New construction of internal space in supergravity theory based on generalized geometry

著者	MURAKI　HISAYOSHI
学位授与機関	Tohoku University
学位授与番号	11301甲第16164号
URL	http://hdl.handle.net/10097/60442

博士論文

New construction of internal space in supergravity theory based on generalized geometry

(一般化幾何学に基づく超重力理論の内部空間の新しい構成)

村木久祥

平成２６年

Contents

Chapter 1

Introduction

String theory has attracted much attention for the last several decades as a promising candidate for the unified theory describing all fundamental interactions.
All interactions except the gravity are formulated as the gauge theory with gauge group $SU(3) \times SU(2) \times U(1)$. In addition to these gauge fields, some fermionic fields of elementary particles called quarks and leptons and a scalar field of Higgs boson altogether form the Standard Model of particle physics. The Standard Model provides a consensual understanding of particle physics at this moment, together with the description of microscopic physics based on quantum theory of fields. However, the rest of the fundamental interactions, *gravity*, is missing from the Standard Model, despite the fact that we experience it in everyday life.
The theory of gravity was proposed by Einstein *99* years ago [1]. It is described by the general theory of relativity which is constructed from Einstein's brilliant insight into the covariance of the physical laws under the general coordinate transformations (the principle of general relativity). To realize his insight, the Riemannian geometry plays a crucial role. The general theory of relativity leads us to recognize that the space-time is no longer the static stage where physical phenomena occur, but itself is a dynamical object described by the space-time metric.
The general theory of relativity has great beauty in its theoretical formalism and still no inconsistent experimental observation with it has been reported. Hence, an application of the field quantization procedure to the metric, that is a field describing the dynamics of the space-time, would be naively expected to provide the quantum theory of gravity. However, this program is known to fall down due to the problem of non-renormalizability caused by the dimensionful Newton constant G_N.
If the quanta mediating the gravitational interaction are described by closed strings rather than point particles, the problem of non-renormalizability can be avoided. This is because a closed string can not shrink to a single point topologically, and hence weakens the ultraviolet divergence stemming from interaction points approaching each other. Indeed, it was recognized that there are oscillation modes which can be interpreted to be gravitons in the closed string spectrum [2, 3].
Furthermore, not only the quanta of gravity but also gauge interactions were dramatically discovered to exist in string theory consistently [4–7]. Ever since this *first superstring revolution* it has been widely expected that string theory will provide the unified theory of all fundamental interactions, especially including quantum gravity.

Since the Riemannian geometry provides us with an intuition about the classical gravity, some kind of geometric notion would be helpful in order to deepen an understanding of the theory of quantum gravity. As above, string theory is expected to provide a description of quantum gravity, and hence the corresponding notion of geometry on the space-time would be "stringy" geometry which could be different from the Riemannian geometry, because the fundamental objects which probe the space-time are strings rather than point particles.

Among various observations concerning stringy geometry, T-duality is one of the most significant features which are distinctive of string theory and one of the main subjects in this dissertation. T-duality concerns the backgrounds in which strings propagate. The backgrounds are specified by a configuration of the space-time metric and the Kalb-Ramond field (B-field) which is a 2nd-rank antisymmetric tensor gauge field.

A typical example of T-duality is given by the case where one of the spacial directions of the space-time is compactified on a circle of radius R [1]. Since the coordinate along the compactified direction has periodicity $2\pi R$, a closed string allows the periodic boundary condition

$$X(\sigma,\tau) = X(\sigma + 2\pi, \tau) + 2\pi R W, \quad W \in \mathbb{Z}\,, \tag{1.1}$$

where X is an embedding function from the point on the string's world-sheet parametrized by (τ, σ) to the coordinate value of the compactified direction, and W denotes a winding number which counts how many times the closed string coils around the circle. Since strings have tension α' called the Regge slope parameter, a closed string with non-zero winding number carries energy. On the other hand, due to the periodicity, the string momentum associated with the compactified direction is discretized as

$$P = \frac{K}{R}, \quad K \in \mathbb{Z}\,, \tag{1.2}$$

just like quantum mechanics of point particle. The integer K is called the Kaluza-Klein number. As the string's winding mode and Kaluza-Klein mode carry energy, both of them contribute to the mass formula for the string as

$$M^2 = \left(\frac{K}{R}\right)^2 + \left(\frac{WR}{\alpha'}\right)^2 + (\text{contribution from other direction}). \tag{1.3}$$

From this analysis one immediately recognizes that the mass spectrum is invariant under the interchange

$$R \quad \longleftrightarrow \quad \widetilde{R} = \frac{\alpha'}{R}. \tag{1.4}$$

This example suggests that the circles of radii R and $\widetilde{R} = \alpha'/R$ are physically equivalent for a string, and actually so it is [8]. This equivalence between the radii R and $\widetilde{R} = \alpha'/R$ is never observed by a point particle, because the appearance of the winding number is a peculiarity of a string.

---

[1] The argument here is a rough sketch and we omit a careful introduction of some terminology and we do not make a precise statement.

The invariance under an interchange of the compactfication radii is also explained from the world-sheet point of view. The action for a string compactified on a circle with radius R is given by

$$S = \int d\tau d\sigma \frac{R^2}{\alpha'} \partial X \bar{\partial} X, \tag{1.5}$$

where $\partial = \partial_\tau - \partial_\sigma$ and $\bar{\partial} = \partial_\tau + \partial_\sigma$. This action has another expression written as

$$S'' = -\int d\tau d\sigma \left(\frac{R^2}{\alpha'} L\bar{L} - \partial Y \bar{L} - \bar{\partial} Y L \right), \tag{1.6}$$

where L and $\bar{L}$ are the Lagrange multipliers. Taking a variation with respect to Y yields the equation of motion $0 = \partial \bar{L} + \bar{\partial} L$ which is solved by

$$L = \partial X, \quad \bar{L} = -\bar{\partial} X, \tag{1.7}$$

for any function X. Substituting this solutions into (1.6) reproduces the original action (1.5). On the other hand, the auxiliary fields L and $\bar{L}$ yield constraints

$$L = \frac{\alpha'}{R^2} \partial X, \quad \bar{L} = \frac{\alpha'}{R^2} \bar{\partial} X, \tag{1.8}$$

leading another action

$$S' = \int d\tau d\sigma \frac{\widetilde{R}^2}{\alpha'} \partial X \bar{\partial} X \,, \quad \text{with} \quad \widetilde{R} = \frac{\alpha'}{R} \,. \tag{1.9}$$

This action S' describes a string propagating on a circle of radius $\widetilde{R} = \alpha'/R$. Since both S (1.5) and S' (1.9) are derived from the same action S'' (1.6), the physical equivalence for a string between the radii R and $\widetilde{R} = \alpha'/R$ would be understood.

The extensions of the argument above based on the world-sheet analysis to the cases where a string propagates in more general metric and B-field are studied by Buscher [9,10]. The relations between backgrounds which are physically equivalent for a string are summarized as the Buscher rule. That is, the Buscher rule gives the backgrounds in the dual description in terms of the original metric and B-field. The caveat is that the Buscher rule is applicable only when there is a direction of isometry which requires the invariance of both the metric and B-field under translations along its direction.

Although T-duality provides us with amazingly rich subjects of string theory, in this dissertation we identify T-duality transformation with the Buscher rule. This is because we are focusing only on the metric and B-field, and hence it is sufficient for our discussions.

A brief summary so far: T-duality is a special feature of considering a string as a probe exploring the space-time. It provides physically equivalent backgrounds for a string. The relation between those backgrounds are given by the Buscher rule which is applicable only when there is an isometry direction.

For a string to propagate consistently under backgrounds, the corresponding configuration of background fields are required to satisfy some conditions [11, 12]. The preferred configurations of those fields are given by the solutions of the ten-dimensional supergravity theory. The part of supergravity action concerning both the metric and B-field (Neveu-Schwarz-Neveu-Schwarz sector fields) is given by[2]

$$\mathcal{L}_{\mathrm{SUGRA}} = \int d^d x \sqrt{-g}\left(R - \frac{1}{12}H^2\right) + \cdots . \tag{1.10}$$

Here R denotes the Ricci scalar constructed from the space-time metric g_{ij} as usual, and H does the field strength of the B-field called H-flux. As the B-field is a 2nd-rank tensor B_{ij}, the H-flux is a 3rd-rank tensor, explicitly given by in components

$$H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}. \tag{1.11}$$

String theory requires the space-time dimension to be $d = 10$ for its consistency, and hence there are extra six spatial dimensions compared to our empirical observation.
One approach to relate the higher dimensional supergravity theories to four-dimensional physics is to compactify the extra six-dimensional space [13]. In general, compactifying the internal space in the presence of fluxes yields the four-dimensional effective theories described by gauged supergravities with gauge algebras

$$\begin{aligned} [e_a, e_b] &= f_{ab}^c e_c + H_{abc} e^c, \\ [e_a, e^b] &= Q_a^{bc} e_c + f_{ac}^b e^c, \\ [e^a, e^b] &= R^{abc} e_c + Q_c^{ab} e^c, \end{aligned} \tag{1.12}$$

called the Kaloper-Myers algebra [14]. Here e_a and e^a are generators associated with the metric $g_{\mu a}$ and the B-field $B_{\mu a}$, where $\mu = 0, 1, \cdots, 3$ denotes the four-dimensional space-time component and $a = 4, \cdots, 9$ does that of the internal space. The algebras above are understood by noticing hat the metric and B-field are decomposed into

$$\begin{aligned} g_{MN} &\to g_{\mu\nu},\, g_{\mu a},\, g_{ab}, \\ B_{MN} &\to B_{\mu\nu},\, B_{\mu a},\, B_{ab}, \end{aligned}$$

where $M = \mu, a$, and then, that the supergravity action contains

$$\begin{aligned} S_{\mathrm{SUGRA}} &= \int_{M_4 \times I_6} \sqrt{-g} g^{MN} g^{KL} g^{PQ} H_{MKP} H_{NLQ} + \cdots \\ &= \int_{M_4} \sqrt{-g}(\cdots + H_{abc} g^{\mu a} g^{\nu b} g^{cd} \partial_\mu B_{\nu d} + \cdots), \end{aligned} \tag{1.13}$$

indicating that H_{abc} plays a role of structure constants.

[2]Here we assume that the scalar field of dilaton to be constant and do not care about the overall constant.

Although algebras associated with coefficients H and f are realized by compactifying on ordinary internal space with conventional fluxes [14, 15], algebras concerning Q and R have never been obtained by such a procedure. Indeed, such algebras have been introduced heuristically, however, it has been expected that they appear from compactifying on "non-geometric backgrounds" which are introduced bellow.

It has been observed that ill-defined "non-geometric" backgrounds are obtained by formal applications of T-duality transformation to well-defined configurations of background fields [16–22].
Here we illustrate an example. Starting with a three-torus with non-zero constant H-flux, a sequence of successive applications of T-duality transformation converts the H-flux into the fluxes of different tensor type, schematically summarized as

$$H_{abc} \longleftrightarrow f_{ab}^{c} \longleftrightarrow Q_{a}^{bc} \longleftrightarrow R^{abc},$$

which are fluxes of tensor type $(0, 3)$, $(1, 2)$, $(2, 1)$ and $(3, 0)$, respectively, and which could be the origins of the missing pieces in the Kaloper-Myers algebra (1.12). Whereas the second flux, called f-flux, can be interpreted as a kind of structure constant [16, 17], the third flux referred to as Q-flux can not be comprehended by such ordinary notion of geometry. Besides, the metric associated with such Q-flux becomes multi-valued. This configuration of the multi-valued metric as well can not be understood by ordinary notion of geometry, but be done by introducing the notion of T-folds which admits T-duality transformations in addition to diffeomorphisms when local patches are glued [18]. As the initial configuration has three manifest periodicities, the third T-duality transformation might be considered and the existence of the flux of tensor type (3,0), named R-flux, is speculated [19]. The nature of R-flux of *tri-vector field* has hardly been understood and only its existence has been suggested by formal arguments. These are why the latter two configurations associated with Q- and R-fluxes are referred to as "non-geometric" in literatures.
Although such ill-defined configurations of backgrounds had hardly been considered in the analysis of supergravity theory, there is *a priori* no reason to forbid them. Moreover, as mentioned above, the non-geometric backgrounds are expected to provide a new variety of supergravity theories which could never have been obtained by dimensional reductions of well-defined configurations of the background fields [16–22].
As it has been mentioned so far, the non-geometric backgrounds are introduced by theoretical considerations and might play roles in both theoretical and phenomenological issues. Nevertheless, neither clear interpretations nor appropriate formulations of such non-geometric objects have been established yet.

Generalized geometry is one of the frameworks which capture some features of string's backgrounds, especially H-flux in mathematical manner. It is a variant of differential geometry, firstly proposed by Hitchin [23] and further developed by Gualtieri [24] and Cavalcanti [25]. Since the metric and B-field are treated on the same footing in generalized geometry, not only the metric but also B-field are regarded as geometrical objects. As a result, the construction of an analogue of Riemannian geometry based on generalized geometry naturally provides the NSNS supergravity action (1.10) with geometric intuition [26, 27]. A brief review on this point is given in this dissertation.

The main part of this dissertation is devoted to the considerations of non-geometric fluxes. A new variant of generalized geometry is proposed, in order to formulate one of the non-geometric fluxes [28]. This novel framework is a kind of dual of the ordinary generalized geometry, and indeed it has analogous structures in mathematical aspects. Since its formulation is based on the Poisson structure of the target space, it is sometimes referred to as Poisson generalized geometry throughout this dissertation. Poisson generalized geometry enables a consistent definition of a *tri-vector field flux* [28]. This flux of *tri-vector field* would be identified with the R-flux which has been neither understood nor formulated in the conventional frameworks, including generalized geometry.
As mentioned above, Poisson generalized geometry is analogous to generalized geometry in mathematical structures. Hence, many objects considered in the ordinary generalized geometry can be transferred to those constructed in Poisson generalized geometry. The latter main part of this dissertation develops investigation in this direction further. A construction of an analogue of Riemannian geometry based on the Poisson generalized geometry is investigated. It is found that the analogues of the connection and the curvature are consistently defined. As the Poisson structure is a fundamental object in the formulation, the resulting geometry is eventually found to be compatible with this Poisson structure in addition to the positive-definite metric. The compatibility condition demands the anti-symmetric part of the connection to be proportional to the derivative of the Poisson tensor.

This dissertation is organized as follows. In chapter 2 we give an overview of string theory. The sigma model which provides an action for a string is studied. Then we see a sigma model has a dual description given by another sigma model. This duality between sigma models is summarized as the Buscher rule which concludes physical equivalence for a string between different backgrounds, referred to as T-duality. We shortly discuss that applications of T-duality transformation to geometric backgrounds provide non-geometric backgrounds. In chapter 3 we present some notion of generalized geometry. We give definitions of the algebraic structures called the Lie algebroid and the Courant algebroid. Then, we introduce the generalized tangent bundle as an example of the Courant algebroid, and shortly discuss its properties. We review on the construction of an analogue of Riemannian geometry in the framework of generalized geometry. Chapter 4 is devoted to the definition of a new variant of generalized geometry, which we call Poisson generalized geometry [28]. The significant objects in this framework are the Lie algebroid and the Courant algebroid both of which are based on the underlying Poisson structure. We briefly discuss how Poisson generalized geometry enables a consistent definition of *tri-vector flux*. In chapter 5, we construct an analogue of Riemannian geometry based on Poisson generalized geometry. We show that an analogue of a connection and a curvature are consistently defined, even in the presence of the tri-vector flux. The geometrical meaning of this connection is intensively investigated. In chapter 6 we give a summary of results and discussions. We especially discuss a construction of the gravity theory based on the analogue of Riemannian geometry based on Poisson generalized geometry and its relation to the supergravity theory. We give a brief review on differential geometry, introducing notations used throughout this dissertation in appendix A. In appendix B, we present some details on calculation.

Chapter 2

String Theory

This chapter gives an overview of string theory, focusing only on what are needed in this dissertation. For more details, see reviews and textbooks [29–33].
Firstly, we review on the sigma model which provides an action for a string propagating in background fields. We give a little analysis of sigma models. In the analysis, we see that the current associated with the gauge transformation of the backgrounds forms the algebra called the Dorfman bracket and the Courant bracket which are introduced mathematically in the next chapter.
Then, we see that a sigma model has another description which is given by another sigma model. The alternative sigma model corresponds to the action for a string propagating in different backgrounds from the original backgrounds. The duality between backgrounds of these sigma models is formulated as the Buscher rule [9, 10]. This duality is referred to as T-duality, see review [34]. We see that there are natural actions of $O(n,n)$ transformation which includes the T-duality transformation as its special case. The $O(n,n)$ transformation is as well one of the main subject in the next chapter.
Finally, we discuss that applications of T-duality transformation to geometric backgrounds yield "non-geometric" backgrounds [18–20].

2.1 Sigma model

In this section, we give actions for a string and for a (charged) particle for comparison, and investigation into them briefly.

2.1.1 Action for point particle

Before introducing the action for a string, we shall recall the action for a point particle, since the point particle's action is analogous to that of string's in some points.

Point particle in curved spacetime

In curved space background characterized by the spacetime metric $g_{\mu\nu}$, it is well known that the action for a point particle is given by the proper length of its trajectory

$$S_{\rm PP} = -m \int d\tau \sqrt{-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu}. \tag{2.1}$$

Here m is the particle's mass, τ parameterizes the world-line, $g_{\mu\nu}(X)$ is the background metric and $\dot{X}^\mu$ $(\mu = 0, 1, \cdots, n)$ denotes $dX^\mu/d\tau$. The function $X^\mu(\tau)$ is an embedding of the world-line into the $(n+1)$-dimensional spacetime. In other words, $X^\mu(\tau = \tau_0)$ gives the coordinate value of particle's position in $(n+1)$-dimensional spacetime referred at $\tau = \tau_0$. The $(n+1)$-dimensional spacetime is often referred to as the target space.

A variation of the action with respect to the field X^μ yields the equation for geodesic

$$0 = \ddot{X}^\rho + \Gamma^\rho_{\mu\nu}\dot{X}^\mu \dot{X}^\nu, \tag{2.2}$$

which minimizes the length of the particle's trajectory. Here we introduce the Christoffel symbol as usually defined by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}). \tag{2.3}$$

Since the all indices of the target space are contracted, it is manifestly invariant under the diffeomorphism of the target space

$$X'^\mu = X'^\mu(X), \quad g'_{\mu\nu}(X') = \frac{\partial X^\rho}{\partial X'^\mu}\frac{\partial X^\sigma}{\partial X'^\nu} g_{\rho\sigma}(X). \tag{2.4}$$

Charged particle in background fields

In general, a point particle has electric charge. In the presence of the external electromagnetic field, i.e. the $U(1)$ gauge field A_μ, the corresponding action for a point particle with electric charge e is given by

$$S_{\rm CPP} = -m \int d\tau \sqrt{-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu} + e \int d\tau \dot{X}^\mu A_\mu(X). \tag{2.5}$$

Especially for the case with flat metric $g_{\mu\nu}(X) = \eta_{\mu\nu}$, the equation of motion reads

$$\frac{d}{d\tau}\left(\frac{m\dot{X}_\mu}{\sqrt{-\dot{X}^2}}\right) = e\dot{X}^\nu \partial_\mu A_\nu - e\dot{X}^\nu \partial_\nu A_\mu. \tag{2.6}$$

If we make a world-line gauge choice as $\tau = X^0$ and take the non-relativistic limit, we easily see that this equation reduces to the familiar equation of motion for charged particle under the action of the Lorentz force

$$\begin{aligned} m\ddot{X}_i &= e\partial_i A_0 - e\partial_0 A_i + e\dot{X}^j(\partial_i A_j - \partial_j A_i) \\ &= eE_i + e\epsilon_{ijk}\dot{X}_j B_k = e[\mathbf{E} + \mathbf{V} \times \mathbf{B}]_i \,. \end{aligned} \tag{2.7}$$

The physics should be independent of the gauge choice of external gauge field A_μ. Under making another gauge choice of gauge field A_μ as

$$A_\mu \to A_\mu + \partial_\mu \lambda, \tag{2.8}$$

the action only changes by total derivative

$$\delta S_{\mathrm{CPP}} = e \int d\tau \dot{X}^\mu \partial_\mu \lambda(X) = e \int d\tau \frac{d}{d\tau} \lambda(X), \tag{2.9}$$

which has no effect on the equation of motion. Hence the physics is actually independent of the gauge choice at least in classical level.

2.1.2 Action for string

The action for a string is formulated as an extension of the action for a point particle. This is given by the Nambu-Goto action which measures the area swept by the string. Though its physical interpretation is clear, it seems hard to be quantized. By introducing the auxiliary field, the problem is avoided for the Polyakov action, which is shown to be equivalent to Nambu-Goto action at least classically. The sigma model is introduced as an extension of the Polyakov action. We give brief analysis of the string action in some special cases.

Nambu-Goto action

Since a string extends along one-spacial direction, it sweeps a two-dimensional surface called world-sheet in the $(n+1)$-dimensional spacetime. The action for a string is given by the area of the world-sheet

$$S_{\mathrm{NG}} = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\det(g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu)}, \tag{2.10}$$

which is referred to as the Nambu-Goto action. Here σ^a $(a = 0, 1)$ is the world-sheet's coordinate. The variation principle extremizes the area of the world-sheet. Although this action has a clear physical interpretation, it is difficult to handle and especially quantize this action, since it involves the square root and hence it is a non-polynomial of the dynamical field X^μ.

Polyakov action

An alternative action equivalent to the Nambu-Goto action is given by the Polyakov action. The Polyakov action has more tractable form than that of the Nambu-Goto action. As a price of its tractability, the Polyakov action requires to introduce the Lorentzian world-sheet metric h_{ab} $(a, b = 0, 1)$ as the auxiliary field:

$$S_{\mathrm{P}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} g_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu, \tag{2.11}$$

where h denotes the determinant of the auxiliary world-sheet metric, $h = \det h_{ab}$.
Taking the variation with respect to the auxiliary field h^{ab} with using $\delta h = -h h_{ab} \delta h^{ab}$, we have the following equation as a constraint

$$0 = \frac{4\pi\alpha'}{\sqrt{-h}} \frac{\delta S_{\mathrm{P}}}{\delta h^{ab}} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2} h_{ab} h^{cd} g_{\mu\nu} \partial_c X^\mu \partial_d X^\nu \,. \tag{2.12}$$

Taking the determinant of both sides, we obtain

$$-\frac{1}{4} h (h^{cd} g_{\mu\nu} \partial_c X^\mu \partial_d X^\nu)^2 = -\det(g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu). \tag{2.13}$$

Finally, taking the square root of both sides, we find that the Lagrangian density of the Polyakov action is equal to that of the Nambu-Goto action:

$$\frac{1}{2}\sqrt{-h} h^{ab} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu = \sqrt{-\det(g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\alpha X^\nu)}. \tag{2.14}$$

It convinces ourselves that the Polyakov action is actually equivalent to the Nambu-Goto action at least classically.

Sigma model

Just like a charged particle couples to the 1-form gauge field A_μ, a string can electrically couple to the 2-form gauge field $B_{\mu\nu}$ referred to as the Kalb-Ramond field, or simply the B-field. In the presence of non-trivial B-field background, the corresponding string action is written as

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma [\sqrt{-h} h^{ab} g_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + \epsilon^{ab} B_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu], \tag{2.15}$$

where ϵ^{ab} is the anti-symmetric tensor with $\epsilon^{01} = 1$. The rest of this chapter is intensively involved in this action.

