Asymptotic behavior and critical regularity of solutions to systems of nonlinear Schrodinger equations with mass resonance

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博　士　論　文

Asymptotic behavior and critical regularity of solutions to systems of nonlinear Schrödinger equations with mass resonance

(質量共鳴をもつ非線形Schrödinger方程式系の解の挙動と正則性)

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Contents

Summary

1.1 System of nonlinear Schrödinger equations

The Schrödinger equation is classified as a dispersive equation by its linear principal part and with adding nonlinear interaction term it is called nonlinear Schrödinger equation. Nonlinear Schrödinger equations arise in various fields of applications such as nonlinear optics, quantum mechanics, and a simplified model in fluid mechanics and they have been developed largely by means of mathematical analysis. In this thesis, we consider systems of nonlinear Schrödinger equations and study the asymptotic behavior of solutions and ill-posedness issue taking the relation between the nonlinear interactions and the effect of coefficients of the linear part into account. Let $u_k(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ $(k = 1, 2, 3)$ be the unknown functions. We consider the system with a particular nonlinear interaction

$$\text{(1.1.1)} \qquad \begin{cases} i\partial_t u_1 + \dfrac{1}{2m_1}\Delta u_1 = \gamma \overline{u}_1 u_2, & t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ i\partial_t u_2 + \dfrac{1}{2m_2}\Delta u_2 = u_1^2, & t \in \mathbb{R},\ x \in \mathbb{R}^n, \end{cases}$$

and as a generalization the three-components system

$$\text{(1.1.2)} \qquad \begin{cases} i\partial_t u_1 + \dfrac{1}{2m_1}\Delta u_1 = \overline{u}_2 u_3, & t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ i\partial_t u_2 + \dfrac{1}{2m_2}\Delta u_2 = \overline{u}_1 u_3, & t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ i\partial_t u_3 + \dfrac{1}{2m_3}\Delta u_3 = u_1 u_2, & t \in \mathbb{R},\ x \in \mathbb{R}^n, \end{cases}$$

where $m_k > 0$ $(k = 1, 2, 3)$ are constants and $\gamma = \pm 1$.

1.2 Mass resonance of a scattering problem for a two-components system

The asymptotic behavior of the solutions to the nonlinear Schrödinger equations is described by the scattering theory, namely, we consider whether nonlinear interaction is negligible or not as time tends to infinity. In particular, we consider the final state problem. Let $e^{\frac{it}{2m}\Delta}$ denote the free Schrödinger evolution group. We seek the solution of the nonlinear Schrödinger equation

$$\text{(1.2.1)} \qquad i\partial_t u + \frac{1}{2m}\Delta u = f(u), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,$$

satisfying the final state condition

$$\lim_{t\to\infty} \left\| u(t) - e^{\frac{it}{2m}\Delta} u_+ \right\| = 0$$

for a given function u_+ belonging to L^2. In the case of the gauge invariant nonlinearity $f(u) = |u|^{p-1}u$, Tsutsumi-Yajima [28] proved that the solution for (1.2.1) converges to a solution of the free Schrödinger equation as $t \to \pm\infty$ if $1 + 2/n < p < 1 + 4/n$. When $1 < p \leq 1 + 2/n$, Barab [1] showed that the solution u for (1.2.1) does not converge to a solution of the free Schrödinger equation as $t \to \pm\infty$. Hence, the exponent $p = 1+2/n$ is the threshold between the existence and the nonexistence of the scattering state. When the exponent is the critical case $p = 1 + 2/n$, Ozawa [24] showed that there exists a modified scattering solution of (1.2.1) which satisfies

$$\left\| u(t) - e^{\frac{it}{2m}\Delta}\mathcal{F}^{-1}e^{-i|\widehat{u}_+|^{\frac{2}{n}}\log t}\mathcal{F}u_+ \right\|_{L^2} \to 0$$

as $t \to \infty$ (cf. Ginibre-Ozawa [7] for higher dimensional cases). We here denote by $\widehat{f}$ or $\mathcal{F}$ the Fourier transform of f. Hence, when we consider the equation (1.2.1), the quadratic nonlinearity $\lambda|u|u$ is critical in two dimensions. For the nonlinear Schrödinger equation with non-gauge invariant nonlinearities such as $u^2, \overline{u}^2, |u|^2$, the asymptotic behavior of the solution is classified in the case of the final state problem (cf. [21, 25]).

