_{2,12}$	$(\frac13,\frac13,-\frac23;-\frac12,\frac12)$
$\gamma_{1,13}$	$(\frac13,-\frac23,\frac13;\frac12,-\frac12)$	$\gamma_{2,13}$	$(\frac13,-\frac23,\frac13;-\frac12,\frac12)$
$\gamma_{1,22}$	$(-\frac23,\frac43,-\frac23;\frac12,-\frac12)$	$\gamma_{2,22}$	$(-\frac23,\frac43,-\frac23;-\frac12,\frac12)$
$\gamma_{1,23}$	$(-\frac23,\frac13,\frac13;\frac12,-\frac12)$	$\gamma_{2,23}$	$(-\frac23,\frac13,\frac13;-\frac12,\frac12)$
$\gamma_{1,33}$	$(-\frac23,-\frac23,\frac43;\frac12,-\frac12)$	$\gamma_{2,33}$	$(-\frac23,-\frac23,\frac43;-\frac12,\frac12)$



Then with our metric, we can express $\gamma_{i,jk}$'s as above. These are the weights of coordinates of V. The picture on the right shows the weights of V and the weights of the upper half are shown in the picture on the left. In the case (a), $\gamma_{i,jk}$ corresponds to the monomial $v_i v_k$ of three variables v_1, v_2, v_3. The (z_{11}, z_{12}, z_{13}) part of the Weyl chamber $\mathfrak{s}^*_{\mathbb{R},+}$ is the lower right region as above. The (z_{21}, z_{22}) part of $\mathfrak{s}^*_{\mathbb{R},+}$ is the lower half of the vertical line. So $\mathfrak{s}^*_{\mathbb{R},+}$ consists of vectors which point down and right and coming toward the reader in the picture on the right.

The set $\mathfrak{B}$ corresponds to the following 13 convex hulls. The case (a) is discussed in [26, pp.198–205] and so we do not include the details here. $S_{\beta_{11}}$, $S_{\beta_{12}}$ and $S_{\beta_{13}}$ are the empty set in all cases and so the cases (a)–(d) all have 10 unstable strata.

strata	convex hull	strata	convex hull	strata	convex hull
S_{β_1}		S_{β_2}		S_{β_3}	
S_{β_4}		S_{β_5}		S_{β_6}	
S_{β_7}		S_{β_8}		S_{β_9}	
$S_{\beta_{10}}$		$S_{\beta_{11}}$		$S_{\beta_{12}}$	
$S_{\beta_{13}}$					

Explicitly, β_j's describe as follows.

$$
\begin{array}{ll}
\beta_1 = (0,0,0;\frac{1}{2},-\frac{1}{2}) & \beta_2 = (-\frac{2}{3},\frac{1}{3},\frac{1}{3};-\frac{1}{2},\frac{1}{2}) \\
\beta_3 = (-\frac{2}{3},-\frac{2}{4},\frac{4}{3};-\frac{1}{2},\frac{1}{2}) & \beta_4 = (-\frac{1}{6},-\frac{1}{6},\frac{1}{3};0,0) \\
\beta_5 = (-\frac{2}{3},\frac{1}{12},\frac{7}{12};-\frac{1}{4},\frac{1}{4}) & \beta_6 = (-\frac{1}{24},-\frac{1}{24},\frac{1}{12};-\frac{1}{8},\frac{1}{8}) \\
\beta_7 = (-\frac{1}{4},0,\frac{1}{4};-\frac{1}{4},\frac{1}{4}) & \beta_8 = (-\frac{2}{21},\frac{1}{21},\frac{1}{21};-\frac{1}{14},\frac{1}{14}) \\
\beta_9 = (-\frac{1}{5},0,\frac{1}{5};-\frac{1}{10},\frac{1}{10}) & \beta_{10} = (-\frac{2}{3},\frac{1}{3},\frac{1}{3};0,0) \\
\beta_{11} = (-\frac{1}{6},-\frac{1}{6},\frac{1}{3};-\frac{1}{2},\frac{1}{2}) & \beta_{12} = (-\frac{2}{3},-\frac{1}{3},\frac{4}{3};-\frac{1}{2},\frac{1}{2}) \\
\beta_{13} = (-\frac{1}{6},-\frac{1}{6},\frac{1}{3};\frac{1}{3},-\frac{1}{3}) &
\end{array}
$$

There are smaller and easier prehomogeneous vector spaces similar to the above examples which have the same set of weights. Consider the following prehomogeneous vector spaces.

(a') $G = \mathrm{GL}_2 \times \mathrm{GL}_2$, $V = \mathrm{Sym}^2\mathrm{Aff}^2 \otimes \mathrm{Aff}^2$.

(b') $G = \mathfrak{R}_{k_1/k}\mathrm{GL}_2 \times \mathrm{GL}_2$, $V = \mathrm{H}_2(k_1) \otimes \mathrm{Aff}^2$.

(c') $G = \mathrm{GL}_2(D) \times \mathrm{GL}_2$, $V = \mathrm{H}_2(D) \otimes \mathrm{Aff}^2$.