Since the all indices of the target space are contracted, it is manifestly invariant under the diffeomorphism of the target space

$$X' = X'^\mu(X), \quad g'_{\mu\nu} = \frac{\partial X^\rho}{\partial X'^\mu} \frac{\partial X^\sigma}{\partial X'^\nu} g_{\rho\sigma}, \quad B'_{\mu\nu} = \frac{\partial X^\rho}{\partial X'^\mu} \frac{\partial X^\sigma}{\partial X'^\nu} B_{\rho\sigma}, \tag{2.16}$$

analogous to the case of a point particle.

H-flux

The field strength of the B-field, called H-flux, is given by an exterior derivative of 2-form B giving a 3-form $H = dB$, in components

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}. \tag{2.17}$$

The B-field also has the gauge transfomations as transformations which preserve the H-flux. It allows us to shift the B-field $B \to B + d\Lambda$, in components

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \,, \tag{2.18}$$

with any 1-form gauge parameter Λ.
In this string case, the physics again should not depend on the gauge choice of the external B-field. Under another gauge choice of the B-field, say $B \to B + d\Lambda$, the difference of the action reads

$$4\pi\alpha'\delta S = 2\int d^2\sigma \partial_a(\epsilon^{ab}\Lambda_\nu\partial_b X^\nu). \tag{2.19}$$

If we assume the world-sheet to be a cylinder and thus assume a closed string, this term becomes trivial, since the (infinitely long) cylinder has no boundary. If we consider the case of open string, due to the existence of the world-sheet's boundaries, we need to introduce the 1-form gauge field on the boundaries to cancel the above boundary term.

2.1.3 A bit of analysis

For simplicity let us consider that $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, +1, \cdots, +1)$, $B_{\mu\nu} = 0$ and make a world-sheet gauge choice as $h_{ab} = \eta_{ab} = (-1, +1)$. Then the sigma-model reduces to

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \partial_a X^\mu \partial^a X_\mu. \tag{2.20}$$

Here raising and lowering of the target space indices are understood to be done by the (inverse of the) flat-Minkowski metric $\eta_{\mu\nu}$ ($\eta^{\mu\nu}$), as usual. The resulting action is the same as the action for the $(n+1)$ Klein-Gordon free-fields defined on two-dimensional Minkowski spacetime $\{\sigma^a\}$.

Equation of motion and boundary condition

Since the variation of the action with respect to the field X^μ yields

$$\delta S = -\frac{1}{2\pi\alpha'}\int d^2\sigma(\partial^a\partial_a X^\mu)\delta X_\mu + \frac{1}{2\pi\alpha'}\int d\sigma^0 \partial_1 X^\mu \delta X_\mu\Big|_{\sigma^1=0}^{2\pi}, \tag{2.21}$$

the equation of motion turns out to be the wave equation

$$\partial^a\partial_a X^\mu = (\partial_1\partial_1 - \partial_0\partial_0)X^\mu = 0. \tag{2.22}$$

The solution is given by a sum of any functions of $\sigma^0 + \sigma^1$ and $\sigma^0 - \sigma^1$

$$X^\mu(\sigma^a) = X_L^\mu(\sigma^0 + \sigma^1) + X_R^\mu(\sigma^0 - \sigma^1). \tag{2.23}$$

The half of the solution X_L (X_R) is referred to as left-mover (right-mover). The boundary term implies the boundary conditions for X^μ.
The boundary term vanishes by imposing the periodic boundary condition on X^μ

$$X^\mu(\sigma^0, \sigma^1 + 2\pi) = X^\mu(\sigma^0, \sigma^1), \tag{2.24}$$

which corresponds to a closed string, or by imposing boundary conditions

$$\delta X^\mu(\sigma^0, 2\pi) = \delta X^\mu(\sigma^0, 0) = 0, \tag{2.25}$$

$$\partial_1 X^\mu(\sigma^0, 2\pi) = \partial_1 X^\mu(\sigma^0, 0) = 0, \tag{2.26}$$

corresponding to an open string with the Dirichlet and Neumann boundary condition, respectively. For later convenience, we note that the Dirichlet condition can be rewritten as

$$\partial_0 X^\mu(\sigma^0, 2\pi) = \partial_0 X^\mu(\sigma^0, 0) = 0. \tag{2.27}$$

Closed-string solution

For the case of a closed string, the field X^μ can be expanded by the plane wave as[1]

$$X^\mu(\sigma^a) = x^\mu + p^\mu \sigma^0 + i \sum_{n\neq 0}^{\infty} \frac{1}{n}\big[\alpha_n^\mu e^{-in(\sigma^0+\sigma^1)} + \tilde{\alpha}_n^\mu e^{-in(\sigma^0-\sigma^1)}\big], \tag{2.28}$$

where α_n^μ and $\hat{\alpha}_n^\mu$ represent the Fourier coefficients. The left-mover and the right-mover are decomposed into

$$X_L^\mu(\sigma^0+\sigma^1) = \frac{x^\mu}{2} + \frac{1}{2}p^\mu(\sigma^0+\sigma^1) + i\sum_{n\neq 0}^{\infty}\frac{1}{n}\alpha_n^\mu e^{-in(\sigma^0+\sigma^1)}, \tag{2.29}$$

$$X_R^\mu(\sigma^0-\sigma^1) = \frac{x^\mu}{2} + \frac{1}{2}p^\mu(\sigma^0-\sigma^1) + i\sum_{n\neq 0}^{\infty}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in(\sigma^0-\sigma^1)}. \tag{2.30}$$

Light-cone coordinates

It is often convenient to use the light-cone coordinates σ^+ and σ^- defined by

$$\sigma^+ = \sigma^0 + \sigma^1\,, \quad \sigma^- = \sigma^0 - \sigma^1. \tag{2.31}$$

By definition we see

$$\partial_+ := \frac{\partial}{\partial\sigma^+} = \frac{\partial\sigma^0}{\partial\sigma^+}\frac{\partial}{\partial\sigma^0} + \frac{\partial\sigma^1}{\partial\sigma^+}\frac{\partial}{\partial\sigma^1} = \frac{1}{2}\big(\partial_0 + \partial_1\big), \tag{2.32}$$

$$\partial_- := \frac{\partial}{\partial\sigma^-} = \frac{1}{2}\big(\partial_0 - \partial_1\big), \tag{2.33}$$

giving $\partial_\pm\sigma^\pm = 1$ and $\partial_\pm\sigma^\mp = 0$. For a general contravariant vector $v^a(a = 0, 1)$, with paying attention to the world-sheet metric being diag$(-1, +1)$, we define

$$v^+ = v^0 + v^1\,, \qquad v^- = v^0 - v^1\,, \tag{2.34}$$

and

$$v_+ = \frac{1}{2}\big(v_0 + v_1\big) = -\frac{1}{2}\big(v^0 - v^1\big)\,, \quad v_- = \frac{1}{2}\big(v_0 - v_1\big) = -\frac{1}{2}\big(v_0 + v_1\big). \tag{2.35}$$

From (2.34) and (2.35) we can easily read off the components of the world-sheet metric in the light-cone coordinates $\{\sigma^+, \sigma^-\}$ as

$$\eta_{++} = \eta_{--} = 0\,, \quad \eta_{+-} = \eta_{-+} = -\frac{1}{2}, \tag{2.36}$$

$$\eta^{++} = \eta^{--} = 0\,, \quad \eta^{+-} = \eta^{-+} = -2. \tag{2.37}$$

[1] We do not care about both the normalization and the mass dimension of the Fourier components to avoid notational complexity.

Using the light-cone coordinates, the Lagrangian density of the action becomes

$$-\partial_0 X^\mu \partial_0 X_\mu + \partial_1 X^\mu \partial_1 X_\mu = -4\,\partial_+ X^\mu \partial_- X_\mu. \tag{2.38}$$

Noting the world-sheet measure becomes $d^2\sigma = d\sigma^0 d\sigma^1 = -1/2 d\sigma^+ d\sigma^-$, we see the action is rewritten as

$$S = \frac{1}{2\pi\alpha'} \int d\sigma^+ d\sigma^-\, \partial_+ X^\mu \partial_- X_\mu. \tag{2.39}$$

The equation of motion, the Dirichlet and Neumann boundary conditions for open string are also rewritten as

$$\partial_+ \partial_- X^\mu = 0, \tag{2.40}$$

$$(\partial_+ + \partial_-) X^\mu |_{\sigma=0,2\pi} = 0, \tag{2.41}$$

$$(\partial_+ - \partial_-) X^\mu |_{\sigma=0,2\pi} = 0, \tag{2.42}$$

respectively. Acting onto the solution X^μ, the derivative operator ∂_+ (∂_-) projects out the right-mover (left-mover):

$$\partial_+ X^\mu = \partial_+ X^\mu_L, \quad \partial_- X^\mu = \partial_- X^\mu_R. \tag{2.43}$$

2.2 Current algebra

As we see in the previous section, the string action (2.15) has symmetries, consisting of the diffeomorphism (2.16) and the B-field gauge transformation (2.18). In this section we examine the current algebra associated with these symmetries. The resulting algebra forms the Dorfman bracket, or equivalently Courant bracket, defined in generalized geometry in the next chapter. The discussion here follows that given by Alekseev and Strobl [35].

The current associated with the infinitesimal diffeomorphism and the B-field gauge transformation is given by

$$\mathcal{J}_{(u,\alpha)}(\sigma) = u^i\big(X(\sigma)\big)p_i(\sigma) + \alpha_i\big(X(\sigma)\big)\partial_\sigma X^i(\sigma), \tag{2.44}$$

where u is any vector field and α is any 1-form. The first term gives the translation when u^i is constant. Hence it is reasonable that the first term generates an infinitiesmal diffeomorphism. From (2.19), it is understood that the second term is the current associated with the B-field gauge transformation.
Since the Poisson bracket is defined by

$$\{F, G\} = \int d\sigma' \left(\frac{\delta F}{\delta X^i(\sigma')} \frac{\delta G}{\delta P_i(\sigma')} - \frac{\delta G}{\delta X^i(\sigma')} \frac{\delta F}{\delta P_i(\sigma')} \right), \tag{2.45}$$

for any functionals F and G, the bracket for the currents is calculated as

$$\begin{aligned}
&\{\mathcal{J}_{(u,\alpha)}(\sigma), \mathcal{J}_{(v,\beta)}(\tau)\} \\
&= \{u^i\big(X(\sigma)\big)p_i(\sigma) + \alpha_i\big(X(\sigma)\big)\partial_\sigma X^i(\sigma), v^i\big(X(\tau)\big)p_i(\tau) + \beta_i\big(X(\tau)\big)\partial_\tau X^i(\tau)\} \\
&= -\big(u^k\partial_k v^i - v^k\partial_k u^i\big)p_i(\sigma)\delta(\sigma-\tau) - \big(u^k\partial_k\beta_i - v^k\partial_k\alpha_i\big)\partial_\sigma X^i(\sigma)\delta(\sigma-\tau) \\
&\quad + \int d\sigma'\alpha_k\big(X(\sigma)\big)\partial_\sigma\delta(\sigma-\sigma')v^k\big(X(\tau)\big)\delta(\tau-\sigma') \\
&\quad - \int d\sigma' u^k\big(X(\sigma)\big)\delta(\sigma-\sigma')\beta_k\big(X(\tau)\big)\partial_\tau\delta(\tau-\sigma').
\end{aligned}$$

With paying attention to $\partial_\sigma\delta(\sigma-\tau) = -\partial_\tau\delta(\sigma-\tau)$, we can rewrite the integrals as

$$\int d\sigma'\alpha_k\big(X(\sigma)\big)\partial_\sigma\delta(\sigma-\sigma')v^k\big(X(\tau)\big)\delta(\tau-\sigma') = -\partial_i\alpha_k v^k\partial_\sigma X^i(\sigma)\delta(\sigma-\tau) + \alpha_k v^k(\tau)\partial_\sigma\delta(\sigma-\tau),$$

and

$$\int d\sigma' u^k\big(X(\sigma)\big)\delta(\sigma-\sigma')\beta_k\big(X(\tau)\big)\partial_\tau\delta(\tau-\sigma') = \partial_i u^k\partial_\sigma X^i\beta_k(\sigma)\delta(\sigma-\tau) - u^k\beta_k(\tau)\partial_\sigma\delta(\sigma-\tau).$$

Then the bracket reads

$$\begin{aligned}
\{\mathcal{J}_{(u,\alpha)}(\sigma), \mathcal{J}_{(v,\beta)}(\tau)\} &= -\big(u^k\partial_k v^i - v^k\partial_k u^i\big)p_i(\sigma)\delta(\sigma-\tau) \\
&\quad - \big(u^k\partial_k\beta_i + \partial_i u^k\beta_k - v^k\partial_k\alpha_i + \partial_i\alpha_k v^k\big)\partial_\sigma X^i(\sigma)\delta(\sigma-\tau) \\
&\quad + \big(\alpha_k v^k + u^k\beta_k\big)(\tau)\partial_\sigma\delta(\sigma-\tau),
\end{aligned} \tag{2.46}$$

here again using $\partial_\tau\delta(\tau-\sigma) = -\partial_\sigma\delta(\tau-\sigma) = -\partial_\sigma\delta(\sigma-\tau)$. With a use of notations widely used in differential geometry[2], we can rewrite the terms above as

$$\big(u^k\partial_k v^i - v^k\partial_k u^i\big)\partial_i = [u,v], \tag{2.47}$$

$$\big(u^k\partial_k\beta_i + \partial_i u^k\beta_k - v^k\partial_k\alpha_i + \partial_i\alpha_k v^k\big)dx^i = \mathcal{L}_u\beta - \mathcal{L}_v\alpha + i_v d\alpha, \tag{2.48}$$

$$\alpha_k v^k + u^k\beta_k = i_u\beta + i_v\alpha, \tag{2.49}$$

and then the algebra is rewritten as

$$\{\mathcal{J}_{(u,\alpha)}(\sigma), \mathcal{J}_{(v,\beta)}(\tau)\} = -\mathcal{J}_{[(u,\alpha),(v,\beta)]}(\sigma)\delta(\sigma-\tau) + \big(i_u\beta + i_v\alpha\big)(\tau)\partial_\sigma\delta(\sigma-\tau). \tag{2.50}$$

Here we introduced a bracket $[\ ,\ ]$ defined by

$$[(u,\alpha),(v,\beta)] = ([u,v], \mathcal{L}_u\beta - \mathcal{L}_v\alpha + i_v d\alpha). \tag{2.51}$$

This bracket is the Dorfman bracket (3.48) used in generalized geometry. Its properties are mentioned in the next chapter. Here we only comment that it is not skew-symmetric.
In the curent algebra (2.50), the left-hand side is manifestly skew-symmetric, while the skew-symmetric property of the right-hand side is not manifest due to the choice of the expression for

[2] A brief introduction for the notation is given in appendix.

the anomalous term.
To rewrite the current algebra in manifestly skew-symmetric manner, notice that the following formula is valid for any function f

$$f(\tau)\partial_\sigma\delta(\sigma-\tau)=\frac{1}{2}f(\tau)\partial_\sigma\delta(\sigma-\tau)-\frac{1}{2}f(\sigma)\partial_\tau\delta(\sigma-\tau)+\frac{1}{2}\partial_\sigma f(\sigma)\delta(\sigma-\tau).$$

With making use of this formula, the current algebra is re-expressed as

$$\{\mathcal{J}_{(u,\alpha)}(\sigma),\mathcal{J}_{(v,\beta)}(\tau)\}=-\mathcal{J}_{((u,\alpha),(v,\beta))}(\sigma)\delta(\sigma-\tau)+(\text{anomalous}), \tag{2.52}$$

with

$$((u,\alpha),(v,\beta)):=\left([u,v],\mathcal{L}_u\beta-\mathcal{L}_v\alpha-\frac{1}{2}d(i_u\beta-i_v\alpha)\right), \tag{2.53}$$

$$(\text{anomalous})=\frac{1}{2}(i_u\beta+i_v\alpha)(\tau)\partial_\sigma\delta(\sigma-\tau)-\frac{1}{2}(i_u\beta+i_v\alpha)(\sigma)\partial_\tau\delta(\sigma-\tau). \tag{2.54}$$

As desired, both the bracket (,) and the anomalous term are manifestly skew-symmetric. The bracket (,) defined here is referred to as the Courant bracket (3.25) which is also used in generalized geometry. Its properties are intensively studied in the next chapter.

2.3 The Buscher rule

In this section we give a derivation of the Buscher rule which provides the duality for sigma-models [Buscher][Duff]. We see that the duality transformation is applicable only when there is a direction of isometry. The existence of the direction of isometry means that the background fields are invariant under transformations along this direction. In the duality transformations, there exist natural actions of $O(n,n)$ transformation group.

2.3.1 One direction of isometry

As a worming-up, we consider the case of both g and B being independent of X^0. Though we assume X^0 as the direction of isometry, there is no intension for "0" to denote the temporal component of the target space. It can be replaced with some other character, say "n", which indicates in general a spacial direction.
As we shall see in a moment, the following action is classically equivalent to (2.15)

$$\begin{aligned}S'=\frac{1}{4\pi\alpha'}\int d^2\sigma[&\sqrt{-h}h^{ab}g_{00}V_aV_b+2\sqrt{-h}h^{ab}g_{0i}V_a\partial_bX^i+\sqrt{-h}h^{ab}g_{ij}\partial_aX^i\partial_bX^j\\&+2\epsilon^{ab}B_{0i}V_a\partial_bX^i+\epsilon^{ab}B_{ij}\partial_aX^i\partial_bX^j+2\epsilon^{ab}\hat{X}^0\partial_aV_b]\,,\end{aligned} \tag{2.55}$$

with $i,j=1,\cdots,n$. Here $\hat{X}^0$ is the Lagrange multiplier leading a constraint

$$\epsilon^{ab}\partial_aV_b=0. \tag{2.56}$$

Following the Poincaré lemma, locally we can find a function X^0 which satisfies $V_a = \partial_a X^0$. Substituting this into (2.55), we can reproduce the original action (2.15).
On the other hand, we find that the equations of motion for V_a reads

$$0 = 4\pi\alpha' \frac{\delta S'}{\delta V_a} = 2\sqrt{-h}h^{ab}g_{00}V_b + 2\sqrt{-h}h^{ab}g_{0i}\partial_b X^i + 2\epsilon^{ab}B_{0i}\partial_b X^i + 2\epsilon^{ab}\partial_b \hat{X}^0. \tag{2.57}$$

Eliminating V_a from (2.55) with a use of the solution of this equation of motion

$$V_a = \frac{1}{g_{00}}\Big(- g_{0i}\partial_a X^i - \frac{h_{ab}}{\sqrt{-h}}\epsilon^{bc}B_{0i}\partial_c X^i - \frac{h_{ab}}{\sqrt{-h}}\epsilon^{bc}\partial_c \hat{X}^0\Big), \tag{2.58}$$

we find that the Lagrangian density becomes

$$\begin{aligned}\sqrt{-h}h^{ab}\frac{1}{g_{00}}\partial_a\hat{X}^0\partial_b\hat{X}^0 + 2\sqrt{-h}h^{ab}\frac{B_{0i}}{g_{00}}\partial_a\hat{X}^0\partial_b X^i + \sqrt{-h}h^{ab}\Big(g_{ij} - \frac{g_{0i}g_{0j} - B_{0i}B_{0j}}{g_{00}}\Big)\partial_a X^i\partial_b X^j + \\ + 2\epsilon^{ab}\frac{g_{0i}}{g_{00}}\partial_a\hat{X}^0\partial_b X^i + \epsilon^{ab}\Big(B_{ij} - \frac{g_{0i}B_{0j} - g_{0j}B_{0i}}{g_{00}}\Big)\partial_a X^i\partial_b X^j.\end{aligned} \tag{2.59}$$

Thus the action (2.55) reduces to the sigma-model

$$\tilde{S} = \frac{1}{4\pi\alpha'}\int d^2\sigma[\sqrt{-h}h^{ab}\tilde{g}_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu + \epsilon^{ab}\tilde{B}_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu] \tag{2.60}$$

with the backgrounds given by

$$\begin{aligned}\tilde{g}_{00} = \frac{1}{g_{00}}, \quad \tilde{g}_{0i} = \frac{B_{0i}}{g_{00}}, \quad \tilde{g}_{ij} = g_{ij} - \frac{g_{0i}g_{0j} - B_{0i}B_{0j}}{g_{00}}, \\ \tilde{B}_{0i} = \frac{g_{0i}}{g_{00}}, \qquad \tilde{B}_{ij} = B_{ij} - \frac{g_{0i}B_{0j} - g_{0j}B_{0i}}{g_{00}}.\end{aligned} \tag{2.61}$$

Since both sigma models (2.15) and (2.60) are obtained from the same action (2.55), the physics described by them should be equivalent. The relations among the background fields (2.61) are referred to as the Buscher rule.

2.3.2 All directions of isometry

As a more special case, let us assume that the background fields g and B are independent of all of the spacetime coordinates X^μ ($\mu = 1, \cdots, n$). The action which corresponds to (2.55) in this case is

$$S'' = \frac{1}{4\pi\alpha'}\int d^2\sigma[\sqrt{-h}h^{ab}g_{\mu\nu}V_a^\mu V_b^\nu + \epsilon^{ab}B_{\mu\nu}V_a^\mu V_b^\nu + 2\epsilon^{ab}\partial_a\hat{X}_\mu V_b^\mu], \tag{2.62}$$

where $\hat{X}_\mu$ are the Lagrange multipliers leading constraints

$$\epsilon^{ab}\partial_a V_b^\mu = 0. \tag{2.63}$$

These constraints are solved by $V_a^\mu = \partial_a X^\mu$ with some functions X^μ due to the Poincaré lemma. Then the action (2.15) is reproduced.
On the other, taking the variations of V_a^μ yields

$$0 = 2\pi\alpha' \frac{\delta S''}{\delta V_a^\mu} = \sqrt{-h} h^{ab} g_{\mu\nu} V_b^\nu + \epsilon^{ab} B_{\mu\nu} V_b^\nu - \epsilon^{ab} \partial_b \hat{X}_\mu, \tag{2.64}$$

and thus

$$\epsilon^{ab} \partial_b \hat{X}_\mu = \sqrt{-h} h^{ab} g_{\mu\nu} V_b^\nu + \epsilon^{ab} B_{\mu\nu} V_b^\nu. \tag{2.65}$$

We would like to algebraically solve this equation with respect to V_a^μ. Making an ansatz

$$V_a^\mu = \frac{1}{\sqrt{-h}} p^{\mu\nu} h_{ab} \epsilon^{bc} \partial_c \hat{X}_\nu + q^{\mu\nu} \partial_a \hat{X}_\nu, \tag{2.66}$$

and then eliminating V_a^μ from (2.65)

$$\epsilon^{ab} \partial_b \hat{X}_\mu = \epsilon^{ab} (g_{\mu\nu} p^{\nu\rho} + B_{\mu\nu} q^{\nu\rho}) \partial_b \hat{X}_\rho + \sqrt{-h} h^{ab} (B_{\mu\nu} p^{\nu\rho} + g_{\mu\nu} q^{\nu\rho}) \partial_b \hat{X}_\rho, \tag{2.67}$$

we find the simultaneous equations

$$\delta_\mu^\rho = g_{\mu\nu} p^{\nu\rho} + B_{\mu\nu} q^{\nu\rho}, \tag{2.68}$$
$$0 = B_{\mu\nu} p^{\nu\rho} + g_{\mu\nu} q^{\nu\rho}. \tag{2.69}$$

They are solved by (here the equations are understood as matrix equations)

$$p = (g - Bg^{-1}B)^{-1}, \tag{2.70}$$
$$q = -g^{-1}B\{(gB^{-1}g - B)g^{-1}B\}^{-1} = (B - gB^{-1}g)^{-1}. \tag{2.71}$$

Note that the matrix p (q) is a(n) (anti-)symmetric matrix. These matrices satisfy the following relations

$$p + q = (g + B)^{-1}, \tag{2.72}$$
$$p = -qgB^{-1} = -B^{-1}gq, \tag{2.73}$$
$$q = -g^{-1}Bp = -pBg^{-1}. \tag{2.74}$$

Thus the action (2.62) reduces to

$$S'' = \frac{1}{4\pi\alpha'} \int d^2\sigma \left[\sqrt{-h} h^{ab} p^{\rho\sigma} \partial_a \hat{X}_\rho \partial_b \hat{X}_\sigma + \epsilon^{ab} q^{\rho\sigma} \partial_a \hat{X}_\rho \partial_b \hat{X}_\sigma \right]. \tag{2.75}$$

Hence we have another sigma model which is physically equivalent to (2.15).