Hereafter, we consider the final state problem for (1.1.1) imposing a final state condition:

Definition 1.2.1. *The final state problem for* (1.1.1) *is the system*

$$\begin{cases} i\partial_t u_1 + \dfrac{1}{2m_1}\Delta u_1 = \gamma\overline{u}_1 u_2, & t \in \mathbb{R},\ x \in \mathbb{R}^n, \\ i\partial_t u_2 + \dfrac{1}{2m_2}\Delta u_2 = u_1^2, & t \in \mathbb{R},\ x \in \mathbb{R}^n \end{cases}$$

together with a specified behavior of the unknown function (u_1, u_2)*, called the final state condition*

$$\|u_k(t) - u_{ka}(t)\|_{L^2} \to 0$$

as $t \to \infty$*, where* $u_{ka}(t)$ *is determined by a time-independent function* u_{k+} *belonging to* $L^2(\mathbb{R}^n)$ *and* $k = 1, 2$*.*

As a typical case, we take $u_{ka}(t) = e^{\frac{it}{2m_k}\Delta}u_{k+}$ for $u_{k+} \in L^2(\mathbb{R}^n)$. Compared with the single nonlinear Schrödinger equations, the asymptotic behavior of solutions to the system (1.1.1) is not yet clear. For higher dimensional cases $n \geq 3$, Hayashi-Li-Ozawa [10] showed the small data scattering for the system (1.1.1). Hayashi-Li-Naumkin [8] showed that the small global solution of (1.1.1) has the same decay rate as the free Schrödinger equation under the mass resonance condition $2m_1 = m_2$ when $n = 2$, while the whole view of the asymptotic behavior of solutions to (1.1.1) mostly remains open. In the case of the final state problem for $n = 2$, Hayashi-Li-Naumkin [9] showed that the three cases occur due to relations between m_1 and m_2 when $\gamma = 1$. If $2m_1 \neq m_2$ and $m_1 \neq m_2$, they showed that there exists a solution of (1.1.1) which converges to a free solution as $t \to \infty$. If $m_1 = m_2$, they proved that there exists a solution which converges to a free solution as $t \to \infty$.

However, in most cases, the asymptotic behavior of solutions is not a free solution. For the case $2m_1 = m_2$, under the condition $|\widehat{u}_{1+}(\xi)| = |\widehat{u}_{2+}(\xi)|$ and $2\arg \widehat{u}_{1+}(\xi) = \arg \widehat{u}_{2+}(\xi)$ for a.e. $\xi \in \mathbb{R}^2$, they showed that there exists a solution (1.1.1) satisfying

$$(1.2.2) \qquad \begin{aligned} &\left\| u_1(t) - e^{\frac{it}{2m_1}\Delta}\mathcal{F}^{-1}D^*(m_1^{-1})e^{\frac{i}{\sqrt{2}}|\widehat{u}_{1+}|\log t}\widehat{u}_{1+}\right\|_{L^2} \to 0, \\ &\left\| u_2(t) - e^{\frac{it}{2m_2}\Delta}\mathcal{F}^{-1}D^*(m_2^{-1})e^{i\sqrt{2}|\widehat{u}_{2+}|\log t}\widehat{u}_{2+}\right\|_{L^2} \to 0 \end{aligned}$$

as $t \to \infty$, where $D^*(m)f(x) = -(m/i)f(mx)$ for $m > 0$. Namely, there exists a modified free solution of (1.1.1) under the mass resonance condition $2m_1 = m_2$. They also showed that the dissipative structure appears under the mass resonance condition $2m_1 = m_2$ when $\gamma = -1$, namely, L^2 norm of both components of a solution of (1.1.1) converges to 0 as $t \to \infty$.