There does not seem to be an analogue of the case (d) above, because the case (d) is based on the "triality". The case (a') is essentially the same as the space of (single) binary quadratic forms by the castling transformation (see [21, p.39]). The global zeta function for the case (a') was needed to determine the pole structure of the global zeta function for the case (a) (see [26]). The global zeta function for the case (b') was considered in [29]. Note that in [29], the structure of $V_k \setminus V_k^{\mathrm{ss}}$ was considered explicitly and by Theorem 1.2, it can now be replaced by the convex hull consideration in [26, pp.153–154]. The density theorem related to the case (b') was proved in [10], [11]. The interpretation of rational orbits of the case (c') was considered in [23].

Let G_1 be $\mathrm{SL}_2 \times \mathrm{SL}_2$, $\mathfrak{R}_{k_1/k}\mathrm{SL}_2 \times \mathrm{SL}_2$ for the cases (a'), (b') respectively. $\mathrm{GL}_2(D)$ can be identified with a subgroup of GL_8. Let $d : \mathrm{GL}_8 \to \mathrm{GL}_1$ be the determinant and $G_1 = \mathrm{Ker}(d)^\circ \times \mathrm{GL}_2$. We use G_1 for the G_1 in Corollary 1.3. In all three cases, let S_1 be the set of diagonal matrices with entries in $k^\times$ of $\mathrm{GL}_2, \mathfrak{R}_{k_1/k}\mathrm{GL}_2, \mathrm{GL}_2(D)$ respectively and S_2 the set of diagonal matrices of SL_2. Then $S = S_1 \times S_2$ is a maximal split torus of G_1. Let $T_0 = \mathrm{GL}_1 \times \mathrm{GL}_1$, $\mathfrak{R}_{k_1/k}\,\mathrm{GL}_1 \times \mathrm{GL}_1$, $\mathrm{GL}_1 \times \mathrm{GL}_1$ for the cases (a')–(c') respectively where two factors are the subgroups of diagonal matrices. Then we are in the situation of Corollary 1.3.

We identify $\mathfrak{s}_{\mathbb{R}}^*$ with

$$\{z = (z_{11}, z_{12}; z_{21}, z_{22}) \in \mathbb{R}^4 \mid z_{11} + z_{12} = 0,\ z_{21} + z_{22} = 0\}.$$

We choose

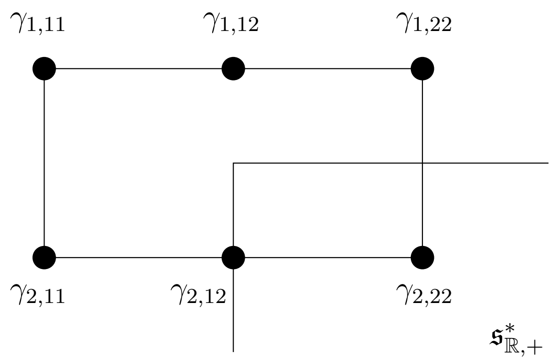
$$\mathfrak{s}^*_{\mathbb{R},+} = \{(z_{11}, z_{12}; z_{21}, z_{22}) \mid z_{11} \leqq z_{12}, z_{21} \leqq z_{22}\}$$

as the Weyl chamber. We use a similar inner product on $\mathfrak{s}_{\mathbb{R}}, \mathfrak{s}^*_{\mathbb{R}}$ as the cases (a)–(d).

We define $\gamma_{i,jk}$ as follows.

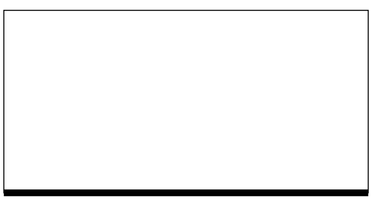
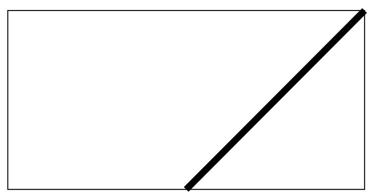
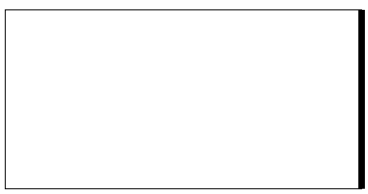
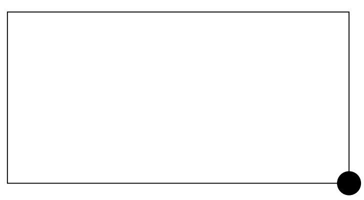
$\gamma_{1,11}$	$(1, -1; \frac{1}{2}, -\frac{1}{2})$	$\gamma_{2,11}$	$(1, -1; -\frac{1}{2}, \frac{1}{2})$
$\gamma_{1,12}$	$(0, 0; \frac{1}{2}, -\frac{1}{2})$	$\gamma_{2,12}$	$(0, 0; -\frac{1}{2}, \frac{1}{2})$
$\gamma_{1,22}$	$(-1, 1; \frac{1}{2}, -\frac{1}{2})$	$\gamma_{2,22}$	$(-1, 1; -\frac{1}{2}, \frac{1}{2})$

Then with our metric, we can express $\gamma_{i,jk}$'s as follows.