The relations between the fields X^μ and $\hat{X}^\mu$ which appear in the action (2.15) and (2.75), respectively, are given by

$$\epsilon^{ab} \partial_b \hat{X}_\mu = \sqrt{-h} h^{ab} g_{\mu\nu} \partial_b X^\nu + \epsilon^{ab} B_{\mu\nu} \partial_b X^\nu, \tag{2.76}$$

from(2.65).Introducing the anti-symmetric tenor $\epsilon_{ab} := -\epsilon^{ab}$, we can rewrite these relations as

$$\epsilon^{ab}\partial_b X^\mu = \sqrt{-h}h^{ab}p^{\mu\nu}\partial_b\hat{X}_\nu + \epsilon^{ab}q^{\mu\nu}\partial_b\hat{X}_\nu. \tag{2.77}$$

By substituting (2.77) into (2.76) and using the formulae for p and q, we obtain

$$\epsilon^{ab}\partial_b\hat{X}_\rho = \sqrt{-h}h^{ab}((g - Bg^{-1}B)_{\rho\lambda}\partial_b X^\lambda + B_{\rho\lambda}g^{\lambda\gamma}\partial_b\hat{X}_\gamma). \tag{2.78}$$

Similarly, substituting (2.76) into (2.77), we equivalently have

$$\epsilon^{ab}\partial_b X^\mu = \sqrt{-h}h^{ab}(-(g^{-1}B)^\mu{}_\nu\partial_b X^\nu + g^{\mu\nu}\partial_b\hat{X}_\nu). \tag{2.79}$$

2.3.3 $O(n,n)$-transformation

Let us introduce a $2n$-dimensional column vector

$$Z^M = \begin{pmatrix} X^\mu \\ \hat{X}_\mu \end{pmatrix}, \tag{2.80}$$

and $2n \times 2n$ matrix Ω

$$\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.81}$$

where 1 denotes $n \times n$ unity matrix. Then the relations between the fields (2.78) and (2.79) can be simply represented as

$$\epsilon^{ab}\Omega_{MN}\partial_b Z^N = \sqrt{-h}h^{ab}G_{MN}\partial_b Z^N, \tag{2.82}$$

with

$$G_{MN} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}. \tag{2.83}$$

Since the matrix Ω is an invariant matrix under actions of any element S in $O(n,n)$ transformation which satisfies $S^T\Omega S = \Omega$, there is a natural action of this transformation on the column vector Z, defined by $Z' = S^{-1}Z$. The action of S^T on (2.82) gives

$$\epsilon^{ab}(S^T)_M{}^L\Omega_{LK}\partial_b Z^K = \epsilon^{ab}\Omega_{MN}(S^{-1})^N{}_K\partial_b Z^K = \epsilon^{ab}\Omega_{MN}\partial_b Z'^N, \tag{2.84}$$

$$\sqrt{-h}h^{ab}(S^T)_M{}^L G_{LK}\partial_b Z^K = \sqrt{-h}h^{ab}(S^T)_M{}^L G_{LK}S^K{}_N\partial_b Z'^N, \tag{2.85}$$

i.e.

$$\epsilon^{ab}\Omega_{LK}\partial_b Z'^K = \sqrt{-h}h^{ab}(S^T)_L{}^P G_{PK}S^K{}_N\partial_b Z'^N. \tag{2.86}$$

Thus the action of an $O(n,n)$ matrix S on the backgrounds are given by

$$G_{MN} \to G'_{MN} = (S^T)_M{}^L G_{LK}S^K{}_N. \tag{2.87}$$

Making the gauge choice as $h_{ab} = \mathrm{diag}(-1,+1)$, we find

$$\begin{pmatrix} \partial_1\hat{X}_\mu \\ \partial_1 X^\mu \end{pmatrix} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}\begin{pmatrix} \partial_0 X^\mu \\ \partial_0\hat{X}_\mu \end{pmatrix}. \tag{2.88}$$

Relation to the Buscher rule

Especially for

$$S = \Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.89}$$

the condition $S^T \Omega S = \Omega$ is satisfied and Ω is an element of $O(n,n)$ transformation group, indeed

$$\Omega^T \Omega \Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega. \tag{2.90}$$

Corresponding to $S = \Omega$, the metric transforms into

$$\begin{aligned} G = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \;\rightarrow\; G' &= \Omega^T \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \Omega \\ &= \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \end{aligned} \tag{2.91}$$

following the rule (2.87). Recalling (2.70), (2.71), (2.72), (2.73) and (2.74), we find the resulting metric can be represented as

$$G' = \begin{pmatrix} p - qp^{-1}q & qp^{-1} \\ -p^{-1}q & p^{-1} \end{pmatrix}. \tag{2.92}$$

We easily see that in this transformation the following replacements take places

$$g \rightarrow p = (g - Bg^{-1}B)^{-1}, \quad B \rightarrow q = -g^{-1}Bp. \tag{2.93}$$

This is the same relations as (2.75) obtained in the dual descriptions of the sigma-models. Hence $O(n,n)$ transformations include the Buscher rule and provide transformations of wider class. The meaning of the other elements in $O(n,n)$ is discussed below.

***SO*(*n*, *n*) generators**

We end this section with a few comments on $O(n,n)$ transformations. Let us consider infinitesimal transformations. Since their successive actions defines an element which is continuously connected to the unity, the resulting element is of $SO(n,n)$. For an infinitesimal transformation $S = 1 + X$, the definition for it to be an element of $SO(n,n)$, $S^T \Omega S = \Omega$, gives

$$\Omega = (1 + X^T)\Omega(1 + X) \sim \Omega + X^T\Omega + \Omega X. \tag{2.94}$$

Hence the generators X must satisfy the condition

$$X^T \Omega = -\Omega X. \tag{2.95}$$

Recalling the definition of Ω and parameterizing the matrix X as

$$X = \begin{pmatrix} A & \eta \\ b & \alpha \end{pmatrix}, \tag{2.96}$$

the condition (2.95) reads

$$\alpha = -A^T, \quad b^T = -b, \quad \eta^T = -\eta. \tag{2.97}$$

The number of the parameter tells us the number of the generators being

$$n^2 + 2 \times \frac{n(n-1)}{2} = \frac{2n(2n-1)}{2}. \tag{2.98}$$

The finite transformation are classified into the following of three types

$$\begin{pmatrix} e^A & 0 \\ 0 & e^{-A^T} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \tag{2.99}$$

where B and β are finite skew-symmetric matrices.
The interpretations of these transformations are as follows. For simplicity taking $g = \eta$, $B = 0$ and infinitesimal transformations parametrized by $A^\mu{}_\nu = \partial_\nu u^\mu$ and $b_{\mu\nu} = \partial_\mu \alpha_\nu$, their actions onto G_{MN} read

$$\begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \xrightarrow{1+A} \begin{pmatrix} \eta_{\mu\nu} + \partial_\mu u_\nu + \partial_\nu u_\mu & 0 \\ 0 & \eta^{\mu\nu} - \partial^\mu u^\nu - \partial^\nu u^\mu \end{pmatrix}, \tag{2.100}$$

$$\begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \xrightarrow{b=d\alpha} \begin{pmatrix} \eta_{\mu\nu} - (d\alpha)_{\mu\rho}\eta^{\rho\sigma}(d\alpha)_{\sigma\nu} & -(d\alpha)_{\mu\rho}\eta^{\rho\nu} \\ \eta^{\mu\rho}(d\alpha)_{\rho\nu} & \eta^{\mu\nu} \end{pmatrix}, \tag{2.101}$$

which correspond to the diffeomorphism generated by the vector field u and the B-field gauge transformation induced by 1-form gauge parameter α, respectively. The β-transformation has no such clear interpretation. The actions of $SO(n, n)$ transformations, especially corresponding to diffeomorphisms and B-field gauge transformations, play important roles as well in the next chapter.

2.4 T-duality

For simple argument, setting $B_{\mu\nu} = 0$ and $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, +1, \cdots, +1)$, furthermore taking a gauge choice as $h^{ab} = \eta^{ab} = \mathrm{diag}(-1, +1)$, we find the relations between the fields (2.78) and (2.79) reduces to

$$\epsilon^{ab}\partial_b \hat{X}_\mu = \partial^a X_\mu, \tag{2.102}$$

$$\epsilon^{ab}\partial_b X^\mu = \partial^a \hat{X}^\mu. \tag{2.103}$$

Since ϵ^{ab} is the anti-symmetric tensor, these equations indicate that the derivative of X^μ with respect to $\sigma^0(\sigma^1)$ is converted to the derivative with respect to $\sigma^1(\sigma^0)$ for $\hat{X}^\mu$. To be more specific, noting that $\epsilon^{01} = -\epsilon^{10} = -1$ and $h^{ab} = \text{diag}(-,+)$, we find

$$\partial_1 \hat{X}_\mu = -\epsilon^{01}\partial_1 \hat{X}_\mu = -\partial^0 X_\mu = \partial_0 X_\mu, \tag{2.104}$$

$$\partial_0 \hat{X}_\mu = \epsilon^{10}\partial_0 \hat{X}_\mu = \partial^1 X_\mu = \partial_1 X_\mu. \tag{2.105}$$

These equations mean the duality which interchanges the roles of derivatives

$$\partial_0 \quad \longleftrightarrow \quad \partial_1 \ , \tag{2.106}$$

implying for a closed string the duality between

$$\partial_+ X_L \ \longleftrightarrow \ \partial_+ \hat{X}_L, \quad \partial_- X_R \ \longleftrightarrow \ -\partial_- \hat{X}_R, \tag{2.107}$$

by (2.43), or equivalently, from more physical point of view

$$\text{momentum} \ \longleftrightarrow \ \text{winding}, \tag{2.108}$$

and for an open string

$$\underset{\text{Dirichlet boundary condition}}{(\partial_+ + \partial_-)X^\mu|_{\sigma=0,2\pi} = 0} \ \longleftrightarrow \ \underset{\text{Neumann boundary condition}}{(\partial_+ - \partial_-)\hat{X}^\mu|_{\sigma=0,2\pi} = 0,} \tag{2.109}$$

by (2.26) and (2.27), or (2.41) and (2.42).
The Dp-branes are characterized as p-dimensional extended objects, to which open strings can attach. This is equivalent to that the D-branes are characterized as hypersurfaces on which open strings have the Dirichlet boundary conditions. Since the duality interchanges the boundary conditions for open strings, its actions on D-branes are summarized as

$$\text{wrapped D}p\text{-brane} \ \longleftrightarrow \ \text{unwrapped D}(p-1)\text{-brane}. \tag{2.110}$$

These dualities are what the T-duality tells us.
Thus the Buscher rule obtained in the previous section is understood to provide a general rule for T-duality transformation with the general metric and B-field.

2.5 Non-geometric backgrounds

In this section we give a brief discussion on how the successive applications of T-duality transformation to the coventional backgrounds provide odd configurations referred to as non-geometric in literatures. For simplicity, the observations here starts with a three-torus with a constant H-flux, for more details and general discussions under general set-up, see [ref]. Since we regard the three-torus as a two-torus fibered over S^1, we firstly review on how a two-torus is characterized geometrically.

2.5.1 Two-torus

This section provides some notions on two-torii. A two-torus T^2 is known to be one of the simplest examples of the Calabi-Yau manifolds. In general, a Calabi-Yau n-fold is a Kähler manifold having n complex dimensions and vanishing the first Chern class. The two-torus is the only compact Calabi-Yau one-fold.

Consider a rectangular torus $T^2 = S^1 \times S^1$ with periodicity $x_1 \sim x_1 + R_1$ and $x_2 \sim x_2 + R_2$. It is convenient to introduce two complex parameters τ and ρ defined by

$$\tau = i\frac{R_1}{R_2}, \tag{2.111}$$

$$\rho = iR_1R_2, \tag{2.112}$$

which describes the complex structure and the size of the torus, respectively. A complex-structure deformation of the torus changes the value of the parameter τ, while a Kähler-structure deformation does ρ.

The rectangular torus is not the most general. There can be an angle θ between one cycle and the other of the two-torus. In general a T^2 is characterized by

$$ds^2 = \sum_{I,J=1}^{2} G_{IJ} dX^I dX^J, \quad B = \frac{1}{2}\sum_{I,J=1}^{2} B_{IJ} dX^I \wedge dX^J. \tag{2.113}$$

Hence its moduli are given by the following four real parameters

$$G_{IJ} = \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix}, \quad B_{IJ} = \begin{pmatrix} 0 & B_{12} \\ -B_{12} & 0 \end{pmatrix}. \tag{2.114}$$

These four real parameters can be traded for two complex parameters τ and ρ, as an extension above:

$$\tau = \tau_1 + i\tau_2 = \frac{G_{12}}{G_{22}} + i\frac{\sqrt{\det G}}{G_{22}}, \tag{2.115}$$

$$\rho = \rho_1 + i\rho_2 = B_{12} + i\sqrt{\det G}. \tag{2.116}$$

The relations between these parameters can be inverted yielding

$$G + B = \frac{\rho_2}{\tau_2}\begin{pmatrix} \tau_1^2 + \tau_2^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix} + \rho_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.117}$$

As is well known, the complex modulus τ determines an equivalent torus when it is replaced with

$$\tau \;\rightarrow\; \frac{a\tau + b}{c\tau + d}, \quad \text{with} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \tag{2.118}$$

which is obtained by successive applications of discrete transformations

$$\tau \rightarrow \tau + 1 \;\text{ and }\; \tau \rightarrow -\frac{1}{\tau}. \tag{2.119}$$

Although it is not so obvious, so does the other modulus ρ. One circumstantial evidence for ρ transforming as an $SL(2,\mathbb{Z})$ modulus is given by noting that they altogether provide the symmetry

$$SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z}) = SO(2,2;\mathbb{Z}), \tag{2.120}$$

i.e. the invariance under (discrete version of) $SO(2,2)$ transformations. As discussed in preceding sections, the invariance under $SO(2,2)$ is realized when there exist two directions of isometry. And indeed the discussion here is the case.

Especially for the case of the rectangular torus with vanishing B-field, since the metric becomes

$$G_{IJ} = R_I^2\,\delta_{IJ}, \tag{2.121}$$

and $\sqrt{G} = R_1R_2$, the parameters τ and ρ reduce to those mentioned above, (2.111) and (2.112). This is equivalent to $\tau_1 = \rho_1 = 0$ and then

$$G + B = \begin{pmatrix} \rho_2\tau_2 & 0 \\ 0 & \rho_2/\tau_2 \end{pmatrix}. \tag{2.122}$$

2.5.2 T-duality and non-geometric backgrounds

This section explains how T-duality transforms the conventional backgrounds into the configurations called non-geometric. We consider a three-torus with a constant H-flux as a starting point.

Three-torus with H-flux Consider the rectangular three-torus

$$ds^2 = R_1^2(dX^1)^2 + R_2^2(dX^2)^2 + R_3^2(dX^3)^2, \tag{2.123}$$

in the presence of a constant H-flux

$$\int_{T^3} H = N \in \mathbb{Z}. \tag{2.124}$$

The periodicities are given by identifications

$$X_I \sim X_I + 1, \quad I = 1,2 \text{ and } 3, \tag{2.125}$$

for each direction. The H-flux can be denoted as

$$H = N dX_1 \wedge dX_2 \wedge dX_3. \tag{2.126}$$

By choosing a gauge choice of the B-field as

$$B = NX_1 dX_2 \wedge dX_3, \tag{2.127}$$

we can conveniently regard this three-torus as a two-torus (X_2, X_3) fibered over $S^1(X_1)$. Then the complex structure and the Kähler structure read

$$\tau = i\frac{R_2}{R_3}, \tag{2.128}$$

$$\rho = NX_1 + iR_2R_3, \tag{2.129}$$

respectively, yielding monodromy of the Kähler structure

$$\rho \to \rho + N, \tag{2.130}$$

under a translation $X_1 \to X_1 + 1$. This is absorbed by an $SL(2,\mathbb{Z})$ transformation of the moduli parameter ρ with monodromy matrix

$$M_0(X_1) = \begin{pmatrix} 1 & NX_1 \\ 0 & 1 \end{pmatrix}_\rho . \tag{2.131}$$

First T-duality transformation We apply the Buscher rule

$$\begin{aligned} \tilde{g}_{00} = \frac{1}{g_{00}}, \quad \tilde{g}_{0i} = \frac{B_{0i}}{g_{00}}, \quad \tilde{g}_{ij} = g_{ij} - \frac{g_{0i}g_{0j} - B_{0i}B_{0j}}{g_{00}}, \\ \tilde{B}_{0i} = \frac{g_{0i}}{g_{00}}, \qquad \tilde{B}_{ij} = B_{ij} - \frac{g_{0i}B_{0j} - g_{0j}B_{0i}}{g_{00}}, \end{aligned} \tag{2.132}$$

regarding X_2-direction as an isometry. Then we find that the metric reads

$$ds^2 = R_1^2(dX_1)^2 + \frac{1}{R_2^2}(dX_2 + NX_1dX_3)^2 + R_3^2(dX_3)^2, \tag{2.133}$$

and the B-field vanishes

$$B = 0. \tag{2.134}$$

The periodicities are understood as follows: for X_3-direction is unchanged $X_3 \sim X_3 + 1$, while

$$(X_1, X_2) \sim (X_1 + 1, X_2 - NX_3), \tag{2.135}$$

which defines a twisted torus or nilmanifold.

Taking a basis of 1-forms $\{\eta^i\}$, $i = 1, 2$ and 3 as

$$\eta^1 = dX_1, \quad \eta^2 = dX_2 + NX_1dX_3, \quad \eta^3 = dX_3, \tag{2.136}$$

and its dual basis of vector fields $\{\eta_i\}$, $i = 1, 2$ and 3 as

$$\eta_1 = \partial_1, \quad \eta_2 = \partial_2, \quad \eta_3 = \partial_3 - NX_1\partial_2, \quad \text{with } \partial_I = \frac{\partial}{\partial X_I}, \tag{2.137}$$

we have

$$\eta^i(\eta_j) = \delta^i_j, \tag{2.138}$$

by construction. Then the metric can be rewritten as

$$ds^2 = (d\eta^1)^2 + (d\eta^2)^2 + (d\eta^3)^2. \tag{2.139}$$

They satisfy the following algebra

$$[\eta_1, \eta_3] = [\partial_1, \partial_3 - NX_1\partial_2] = -N\partial_2 = f^2_{13}\eta_2, \tag{2.140}$$

$$[\eta^2, \eta_1] = [dY_2 + NX_1dX_3, \partial_1] = -NdX_3 = f^2_{13}\eta^3, \tag{2.141}$$

$$[\eta^2, \eta_3] = -f^2_{13}\eta^1, \tag{2.142}$$

with $f^2_{13} = -N$.

Second T-duality transformation Since both the metric (2.133) and the B-field (2.134) are independent of any shift along X_3-direction. This z-direction is left as a direction of isometry. By performing the Buscher rule, we find that the resulting metric reads

$$ds^2 = dX_1^2 + \frac{1}{1+N^2X_1^2}(dX_2^2 + dX_3^2), \tag{2.143}$$

and the B-field does

$$B = \frac{-NX_1 dX_2 \wedge dX_3}{1+N^2X_1^2}. \tag{2.144}$$

Though the isometry with respect to X_1-coordinate is no longer manifest, introducing the notion of T-fold [18] enables to restore the notion of periodicity in the resulting geometry.