We show the existence of a solution to (1.1.1) whose asymptotic behavior is different from (1.2.2) under the mass resonance condition $2m_1 = m_2$ when $\gamma = 1$. When the parameter $\gamma = 1$, the L^2 conservation law becomes

$$(1.2.3) \qquad Q(u_1, u_2) = \|u_1(t)\|_{L^2}^2 + \|u_2(t)\|_{L^2}^2,$$

that is, the L^2 conservation law is given by the sum of each component. Therefore, L^2 norm of each component is not necessarily conserved. Indeed, it possibly occurs that the L^2 norm of the each component may be interact with each other. As a result, one may expect that the each component of the solution has different L^2 norm along the time trajectory. We call this the L^2 *transition phenomenon* for the system (1.1.1). We see from the proof of [9] that the asymptotic behavior of solutions of (1.1.1) is determined by the system of ordinary differential equations

$$(1.2.4) \qquad \begin{cases} i\partial_\tau \phi_1 = \overline{\phi}_1\phi_2, \\ i\partial_\tau \phi_2 = \phi_1^2. \end{cases}$$

We here use the particular solution of (1.2.4)

$$\phi_1(\tau) = \operatorname{sech}(\tau), \quad \phi_2(\tau) = \frac{1}{i}\tanh(\tau)$$

to describe the L^2 transition phenomenon. To state our result, we introduce several notations. For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^n)$ the inhomogeneous Sobolev spaces

$$H^s = H^s(\mathbb{R}^n) := \{f \in \mathcal{S}' \ ; \ \|f\|_{H^s} := \|(1+|\xi|^2)^{s/2}\widehat{f}\|_{L^2} < \infty\}$$

and by $\dot{H}^s(\mathbb{R}^n)$ the homogeneous Sobolev spaces

$$\dot{H}^s = \dot{H}^s(\mathbb{R}^n) := \{f \in \mathcal{S}'/\mathcal{P} \ ; \ \|f\|_{\dot{H}^s} := \||\xi|^s\widehat{f}\|_{L^2} < \infty\}.$$

We state the asymptotic behavior of a solution to the two-components system (1.1.1).

Theorem 1.2.2. *Let $2m_1 = m_2$, $\gamma = 1$, $1 < s \le 2$ and $1/2 < \alpha < s/2$. For some small $\eta > 0$, let $(u_{1+}, u_{2+}) \in (H^{0,s}(\mathbb{R}^2))^2$ with $\||\widehat{u}_{1+}|\|_{H^s} + \||\widehat{u}_{2+}|\|_{H^s} < \eta$. Assume that*

$$\begin{aligned} |\widehat{u}_{1+}(\xi)| &= |\widehat{u}_{2+}(\xi)|, \\ 2\arg \widehat{u}_{1+}(\xi) &= \arg \widehat{u}_{2+}(\xi), \end{aligned} \qquad \text{for a.e. } \xi \in \mathbb{R}^2,$$

and $\arg \widehat{u}_{1+}$, $\arg \widehat{u}_{2+} \in \dot{H}^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. *Then there exists $T > 0$ and (1.1.1) has a unique solution $(u_1, u_2) \in (C([T,\infty); L^2) \cap L^q(T,\infty; L^p))^2$ satisfying*

$$\left\| u_1(t) - e^{\frac{it}{2m_1}\Delta} \mathcal{F}^{-1} D^* \left(m_1^{-1}\right) \operatorname{sech}(|\widehat{u}_{1+}| \log t) \mathcal{F} u_{1+} \right\|_{L^2} = O(t^{-\alpha}) \quad \text{as} \quad t \to \infty,$$

$$\left\| u_2(t) - e^{\frac{it}{2m_2}\Delta} \mathcal{F}^{-1} D^* \left(m_2^{-1}\right) \frac{1}{i} \tanh(|\widehat{u}_{2+}| \log t) \mathcal{F} u_{2+} \right\|_{L^2} = O(t^{-\alpha}) \quad \text{as} \quad t \to \infty,$$

where $2/q = 1 - 2/p$, $2 < q < \infty$.