These are the weights of coordinates of V. The Weyl chamber $\mathfrak{s}^*_{\mathbb{R},+}$ is the lower right region as above.

The set $\mathfrak{B}$ corresponds to the following 4 convex hulls. The stratum S_{β_3} is the empty set in all cases and so the cases (a'), (b'), (c') all have 3 unstable strata.

strata	convex hull	strata	convex hull
S_{β_1}		S_{β_2}	
S_{β_3}		S_{β_4}	

Explicitly, β_j's describe as follows.

$$\beta_1 = (0,0;-\tfrac{1}{2},\tfrac{1}{2}) \qquad \beta_2 = (-\tfrac{1}{4},\tfrac{1}{4};-\tfrac{1}{4},\tfrac{1}{4})$$
$$\beta_3 = (-1,1;0,0) \qquad \beta_4 = (-1,1;-\tfrac{1}{2},\tfrac{1}{2})$$

We summarize for the GIT stratification in the case (a') (see [26, pp.152–155]). $W = \mathrm{Sym}^2\,\mathrm{Aff}^2$ identify with the space of quadratic forms with variable $v = (v_1, v_2)$.

We consider the space $V = \mathrm{Sym}^2\,\mathrm{Aff}^3 \otimes \mathrm{Aff}^2 = \mathrm{Sym}^2\,\mathrm{Aff}^3 \oplus \mathrm{Sym}^2\,\mathrm{Aff}^3$. An element $x \in V$ can be written as

$$x = (x_1, x_2) \in V, \quad x_i(v) = x_{i\,11}v_1^2 + x_{i\,12}v_1v_2 + x_{i\,22}v_2^2 \quad (i = 1, 2).$$

Then, we use the coordinate $x = (x_{i\,jk})$ on $V = \mathrm{Sym}^2\,\mathrm{Aff}^2 \otimes \mathrm{Aff}^2$ on which S acts diagonally. Then Z_β, W_β as follows.

- $Z_{\beta_1} = \{(x_{i\,jk}) \mid x_{1\,jk} = 0 \text{ for } j,k = 1,2\}, W_{\beta_1} = \{0\}$.
- $Z_{\beta_2} = \{(x_{i\,jk}) \mid x_{1\,11} = x_{1\,12} = x_{2\,11} = x_{2\,22} = 0\}, W_{\beta_2} = \{(x_{i\,jk}) \mid x_{i\,jk} = 0 \text{ for } (i,j,k) \neq (2,2,2)\}$.
- $Z_{\beta_3} = \{(x_{i\,jk}) \mid x_{i\,11} = x_{i\,12} = 0 \text{ for } i = 1,2\}, W_{\beta_3} = \{0\}$.
- $Z_{\beta_4} = \{(x_{i\,jk}) \mid x_{i\,jk} = 0 \text{ for } (i,j,k) \neq (2,2,2)\}, W_{\beta_4} = \{0\}$.

We put $M_{1\,\beta} = G_1 \cap M_\beta$.

β	λ_β	$M_{1\,\beta}$
$\beta_1 = (0,0;-\frac{1}{2},\frac{1}{2})$	$(a_2(t^{-1},t), I_2)$	$\{e\} \times \mathrm{SL}_2$
$\beta_2 = (-\frac{1}{4},\frac{1}{4};-\frac{1}{4},\frac{1}{4})$	$(a_2(t^{-1},t), a_2(t^{-1},t))$	$\{(a_2(t^{-1},t), a_2(t,t^{-1}))\}$
$\beta_3 = (-1,1;0,0)$	$(I_2, a_2(t^{-1},t))$	$\mathrm{SL}_2 \times \{e\}$
$\beta_4 = (-1,1;-\frac{1}{2},\frac{1}{2})$	$(a_2(t^{-2},t^2), a_2(t^{-1},t))$	$\{(a_2(t^{-1},t), a_2(t^2,t^{-2}))\}$

By the above table, $Z_{\beta_2}^{\mathrm{ss}} = \{(x_{i\,jk}) \in Z_{\beta_2} \mid x_{1\,22}, x_{2\,12} \neq 0\}$. We identify Z_{β_1} with W. Then $Z_{\beta_1}^{\mathrm{ss}} = \{(x_{i\,jk}) \in Z_{\beta_1} \mid x_{2\,12}^2 - 4x_{2\,11}x_{2\,22} \neq 0\}$. Since $M_{1\,\beta_4}$ acts trivially on Z_{β_4}, $Z_{\beta_4}^{\mathrm{ss}} = Z_{\beta_4} \setminus \{0\}$. The vector space Z_{β_3} is a standard representation of $M_{1\,\beta_3}$. Therefore, $Z_{\beta_3}^{\mathrm{ss}} = \emptyset$.

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