2.5.3 $O(d,d)$-invariant action

A construction of an $O(d,d)$-invariant action has been investigated [36–42] and [43–46]. Their resulting gravity theory is shown to be physically equivalent to the original low-energy supergravity theory of the Neveu-Schwarz-Neveu-Schwarz (NSNS) sector

$$\mathcal{L}_{\text{NSNS}} = \sqrt{|g|}\left(R - \frac{1}{12}H_{ijk}H^{ijk}\right). \tag{2.145}$$

Although their considerations are done in the framework of doubled field theory (DFT), giving an introduction to DFT is beyond the scope of this dissertation, see [47–51] for more details about DFT. In [36–42], the dynamical variables are given by two doublets[3]

$$(g_{ij}, B_{ij}) \quad \text{and} \quad (G_{ij}, \beta^{ij}). \tag{2.146}$$

The former doublet (g_{ij}, B_{ij}) consists of the conventional NSNS sector fields. The latter doublet is related to the former by

$$\tilde{\mathcal{E}}_{ij}^{-1} = \mathcal{E}_{ij} = g_{ij} + B_{ij}, \tag{2.147}$$

$$\mathcal{E}^{ij} = G^{ij} + \beta^{ij}. \tag{2.148}$$

These relations are realized as a T-duality transformation (2.70) and (2.71) .
DFT treats not only the conventional coordinate x^i but also a dual coordinate $\tilde{x}_i$ to accomplish an invariant theory under $O(n,n)$ transformations which include T-duality transformations. The doubled coordinates X^M are introduced by $X^M = (\tilde{x}_i, x^i)$. To remove the redundancy of these doubled coordinate dependence in the formulation, the "strong constraint" or "section condition" should be imposed

$$\eta^{MN}\partial_M\partial_N = 0, \quad \eta^{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.149}$$

[3] To be precise, there is the dilation field in addition to the metric and the B-field.

on any fields, gauge parameters and etc. where $\partial_M = (\tilde{\partial}^i, \partial_i)$. Imposing this condition is equivalent to imposing

$$\partial_i \tilde{\partial}^i = 0. \tag{2.150}$$

The gauge transformation in DFT is parametrized by $\xi^M = (\tilde{\xi}_i, \xi^i)$ as

$$\begin{aligned}\delta \mathcal{E}_{ij} =& \mathcal{L}_\xi \mathcal{E}_{ij} + \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i \\ &+ \mathcal{L}_{\tilde{\xi}} \mathcal{E}_{ij} - \mathcal{E}_{ik}(\tilde{\partial}^k \xi^l - \tilde{\partial}^l \xi^k)\mathcal{E}_{lj},\end{aligned} \tag{2.151}$$

where

$$\mathcal{L}_\xi \mathcal{E}_{ij} = \xi^k \partial_k \mathcal{E}_{ij} + \partial_i \xi^k \mathcal{E}_{kj} + \partial_j \xi^k \mathcal{E}_{ki}, \tag{2.152}$$

$$\mathcal{L}_{\tilde{\xi}} \mathcal{E}_{ij} = \tilde{\xi}_k \tilde{\partial}^k \mathcal{E}_{ij} - \tilde{\partial}^k \tilde{\xi}_i \mathcal{E}_{kj} - \tilde{\partial}^k \tilde{\xi}_j \mathcal{E}_{ki}. \tag{2.153}$$

The difference of the signs in front of the gauge parameters are chosen to make $\mathcal{E}_{ij}$ a covariant tensor with respect to x^i whereas to make $\mathcal{E}_{ij}$ a contravariant tensor with respect to $\tilde{x}_i$.
In [36–42], a construction of an invariant theory under these gauge transformations is accomplished in somehow a heuristic manner. A derivative operator is defined by

$$\tilde{D}^i = \tilde{\partial}^i - \beta^{ij}\partial_j, \tag{2.154}$$

with the anti-symmetric tensor β. This derivative operator has a non-trivial commutator

$$[\tilde{D}^i, \tilde{D}^j] = -R^{ijk}\partial_k - Q_k^{\ ij}\tilde{D}^k, \tag{2.155}$$

giving

$$R^{ijk} = 3\tilde{D}^{[i}\beta^{jk]} = 3(\tilde{\partial}^{[i}\beta^{jk]} + \beta^{l[i}\partial_l \beta^{jk]}), \tag{2.156}$$

$$Q_k^{\ ij} = \partial_k \beta^{ij}. \tag{2.157}$$

A covariant derivative with upper indices is introduced by

$$\tilde{\nabla}^i V^j = \tilde{D}^i V^j - \breve{\Gamma}^{ij}_k V^k, \quad \tilde{\nabla}^i V_j = \tilde{D}^i V_j + \breve{\Gamma}^{ik}_j V_k. \tag{2.158}$$

The connection coefficients are given by

$$\breve{\Gamma}^{(ij)}_k = \breve{\Gamma}^{ij}_k - G_{kl}(G^{mi}\breve{\Gamma}^{[jl]}_m + G^{mj}\breve{\Gamma}^{[il]}_m), \tag{2.159}$$

where

$$\breve{\Gamma}^{ij}_k = \frac{1}{2}G_{kl}(\tilde{D}^i G^{jl} + \tilde{D}^j G^{il} - \tilde{D}^l G^{ij}), \tag{2.160}$$

$$\breve{\Gamma}^{[ij]}_k = -\frac{1}{2}Q_k^{\ ij}. \tag{2.161}$$

Hence, explicitly they read

$$\begin{aligned}\breve{\Gamma}^{ij}_{k} &= \frac{1}{2}G_{kl}[(\tilde{\partial}^i - \beta^{im}\partial_m)G^{jl} + (\tilde{\partial}^j - \beta^{jm}\partial_m)G^{il} - (\tilde{\partial}^l - \beta^{lm}\partial_m)G^{ij}] \\ &= \widetilde{\Gamma}^{ij}_{k} - \frac{1}{2}G_{kl}(\beta^{im}\partial_m G^{jl} + \beta^{jm}\partial_m G^{il} - \beta^{lm}\partial_m G^{ij}) \qquad (2.162)\\ \breve{\Gamma}^{[ij]}_{k} &= -\frac{1}{2}\partial_k\beta^{ij}, \qquad (2.163)\end{aligned}$$

where $\widetilde{\Gamma}^{ij}_{k}$ is introduced by

$$\widetilde{\Gamma}^{ij}_{k} = \frac{1}{2}G_{kl}(\tilde{\partial}^i G^{jl} + \tilde{\partial}^j G^{il} - \tilde{\partial}^l G^{ij}). \qquad (2.164)$$

Thus the connection coefficients read

$$\begin{aligned}\breve{\Gamma}^{(ij)}_{k} &= \tilde{\Gamma}^{ij}_{k} - \frac{1}{2}G_{kl}(\beta^{im}\partial_m G^{jl} + \beta^{jm}\partial_m G^{il} - \beta^{lm}\partial_m G^{ij} - G^{mi}\partial_m\beta^{jl} - G^{mj}\partial_m\beta^{il}), \qquad (2.165)\\ \breve{\Gamma}^{[ij]}_{k} &= -\frac{1}{2}\partial_k\beta^{ij}. \qquad (2.166)\end{aligned}$$

Since the construction of this connection is based on the doubled gauge transformations and the requirement that the theory should be covariant, so far the geometrical meaning of the connection has been less clear. Eventually, the meaning of the connection will be demystified in this dissertation from the viewpoint of the algebroid.

Chapter 3

Generalized geometry

In the previous chapter, we observed that the algebra of the currents associated with the gauge transformations of the background fields forms the Dorfman bracket, or equivalently Courant bracket and there are natural actions of $O(n,n)$ transformations on background fields. Generalized geometry provides a framework that formulates these observations in mathematical manner. In this chapter, we present some notion of generalized geometry, focussing only on what are needed in this dissertation. For systematic introduction, see [24], for review on mathematical aspects, see [27], and review for physical applications, see [52, 53]. Firstly, we give definitions of the algebraic structures, called the Lie algebroid and the Courant algebroid. Then we introduce the generalized tangent bundle as an example of the Courant algebroid on which generalized geometry is defined. We briefly discuss some properties of the generalized tangent bundle. Finally, we review on the construction of an analogue of Riemannian geometry in the framework of generalized geometry.

3.1 Lie algebroid

In this section, we introduce the notion of Lie algebroid [54, 55]. The Lie algebroid is the most fundamental structure to define (analogue of) generalized geometry.

Definition

A Lie algebroid is defined by a triple $(E, [\cdot,\cdot]_E, \rho)$. Here E denotes a vector bundle $\pi : E \to M$ over a base manifold M. We represent the set of the sections of this vector bundle E as $\Gamma(E)$, as usual. The bracket $[\cdot,\cdot]_E$ is a Lie bracket *i.e.* a skew-symmetric bi-linear map

$$[\,\cdot,\,\cdot]_E : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E), \tag{3.1}$$

satisfying the Jacobi identity. And ρ is an anchor map $\rho : E \to TM$, such that 1) the induced map $\rho : \Gamma(E) \to \Gamma(TM)$ defines a Lie-algebra homomorphism and 2) it satisfies the Leibniz rule

$$[X, fY]_E = f[X,Y]_E + (\rho(X)f)Y, \tag{3.2}$$

for any $X, Y \in \Gamma(E)$ and $f \in C^\infty(M)$. Here $\rho(X)f$ denotes the action of the vector field $\rho(X) \in \Gamma(TM)$ on the function $f \in C^\infty(M)$ resulting in the derived function of f along the vector field $\rho(X)$.

Tangent bundle as a Lie algebroid

The tangent bundle TM is a trivial example of the Lie algebroid. The identity map $\mathrm{id} : \Gamma(TM) \to \Gamma(TM)$ defines the anchor map and the Lie bracket is defined by the commutator

$$[X, Y]_{TM} = XY - YX, \quad \text{where} \quad X, Y \in \Gamma(TM), \tag{3.3}$$

as usual. One can easily show that the triple $(TM, [\,\cdot\,,\,\cdot\,]_{TM}, \mathrm{id})$ actually satisfies the conditions to be a Lie algebroid mentioned above. We will simply denote this Lie algebroid $(TM, [\,\cdot\,,\,\cdot\,]_{TM}, \mathrm{id})$ as TM.

Lie algebroid of a Poisson manifold

Poisson bivector Let (M, θ) be a Poisson manifold equipped with a Poisson bivector $\theta \in \Gamma(\wedge^2 TM)$. In some local coordinate system $\{x^m\}$, it is represented in components as

$$\theta = \frac{1}{2}\theta^{ij}\partial_i \wedge \partial_j. \tag{3.4}$$

The Poisson bivector defines the Poisson bracket as

$$\{f, g\} = \theta^{ij}\partial_i f \partial_j g = \theta^{ij}\partial_i(df)\partial_j(dg) =: i_{dg} i_{df}\theta. \tag{3.5}$$

The Poisson bivector θ satisfies $[\theta, \theta]_S = 0$, where $[\cdot, \cdot]_S$ is the Schouten-Nijenhuis bracket, an extension of the Lie bracket $[\,\cdot\,,\,\cdot\,]_{TM}$ to the bracket acting on poly-vector fields $\Gamma(\wedge^\bullet TM)$. This is calculated as

$$\begin{aligned}
&[\theta^{ij}\partial_i \wedge \partial_j, \theta^{kl}\partial_k \wedge \partial_l]_S \\
&= [\theta^{ij}\partial_i, \theta^{kl}\partial_k] \wedge \partial_j \wedge \partial_l - [\theta^{ij}\partial_i, \partial_l] \wedge \partial_j \wedge \theta^{kl}\partial_k - [\partial_j, \theta^{kl}\partial_k] \wedge \theta^{ij}\partial_i \wedge \partial_l \\
&= (\theta^{ij}\partial_i\theta^{kl}\partial_k - \theta^{kl}\partial_k\theta^{ij}\partial_i) \wedge \partial_j \wedge \partial_l + \theta^{kl}\partial_l\theta^{ij}\partial_i \wedge \partial_j \wedge \partial_k - \theta^{ij}\partial_j\theta^{kl}\partial_k \wedge \partial_i \wedge \partial_l \\
&= 4\theta^{kl}\partial_l\theta^{ij}\partial_i \wedge \partial_j \wedge \partial_k.
\end{aligned} \tag{3.6}$$

Thus the condition $[\theta, \theta]_S = 0$ implies, in terms of components,

$$\theta^{l[k}\partial_l\theta^{ij]} = 0. \tag{3.7}$$

This condition guarantees that the Poisson bracket $\{\cdot, \cdot\}$ satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \tag{3.8}$$

for any smooth functions $f, g, h \in C^\infty(M)$.

Lie algebroid A Lie algebroid of a Poisson manifold [56] is defined by a triple $(T^*M, [\cdot,\cdot]_\theta, \theta)$. Here T^*M denotes the cotangent bundle. The anchor map is induced by the Poisson bi-vector θ by regarding it as a map $\theta : \Gamma(T^*M) \to \Gamma(TM)$ i.e.

$$\theta(\xi) = i_\xi \theta, \quad \text{for} \quad \xi \in \Gamma(T^*M), \tag{3.9}$$

in components

$$i_\xi \theta = i_{\xi_k dx^k}\left(\frac{1}{2}\theta^{ij}\partial_i \wedge \partial_j\right) = \xi_i \theta^{ij}\partial_j. \tag{3.10}$$

The Lie bracket $[\cdot,\cdot]_\theta$ is defined by the Koszul bracket:

$$[\xi,\eta]_\theta = \mathcal{L}_{\theta(\xi)}\eta - i_{\theta(\eta)}d\xi, \tag{3.11}$$

in components

$$\begin{aligned}
[\xi,\eta]_\theta &= (di_{\theta(\xi)} + i_{\theta(\xi)}d)\eta - i_{\theta(\eta)}d\xi \\
&= d(\xi_i\theta^{ij}\eta_j) + i_{\xi_i\theta^{ij}\partial_j}(\partial_k\eta_l)dx^k \wedge dx^l - i_{\eta_i\theta^{ij}\partial_j}(\partial_k\xi_l)dx^k \wedge dx^l \\
&= \partial_k(\xi_i\theta^{ij}\eta_j)dx^k + \xi_i\theta^{ij}(\partial_j\eta_k)dx^k - \xi_i\theta^{ij}(\partial_k\eta_j)dx^k - \eta_i\theta^{ij}(\partial_j\xi_k)dx^k + \eta_i\theta^{ij}(\partial_k\xi_j)dx^k \\
&= (\xi_i\eta_j\partial_k\theta^{ij} + \xi_i\theta^{ij}\partial_j\eta_k - \eta_i\theta^{ij}\partial_j\xi_k)dx^k.
\end{aligned} \tag{3.12}$$

The skew-symmetry is manifest if we rewrite the bracket as

$$[\xi,\eta]_\theta = \mathcal{L}_{\theta(\xi)}\eta - \mathcal{L}_{\theta(\eta)}\xi + d(i_\xi i_\eta\theta). \tag{3.13}$$

We will often denote this Lie algebroid $(T^*M, [\cdot,\cdot]_\theta, \theta)$ as $(T^*M)_\theta$ for short.

3.2 Courant algebroid

In this section we give the definition of the Courant algebroid. The Courant algebroids constructed from the Lie algebroid TM and $(T^*M)_\theta$, introduced in the previous section, are the main objects, which are considered in the generalized geometry and the Poisson generalized geometry, respectively. We explain the former as an example of the Courant algebroid.

Definiftion

A Courant algebroid is defined by a quadruple $(E, [\cdot,\cdot]_E, \rho, \langle\cdot,\cdot\rangle_E)$, where E is a vector bundle over a base manifold M, the bracket $[\cdot,\cdot]_E$ is a skew-symmetric bi-linear bracket *i.e.*

$$[\cdot,\cdot]_E : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E), \tag{3.14}$$

ρ is a bundle map $\rho : E \to TM$ and $\langle\cdot,\cdot\rangle_E$ denotes non-degenerate symmetric product on $\Gamma(E)$,

$$\langle\cdot,\cdot\rangle_E : \Gamma(E) \otimes \Gamma(E) \to C^\infty(M). \tag{3.15}$$

They are required to satisfy the following five conditions: 1) for $e_1, e_2 \in \Gamma(E)$ the induced map $\Gamma(E) \to \Gamma(TM)$ satisfies

$$\rho[e_1, e_2]_E = [\rho(e_1), \rho(e_2)]_{TM}, \tag{3.16}$$

2) and for any function $f \in C^\infty(M)$,

$$[e_1, fe_2]_E = f[\rho(e_1), \rho(e_2)]_{TM} + (\rho(e_1)f)e_2 - \langle e_1, e_2\rangle \mathcal{D}f, \tag{3.17}$$

where the map $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined by $\mathcal{D} = \rho^* d$. Since ρ^* is the pull-back of ρ which defines a map: $\Gamma(T^*M) = \Gamma^*(TM) \to \Gamma(E)$ and d is the de Rham differential: $C^\infty(M) \to \Gamma(T^*M)$, the map $\mathcal{D}$ is defined, which means that

$$\langle \mathcal{D}f, e\rangle = \langle \rho^* df, e\rangle = \langle df, \rho(e)\rangle = \rho(e)f. \tag{3.18}$$

3)The map $\mathcal{D}$ is required to satisfy

$$\rho \circ \mathcal{D} = 0, \tag{3.19}$$

which induces, for any functions $f, g \in C^\infty(M)$,

$$\langle \mathcal{D}f, \mathcal{D}g\rangle = \langle \rho^* df, \mathcal{D}g\rangle = \rho(\mathcal{D}g)f = 0. \tag{3.20}$$

And 4) for $e, e_1, e_2 \in \Gamma(E)$,

$$\rho(e)\langle e_1, e_2\rangle = \langle [e, e_1] + \mathcal{D}\langle e, e_1\rangle, e_2\rangle + \langle e_1, [e, e_2] + \mathcal{D}\langle e, e_2\rangle\rangle, \tag{3.21}$$

finally, 5)

$$\mathcal{D}\mathcal{T}(e, e_1, e_2) = \mathcal{J}(e, e_1, e_2). \tag{3.22}$$

Here $\mathcal{T}$ and $\mathcal{J}$ are defined, respectively, as follows

$$\mathcal{T}(e, e_1, e_2) = \frac{1}{3}(\langle [e, e_1], e_2\rangle + \langle [e_1, e_2], e\rangle + \langle [e_2, e], e_1\rangle), \tag{3.23}$$

$$\mathcal{J}(e, e_1, e_2) = [[e, e_1], e_2] + [[e_1, e_2], e] + [[e_2, e], e_1]. \tag{3.24}$$

Generalized tangent bundle as Courant algebroid

Let M be a n-dimensional smooth manifold, TM be the tangent and T^*M be the cotangent bundle, respectively. The formal sum of these bundles, $TM \oplus T^*M$ is referred to as the generalized tangent bundle. A section of $TM \oplus T^*M$ is called a generalized vector field, consists of a sum of a vector field $X \in \Gamma(TM)$ and a 1-form $\xi \in \Gamma(T^*M)$, $X + \xi \in \Gamma(TM \oplus T^*M)$. So that the dimension of the fiber is $2n$.

The quadruple $(TM \oplus T^*M, [\cdot, \cdot], \rho, \langle \cdot, \cdot\rangle)$ defines a Courant algebroid, where the bracket, the anchor map and the bilinear form are given, respectively, by for $X + \xi, Y + \eta \in \Gamma(TM \oplus T^*M)$,

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi), \tag{3.25}$$

$$\rho(X + \xi) = X, \tag{3.26}$$

$$\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(i_X\eta + i_Y\xi). \tag{3.27}$$

Since the anchor map ρ acts as a projection $\rho : \Gamma(TM \oplus T^*M) \to \Gamma(TM)$, the induced map ρ^* defines an injection $\rho^* : \Gamma(T^*M) \to \Gamma(TM \oplus T^*M)$. Thus, the condition 1) and 3) are easily understood. Following the definitions, straightforward calculations yield, for $u, v, w \in \Gamma(TM \oplus T^*M)$,

$$[u, fv] = f[u, v] + (\rho(u)f)v - \langle u, v\rangle df \,, \tag{3.28}$$

$$\rho(u)\langle v, w\rangle = \langle [u, v] + d\langle u, v\rangle, w\rangle + \langle v, [u, w] + d\langle u, w\rangle\rangle. \tag{3.29}$$

They correspond to the conditions 2) and 4), respectively. Finally, the condition 5) is shown as follows. With noting that

$$\begin{aligned}
&\mathcal{DT}(X_1 + \xi_1, X_2 + \xi_2, X_3 + \xi_3)\\
&= \frac{1}{6}\left[di_{[X_1,X_2]} + di_{X_1}\left(\mathcal{L}_{X_2} - \frac{1}{2}di_{X_2}\right) - di_{X_2}\left(\mathcal{L}_{X_1} - \frac{1}{2}di_{X_1}\right)\right]\xi_3\\
&\qquad + (1 \to 2 \to 3 \to 1) + (1 \to 3 \to 2 \to 1),
\end{aligned}$$

while

$$\begin{aligned}
&\mathcal{J}(X_1 + \xi_1, X_2 + \xi_2, X_3 + \xi_3)\\
&= \frac{1}{2}\left[- di_{[X_1,X_2]} + \mathcal{L}_{X_1}di_{X_2} - \mathcal{L}_{X_2}di_{X_1} + di_{X_1}\left(\mathcal{L}_{X_2} - \frac{1}{2}di_{X_2}\right) - di_{X_2}\left(\mathcal{L}_{X_1} - \frac{1}{2}di_{X_1}\right)\right]\xi_3\\
&\qquad + (1 \to 2 \to 3 \to 1) + (1 \to 3 \to 2 \to 1),
\end{aligned}$$

thus what we have to show is $A = B$, where

$$\begin{aligned}
A &= \frac{1}{3}\left[di_{[X_1,X_2]} + di_{X_1}\left(\mathcal{L}_{X_2} - \frac{1}{2}di_{X_2}\right) - di_{X_2}\left(\mathcal{L}_{X_1} - \frac{1}{2}di_{X_1}\right)\right],\\
B &= -di_{[X_1,X_2]} + \mathcal{L}_{X_1}di_{X_2} - \mathcal{L}_{X_2}di_{X_1} + di_{X_1}\left(\mathcal{L}_{X_2} - \frac{1}{2}di_{X_2}\right) - di_{X_2}\left(\mathcal{L}_{X_1} - \frac{1}{2}di_{X_1}\right),
\end{aligned}$$

or equivalently, $\tilde{A} = \tilde{B}$, with

$$\begin{aligned}
\tilde{A} &= 4di_{[X_1,X_2]}\\
\tilde{B} &= 3\mathcal{L}_{X_1}di_{X_2} - 3\mathcal{L}_{X_2}di_{X_1} + di_{X_1}(2\mathcal{L}_{X_2} - di_{X_2}) - di_{X_2}(2\mathcal{L}_{X_1} - di_{X_1}).
\end{aligned}$$

Thus it is actually shown to be valid:

$$\begin{aligned}
\tilde{A} &= 2di_{[X_1,X_2]} - 2di_{[X_2,X_1]} = 2d(\mathcal{L}_{X_1}i_{X_2} - i_{X_2}\mathcal{L}_{X_1})2 - d(\mathcal{L}_{X_2}i_{X_1} - i_{X_1}\mathcal{L}_{X_2})\\
&= 2di_{X_1}di_{X_2} - 2di_{X_2}(di_{X_1} + i_{X_1}d) - 2di_{X_2}di_{X_1} + 2di_{X_1}(di_{X_2} + i_{X_2}d)\\
&= 4di_{X_1}di_{X_2} - 4di_{X_2}di_{X_1} + 4di_{X_1}i_{X_2}d,\\
\tilde{B} &= 3di_{X_1}di_{X_2} - 3di_{X_2}di_{X_1} + 2di_{X_1}i_{X_2}d + di_{X_1}di_{X_2} - 2di_{X_2}i_{X_1}d - di_{X_2}di_{X_1} = \tilde{A}.
\end{aligned}$$

3.3 Generalized geometry

In the rest of this chapter, we are denoted in the generalized tangent bundle, i.e. the Courant algebroid $TM \oplus T^*M$ defined as above: Its sections consist of the sum $u+\xi$ of vector field $u \in \Gamma(TM)$ and 1-form $\xi \in \Gamma(T^*M)$, equipped with the Courant bracket, the bilinear form, and the anchor map $\rho : TM \oplus T^*M \to TM$.

3.3.1 Some properties

Bilinear form under $SO(n,n)$-transformation

The bilinear form (3.27) can be represented as

$$\langle u+\xi, v+\eta\rangle = \frac{1}{2}\begin{pmatrix} u \\ \xi \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v \\ \eta \end{pmatrix}. \tag{3.30}$$

Thus the action of $SO(n,n)$ is naturally considered. Since

$$\begin{pmatrix} u \\ \xi \end{pmatrix} \to \begin{pmatrix} u' \\ \xi' \end{pmatrix} = S^{-1}\begin{pmatrix} u \\ \xi \end{pmatrix}, \tag{3.31}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to S^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.32}$$

the bilinear form is invariant under $SO(n,n)$ transformation

$$\langle u'+\xi', v'+\eta'\rangle = \langle u+\xi, v+\eta\rangle. \tag{3.33}$$

The $SO(n,n)$ transformation can be classified as (2.99)

$$\begin{pmatrix} e^A & 0 \\ 0 & e^{-A^T} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}. \tag{3.34}$$

The actions of these transformations on generalized vectors are represented as,

$$\begin{pmatrix} u' \\ \xi' \end{pmatrix} = \begin{pmatrix} 1+\mathcal{L}_v & 0 \\ 0 & 1+\mathcal{L}_v \end{pmatrix}\begin{pmatrix} u \\ \xi \end{pmatrix} = \begin{pmatrix} u^\mu - u^\nu\partial_\nu v^\mu \\ \xi_\mu + \xi_\nu\partial_\mu v^\nu \end{pmatrix}, \tag{3.35}$$

$$\begin{pmatrix} u' \\ \xi' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} u \\ \xi \end{pmatrix} = \begin{pmatrix} u^\mu \\ \xi_\mu + B_{\nu\mu}u^\nu \end{pmatrix}, \tag{3.36}$$

$$\begin{pmatrix} u' \\ \xi' \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ \xi \end{pmatrix} = \begin{pmatrix} u^\mu + \beta^{\nu\mu}\xi_\nu \\ \xi_\mu \end{pmatrix}, \tag{3.37}$$

respectively. The first is surely a diffeomorphism generated by a vector field v. The second is understood to correspond to the B-transformation as follows: recalling that the generalized tangent vector is a parameter of the current, we find

$$\begin{aligned}\mathcal{J}_{(u,\xi)} &= u^\mu p_\mu + \xi_\mu \partial_\sigma X^\mu \\ &\to u^\mu(p_\mu + B_{\mu\nu}\partial_\sigma X^\nu) + \xi_\mu\partial_\sigma X^\mu = u^\mu \Pi_\mu + \xi_\mu \partial_\sigma X^\mu.\end{aligned} \tag{3.38}$$

And the last one can be understood to yield

$$\begin{aligned}\mathcal{J}_{(u,\xi)} &= u^\mu p_\mu + \xi_\mu \partial_\sigma X^\mu \\ &\to u^\mu p_\mu + \xi_\mu(\partial_\sigma X^\mu + \beta^{\mu\nu}p_\nu),\end{aligned} \tag{3.39}$$

but it does not have clear interpretation.