Finally, we mention to a related result. Katayama-Matoba-Sunagawa [16] studied the following system of nonlinear wave equations

$$\text{(1.2.5)} \qquad \begin{cases} \partial_t^2 w_1 - \Delta w_1 = -w_1 w_2, & t \in \mathbb{R},\ x \in \mathbb{R}^3, \\ \partial_t^2 w_2 - \Delta w_2 = w_1^2, & t \in \mathbb{R},\ x \in \mathbb{R}^3. \end{cases}$$

They showed the global existence of the solution to (1.2.5). Moreover, they proved that the energy of the first component w_1 converges to 0 and the second component w_2 obtain the total energy as time tends to infinity by using the secant and tangent hyperbolic functions.

1.3 Mass resonance of a scattering problem for a three-components system

We consider the three-components system (1.1.2), which is a generalization of the two-components system (1.1.1). In the previous section, we discussed the asymptotic behavior of solutions to the two-components system (1.1.1). Combining the L^2 conservation law of the system (1.1.1)

$$\|u_1(t)\|_{L^2}^2 + \|u_2(t)\|_{L^2}^2 = \text{constant},$$

we constructed a solution of (1.1.1) whose second component u_2 obtains the total charge of the system (1.1.1) as $t \to \infty$. Although we expected that (1.1.1) admits a solution (u_1, u_2) which have transition of the L^2 norm between the first and the second component of the solution periodically in time, it was not clear that such a solution to the system (1.1.1) actually exists.

For the three-components system (1.1.2), the corresponding mass resonance condition is given by $m_1 + m_2 = m_3$. The two-components system (1.1.1) can be considered as a degenerate system of the three-components system (1.1.2). Indeed, regarding $u_1 = u_2$ and $m_1 = m_2$, we see that (1.1.2) corresponds to (1.1.1). Hence, we may expect that the asymptotic behavior of the solution in Theorem 1.2.2 is a degenerate situation from the three-components system (1.1.2). Along their idea, we construct a solution to (1.1.2) which has the charge transition among the three-components of the solution due to the L^2 conservation laws

$$Q_1(u_1, u_2, u_3) := \|u_1(t)\|_{L^2}^2 + \|u_3(t)\|_{L^2}^2, \quad Q_2(u_1, u_2, u_3) := \|u_2(t)\|_{L^2}^2 + \|u_3(t)\|_{L^2}^2.$$

As we saw for the two-components system (1.1.1), the system of ordinary differential equations (1.2.4) plays an important role. We see that the asymptotic behavior of solutions to (1.1.2) can be approximated by the system of ordinary differential equations

$$\begin{cases} \partial_\tau \phi_1 = -\alpha^2 \overline{\phi}_2 \phi_3, \\ \partial_\tau \phi_2 = -\overline{\phi}_1 \phi_3, \\ \partial_\tau \phi_3 = \phi_1 \phi_2. \end{cases} \tag{1.3.1}$$

We use a particular solution of (1.3.1), namely, the Jacobi elliptic functions. To state our main result, we recall the definition of the Besov spaces.