Courant bracket under $SO(n,n)$-transformation

Here we investigate responses of the Courant bracket to $SO(n,n)$-transformations.

Diffeomorphism In the ordinary differential geometry TM, a diffeomorphism $\varphi : M \to M$ induces an automorphism $\varphi_* : TM \to TM$, so that the symmetry of the Lie algebroid TM consists of diffeomorphisms, Diff(M). An infinitesimal diffeomorphism generated by a vector field $w = w^\mu \partial_\mu$, and its action on $\Gamma(TM)$ is represented by the Lie derivative $\mathcal{L}_w$, as $\mathcal{L}_w u = [w, u]$ for $u \in \Gamma(TM)$.
For the generalized tangent bundle $TM \oplus T^*M$, a diffeomorphism $\varphi : M \to M$ induces an automorphism $\varphi_* \oplus \varphi^* : TM \oplus T^*M \to TM \oplus T^*M$. Its infinitesimal form can again be represented by the Lie derivative. Under an infinitesimal diffeomorphism the bracket responds as

$$\begin{aligned}
&[(1+\mathcal{L}_w)(u+\xi), (1+\mathcal{L}_w)(v+\eta)] \\
&= [u+\xi, v+\eta] + [\mathcal{L}_w u, v] + [u, \mathcal{L}_w v] + \mathcal{L}_{\mathcal{L}_w u}\eta + \mathcal{L}_u \mathcal{L}_w \eta - \mathcal{L}_{\mathcal{L}_w v}\xi - \mathcal{L}_v \mathcal{L}_w \xi \\
&\quad - \frac{1}{2} d(i_{\mathcal{L}_w u}\eta + i_u \mathcal{L}_w \eta - i_{\mathcal{L}_w v}\xi - i_v \mathcal{L}_w \xi) \\
&= [u+\xi, v+\eta] + [w, [u,v]] + \mathcal{L}_w \mathcal{L}_u \eta - \mathcal{L}_w \mathcal{L}_u \eta - \frac{1}{2}\mathcal{L}_w d(i_u \eta - i_v \xi) \\
&= (1+\mathcal{L}_w)[u+\xi, v+\eta],
\end{aligned} \tag{3.40}$$

with using the Cartan formulae. Thus the Courant bracket closes under diffeomorphisms.

B-transformation One of the significant properties of the Courant bracket is that it is isomorphic under the B-transformation e^B with closed 2-form, $dB = 0$: For arbitrary generalized tangent vectors $u = X + \xi$ and $v = Y + \eta$,

$$\begin{aligned}
[e^B(u), e^B(v)] &= [u + B(X), v + B(Y)] \\
&= [u,v] + i_{[X,Y]}B - i_X i_Y dB \\
&= e^B([u,v]) + i_Y i_X dB.
\end{aligned} \tag{3.41}$$

For later convenience and actually we will be interested in the case of $dB \neq 0$, we retain here the term involving dB.
Note that an H-twisted bracket is defined by

$$[u,v]_H = [u,v] + i_Y i_X H. \tag{3.42}$$

Using this bracket, for the case with $dB \neq 0$ the B-transformed bracket is represented as

$$e^{-B}[e^B(u), e^B(v)] = [u,v]_{dB}. \tag{3.43}$$

β-transformation The β-transformation yields, for $u = X + \xi$ and $v = Y + \eta$,

$$\begin{aligned}
[e^\beta(u), e^\beta(v)] &= [u + \beta(\xi), v + \beta(\eta)] \\
&= [u,v] + [u, \beta(\eta)] + [\beta(\xi), v] - [\beta(\xi), \beta(\eta)] \\
&= [u,v] - \mathcal{L}_{\beta(\eta)}(X+\xi) + \mathcal{L}_{\beta(\xi)}(Y+\eta) - \mathcal{L}_{\beta(\xi)}\beta(\eta),
\end{aligned} \tag{3.44}$$

while

$$e^{\beta}([u,v]) = [u,v] + \beta\Big(\mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_y\xi)\Big) \tag{3.45}$$

So that

$$\begin{aligned}&[e^{\beta}(u), e^{\beta}(v)] - e^{\beta}([u,v])\\ &= -\mathcal{L}_{\beta(\eta)}(X+\xi) + \mathcal{L}_{\beta(\xi)}(Y+\eta) - \mathcal{L}_{\beta(\xi)}\beta(\eta) - \beta\Big(\mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_y\xi)\Big).\end{aligned} \tag{3.46}$$

This bracket can not close only using the choice of the bi-vector field β. However, we can do it by selecting the form of generalized vectors, as mentioned later.

Generalized Lie derivative As mentioned above, the symmetry of the Courant algebroid $TM \oplus T^*M$ turns out to be a semi-direct product $\mathrm{Diff}(M) \times \Omega^2_{\mathrm{closed}}(M)$, of the diffeomorphism and the B-field transformations. We call an element of this group as generalized diffeomorphism.
An infinitesimal generalized diffeomorphism is represented by a generalized Lie derivative $\mathcal{L}_{(w,b)}$, which acts on $u+\xi \in \Gamma(TM \oplus T^*M)$ as

$$\mathcal{L}_{(w,b)}(u+\xi) := \mathcal{L}_w(u+\xi) + i_u b. \tag{3.47}$$

Especially when b is exact, $b = -d\Lambda$, it reduces to the Dorfman bracket. We represent the generalized Lie derivative of such a case as $\mathcal{L}_{w+\Lambda}$,

$$\mathcal{L}_{w+\Lambda}(u+\xi) = \mathcal{L}_w(u+\xi) - i_u d\Lambda = [w+\Lambda, u+\xi]_D. \tag{3.48}$$

3.3.2 Dirac structure

A Dirac structure is a subbundle $L \subset TM \oplus T^*M$ satisfying the following three- conditions: 1) it is involutive under the Dorfman bracket $[u+\xi, v+\eta] \in L$ for $u+\xi, v+\eta \in \Gamma(L)$, 2) it is isotropic under the canonical inner product, which means $\langle u+\xi, v+\eta\rangle = 0$ for $u+\xi, v+\eta \in \Gamma(L)$ and 3) it has the maximal rank i.e. the fibre dimension of L is n, half of that of $TM \oplus T^*M$.
Since the Dorfman bracket on $\Gamma(L)$ beceoms skew-symmetric, a Dirac structure defines a Lie algebroid by definition.
It is immediately understood that a generalized diffeomorphism (3.48) by an element of L is a symmetry of the Dirac structure L. Indeed, the action $\mathcal{L}_{w+\Lambda}$ for $w+\Lambda \in \Gamma(L)$ on a section $u+\xi \in \Gamma(L)$ results in again on L due to the involutiveity $\mathcal{L}_{w+\Lambda}(u+\xi) = [w+\Lambda, u+\xi]_D \in L$. We call it as a L-diffeomorphism.
Trivial examples of the Dirac structure are TM and T^*M. Another examples are obtained by B-transformation of TM and β-transformation of T^*M, which we describe below.

B-transformation of TM

Given an arbitrary 2-form $\omega \in \Gamma(\wedge^2 T^*M)$, B-transformation of TM with ω defines a subbundle $L_\omega = e^{\omega}(TM)$, which is denoted by

$$L_\omega = \{e^{\omega}(u) = u + \omega(u) \big| u \in \Gamma(TM)\}. \tag{3.49}$$

Here the 2-form ω is regarded as a map $\Gamma(TM) \to \Gamma(T^*M)$. It is defined by

$$\omega(u) := \omega(u, \cdot) = i_u\omega = \omega_{\mu\nu}u^\mu dx^\nu, \tag{3.50}$$

where the last expression is the representation in some local coordinates. There the 2-form ω is represented as $\omega = \frac{1}{2}\omega_{\mu\nu}dx^\mu \wedge dx^\nu$.
The subbundle L_ω defines a Dirac structure if and only if ω is a closed 2-form, $d\omega = 0$. This is because the B-transformation generated by a closed 2-form is a symmetry of the generalized tangent bundle.

β-transformation of T^*M

Given an arbitrary bi-vector $\theta \in \Gamma(\wedge^2 TM)$, a β-transformation of T^*M defines a subbundle $L_\theta = e^\theta(TM)$,

$$L_\theta = \{e^\theta(\xi) = \xi + \theta(\xi)\big|\xi \in \Gamma(T^*M)\}. \tag{3.51}$$

Here, the bivector θ is considered as a map $\Gamma(T^*M) \to \Gamma(TM)$, defined by

$$\theta(\xi) := \theta(\xi, \cdot) = \iota_\xi\theta = \theta^{\mu\nu}\xi_\mu\partial_\nu, \tag{3.52}$$

where the last expression is a local expression written in local coordinates.
The subbundle L_θ defines a Dirac structure if and only if θ is a Poisson bivector i.e. $[\theta, \theta]_S = 0$ where $[\cdot, \cdot]_S$ is the Schouten-Nijenhuis bracket. We again mention that this condition to be a Poisson bivector is the same condition for the Jacobi identity of the Poisson bracket $\{f, g\} = \theta(df, dg)$ for $f, g \in C^\infty(M)$ to be satisfied.

Two ways to describe the same Dirac structure

In the last two examples, the assumptions are equivalent to the requirement that the spacetime (M, ω) is equipped with a symplectic form ω. In the generalized geometry, there are two possibilities L_θ and L_ω which define the same Dirac structure, as we discuss in the following.
As mentioned above, any element of L_ω can be represented by using a vector $u \in \Gamma()$ as $u + \omega(u)$, and any element of L_θ can be represented by using 1-form $\xi \in \Gamma(T^*M)$ as $\xi + \theta(\xi)$. If the two Dirac structures define the same subbundle $L_\theta = L_\omega$, there must be one to one correspondence between these two representations:

$$\xi + \theta(\xi) = u + \omega(u). \tag{3.53}$$

Comparing the vectors and forms in both sides, we get

$$\xi = \omega(u), \ u = \theta(\xi). \tag{3.54}$$

Substituting the first relation into the second equation yields

$$u = \theta(\omega(u)). \tag{3.55}$$

Since u is arbitrary, we have a relation between matrices

$$\theta^{\mu\nu} = (\omega_{\mu\nu})^{-1}. \tag{3.56}$$

This relation is also pointed out in [57]. In this example, it gives a rather trivial statement that a symplectic structure defines a Poisson bivector as its inverse. With this setting we have the two descriptions of the same Dirac structure, the one is by TM and the other is T^*M. In [58] the Dirac structure is identified with the D-brane, and in [59] the semiclassical Seiberg-Witten map is constructed.

3.4 Generalized Riemannian geometry

In this section we give a review on the construction of an analogue of Riremannian geometry based on the Courant bracket. The discussion here follow along [27].

3.4.1 Positive definite subbundle

Since the bilinear form has the signature (n, n), we can decompose the space of generalized tangent vectors into a maximally positive definite subbundle $C_+ \subset TM \oplus T^*M$ and a maximally negative definite subbundle $C_- \subset TM \oplus T^*M$. The elements in C_+ are defined by a graph of a map $g + B : \Gamma(TM) \to \Gamma(T^*M)$,

$$C_+ = \{X + (g + B)(X) \mid X \in \Gamma(TM)\}. \tag{3.57}$$

Here g denotes the positive-definite Riemannian metric and B denotes any 2-form. Similarly, the orthogonal complement C_-, which has negative definite inner product, is given by

$$C_- = \{X + (-g + B)(X) \mid X \in \Gamma(TM)\}. \tag{3.58}$$

Especially for the case with $B = 0$, we define an extension map $^{\pm} : \Gamma(TM) \to C_\pm$ as

$$X^\pm = X \pm g(X) \in C_\pm. \tag{3.59}$$

So that we have

$$X^+ - X^- = 2g(X). \tag{3.60}$$

Then we define a projection operator $\pi_\pm : \Gamma(TM) \oplus \Gamma(T^*M) \to C_\pm$ as

$$\pi_\pm(2g(X)) = \pm X^\pm. \tag{3.61}$$

To get the formula for an arbitrary 1-form ξ, we rewtire the 1-form $g(X)$ as $\xi = g(X)$, then we find

$$\begin{aligned} 2\pi_\pm(\xi) &= \pm(X \pm g(X)) = \pm(g^{-1}(\xi) \pm \xi) \\ &= \xi \pm g^{-1}(\xi) =: \xi^\pm. \end{aligned} \tag{3.62}$$

For the consistency to be the projection operators, they should satisfy

$$\pi_\pm(X^\pm) = X^\pm, \tag{3.63}$$
$$\pi_\mp(X^\pm) = 0. \tag{3.64}$$

From the latter condition we can read off how we should define the actions of the projection operators onto a vector field X as

$$\begin{aligned} 0 = \pi_\pm(X^\mp) &= \pi_\pm(X) \mp \pi_\pm(g(X)) \\ &= \pi_\pm(X) \mp \frac{1}{2}(g(X) \pm g^{-1}(g(X))), \end{aligned} \tag{3.65}$$

i.e.

$$\pi_\pm(X) = \frac{1}{2}(X \pm g(X)) \in C_\pm. \tag{3.66}$$

For a consistency check, we calculate the left-hand side of the former condition:

$$\begin{aligned} \pi_\pm(X^\pm) &= \pi_\pm(X) \pm \pi_\pm(g(X)) \\ &= \frac{1}{2}(X \pm g(X)) \pm \left(\pm \frac{1}{2} X^\pm \right) = X^\pm. \end{aligned} \tag{3.67}$$

Here we use the (3.66) and (3.61). Thus the projection operators are well-defined.
To extend the discussion in the case of $B \neq 0$, we successively apply a B-transformation e^B in addition to the extension map $^\pm$:

$$X \longmapsto e^B X^\pm = X + (\pm g + B)(X) \in C_\pm, \tag{3.68}$$
$$\xi \longmapsto e^B \xi^\pm = \xi \pm g^{-1}(\xi) \pm B(g^{-1}(\xi)) = \pm[g^{-1}(\xi) + (\pm g + B)(g^{-1})(\xi)] \in C_\pm. \tag{3.69}$$

3.4.2 Generalized connection

To construct Riemannian geometry, we are devoted in the positive-definite subbubdle C_+ defined in the previous subsection. Firstly, we define a connection on C_+. Since a connection ∇ is induced by a vector field, it should be defined as a map

$$\nabla : C_+ \to T^*M \otimes C_+. \tag{3.70}$$

A connection $\nabla : C_+ \to T^*M \otimes C_+$ is defined by

$$\nabla_X u = \pi_+(e^{-B}[e^B X^-, e^B u]) = \pi_+([X^-, u]_{dB}), \tag{3.71}$$

for $X \in \Gamma(TM)$ and $u = Y + g(Y) = Y^+$ i.e. $e^B u = Y + (g + B)(Y) \in C_+$. Notice that the property of the Courant bracket under a B-transformation leads, for $u = Y + g(Y) = Y^+$,

$$\nabla_X u = \pi_+([X^-, Y^+] + i_Y i_X dB), \tag{3.72}$$

as explained in the above. We can check for this connection satisfying the Leibniz rule: For an extension $X^- = X - g(X)$ and a positive definite generalized vector $u = Y + g(Y) = Y^+$, and arbitrary functions f and h, with noting (3.28) and $\langle X^-, Y^+\rangle = 0$, we have

$$\begin{aligned}\nabla_{fX}(hu) &= \pi_+([fX^-, hY^+] + fhi_Y i_X dB)\\ &= \pi_+(fh[X^-, Y^+] + f(Xh)Y^+ - h(Yf)X^- + fhi_Y i_X dB)\\ &= \pi_+(fh[X^-, Y^+] + fhi_Y i_X dB + f(Xh)Y^+)\\ &= fh\nabla_X u + f(Xh)u. \end{aligned} \tag{3.73}$$

From the second line to the third we use a property that the operator π_+ projects out X^-.

3.4.3 Generalized curvature

A curvature on is constructed by taking a commutator of the connection. A curvature on the positive-definite subbundle C_+ is defined by

$$R(X, Y)u := (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})u, \tag{3.74}$$

where $X, Y \in \Gamma(TM)$ and $u = Z + g(Z) = Z^+$ with some vector field $Z \in \Gamma(TM)$. We mention that $[X, Y]$ denotes the usual Lie bracket, or equivalently we can say it as the Courant bracket, because by definition it reduces to the Lie bracket with the case of both X and Y being vector fields. We can easily show that this curvature satisfies the tensor property,

$$R(fX, gY)hu = fghR(X, Y)u, \tag{3.75}$$

see appendix B.1.

3.4.4 Local expressions

In terms of components in some local coordinates $\{x^i\}$, we have

$$\nabla_{\partial_i}(\partial_j)^+ = \pi_+([(\partial_i)^-, (\partial_j)^+] + i_{\partial_j} i_{\partial_i} dB). \tag{3.76}$$

With some calculations

$$\begin{aligned}[(\partial_i)^-, (\partial_j)^+] &= [\partial_i - g_{il}dx^l, \partial_j + g_{jk}dx^k]\\ &= (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})dx^k, \end{aligned} \tag{3.77}$$

$$\begin{aligned} i_{\partial_j} i_{\partial_i} dB &= i_{\partial_j} i_{\partial_i}\left(\frac{1}{2}\partial_k B_{lm} dx^k \wedge dx^l \wedge dx^m\right)\\ &= (\partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij})dx^k, \end{aligned} \tag{3.78}$$

we find that the coefficient of the connection is

$$\nabla_{\partial_i}(\partial_j)^+ = \pi_+\big([2\Gamma_{kij} + H_{kij}]dx^k\big). \tag{3.79}$$

Here we define this coefficients as

$$\Gamma_{kij} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}), \tag{3.80}$$

$$H_{kij} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}. \tag{3.81}$$

These are very the Christoffel symbol constructed by the positive-definite Riemannian metric $g_{\mu\nu}$, and the H-flux, which is the 3-form field strength of the 2-form B-field, respectively.

For the case with $B = 0$

We would like to calculate the connection and the curvature in components of local coordinates $\{x^i\}$. Firstly we set $B = 0$ for simplicity. From the above calculation, we have

$$\begin{aligned}\nabla_{\partial_i}(\partial_j)^+ &= \pi_+(2\Gamma_{kij}dx^k) = \Gamma_{kij}(dx^k + g^{kl}\partial_l) \\ &= g^{kl}\Gamma_{kij}(\partial_l + g_{lm}dx^m) = \Gamma^l_{ij}(\partial_l)^+.\end{aligned} \tag{3.82}$$

To obtain the curvature we have to compute

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)(\partial_k)^+. \tag{3.83}$$

Here we denote $\nabla_i = \nabla_{\partial_i}$ and use $\nabla_{[\partial_i,\partial_j]} = 0$. By using (3.73) and (3.82) we obtain

$$\begin{aligned}\nabla_i\nabla_j(\partial_k)^+ &= \nabla_i\Big(\Gamma^m_{jk}(\partial_m)^+\Big) \\ &= (\partial_i\Gamma^m_{jk})(\partial_m)^+ + \Gamma^m_{jk}\Gamma^p_{im}(\partial_p)^+ = (\partial_i\Gamma^m_{jk} + \Gamma^l_{jk}\Gamma^m_{il})(\partial_m)^+,\end{aligned} \tag{3.84}$$

and then

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)(\partial_k)^+ = (\partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^l_{jk}\Gamma^m_{il} - \Gamma^l_{ik}\Gamma^m_{jl})(\partial_m)^+ =: R^m_{kij}(\partial_m)^+. \tag{3.85}$$

Here the coefficient turns out to the celebrated Riemann curvature tensor:

$$R^m_{kij} = \partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^l_{jk}\Gamma^m_{il} - \Gamma^l_{ik}\Gamma^m_{jl}. \tag{3.86}$$

For the case with $B \neq 0$

Under the situation with $B \neq 0$ the discussion above is also applicable. Since we have (3.79), performing the projection operator gives

$$\begin{aligned}\nabla_i\partial^+_j &= \pi_+\Big([2\Gamma_{kij} + H_{kij}]dx^k\Big) = (2\Gamma_{kij} + H_{kij})\pi_+(dx^k) \\ &= \frac{1}{2}(2\Gamma_{kij} + H_{kij})(dx^k + g^{km}\partial_m) = \frac{1}{2}(2\Gamma_{kij} + H_{kij})g^{km}(\partial_m + g_{ml}dx^l) \\ &= \frac{1}{2}(2\Gamma^m_{ij} + g^{mk}H_{kij})(\partial_m)^+ =: \Xi^m_{ij}(\partial_m)^+,\end{aligned} \tag{3.87}$$

where we define Ξ^l_{ij} as an extended object of the Christoffel symbol by

$$\Xi^l_{ij} := \Gamma^m_{ij} + \frac{1}{2} g^{mk} H_{kij}. \tag{3.88}$$

Then the curvature is calculated straightforwardly,

$$\begin{aligned}(\nabla_i \nabla_j - \nabla_j \nabla_i)(\partial_k)^+ &= (\partial_i \Xi^m_{jk} - \partial_j \Xi^m_{ik} + \Xi^l_{jk} \Xi^m_{il} - \Xi^l_{ik} \Xi^m_{jl})(\partial_m)^+ \\ &= \Upsilon^m_{kij} (\partial_m)^+, \end{aligned} \tag{3.89}$$

where we introduce $\Upsilon^m_{kij} = \partial_i \Xi^m_{jk} - \partial_j \Xi^m_{ik} + \Xi^l_{jk} \Xi^m_{il} - \Xi^l_{ik} \Xi^m_{jl}$. Explicitly, in terms of the components, see appendix B.1, we obtain a tensor expression of this curvature as

$$\Upsilon^m_{kij} = R^m_{kij} + \frac{1}{4} g^{ln} H_{njk} g^{mp} H_{pil} - \frac{1}{4} g^{ln} H_{nik} g^{mp} H_{pjl} + \frac{1}{2} g^{ml} \nabla_i H_{ljk} - \frac{1}{2} g^{ml} \nabla_j H_{lik}. \tag{3.90}$$

The Ricci tensor is obtained by taking a contraction between an upper index and a lower index

$$\begin{aligned}\Upsilon_{kj} &:= \Upsilon^m_{kmj} \\ &= R^m_{kmj} + \frac{1}{4} g^{ln} H_{njk} g^{mp} H_{pml} - \frac{1}{4} g^{ln} H_{nmk} g^{mp} H_{pjl} + \frac{1}{2} g^{ml} \nabla_m H_{ljk} - \frac{1}{2} g^{ml} \nabla_j H_{lmk} \\ &= R^m_{kmj} - \frac{1}{4} g^{ln} H_{nmk} g^{mp} H_{pjl} + \frac{1}{2} g^{ml} \nabla_m H_{ljk}. \end{aligned} \tag{3.91}$$

Furthermore the Ricci scalar is given by contracting between the metric and the Ricci scalar

$$\begin{aligned}\Upsilon &:= g^{kj} \Upsilon_{kj} = g^{kj} \Upsilon^m_{kmj} \\ &= R - \frac{1}{4} g^{ln} g^{kj} H_{nmk} g^{mp} H_{pjl} + \frac{1}{2} g^{kj} g^{ml} \nabla_m H_{ljk} = R - \frac{1}{4} H^{lpj} H_{pjl} = R - \frac{1}{4} H^2, \end{aligned} \tag{3.92}$$

where R denotes the usual Ricci scalar constructed by the Riemannian metric g. The generalized Ricci scalar Υ involves the term squared of the H-flux and is the same as the Einstein-Hilbert action of the supergravity.