Definition. Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity. Namely, let $\widehat{\phi} \in \mathcal{S}(\mathbb{R}^n)$ be a radial non-negative function satisfying $\operatorname{supp} \widehat{\phi} \subset \{\xi \in \mathbb{R}^n \ ; \ 2^{-1} \leq |\xi| \leq 2\}$ and

$$\sum_{j\in\mathbb{Z}} \widehat{\phi}_j(\xi) \equiv 1$$

for all $\xi \neq 0$, where $\widehat{\phi}_j(\xi) = \widehat{\phi}(\xi/2^j)$ for $j \in \mathbb{Z}$. We set $\widehat{\Phi}(\xi) = 1 - \sum_{j\geq 0} \widehat{\phi}_j(\xi)$. Then for $s \in \mathbb{R}$, $1 \leq p, \sigma \leq \infty$, the inhomogeneous Besov spaces $B^s_{p,\sigma}$ is defined by

$$B^s_{p,\sigma}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'; \|f\|_{B^s_{p,\sigma}} \equiv \left(\|\Phi * f\|^\sigma_{L^p} + \sum_{j\geq 0} 2^{js\sigma} \|\phi_j * f\|^\sigma_{L^p} \right)^{1/\sigma} < \infty \right\}.$$

We state the asymptotic behavior of a solution to the three-components system (1.1.2).

Theorem 1.3.1. *Let $n = 2$, $m_1 + m_2 = m_3$, $s > 1$ and $0 < \alpha \leq 1$. Assume that ω, $\theta \in \dot{B}^1_{2,1} \cap \dot{B}^s_{2,1}$ and $\|\omega\|_{\dot{B}^1_{2,1}} < \eta_0$, where η_0 is sufficiently small. Then there exists $T > 0$ such that the system* (1.1.2) *admits a unique solution (u_1, u_2, u_3) satisfying*

$$u_k \in C([T, \infty); L^2), \quad \sum_{j\in\mathbb{Z}} 2^j \|e^{\frac{it}{2m_k}\Delta} \widehat{\phi}_j e^{-\frac{it}{2m_k}\Delta} u_k(t)\|_{L^2} \in C([T, \infty))$$

for $k = 1, 2, 3$ and the following asymptotic behavior

$$\begin{aligned} &\left\| u_1(t) - e^{\frac{it}{2m_1}\Delta} \mathcal{F}^{-1} D^*(m_1^{-1}) \operatorname{dn}(|\widehat{u}_{1+}| \log t, \alpha) \widehat{u}_{1+} \right\|_{L^2} \to 0, \\ &\left\| u_2(t) - e^{\frac{it}{2m_2}\Delta} \mathcal{F}^{-1} D^*(m_2^{-1}) \operatorname{cn}(\alpha^{-1} |\widehat{u}_{2+}| \log t, \alpha) \widehat{u}_{2+} \right\|_{L^2} \to 0, \\ &\left\| u_3(t) - e^{\frac{it}{2m_3}\Delta} \mathcal{F}^{-1} D^*(m_3^{-1}) (i^{-1}) \operatorname{sn}(\alpha^{-1} |\widehat{u}_{3+}| \log t, \alpha) \widehat{u}_{3+} \right\|_{L^2} \to 0, \end{aligned} \tag{1.3.2}$$

as $t \to \infty$, where $\widehat{u}_{1+}(\xi) = \omega(\xi) e^{i\theta(\xi)}$, $\widehat{u}_{2+}(\xi) = \alpha\omega(\xi) e^{i\theta(\xi)}$, and $\widehat{u}_{3+}(\xi) = \alpha\omega(\xi) e^{2i\theta(\xi)}$.

Since the Jacobi elliptic function $\mathrm{sn}(\tau, \alpha)$ has the period $4K(\alpha)$ which is given by the elliptic integral of first kind, Theorem 1.3.1 shows that we are able to construct a solution which has the mass transition phenomenon with the period $4K(\alpha)$ given by a parameter $\alpha \in [0, 1)$. In the study of the two-components system, the hyperbolic secant function $\mathrm{sech}(t)$ and the hyperbolic tangent function $\tanh(t)$ are used to describe the asymptotic behavior of the solution. These hyperbolic functions are special cases of the Jacobi elliptic functions $\mathrm{sn}(t, \alpha), \mathrm{cn}(t, \alpha), \mathrm{dn}(t, \alpha)$, since $\mathrm{cn}(t, 1) = \mathrm{dn}(t, 1) = \mathrm{sech}(t)$ and $\mathrm{sn}(t, 1) = \tanh(t)$. Indeed, the asymptotic profile (1.3.2) for a solution to (1.1.2) corresponds to those of (1.1.1) in Theorem 1.2.2 by putting $\alpha = 1$ and $m_1 = m_2$. Hence, we see that the three-components system (1.1.2) actually describes a generalized situation from the two-components system (1.1.1). This gives a justification to the formal observation in [12] from the view point of the asymptotic behavior of solutions of (1.1.2).

1.4 Mass resonance and ill-posedness issue for a two-components system

We investigate how the mass resonance influences the critical regularity of solutions of a system of nonlinear Schrödinger equations. We consider the ill-posedness issue for the initial value problem of the system

$$\begin{cases} i\partial_t u_1 + \dfrac{1}{2m_1}\Delta u_1 = \overline{u}_1 u_2, & t \in \mathbb{R}, \quad x \in \mathbb{R}^2, \\ i\partial_t u_2 + \dfrac{1}{2m_2}\Delta u_2 = u_1^2, & t \in \mathbb{R}, \quad x \in \mathbb{R}^2, \\ u_1(0, x) = \psi_1(x),\ u_2(0, x) = \psi_2(x), & x \in \mathbb{R}^2. \end{cases} \tag{1.4.1}$$

The notion of the well-posedness consists of the existence, uniqueness, and continuous dependence on the initial data of the solution. We particularly focus on the failure of the continuous dependence on the initial data to show the ill-posedness result.

In Section 1.2, we studied the asymptotic behavior of solutions to the two-components system (1.1.1). It is known that the initial value problem and the final state problem of the single nonlinear Schrödinger equation correspond to each other by means of the pseudo conformal transformation. By introducing the pseudo conformal transformation for (1.4.1)

$$\begin{cases} u_{1p}(t, x) = e^{\frac{im_1|x|^2}{2t}} t^{-\frac{n}{2}} \overline{u}_1\left(\dfrac{1}{t}, \dfrac{x}{t}\right), \\ u_{2p}(t, x) = e^{\frac{im_2|x|^2}{2t}} t^{-\frac{n}{2}} \overline{u}_2\left(\dfrac{1}{t}, \dfrac{x}{t}\right), \end{cases}$$

we see that the system (1.4.1) is transformed into

$$\begin{cases} i\partial_t u_{1p} + \dfrac{1}{2m_1}\Delta u_{1p} = t^{n/2-2} e^{\frac{i(2m_1-m_2)|x|^2}{2t}} \overline{u}_{1p} u_{2p}, & t \in \mathbb{R}\setminus\{0\},\ x \in \mathbb{R}^n, \\ i\partial_t u_{2p} + \dfrac{1}{2m_2}\Delta u_{2p} = t^{n/2-2} e^{\frac{i(-2m_1+m_2)|x|^2}{2t}} u_{1p}^2, & t \in \mathbb{R}\setminus\{0\},\ x \in \mathbb{R}^n. \end{cases} \tag{1.4.2}$$

Under the mass resonance condition $2m_1 = m_2$, the system (1.4.1) is invariant when $n = 4$. The system (1.4.1) is not invariant in other cases. However, we may treat the difference as a small error. Hence, we expect that the mass resonance phenomenon also appears in the study of the initial value problem, that is, the ill-posedness issue for (1.4.1). Due to the scaling argument, we see that the Sobolev space $H^{-1}(\mathbb{R}^2)$ is the critical space for quadratic nonlinear Schrödinger equations. The local well-posedness and ill-posedness for single nonlinear Schrödinger equations are extensively studied (cf. [2, 3, 4, 6, 13, 17, 19, 20, 26, 27]). In Section 1.2, we showed that the parameter m_1 and m_2 influence the asymptotic behavior for solutions to (1.1.1). More precisely, the situation is different in the following three cases: (i) $2m_1 \neq m_2$ and $m_1 \neq m_2$, (ii) $m_1 = m_2$, and (iii) $2m_1 = m_2$. We show the actual threshold of the ill-posedness depends on the relation between m_1 and m_2 as the case of the asymptotic behavior of solutions.