Chapter 4

Poisson generalized geometry

In this chapter we define a new Courant algebroid [28]. This algebroid is a dual of the algebroid used in generalized geometry in a sense that the roles of the tangent and the cotangent bundles are interchanged. We discuss some properties of the new Courant algebroid, which provide the bases of the next chapter.

4.1 New Courant algebroid

In this section, after recalling some notions of the Lie algebroid $(T^*M)_\theta$ of a Poisson manifold introduced in the beginning of the previous chapter, we give a definition a new Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$. The corresponding bracket is different from that of the Courant algebroid $TM \oplus T^*M$. Here we investigate some properties of the new Courant algebroid.

4.1.1 Lie algebroid of a Poisson manifold

Let (M, θ) be a Poisson manifold equipped with a Poisson bivector $\theta \in \wedge^2 TM$. The Poisson bivector θ satisfies $[\theta, \theta]_S = 0$, where $[\cdot, \cdot]_S$ is the Schouten-Nijenhuis bracket. A Lie algebroid of a Poisson manifold [56] is defined by a triple $(T^*M,\ \theta,\ [\cdot, \cdot]_\theta)$, where T^*M is the cotangent bundle, θ is the anchor map, and $[\cdot, \cdot]_\theta$ denotes the Lie bracket defined by the Koszul bracket as

$$[\xi, \eta]_\theta = \mathcal{L}_{\theta(\xi)}\eta - i_{\theta(\eta)}d\xi, \tag{4.1}$$

for $\xi, \eta \in \Gamma(T^*M)$. We denote this Lie algebroid as $(T^*M)_\theta$ for short.
we can define the Cartan differential calculus on the space of polyvectors $\Gamma(\wedge^\bullet TM)$. The "exterior derivative" $d_\theta : \Gamma(\wedge^p TM) \to \Gamma(\wedge^{p+1} TM)$ is defined as $d_\theta = [\theta, \cdot]_S$. In particular, for a function $f \in C^\infty(M) = \Gamma(\wedge^0 TM)$, its action is defined by

$$d_\theta f = [\theta, f]_S = -\theta(df). \tag{4.2}$$

And the "interior derivative" is defined by the contraction between 1-form and poly-vectors. Then we can defined the "Lie derivative" as follows. The actions of the Lie derivative $\mathcal{L}_\zeta$, where $\zeta \in$

$\Gamma(T^*M)$, on a function f, a 1-form ξ and a vector field X are given by

$$\begin{aligned}\mathcal{L}_\zeta f &:= i_\zeta d_\theta f,\\ \mathcal{L}_\zeta \xi &:= [\zeta,\xi]_\theta,\\ \mathcal{L}_\zeta X &:= (d_\theta i_\zeta + i_\zeta d_\theta)X,\end{aligned} \tag{4.3}$$

respectively. They are shown to satisfy following Cartan relations on the space of polyvectors $\Gamma(\wedge^\bullet TM)$,

$$\{i_\xi, i_\eta\} = 0, \quad \{d_\theta, i_\xi\} = \mathcal{L}_\xi, \quad [\mathcal{L}_\xi, i_\eta] = i_{[\xi,\eta]_\theta}, \quad [\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]_\theta}, \quad [d_\theta, \mathcal{L}_\xi] = 0. \tag{4.4}$$

4.1.2 Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$

Let us consider a vector bundle $TM \oplus T^*M$ equipped with a canonical inner product

$$\langle X+\xi, Y+\eta\rangle = \tfrac{1}{2}(i_X\eta + i_Y\xi), \tag{4.5}$$

anchor map $\rho : TM \oplus T^*M \to TM$,

$$\rho(X+\xi) = \theta(\xi), \tag{4.6}$$

and a skew-symmetric bracket

$$[X+\xi, Y+\eta] = [\xi,\eta]_\theta + \mathcal{L}_\xi Y - \mathcal{L}_\eta X - \frac{1}{2} d_\theta(i_\xi Y - i_\eta X). \tag{4.7}$$

Then we can show that the quadruple $(TM \oplus T^*M, \langle\cdot,\cdot\rangle, \rho, [\cdot,\cdot])$ defines a Courant algebroid. To show this, note that a Lie bialgebroid $A \oplus A^*$ is always a Courant algebroid [54], and the above Courant algebroid is of this type. This means that the first Lie algebroid $(TM)_0 := (TM,\, a = 0,\, [\cdot,\cdot] = 0)$ is a tangent bundle with the vanishing Lie bracket and the vanishing anchor map, while the second $(T^*M)_\theta = (T^*M,\, \theta,\, [\cdot,\cdot]_\theta)$ is the Lie algebroid of a Poisson manifold explained in the previous subsection. We denote this Courant algebroid as $(TM)_0 \oplus (T^*M)_\theta$ for short.
We can show, see appendix B.2, that the bracket (4.7) satisfies the following equations

$$[u, fv] = f[u,v] + (\mathcal{L}_\xi f)v - (d_\theta f)\langle u,v\rangle, \tag{4.8}$$

$$\mathcal{L}_\xi\langle v,w\rangle = \langle [u,v] + d_\theta\langle u,v\rangle, w\rangle + \langle v, [u,w] + d_\theta\langle u,w\rangle\rangle, \tag{4.9}$$

where $u = X+\xi$, $v = Y+\eta$, $w = Z+\zeta \in (TM)_0 \oplus (T^*M)_\theta$, and f is a smooth function. These properties are crucial in defining a connection which is compatible with $O(d,d)$-invariant inner product $\langle\cdot,\cdot\rangle$ as we shall see in the following section. Note that in the second equation, while the left-hand side depends only on ξ, the right-hand side seems to depend on X as well. Despite of its appearance, we can check that the cancellation between the terms involving X does occur in the right-hand side. Thus its appearance is really dummy and the right-hand side is indeed independent of X.

To make a comparison between the new Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$ and $TM \oplus T^*M$ used in generalized geometry, we give some comments. In the standard case $TM \oplus T^*M$, the anchor map is given by $\rho(X+\xi) = X$ and the Courant bracket $[\cdot,\cdot]_C$ is defined by

$$[X+\xi, Y+\eta]_C = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi). \tag{4.10}$$

The Courant algebroid $TM \oplus T^*M$ can be considered as an extension of the Lie algebroid TM, and in fact it is a Lie bialgebroid $(TM, \mathrm{id}, [\cdot,\cdot]_{TM}) \oplus (T^*M, 0, 0)$.
In our new Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$, the roles of TM and T^*M are interchanged: the underlying Lie bialgebroid is $(T^*M, \theta, [\cdot,\cdot]_\theta) \oplus (TM, 0, 0)$, the anchor map (4.6) picks up only T^*M-part, the Courant bracket (4.7) is written in terms of the operators defined by $(T^*M)_\theta$ only. In this way, our Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$ can be considered as an extension of the Lie algebroid $(T^*M)_\theta$.
As a consequence, in the Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$, the Poisson Lie algebroid $(T^*M)_\theta$ governs the differential geometry, and the resulting differential geometry is quite different from the one governed by the Lie algebroid TM. However, we can proceed to formulate an analogue of the generalized geometry exactly in the same manner as the standard generalized tangent bundle.
Some comments are in order: First, the standard Courant bracket (4.10) and the new bracket (4.7) can be considered as complementary parts in the Roytenberg bracket [55, 60]:

$$\begin{aligned}[X+\xi, Y+\eta]_{\mathrm{Roy}} &= [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) \\ &\quad + [\xi,\eta]_\theta + \mathcal{L}_\xi Y - \mathcal{L}_\eta X + \frac{1}{2}d_\theta(i_X\eta - i_Y\xi).\end{aligned} \tag{4.11}$$

Note that the Roytenberg bracket is the bracket for a Lie bialgebroid $TM \oplus (T^*M)_\theta$ and not for the present Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$.
Secondly, in general, an anchor map $\rho : E \to TM$ of a Courant algebroid E induces a natural differential operator $D : C^\infty(M) \to \Gamma(E)$ defined by $\langle Df, A\rangle = 1/2\rho(A)f$, for arbitrary function $f \in C^\infty(M)$ and section $A \in \Gamma(E)$. In our case,

$$\langle Df, X+\xi\rangle = \frac{1}{2}\theta(\xi)\cdot f = \frac{1}{2}\theta(\xi, df) = -\frac{1}{2}i_\xi\theta(df), \tag{4.12}$$

implies that $Df = d_\theta f = -\theta(df) \in \Gamma(TM)$.
Finally, in [59], the same Lie algebroid $(T^*M)_\theta$ is used but in a different context. It appears as a Dirac structure in the standard generalized tangent bundle $TM \oplus T^*M$.

4.1.3 Symmetry of $(TM)_0 \oplus (T^*M)_\theta$

As is mentioned in the previous chapter, the symmetry of the generalized tangent bundle $TM \oplus T^*M$ consists of diffeomorphisms generated by vector fields and B-transformations induced by closed 2-forms. Here we investigate the symmetry of the new Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$. It turns out to be a direct product of β-diffeomorphisms and β-transformations.

Let us define the following two transformations acting on a section $X + \xi \in \Gamma((TM)_0 \oplus (T^*M)_\theta)$:
1) β-diffeomorphism: for a 1-form $\zeta \in \Gamma(T^*M)$, we define

$$\mathcal{L}_\zeta(X + \xi) = \mathcal{L}_\zeta X + \mathcal{L}_\zeta \xi, \tag{4.13}$$

by the diagonal actions of the Lie derivative $\mathcal{L}_\zeta$ given by (4.3).
2) β-transformation: for a bivector $\beta \in \Gamma(\wedge^2 TM)$, we define

$$e^\beta(X + \xi) = X + \xi + i_\xi \beta. \tag{4.14}$$

The β-transformation here follows a widely-used definition in the context of $TM \oplus T^*M$. As is mentioned in the previous chapter, it turns out *not* to be a symmetry of the Courant bracket of $TM \oplus T^*M$. However, the β-transformation *is* indeed a symmetry of the new Courant bracket $(TM)_0 \oplus (T^*M)_\theta$ as we shall see in a minute.
The β-diffeomorphism is a natural object for the Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$ as follows. It is instructive to rewrite (4.3) following [43–46] as

$$\begin{aligned}
&\mathcal{L}_\zeta f = \mathcal{L}_{\theta(\zeta)} f,\\
&\mathcal{L}_\zeta \xi = \mathcal{L}_{\theta(\zeta)} \xi - i_{\theta(\xi)} d\zeta,\\
&\mathcal{L}_\zeta X = \mathcal{L}_{\theta(\zeta)} X + \theta(i_X d\zeta).
\end{aligned} \tag{4.15}$$

The third equation of (4.15) is proven in the appendix B.2. In the above expressions, the terms of the ordinary Lie derivative $\mathcal{L}_{\theta(\zeta)}$ represent a diffeomorphism generated by a vector field $\theta(\zeta)$. The term $i_{\theta(\xi)} d\zeta$ in the second equation is a B-transformation with $d\zeta$ of a β-transformed vector $\theta(\xi)$, while the term $\theta(i_X d\zeta)$ in the third equation is a β-transformation of a B-transformation with $d\zeta$. Therefore, the β-diffeomorphism is a rather complicated combination of a diffeomorphism, a B-transformation and a β-transformation from the viewpoint of $TM \oplus T^*M$. And it is no longer a symmetry of the Courant bracket of $TM \oplus T^*M$.
It is worth mentioning that if the parameter ζ is an exact 1-form, say $\zeta = dh$, the β-diffeomorphism (4.3) reduces to the ordinary diffeomorphism generated by the Hamilton vector field $X_h = \theta(dh)$:

$$\mathcal{L}_{dh} X = \mathcal{L}_{X_h} X, \quad \mathcal{L}_{dh} \xi = \mathcal{L}_{X_h} \xi. \tag{4.16}$$

Such exact 1-forms form a subgroup of the group of β-diffeomorphisms.
We are now ready to study the symmetry of the new Courant algebroid. For an infinitesimal β-diffeomorphism $\mathcal{L}_\zeta$, we can show that

$$\begin{aligned}
&\langle \mathcal{L}_\zeta A, B\rangle + \langle A, \mathcal{L}_\zeta B\rangle = \mathcal{L}_\zeta \langle A, B\rangle,\\
&\rho(\mathcal{L}_\zeta A) = \mathcal{L}_\zeta \rho(B),\\
&[\mathcal{L}_\zeta A, B] + [A, \mathcal{L}_\zeta B] = \mathcal{L}_\zeta [A, B].
\end{aligned} \tag{4.17}$$

The first and the third equations are satisfied by an arbitrary ζ, while the second equation holds when the vector field $\theta(\zeta)$ is d_θ-closed. The proofs of the above relations are given in the appendix

B.2. And for a β-transformation e^β, we can also show that

$$\begin{aligned}&\langle e^\beta A, e^\beta B\rangle = \langle A, B\rangle,\\ &\rho(e^\beta A) = \rho(B),\\ &[e^\beta A, e^\beta B] = e^\beta[A, B].\end{aligned} \tag{4.18}$$

Here the first and the second equations are satisfied by an arbitrary β, while the last holds when the bivector field β is d_θ-closed. First two equations are obvious to hold. We give the proof of the third equation in the appendix B.2.
In summary, a β-diffeomorphism $\mathcal{L}_\zeta$ is a symmetry if $\mathcal{L}_\zeta\theta = 0$ and a β-transformation e^β is a symmetry if $d_\theta\beta = 0$. In particular, for construction of R-fluxes, it is essential that the β-transformations are the symmetry of the new bracket, as we shall see in the next section.
We end this section with a few remarks. As in the case of B-transformation, we call the particular case of a β-transformation $e^{d_\theta Z}$ with a d_θ-exact bivector $\beta = d_\theta Z$, a β-gauge transformation. Similar to the Courant bracket of $TM \oplus T^*M$, the action of a pair $(\zeta, \beta) = (\zeta, -d_\theta Z)$ can be written as

$$\begin{aligned}\mathcal{L}_{(\zeta, -d_\theta Z)}(X+\xi) &= [\zeta, \xi]_\theta + \mathcal{L}_\zeta X - i_\xi d_\theta Z\\ &= (\zeta + Z)\circ(X+\xi),\end{aligned} \tag{4.19}$$

where in the last line, the symbol $\circ$ denotes the analogue of the Dorfman bracket [1]. Hence, a β-gauge transformation is an inner transformation.
It is also worth to note that the β-transformation does not yield a naive shift $\theta \to \theta + \beta$ of the bivector θ. Here the situation is different from the case in the paper [59], where $(T^*M)_\theta$ is regarded as a Dirac structure in $TM \oplus T^*M$, and the β-transformation is required to preserve the Dirac structure. In that case the β-transformation indeed results in a shift $\theta \to \theta + \beta$, and the Maurer-Cartan type condition for β has to be satisfied.

4.1.4 Dirac structure

A Dirac structure L is defined in the same manner as in the standard generalized geometry. That is, a Dirac structure $L \subset (TM)_0 \oplus (T^*M)_\theta$ is a maximally isotropic subbundle, and is involutive with respect to the new bracket $[L, L] \subset L$. There are always two Dirac structures independent of the choice of a Poisson bivector θ:

1. $L = (T^*M)_\theta$. Its bracket $[\cdot,\cdot]_\theta$ is a Lie bracket. because of $\rho(L) = \theta(T^*M)$, the dimension of the leaf equals to the rank of θ.

2. $L = (TM)_0$. Its bracket vanishes, and $\rho(L) = 0$. All leaves are 0-dimensional.

Contrary to the standard generalized geometry, even a simple subbundle such as $L = \mathrm{span}\{\partial_a, dx^i\}$ is not necessarily a Dirac structure, depending on the choice of the Poisson bivector. Nevertheless, we can say some general statements analogous to those given in the standard generalized geometry:

[1]The skew-symmetrization of $\circ$ gives the new bracket (4.7)

1. Let $\Delta \subset T^*M$ be a subbundle of T^*M such that it is involutive $[\Delta, \Delta]_\theta \subset \Delta$ with respect to the Koszul bracket. Then, $L = \Delta \oplus \mathrm{Ann}(\Delta)$ is a Dirac structure of $(TM)_0 \oplus (T^*M)_\theta$.
   Proof. L is apparently maximally isotropic. The involutive condition $[L, L] \subset L$ reduces to $[\Delta, \mathrm{Ann}(\Delta)] \subset \mathrm{Ann}(\Delta)$, because $[\Delta, \Delta]_\theta \subset \Delta$ and $[\mathrm{Ann}(\Delta), \mathrm{Ann}(\Delta)] = 0$. This condition means for arbitrary $\xi, \eta \in \Delta$ and $X \in \mathrm{Ann}(\Delta)$
   $$0 = \langle [\xi, X], \eta \rangle, \tag{4.20}$$
   but it is rewritten as
   $$0 = \langle [\xi, \eta]_\theta, X \rangle, \tag{4.21}$$
   which is automatically satisfied by definition. *(End of the proof)*

2. Given a Dirac structure L, its deformation $L_\mathcal{F} = e^\mathcal{F} L$ by a L-2 form $\mathcal{F} \in \wedge^2 L^*$ is still a Dirac structure iff the Maurer-Cartan type equation $d_L \mathcal{F} + \frac{1}{2}[\mathcal{F}, \mathcal{F}]_{L^*} = 0$ is satisfied [54]. For $L = (T^*M)_\theta$, $\mathcal{F}$ is a bivector such that $d_\theta \mathcal{F} = 0$. For $L = (TM)_0$, $\mathcal{F}$ is a 2-form such that $[\mathcal{F}, \mathcal{F}]_\theta = 0$ ($[\cdot, \cdot]_\theta$ is the Gerstenhaber bracket extending the Koszul bracket.).

In the papers [58, 59], Dirac structures are identified with D-branes (with fluctuations). It is interesting to investigate the Dirac structures here in this context.

4.2 Proposal of R-flux

In this section, we propose a definition of R-fluxes by a set of data $(R, \beta_i, \alpha_{ij})$, where $R \in \wedge^3 TM$, $\beta_i \in \wedge^2 TU_i$ and $\alpha_{ij} \in TU_{ij}$, such that

$$\begin{aligned} R|_{U_i} &= d_\theta \beta_i, \\ \beta_j - \beta_i|_{U_{ij}} &= d_\theta \alpha_{ij}. \end{aligned} \tag{4.22}$$

Here $\{U_i\}$ denotes a good open covering of M and $U_{ij} = U_i \cap U_j$. It follows from (4.22) that R is a global 3-vector on M and is d_θ-closed: $d_\theta R = 0$. Local bivectors $\{\beta_i\}$ are gauge potentials for the R-flux, the analogue of B-fields for H-fluxes, and correspondingly, there is the local β-gauge symmetry of the form

$$\beta_i \mapsto \beta_i + d_\theta \Lambda_i, \quad \alpha_{ij} \mapsto \alpha_{ij} + \Lambda_i - \Lambda_j, \tag{4.23}$$

for an arbitrary gauge parameter $\Lambda_i \in TU_i$. In particular, since the R-flux is invariant under the gauge symmetry, it is Abelian.
This proposal is based on the mathematical correspondence between the standard generalized tangent bundle $TM \oplus T^*M$ and our new Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$. In the following we show that this R-flux is exactly the $(TM)_0 \oplus (T^*M)_\theta$-analogue of an H-flux in $TM \oplus T^*M$. We give a review on the global definition of the H-flux in the appendix B.2.

Recall that in the new Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$, comparing with $TM \oplus T^*M$, $(T^*M)_\theta$ play the role of TM. Thus, an H-twisting of $TM \oplus T^*M$ (B.34) corresponds to a twisting of $(TM)_0 \oplus (T^*M)_\theta$ satisfying the exact sequence

$$0 \to (TM)_0 \xrightarrow{\pi^*} E \xrightarrow{\pi} (T^*M)_\theta \to 0. \tag{4.24}$$

We emphasize that the bundle map π is *not* an anchor map, thus the meaning of the exactness of (4.24) is different from the standard exact Courant algebroid.
In the following subsection we show

1. Given a data (R, β, α) in (4.22) we can construct a Courant algebroid E that satisfies the exact sequence (4.24). It is classified by Poisson cohomology $[R] \in H^3_\theta(M)$.

2. E is isomorphic to the untwisted Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$ with the R-twisted bracket.

3. E is a quasi-Lie bialgebroid $((TM)_0, \delta = 0, \phi = R)$.

Each statement has its analog in the case of H-fluxes [61, 62], here our logic follows in part along with the presentation by [62].

4.2.1 Gluing by local β-gauge transformation

We follow the argument of [62] for H-fluxes, by replacing the role of TM with that of T^*M, and B-transformations with β-transformations.
Let (M, θ) be a d-dimensional Poisson manifold with a d_θ-closed 3-vector $R \in \wedge^3 TM$. We assume a trivialization of R, that is, an open cover $\{U_i\}$ of M equipped with local bivectors $\beta_i \in \wedge^2 TU_i$ and vectors $\alpha_{ij} \in TU_{ij}$ such that (4.22) is satisfied. Given such a trivialization, a Courant algebroid E is constructed in the following way. First, over each open set U_i, we can consider a Courant algebroid $E_i = (TU_i)_0 \oplus (T^*U_i)_\theta$, equipped with the anchor map ρ_i, the inner product $\langle \cdot, \cdot \rangle_i$ and the bracket $[\cdot, \cdot]_i$ defined by

$$\begin{aligned} &\rho_i(\xi) = \theta(\xi), \quad \langle X + \xi, Y + \eta \rangle_i = \frac{1}{2}(i_X \eta - i_Y \xi), \\ &[X + \xi, Y + \eta]_i = [\xi, \eta]_\theta + \mathcal{L}_\xi Y - \mathcal{L}_\eta X + \frac{1}{2} d_\theta (i_X \eta - i_Y \xi), \end{aligned} \tag{4.25}$$

for $X + \xi, Y + \eta \in (TU_i)_0 \oplus (T^*U_i)_\theta$. On the intersection U_{ij}, E_i and E_j are glued by a β-gauge transformation generated by α_{ij}, that is the transition function is

$$\begin{aligned} &G_{ij} : U_{ij} \to O(d, d), \\ &G_{ij}(x) = \begin{pmatrix} 1 & -d_\theta \alpha_{ij}(x) \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{4.26}$$

It defines the equivalence relation $\sim$ between $X + \xi \in E_j|_{U_{ij}}$ and

$$G_{ij}(X + \xi) = X + \xi - d_\theta \alpha_{ij}(\xi) \in E_i|_{U_{ij}}. \tag{4.27}$$

Such G_{ij} satisfies the cocycle condition

$$G_{ij}G_{jk} = G_{ik}, \tag{4.28}$$

on U_{ijk} due to (4.22). Therefore, it defines the vector bundle over M by

$$E = \coprod_{x\in M} (TU_i)_0 \oplus (T^*U_i)_\theta / \sim . \tag{4.29}$$

Since a β-gauge transformation preserves the anchor map, the inner product and the bracket (see (4.18)), they all are globally well-defined on the quotient. For example, the bracket on U_i and U_j are related by

$$[G_{ij}(X+\xi), G_{ij}(Y+\eta)]_i = G_{ij}([X+\xi, Y+\eta]_j). \tag{4.30}$$

Therefore, the vector bundle E is in fact a Courant algebroid.
It is apparent that E satisfies the exact sequence (4.24). Here the map π is induced by the projection $(TU_i)_0 \oplus (T^*U_i)_\theta \to (T^*U_i)_\theta$ to the second factor and π^* is the inclusion.
As in the case of H-twist, the set of bivectors $\{\beta_i\}$ induces a bundle map $s : (T^*M)_\theta \to E$, locally defined by a β-transform as

$$s(\xi) = e^{\beta_i}(\xi) = \xi + \beta_i(\xi) \tag{4.31}$$

for $\xi \in T^*U_i$. It follows form (4.22) that $s((T^*M)_\theta)$ is globally well-defined as a vector bundle over M. This map s is in fact an isotropic splitting, since it satisfies $\pi \circ s(\xi) = \xi$ and $\langle s(\xi), s(\eta)\rangle = 0$ for all $\xi, \eta \in (T^*M)_\theta$. Therefore, s induces the isotropic splitting $E = \pi^*((TM)_0) \oplus s((T^*M)_\theta)$ of E, and any section $A \in E$ can be uniquely expressed for $X \in TM$ and $\xi \in T^*M$ as

$$A = X + s(\xi). \tag{4.32}$$

4.2.2 R-twisted bracket

From this splitting, the structure of the Courant algebroid in $E = \pi^*((TM)_0) \oplus s((T^*M)_\theta)$ is translated to that in $(TM)_0 \oplus (T^*M)_\theta$. Since s is a β-transformation e^{β_i} locally, the anchor map and the inner product is unchanged from $(TM)_0 \oplus (T^*M)_\theta$ (see (4.18)):

$$\rho(X + s(\xi)) = \theta(\xi) = \rho(X+\xi), \quad \langle X + s(\xi), Y + s(\eta)\rangle = \langle X+\xi, Y+\eta\rangle. \tag{4.33}$$

The bracket on $\pi^*((TM)_0) \oplus s((T^*M)_\theta)$ is our bracket of sections of the form (4.32). We compute it locally as (see (4.18))

$$\begin{aligned}
[X + s(\xi), Y + s(\eta)] &= [e^{\beta_i}(X+\xi), e^{\beta_i}(Y+\eta)] \\
&= e^{\beta_i}[X+\xi, Y+\eta] + [\theta, \beta_i]_S(\xi, \eta) \\
&= s([\xi,\eta]_\theta) + \mathcal{L}_\xi Y - \mathcal{L}_\eta X + \frac{1}{2} d_\theta(i_X\eta - i_Y\xi) + (d_\theta\beta_i)(\xi,\eta).
\end{aligned} \tag{4.34}$$

Hence, if we define the R-twisted bracket by

$$[X+\xi, Y+\eta]_R := [X+\xi, Y+\eta] + R(\xi, \beta), \tag{4.35}$$

then we have

$$[X+s(\xi), Y+s(\eta)] = (\pi^* \oplus s)([X+\xi, Y+\eta]_R). \tag{4.36}$$

Therefore, as a Courant algebroid, $E = \pi^*((TM)_0) \oplus s((T^*M)_\theta)$ is equivalent to $(TM)_0 \oplus (T^*M)_\theta$ but with the R-twisted bracket.

Chapter 5

Poisson Generalized Riemannian geometry

In this chapter we construct an analogue of Riemannian geometry based on the new Courant algebroid. Firstly, we define positive-definite subbundle. Since the inner product in the Courant algebra $(TM)_0 \oplus (T^*M)_\theta$ is the same as that in $TM \oplus T^*M$, the positive-definite subbundle in $(TM)_0 \oplus (T^*M)_\theta$ is so too. We shall show that a connection and a curvature are consistently defined on the positive-definite subbundle, even in the presence of the R-flux. The final section provides a construction of the gravity theory based on the framework defined and investigated in the other part of this chapter.

5.1 Positive-definite subbundle

As in the standard Courant algebroid, we define a generalized Riemannian structure as a maximal-positive definite subbundle $C_+ \subset (TM)_0 \oplus (T^*M)_\theta$. Since this definition depends only on $TM \oplus T^*M$ as a vector bundle, and the bilinear form is independent of the bracket, a generalized Riemannian structure C_+ of the standard tangent bundle becomes automatically that of the new Courant algebroid. In other words, two Courant algebroids share the same C_+. Therefore, C_+ is given by a graph of a map $g + B : TM \to T^*M$,

$$C_+ = \{X + (g + B)(X) \mid X \in TM\}. \tag{5.1}$$

As is emphasized in our previous papers [58, 59], however, there are various ways to represent C_+ as graphs. In particular, C_+ can be seen from T^*M as

$$C_+ = \{\xi + (G + \beta)(\xi) \mid \xi \in T^*M\}, \tag{5.2}$$

where $G + \beta = (g + B)^{-1} : T^*M \to TM$ is the inverse map. Two representations (5.1) and (5.2) of C_+ are equivalent if the fluxes are absent.

However, in the presence of the fluxes, the situation is changed. In the presence of an H-flux, the representation (5.1) is natural, since an H-twisting requires to replace B with a local 2-form

B_i while it does not affect the symmetric part g. In other words, a Riemannian manifold (M, g) is unchanged regardless of the presence of H-fluxes. This compatibility of the generalized metric with H-twisting is emphasized in [27]. On the other hand, in the different representation (5.2), an H-twisting affects both the symmetric part G and the skew-symmetric part β, so that G is non-trivially glued by local B-gauge transformations.
Similarly, in the presence of a R-flux, the representation (5.2) is natural, since now a R-twisting affects only β, kept fixed the symmetric part G. Here G is a fiber metric on T^*M defining a Riemannian manifold.
In the case with $\beta = 0$, we make a choice of the positive-definite subbundle C_+ as

$$C_+ = \{X + G^{-1}(X) | X \in TM\} = \{\xi + G(\xi) | \xi \in T^*M\}. \tag{5.3}$$

Recalling some definitions may be helpful: The projection operators $\pi_\pm$ and extension maps $^\pm$ give

$$\pi_\pm(X) = \frac{1}{2}(X \pm G^{-1}(X)) = \frac{1}{2}X^\pm, \tag{5.4}$$

$$\pi_\pm(\xi) = \frac{1}{2}(\pm G(\xi) + \xi) = \frac{1}{2}\xi^\pm, \tag{5.5}$$

for every vector field X and 1-form ξ.
To make the discussion in the case with $\beta \neq 0$, we perform a successive β-transformation to the positive-definite subbundle of trivial β:

$$e^\beta(\xi^+) = \xi + (G + \beta)(\xi) \in C_+. \tag{5.6}$$

In the rest of this section we assume $\beta = 0$ for simplicity. We shall discuss about the extension to the case with $\beta \neq 0$ in the next section.

5.2 In the absence of R-flux

5.2.1 Connection, Torsion and Curvature

Here we shall define a connection, a torsion and a curvature on positive-definite subbundle $C_+^{\beta=0}$. As mentioned above, here we assume $\beta = 0$ for making the argument simple. The definitions are based on the algebraic structure of the Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$.

Poisson generalized connection We define a bilinear map $\nabla : C_+^{\beta=0} \to TM \otimes C_+^{\beta=0}$ as

$$\nabla_\xi u := \pi_+\big([\xi^-, u]\big), \tag{5.7}$$

where $\xi \in T^*M$ and $u \in C_+^{\beta=0}$. Then this map satisfies the following properties

$$\nabla_{f\xi} u = f\nabla_\xi u, \tag{5.8}$$

$$\nabla_\xi(fu) = f\nabla_\xi u + (\mathcal{L}_\xi f)u, \tag{5.9}$$

where f denotes any smooth function. The proofs of these relations are given in appendix B.2. Hence the map ∇ defines a connection.

Furthermore, we can show that this is compatible with the canonical $O(d,d)$-invariant inner product $\langle\cdot,\cdot\rangle$:

$$\mathcal{L}_\xi\langle u,v\rangle = \langle\nabla_\xi u,v\rangle + \langle u,\nabla_\xi v\rangle, \tag{5.10}$$

for $u,v\in C_+^{\beta=0}$, see appendix B.2.

Poisson generalized torsion A torsion can also be defined in a parallel manner with the generalized geometry:

$$\bar{T}(\xi,\eta) = \nabla_\xi\eta^+ - \nabla_\eta\xi^+ - ([\xi,\eta]_\theta)^+, \tag{5.11}$$

as is easily shown that

$$\bar{T}(f\xi,g\eta) = fg\bar{T}(\xi,\eta), \tag{5.12}$$

with a use of the formula (**??**).

Poisson generalized curvature We define a curvature by

$$\bar{R}(\xi,\eta)u = (\nabla_\xi\nabla_\eta - \nabla_\eta\nabla_\xi - \nabla_{[\xi,\eta]_\theta})u, \tag{5.13}$$

for any 1-forms ξ, η and $u\in C_+^{\beta=0}$. Note that the Koszul bracket $[\cdot,\cdot]_\theta$, which is the Lie bracket of the Poisson geometry $(T^*M)_\theta$, appears in the third term. We can show that this curvature actually satisfies the following tensorial property

$$\bar{R}(f\xi,g\eta)hu = fgh\bar{R}(\xi,\eta)u, \tag{5.14}$$

see appendix B.2.

5.2.2 Local expressions

The definitions above are quite abstract and indeed their proofs are based only on the algebraic properties of the Courant algebroid $(TM)_0\oplus(T^*M)_\theta$. In this subsection, we present local expressions of the objects introduced in the previous subsection on some local coordinates $\{x^\mu\}$.

Poisson generalized connection Firstly, we shall calculate the connection (5.7)

$$\nabla_{dx^i}(dx^j)^+ = \pi_+([(dx^i)^-,(dx^j)^+]) = \pi_+([dx^i - G^{ik}\partial_k, dx^j + G^{jl}\partial_l]). \tag{5.15}$$

The bracket under the projection operator becomes as follows, see appendix B.2,

$$\begin{aligned}&[dx^i - G^{ik}\partial_k, dx^j + G^{jl}\partial_l]\\ &= \partial_k\theta^{ij}dx^k + [\theta^{mn}(\partial_m G^{ji}) - \theta^{mi}(\partial_m G^{jn}) - \theta^{mj}(\partial_m G^{in}) - G^{jl}(\partial_l\theta^{in}) - G^{il}(\partial_l\theta^{jn})]\partial_n.\end{aligned} \tag{5.16}$$

Notice that the coefficient in front of 1-form dx^k is skew-symmetric under an exchange bewteen the indices i and j, while that of the vector field ∂_n is symmetric. We denote these coefficients as

$$\bar{\Gamma}_k^{\{ij\}} = \frac{1}{2}[\theta^{mn}(\partial_m G^{ji}) - \theta^{mi}(\partial_m G^{jn}) - \theta^{mj}(\partial_m G^{in}) - G^{jl}(\partial_l \theta^{in}) - G^{il}(\partial_l \theta^{jn})]G_{nk}, \tag{5.17}$$

$$\bar{\Gamma}_k^{[ij]} = \frac{1}{2}(\partial_k \theta^{ij}) = \frac{1}{2}G^{nl}(\partial_l \theta^{ij})G_{nk}. \tag{5.18}$$

Then the connection can be represented as

$$\nabla_{dx^i}(dx^j)^+ = (\bar{\Gamma}_k^{\{ij\}} + \bar{\Gamma}_k^{[ij]})(dx^k)^+ =: \bar{\Gamma}_k^{ij}(dx^k)^+. \tag{5.19}$$

The meaning of this connection is intensively studied in the next section.

Poisson generalized torsion The local expression for the torsion tensor is obtained by computing

$$\nabla_{dx^i}(dx^j)^+ - \nabla_{dx^j}(dx^i)^+ - ([dx^i, dx^j])^+. \tag{5.20}$$

It is worth noting that $[dx^i, dx^j] = \partial_k \theta^{ij} dx^k \neq 0$ with our Courant algebroid. This is significantly different point from the standard one, as $[\partial_i, \partial_j]_{TM} = 0$. And then

$$\bar{T}^{ij} := \bar{T}(dx^i, dx^j) = (\bar{\Gamma}_k^{ij} - \bar{\Gamma}_k^{ji} - \partial_k \theta^{ij})(dx^k)^+ = (2\bar{\Gamma}_k^{[ij]} - \partial_k \theta^{ij})(dx^k)^+ = 0. \tag{5.21}$$

Thus, when $\beta = 0$, the torsion tensor is identically zero

$$\bar{T} = 0, \tag{5.22}$$

which is guaranteed by the multi-linearity (5.21). Although the connection non-trivially has anti-symmetric part, it is torsion free.

Poisson generalized curvature The local expression for the curvature is obtained by computing

$$(\nabla_{dx^i}\nabla_{dx^j} - \nabla_{dx^j}\nabla_{dx^i} - \nabla_{[dx^i, dx^j]})(dx^k)^+. \tag{5.23}$$

Using the Leibniz rule and the linearity of the covariant derivative, we have

$$\nabla_{dx^i}\nabla_{dx^j} dx^k = \nabla_{dx^i}\bar{\Gamma}_l^{jk}(dx^l)^+ = (\mathcal{L}_{dx^i}\bar{\Gamma}_l^{jk} + \bar{\Gamma}_m^{jk}\bar{\Gamma}_l^{im})(dx^l)^+, \tag{5.24}$$

and, as mentioned above,

$$\nabla_{[dx^i, dx^j]}(dx^k)^+ = (\partial_n \theta^{ij})\bar{\Gamma}_l^{nk}(dx^l)^+. \tag{5.25}$$

By combining these results, we find that the curvature is

$$\begin{aligned}
\bar{R}_l^{kij}(dx^l)^+ &:= (\nabla_{dx^i}\nabla_{dx^j} - \nabla_{dx^j}\nabla_{dx^i} - \nabla_{[dx^i, dx^j]})(dx^k)^+ \\
&= (\theta^{im}\partial_m \bar{\Gamma}_l^{jk} - \theta^{jm}\partial_m \bar{\Gamma}_l^{ik} - \partial_n \theta^{ij}\bar{\Gamma}_l^{nk} + \bar{\Gamma}_m^{jk}\bar{\Gamma}_l^{im} - \bar{\Gamma}_m^{ik}\bar{\Gamma}_l^{jm})(dx^l)^+ \\
&=: \Pi_l^{kij}(dx^l)^+.
\end{aligned} \tag{5.26}$$

Here we introduce a symbol Π_l^{kij} to denote the coefficients

$$\Pi_l^{kij} = \theta^{im}\partial_m \bar{\Gamma}_l^{jk} - \theta^{jm}\partial_m \bar{\Gamma}_l^{ik} - \partial_n \theta^{ij}\bar{\Gamma}_l^{nk} + \bar{\Gamma}_m^{jk}\bar{\Gamma}_l^{im} - \bar{\Gamma}_m^{ik}\bar{\Gamma}_l^{jm}. \tag{5.27}$$

This object can be interpreted as an analogue of the Riemann curvature tensor.

Ricci tensor and Ricci scalar The Ricci tensor is obtained by a contraction between an upper and a lower indices of the Riemann curvature tensor

$$\bar{R}^{kj} = \bar{R}_l^{klj}. \tag{5.28}$$

And the Ricci scalar is constructed by a contraction of the Ricci tensor with the metric

$$\bar{R} = G_{kj}\bar{R}^{kj}. \tag{5.29}$$

5.2.3 Rewriting in covariant manner

To get more insights into the connection $\bar{\Gamma}_k^{ij}$, we rewrite it in manifestly covariant manner. Using the ordinary Levi-Civita connection Γ_{ij}^k, defined as usual,

$$\Gamma_{ij}^k = \frac{1}{2}G^{kl}(\partial_i G_{lj} + \partial_j G_{li} - \partial_l G_{ij}), \tag{5.30}$$

we can rewrite the derivative of the metric G as

$$\partial_m G^{ij} = -\Gamma_{ml}^i G^{lj} - \Gamma_{ml}^j G^{il}. \tag{5.31}$$

We introduce the usual Levi-Civita connection as ∇_i. Then the symmetric part of the connection reads

$$2\bar{\Gamma}_k^{\{ij\}} = [\Gamma_{ml}^i \theta^{mj} G^{ln} + \Gamma_{ml}^j \theta^{mi} G^{ln} - G^{jl}\nabla_l \theta^{in} - G^{il}\nabla_l \theta^{jn}]G_{nk}, \tag{5.32}$$

while the antisymmetric part does

$$2\bar{\Gamma}_k^{[ij]} = G^{nl}[\nabla_l \theta^{ij} - \Gamma_{ml}^i \theta^{mj} + \Gamma_{lm}^j \theta^{mi}]G_{nk}. \tag{5.33}$$

By summing them up we find that the connection (5.19) is rewritten as

$$2\bar{\Gamma}_k^{\{ij\}} + 2\bar{\Gamma}_k^{[ij]} = 2\Gamma_{mk}^j \theta^{mi} + \nabla_k \theta^{ij} - G^{jl}\nabla_l(\theta^{in}G_{nk}) - G^{il}\nabla_l(\theta^{jn}G_{nk}), \tag{5.34}$$

i.e.

$$\bar{\Gamma}_k^{ij} = \Gamma_{mk}^j \theta^{mi} + \frac{1}{2}[\nabla_k \theta^{ij} - \nabla^j(\theta^{in}G_{nk}) - \nabla^i(\theta^{jn}G_{nk})] =: \Gamma_{mk}^j \theta^{mi} + K_k^{ij}, \tag{5.35}$$

where we introduce a tensor K_k^{ij} defined as

$$K_k^{ij} = \frac{1}{2}G_{kn}[\nabla^n \theta^{ij} - \nabla^i \theta^{jn} + \nabla^j \theta^{ni}] =: G_{kn}K^{nij}, \tag{5.36}$$

which can be interpreted as an analogue of "contorsion tensor". Here raising and lowering of indices are done by the metric G, and pay attention to $\nabla^i \neq \nabla_{dx^i}$.

Thus we can rewrite the connection (5.19) as

$$\nabla_{dx^i}(dx^j)^+ = (\Gamma_{km}^j \theta^{mi} + K_k^{ij})(dx^k)^+ = \bar{\Gamma}_k^{ij}(dx^k)^+. \tag{5.37}$$

After some lengthy computation, see appendix B.2, we obtain the covariant expression of the curvature as

$$\Pi_l^{kij} = \theta^{im}\theta^{nj}R^k_{lmn} - (\nabla_n\theta^{ij})K_l^{nk} + \theta^{nj}\nabla_n K_l^{ik} - \theta^{ni}\nabla_n K_l^{jk} + K_m^{jk}K_l^{im} - K_m^{ik}K_l^{jm}, \tag{5.38}$$

where R^k_{lmn} and ∇ denote the Riemann curvature tensor and the Levi-Citiva connection constructed by the metric G, respectively.

An analogue of the Ricci tensor is constructed by contracting an upper index with a lower index as usual:

$$\begin{aligned}\Pi^{kj} &:= \Pi_l^{klj} \\ &= \theta^{lm}\theta^{nj}R^k_{lmn} - (\nabla_n\theta^{lj})K_l^{nk} + \theta^{nj}\nabla_n K_l^{lk} - \theta^{nl}\nabla_n K_l^{jk} + K_m^{jk}K_l^{lm} - K_m^{lk}K_l^{jm}.\end{aligned} \tag{5.39}$$

Notice that

$$2K_l^{lk} = 2\nabla_l\theta^{lk}, \tag{5.40}$$

where we used $\theta^{lp}G_{pl} = 0$. Hence we obtain an analogue of the Ricci tensor

$$\begin{aligned}\Pi^{kj} &= \Pi_l^{klj} \\ &= \theta^{lm}\theta^{nj}R^k_{lmn} - (\nabla_n\theta^{lj})K_l^{nk} + \theta^{nj}\nabla_n\nabla_l\theta^{lk} - \theta^{nl}\nabla_n K_l^{jk} + K_m^{jk}\nabla_l\theta^{lm} - K_m^{lk}K_l^{jm}.\end{aligned} \tag{5.41}$$

Finally an analogue of the Ricci scalar is defined by taking the contraction between the metric G and the Ricci tensor

$$\begin{aligned}\Pi &= G_{kj}\Pi^{kj} \\ &= \theta^{lm}\theta^{nj}R_{jlmn} - G_{kj}(\nabla_n\theta^{lj})K_l^{nk} + \theta^{nj}G_{kj}\nabla_n\nabla_l\theta^{lk} - G_{kj}\theta^{nl}\nabla_n K_l^{jk} + G_{kj}K_m^{jk}\nabla_l\theta^{lm} - G_{kj}K_m^{lk}K_l^{jm}.\end{aligned} \tag{5.42}$$

Notice that

$$2G_{kj}K_l^{jk} = -2G_{pl}\nabla_j(\theta^{jp}), \tag{5.43}$$

where we used $G_{kj}\nabla_l\theta^{jk} = \nabla_l(G_{kj}\theta^{jk}) = 0$. Then we find that the analogue of the Ricci scalar results in

$$\Pi = \theta^{lm}\theta^{nj}R_{jlmn} + 2\theta_{nm}\nabla^n\nabla_l\theta^{lm} - \nabla^n\theta_{nm}\nabla_l\theta^{lm}. \tag{5.44}$$

Here raising and lowering of indices are done by the metric G.