We state the ill-posedness result for the two-components system (1.1.1).

Theorem 1.4.1. *Let $n = 2$. We show the following ill-posedness results depending on the relations of m_1 and m_2:*

(1) *Let $2m_1 \neq m_2$ and $m_1 \neq m_2$. For any fixed $s \leq -1$, there exist a sequence of time $\{T_N\}_{N\in\mathbb{N}}$ with $T_N \to 0$ $(N \to \infty)$ and a sequence of the initial data $\{\psi_N\}_{N\in\mathbb{N}} \subset L^2(\mathbb{R}^2)$ with $\|\psi_N\|_{H^s} = 0$ $(N \to \infty)$ such that the corresponding sequence of the solution $\{u_{1N}\}_{N\in\mathbb{N}}$, $\{u_{2N}\}_{N\in\mathbb{N}}$ to (1.4.1) with $u_{1N}(0,x) = \psi_N(x)$ and $u_{2N}(0,x) = \psi_N(x)$ satisfies*

$$\lim_{N\to\infty} \|u_{1N}(T_N)\|_{H^s} = \infty, \quad \lim_{N\to\infty} \|u_{2N}(T_N)\|_{H^s} = \infty.$$

(2) *Let $m_1 = m_2$, $\sigma > 4$. There exist a sequence of time $\{T_N\}_{N\in\mathbb{N}}$ with $T_N \to 0$ $(N \to \infty)$ and a sequence of the initial data $\{\psi_N\}_{N\in\mathbb{N}} \subset L^2(\mathbb{R}^2)$ with $\|\psi_N\|_{B^{-1/4}_{2,\sigma}} \to 0$ $(N \to \infty)$ such that the corresponding sequence of the solution $\{u_{1N}\}_{N\in\mathbb{N}}$, $\{u_{2N}\}_{N\in\mathbb{N}}$ to (1.4.1) with $u_{1N}(0,x) = \psi_N(x)$ and $u_{2N}(0,x) = \psi_N(x)$ satisfies*

$$\lim_{N\to\infty} \|u_{1N}(T_N)\|_{B^{-1/4}_{2,\sigma}} = \infty.$$

(3) *Let $2m_1 = m_2$. If $s \in (-1/2, 0)$, then the data-solution map $(\psi_1, \psi_2) \to (u_1(t), u_2(t))$ is not uniformly continuous, where $(u_1(t), u_2(t))$ is the solution of (1.4.1) with the initial data (ψ_1, ψ_2).*

In order to prove the first case and second case in Theorem 1.4.1, we modify the construction of the initial data in [13] and [15]. On the other hand, (1.4.1) has the Galilei invariance under the mass resonance condition $2m_1 = m_2$: If $(u_1(t,x), u_2(t,x))$ solves (1.4.1), then so does

$$\begin{cases} u_{1g}(t,x) = u_1(t, x - t\eta_1) e^{i(x\cdot\eta_1 - |\eta_1|^2 \frac{t}{2m_1})}, \\ u_{2g}(t,x) = u_2(t, x - t\eta_2) e^{i(x\cdot\eta_2 - |\eta_2|^2 \frac{t}{2m_2})}, \end{cases}$$

where $\eta_1, \eta_2 \in \mathbb{R}^n$ denote the moment parameter satisfying $2\eta_1 = \eta_2$. Hence, we are able to adapt the method due to Kenig-Ponce-Vega [18] to show the third case in Theorem 1.4.1.

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