5.3 Interpretation of the connection

In this section we try to clarify the geometrical meaning of the connection presented in the last two sections. We investigate here what kind of geometry the connection $\bar{\Gamma}_k^{ij}$ defines. To this end, we extend the connection to that acting on 1-form $\{dx^i\}$, *not* $\{(dx^i)^+\}$[1] and forget the results above for a while:

$$\nabla_{dx^i} dx^j = \bar{\Omega}^{ij}_k dx^k. \tag{5.45}$$

We define a connection for any 1-forms ξ, η and any functions f, g,

$$\nabla_{f\xi}(g\eta) := fg\nabla_\xi \eta + f(\mathcal{L}_\xi g)\eta, \tag{5.46}$$

especially for $\xi = \xi_i dx^i$ and $\eta = \eta_i dx^i$

$$\nabla_\xi \eta = \xi_i \eta_j \nabla_{dx^i} dx^j + \xi_i (\mathcal{L}_{dx^i} \eta_j) dx^j = (\xi_i \eta_j \bar{\Omega}^{ij}_k + \xi_i \theta^{ij} \partial_j \eta_k) dx^k. \tag{5.47}$$

We require that the covariant derivative should be compatible with the metric G in the sense of what follows:

$$\mathcal{L}_\xi G(\eta, \zeta) = G(\nabla_\xi \eta, \zeta) + G(\eta, \nabla_\xi \zeta). \tag{5.48}$$

We calculate straightforwardly the left-hand side

$$\mathcal{L}_\xi (G^{ij} \eta_i \zeta_j) = \xi_k \theta^{kl} \partial_l (G^{ij} \eta_i \zeta_j), \tag{5.49}$$

and the right-hand side

$$\begin{aligned} & G(\nabla_\xi \eta, \zeta) + G(\eta, \nabla_\xi \zeta) \\ & = G((\xi_i \eta_j \bar{\Omega}^{ij}_k + \xi_i \theta^{ij} \partial_j \eta_k) dx^k, \zeta) + G(\eta, (\xi_i \zeta_j \bar{\Omega}^{ij}_k + \xi_i \theta^{ij} \partial_j \zeta_k) dx^k) \\ & = \xi_k \eta_i \zeta_j (G^{il} \bar{\Omega}^{kj}_l + G^{jl} \bar{\Omega}^{ki}_l) + \xi_k G^{ij} \zeta_j \theta^{kl} \partial_l \eta_i + \xi_k G^{ij} \eta_i \theta^{kl} \partial_l \zeta_j. \end{aligned} \tag{5.50}$$

Hence, we impose

$$\theta^{kl} \partial_l G^{ij} - \bar{\Omega}^{ki}_l G^{lj} - \bar{\Omega}^{kj}_l G^{il} = 0. \tag{5.51}$$

Permuting the indices of (5.51) cyclicly, we have

$$\theta^{il} \partial_l G^{jk} - \bar{\Omega}^{ij}_m G^{mk} - \bar{\Omega}^{ik}_m G^{jm} = 0, \tag{5.52}$$

$$\theta^{jl} \partial_l G^{ki} - \bar{\Omega}^{jk}_m G^{mi} - \bar{\Omega}^{ji}_m G^{km} = 0. \tag{5.53}$$

Furthermore, taking a sum $-(5.51) + (5.52) + (5.53)$, we have

$$-\theta^{kl} \partial_l G^{ij} + \theta^{il} \partial_l G^{jk} + \theta^{jl} \partial_l G^{ki} = -(\bar{\Omega}^{ki}_m - \bar{\Omega}^{ik}_m) G^{mj} - (\bar{\Omega}^{kj}_m - \bar{\Omega}^{jk}_m) G^{im} + (\bar{\Omega}^{ij}_m + \bar{\Omega}^{ji}_m) G^{mk}. \tag{5.54}$$

We can define an analogue of the torsion tensor as

$$\bar{T}(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]_\theta, \tag{5.55}$$

[1] For Riemannian geometry, the Christoffel symbol defines the Levi-Civita connection on the basis vectors $\{\partial_i\}$ as $\nabla_i \partial_j = \Gamma^k_{ij} \partial_k$. Here we make an analogous discussion.

for any 1-forms ξ and η. Since the right-hand side is manifestly skew-symmetric, $\bar{T}$ defines a skew-symmetric map. We can check that this map defines a tensor. As a consistency check, we can easily show that

$$\begin{aligned}\bar{T}(f\xi, g\eta) &= \nabla_{f\xi}(g\eta) - \nabla_{g\eta}(f\xi) - [f\xi, g\eta]_\theta \\ &= fg\nabla_\xi\eta + f(\mathcal{L}_\xi g)\eta - gf\nabla_\eta\xi - g(\mathcal{L}_\eta f)\xi - fg[\xi,\eta]_\theta - f(\mathcal{L}_\xi g)\eta + g(\mathcal{L}_\eta f)\xi \\ &= fg(\nabla_\xi\eta - \nabla_\eta\xi - [\xi,\eta]_\theta) = fg\bar{T}(\xi,\eta),\end{aligned} \tag{5.56}$$

for arbitrary function f and g. Thus $\bar{T}$ indeed defines a skew-symmetric tensor. Torsion-free condition claims

$$\begin{aligned}\bar{T}^{ij} := \bar{T}(dx^i, dx^j) &= \nabla_{dx^i}dx^j - \nabla_{dx^j}dx^i - [dx^i, dx^j]_\theta \\ &= (\bar{\Omega}^{ij}_k - \bar{\Omega}^{ji}_k - \partial_k\theta^{ij})dx^k = 0,\end{aligned} \tag{5.57}$$

i.e.

$$\bar{\Omega}^{ij}_k - \bar{\Omega}^{ji}_k = \partial_k\theta^{ij}, \tag{5.58}$$

giving (5.54)

$$(\bar{\Omega}^{ij}_m + \bar{\Omega}^{ji}_m)G^{mk} = \theta^{mk}\partial_m G^{ij} - \theta^{mi}\partial_m G^{jk} - \theta^{mj}\partial_m G^{ki} - G^{jm}\partial_m\theta^{ik} - G^{im}\partial_m\theta^{jk} \tag{5.59}$$

These are very the same as $\bar{\Gamma}^{ij}_k$. Hence, the connection $\bar{\Omega}^{ij}_k = \bar{\Gamma}^{ij}_k$ is obtained as the solution of both the condition for the compatibility with the metric G^{ij} and the torsion-free.

Note that we obtain an interesting observation about the Poisson bi-vector θ. As we have

$$\nabla_{dx^k}\theta^{ij} = \theta^{km}\partial_m\theta^{ij} - \Omega^{ki}_m\theta^{mj} - \Omega^{kj}_m\theta^{im}, \tag{5.60}$$

permuting the indices cyclicly,

$$\nabla_{dx^i}\theta^{jk} = \theta^{im}\partial_m\theta^{jk} - \Omega^{ij}_m\theta^{mk} - \Omega^{ik}_m\theta^{jm}, \tag{5.61}$$

$$\nabla_{dx^j}\theta^{ki} = \theta^{jm}\partial_m\theta^{ki} - \Omega^{jk}_m\theta^{mi} - \Omega^{ji}_m\theta^{km} \tag{5.62}$$

and then taking a sum we have

$$\begin{aligned}&\nabla_{dx^k}\theta^{ij} + \nabla_{dx^i}\theta^{jk} + \nabla_{dx^j}\theta^{ki} \\ &= \theta^{km}\partial_m\theta^{ij} + \theta^{im}\partial_m\theta^{jk} + \theta^{jm}\partial_m\theta^{ki} - \Omega^{ki}_m\theta^{mj} - \Omega^{kj}_m\theta^{im} - \Omega^{ij}_m\theta^{mk} - \Omega^{ik}_m\theta^{jm} - \Omega^{jk}_m\theta^{mi} - \Omega^{ji}_m\theta^{km} \\ &= \theta^{mj}(\Omega^{ik}_m - \Omega^{ki}_m) + \theta^{im}(\Omega^{jk}_m - \Omega^{kj}_m) - \theta^{mk}(\Omega^{ij}_m - \Omega^{ji}_m) \\ &= \theta^{mj}\partial_m\theta^{ik} + \theta^{im}\partial_m\theta^{jk} - \theta^{mk}\partial_m\theta^{ij} = -\theta^{mj}\partial_m\theta^{ki} - \theta^{mi}\partial_m\theta^{jk} - \theta^{mk}\partial_m\theta^{ij} = 0,\end{aligned} \tag{5.63}$$

with a use of the Poisson condition $\theta^{m[i}\partial_m\theta^{jk]} = 0$. Hence, the covariantized Poisson condition

$$\nabla_{dx^{[i}}\theta^{jk]} = 0 \tag{5.64}$$

is satisfied.

5.4 In the presence of R-flux

This section examines the extension of the construction of Riemann geometry in the previous section to the case in the presence of R-flux. Because the R-flux is found to enter differently from the metric and the Poisson tensor, the extension is done straightforwardly.

5.4.1 Connection, Torsion and Curvature

Poisson generalized connection In order to incorporate an R-flux into our formalism, we make a definition of a connection under the presence of β as

$$\nabla_\xi u = \pi_+(e^{-\beta}[e^\beta(\xi^-), e^\beta u]), \tag{5.65}$$

with $\xi \in \Gamma(T^*M)$ and $u \in C_+^{\beta=0}$. Here we introduced extensions $^\pm$ as $\xi^\pm = \xi \pm G(\xi) \in C_\pm^{\beta=0}$, and hence $e^\beta \xi^\pm = \xi + (\pm G + \beta)(\xi) \in C_\pm$. Recall that our bracket satisfies, for $u = X + \xi$ and $v = Y + \eta$,

$$[e^\beta(u), e^\beta(v)] = [u + \beta(\xi), v + \beta(\eta)] = e^\beta[u, v] + i_\eta i_\xi d_\theta \beta. \tag{5.66}$$

Hence we have

$$\nabla_\xi u = \pi_+([\xi^-, u] + i_\eta i_\xi d_\theta \beta) = \pi_+([\xi^-, u]) + \pi_+(i_\eta i_\xi d_\theta \beta), \tag{5.67}$$

where $u|_{1-\mathrm{form}} = \eta$. It is noteworthy that the first term in the right-hand side is the same as that of trivial β and the R-flux separately appears as the second term. Thus, using the results of the case of trivial β, the Leibniz rule of this connection

$$\nabla_{f\xi}(gu) = fg\nabla_\xi u + f(\mathcal{L}_\xi g)u \tag{5.68}$$

is easily shown to be satisfied.

Poisson generalized torsion A torsion is also defined by

$$\bar{T}(\xi, \eta) = \nabla_\xi \eta^+ - \nabla_\eta \xi^+ - ([\xi, \eta]_\theta)^+, \tag{5.69}$$

and it is easily shown that

$$\bar{T}(f\xi, g\eta) = fg\bar{T}(\xi, \eta). \tag{5.70}$$

Poisson generalized curvature A curvature is given by

$$\bar{R}(\xi, \eta)u = (\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi,\eta]_\theta})u, \tag{5.71}$$

for any 1-forms ξ, η and $u \in C_+$. The proof of the tensor property

$$\bar{R}(f\xi, g\eta)hu = fgh\bar{R}(\xi, \eta)u, \tag{5.72}$$

is done in a parallel manner with the case of $\beta = 0$, using only the Leibniz property of the connection

5.4.2 Local expressions

Poisson generalized connection In the local coordinates $\{x^i\}$ the connection reads

$$\nabla_{dx^i}(dx^j)^+ = \pi_+([(dx^i)^-, (dx^j)^+]) + \pi_+(i_{dx^j} i_{dx^i} d_\theta \beta). \tag{5.73}$$

The first term is nothing but (5.16) except for the replacement of g with G:

$$\begin{aligned} &[(dx^i)^-, (dx^j)^+] = [dx^i - G^{ik}\partial_k, dx^j + G^{jl}\partial_l] \\ &= \partial_k\theta^{ij} dx^k + [\theta^{mn}(\partial_m G^{ji}) - \theta^{mi}(\partial_m G^{jn}) - \theta^{mj}(\partial_m G^{in}) - G^{jl}(\partial_l\theta^{in}) - G^{il}(\partial_l\theta^{jn})]\partial_n. \end{aligned} \tag{5.74}$$

The different point of the connection from the case with $\beta = 0$ is the presence of the term involving $d_\theta\beta$. We give explicit form of this term. Note that in our formulation [28] an R-flux is given by a Poisson exterior derivative with d_θ of a bi-vector field potential β. It results in a tri-vector field:

$$d_\theta\beta = [\theta, \beta] = [\frac{1}{2}\theta^{nm}\partial_n \wedge \partial_m, \frac{1}{2}\beta^{ij}\partial_i \wedge \partial_j] =: \frac{1}{3!}\mathcal{R}^{abc}\partial_a \wedge \partial_b \wedge \partial_c. \tag{5.75}$$

Then we find

$$\begin{aligned} &i_{dx^j} i_{dx^i} d_\theta\beta \\ &= [\theta^{nm}\partial_m\beta^{ij} + \theta^{im}\partial_m\beta^{jn} + \theta^{jm}\partial_m\beta^{ni} + \beta^{nm}\partial_m\theta^{ij} + \beta^{im}\partial_m\theta^{jn} + \beta^{jm}\partial_m\theta^{ni}]\partial_n \\ &= \mathcal{R}^{ijn}\partial_n, \end{aligned} \tag{5.76}$$

Here again we define the coefficients of the connection as

$$2\bar{\Gamma}^{\{ij\}}_k = G_{nk}[\theta^{mn}\partial_m G^{ji} - \theta^{mj}\partial_m G^{in} - \theta^{mi}\partial_m G^{jn} - G^{im}\partial_m\theta^{jn} - G^{jl}\partial_l\theta^{in}], \tag{5.77}$$

$$2\bar{\Gamma}^{[ij]}_k = \partial_k\theta^{ij}, \tag{5.78}$$

$$\mathcal{R}^{ijn} = \theta^{nm}\partial_m\beta^{ij} + \theta^{im}\partial_m\beta^{jn} + \theta^{jm}\partial_m\beta^{ni} + \beta^{nm}\partial_m\theta^{ij} + \beta^{im}\partial_m\theta^{jn} + \beta^{jm}\partial_m\theta^{ni}. \tag{5.79}$$

Then we see that the connection can be written as

$$\nabla_{dx^i}(dx^j)^+ = \left(\bar{\Gamma}^{ij}_k + \frac{1}{2}\mathcal{R}^{ijn}G_{nk}\right)(dx^k)^+. \tag{5.80}$$

Poisson generalized torsion The local expression for torsion is read straightforwardly

$$\bar{T}^{ij} := \nabla_{dx^i}(dx^j)^+ - \nabla_{dx^j}(dx^i)^+ - ([dx^i, dx^j]_\theta)^+ = \mathcal{R}^{ijn}G_{nk}(dx^k)^+. \tag{5.81}$$

Hence the R-flux appears as the torsion. This is parallel to the H-flux appearing the torsion of generalized connection.

Poisson generalized curvature In the local coordinates $\{x^i\}$, the Riemann curvature reads

$$(\nabla_{dx^i}\nabla_{dx^j} - \nabla_{dx^j}\nabla_{dx^i} - \nabla_{[dx^i, dx^j]})(dx^k)^+. \tag{5.82}$$

Using the Leibniz rule and the linearity of the covariant derivative, we have

$$\begin{aligned}
&\nabla_{dx^i}\nabla_{dx^j}dx^k\\
&=\nabla_{dx^i}\left(\bar\Gamma^{jk}_l+\frac{1}{2}\mathcal{R}^{jkn}G_{nl}\right)(dx^l)^+\\
&=\left\{\mathcal{L}_{dx^i}\left(\bar\Gamma^{jk}_l+\frac{1}{2}\mathcal{R}^{jkn}G_{nl}\right)+\left(\bar\Gamma^{jk}_m+\frac{1}{2}\mathcal{R}^{jkn}G_{nm}\right)\left(\bar\Gamma^{im}_l+\frac{1}{2}\mathcal{R}^{imp}G_{pl}\right)\right\}(dx^l)^+,
\end{aligned}\tag{5.83}$$

and by definition

$$\nabla_{[dx^i,dx^j]}(dx^k)^+=(\partial_n\theta^{ij})\nabla_{dx^n}(dx^k)^+=(\partial_n\theta^{ij})\left(\bar\Gamma^{nk}_l+\frac{1}{2}\mathcal{R}^{nkp}G_{pl}\right)(dx^l)^+.\tag{5.84}$$

Combining these equations, we find that the curvature is

$$\begin{aligned}
&(\nabla_{dx^i}\nabla_{dx^j}-\nabla_{dx^j}\nabla_{dx^i}-\nabla_{[dx^i,dx^j]})(dx^k)^+=:\bar R^{kij}_l(dx^l)^+\\
=&\Big\{\Pi^{kij}_l+\frac{1}{2}\theta^{im}\partial_m(\mathcal{R}^{jkn}G_{nl})+\frac{1}{2}\mathcal{R}^{jkn}G_{nm}\bar\Gamma^{im}_l+\frac{1}{2}\bar\Gamma^{jk}_m\mathcal{R}^{imp}G_{pl}+\frac{1}{4}\mathcal{R}^{jkn}G_{nm}\mathcal{R}^{imp}G_{pl}-\\
&-\frac{1}{2}\theta^{jm}\partial_m(\mathcal{R}^{ikn}G_{nl})-\frac{1}{2}\mathcal{R}^{ikn}G_{nm}\bar\Gamma^{jm}_l-\frac{1}{2}\bar\Gamma^{ik}_m\mathcal{R}^{jmp}G_{pl}-\frac{1}{4}\mathcal{R}^{ikn}G_{nm}\mathcal{R}^{jmp}G_{pl}-\\
&-\frac{1}{2}(\partial_n\theta^{ij})\mathcal{R}^{nkp}G_{pl}\Big\}(dx^l)^+,
\end{aligned}\tag{5.85}$$

with

$$\Pi^{kij}_l=(\theta^{im}\partial_m\bar\Gamma^{jk}_l-\theta^{jm}\partial_m\bar\Gamma^{ik}_l-\partial_n\theta^{ij}\bar\Gamma^{nk}_l+\bar\Gamma^{jk}_m\bar\Gamma^{im}_l-\bar\Gamma^{ik}_m\bar\Gamma^{jm}_l),\tag{5.86}$$

as already introduced in the case with $\beta=0$.

Poisson generalized Ricci tensor The Ricci tensor $\bar R^{kj}$ is obtained by taking a contraction

$$\begin{aligned}
\bar R^{kj}:=&\bar R^{klj}_l\\
=&\Pi^{kj}+\frac{1}{2}\theta^{lm}\partial_m(\mathcal{R}^{jkn}G_{nl})+\frac{1}{2}\mathcal{R}^{jkn}G_{nm}\bar\Gamma^{lm}_l-\\
&-\frac{1}{2}\mathcal{R}^{lkn}G_{nm}\bar\Gamma^{jm}_l-\frac{1}{2}\bar\Gamma^{lk}_m\mathcal{R}^{jmp}G_{pl}-\frac{1}{4}\mathcal{R}^{lkn}G_{nm}\mathcal{R}^{jmp}G_{pl}-\frac{1}{2}(\partial_n\theta^{lj})\mathcal{R}^{nkp}G_{pl}.
\end{aligned}\tag{5.87}$$

Poisson generalized Ricci scalar Finally, the contraction between the Ricci tensor and the metric yields the Ricci scalar $\bar R$

$$\bar R:=G_{kj}\bar R^{kj}=\Pi-\frac{1}{4}\mathcal{R}^2,\tag{5.88}$$

with

$$\Pi=G_{kj}(\theta^{lm}\partial_m\bar\Gamma^{jk}_l-\theta^{jm}\partial_m\bar\Gamma^{lk}_l-\partial_n\theta^{lj}\bar\Gamma^{nk}_l+\bar\Gamma^{jk}_m\bar\Gamma^{lm}_l-\bar\Gamma^{lk}_m\bar\Gamma^{jm}_l),\tag{5.89}$$

$$\mathcal{R}^2=G_{kj}G_{lp}G_{nm}\mathcal{R}^{kln}\mathcal{R}^{jpm},\tag{5.90}$$

where R_{ijkl} denotes the Riemann curvature tensor constructed of G_{ij} and $\mathcal{R}^{ink}$ does the R-flux, defined by $R = d_\theta \beta$. Since the Ricci scalar $\bar{R}$ is a sum of the Ricci scalar Π which is obtained in the absence of R-flux in the previous section and the square of the R-flux. This result (5.88) is parallel to that of generalized Riemannian geometry, which is the sum of the ordinary Ricci scalar and the square of the H-flux (3.92).

Chapter 6

Conclusion and Discussion

In this dissertation we addressed the issue of formulating non-geometric fluxes. We proposed a new variant of generalized geometry to formulate the R-flux which is one of the non-geometric fluxes [28]. This novel framework, which we referred to as Poisson generalized geometry, was a kind of dual of the ordinary generalized geometry and was based on the Poisson structure θ of the target space. It was a Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$ equipped with a new bracket

$$[X+\xi, Y+\eta] = [\xi,\eta]_\theta + \mathcal{L}_\xi Y - \mathcal{L}_\eta X - \frac{1}{2}d_\theta(i_\xi Y - i_\eta X), \tag{6.1}$$

where the roles of the vector field and 1-form were interchanged compared to the standard generalized geometry which is equipped with the following Courant bracket

$$[X+\xi, Y+\eta]_C = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi). \tag{6.2}$$

The relevant symmetries to the novel framework consisted of β-deformations and β-diffeomorphisms, which terminology is introduced in refs. [36–42]. In our formulation, the R-flux was realized as a twisting of the Courant algebroid $(TM)_0 \oplus (T^*M)_\theta$ via β-transformations:

$$[e^\beta(X+\xi), e^\beta(Y+\eta)] = e^\beta[X+\xi, Y+\eta] + i_\eta i_\xi d_\theta\beta, \tag{6.3}$$

where the R-flux was given by $R = d_\theta\beta$. These were parallel to those of generalized geometry: the relevant symmetries to generalized geometry are given by B-transformations and diffeomorphisms and the H-flux is realized as a twisting of the standard Courant algebroid $TM \oplus T^*M$ via B-transformations:

$$[e^B(X+\xi), e^B(Y+\eta)]_C = e^B[X+\xi, Y+\eta]_C + i_Y i_X dB, \tag{6.4}$$

where the H-flux is given by $H = dB$.

In the latter part of this dissertation we constructed an analogue of Riemannian geometry based on the Poisson generalized geometry. Our construction of Poisson generalized Riemannian geometry had its basis on the differential geometry of $(T^*M)_\theta$. It was found that the analogues of the connection and the curvature were consistently defined. The resulting geometry was found to be compatible with both the Poisson structure θ and the positive-definite metric G: The connection was given by

$$\nabla_{dx^i}(dx^j)^+ = \left(\bar{\Gamma}^{\{ij\}}_k + \bar{\Gamma}^{[ij]}_k + \frac{1}{2}R^{ijn}G_{nk}\right)(dx^k)^+, \tag{6.5}$$

with

$$\bar{\Gamma}^{\{ij\}}_k = \frac{1}{2}G_{nk}[\theta^{mn}\partial_m G^{ji} - \theta^{mj}\partial_m G^{in} - \theta^{mi}\partial_m G^{jn} - G^{im}\partial_m\theta^{jn} - G^{jl}\partial_l\theta^{in}], \tag{6.6}$$

$$\bar{\Gamma}^{[ij]}_k = \frac{1}{2}\partial_k\theta^{ij}, \tag{6.7}$$

$$R^{ijn} = \theta^{nm}\partial_m\beta^{ij} + \theta^{im}\partial_m\beta^{jn} + \theta^{jm}\partial_m\beta^{ni} + \beta^{nm}\partial_m\theta^{ij} + \beta^{im}\partial_m\theta^{jn} + \beta^{jm}\partial_m\theta^{ni}. \tag{6.8}$$

The R-flux, $R = d_\theta\beta$, arose in the connection and was interpreted as a torsion tensor. The analogue of Ricci scalar Ξ read

$$\Xi := \Pi - \frac{1}{4}R^2, \tag{6.9}$$

with

$$\Pi = G_{kj}(\theta^{lm}\partial_m\bar{\Gamma}^{jk}_l - \theta^{jm}\partial_m\bar{\Gamma}^{lk}_l - \partial_n\theta^{lj}\bar{\Gamma}^{nk}_l + \bar{\Gamma}^{jk}_m\bar{\Gamma}^{lm}_l - \bar{\Gamma}^{lk}_m\bar{\Gamma}^{jm}_l) \tag{6.10}$$

$$= \theta^{lm}\theta^{nj}R_{jlmn} + 2\theta_{nm}\nabla^n\nabla_l\theta^{lm} - \nabla^n\theta_{nm}\nabla_l\theta^{lm}, \tag{6.11}$$

$$R^2 = G_{kj}G_{lp}G_{nm}R^{kln}R^{jpm}, \tag{6.12}$$

where R_{ijkl} denotes the Riemann curvature tensor constructed of G_{ij} and R^{ink} does the R-flux, defined by $R = d_\theta\beta$. These results were similar to the case of generalized geometry where the H-flux appears as a torsion tensor and the generalized Ricci scalar is

$$\Upsilon = R - \frac{1}{4}H^2, \tag{6.13}$$

where R is the ordinary Ricci scalar constructed from g_{ij} and $H^2 = g^{ij}g^{kl}g^{mn}H_{ikm}H_{jln}$ with $H = dB$.

We end this dissertation with a few comments on the relation between string theory and the geometry which was constructed in this dissertation.
In order to write down the Einstein-Hilbert action, we should define an invariant measure. One natural choice of invariant measure would be

$$\sqrt{\det G_{ij}}dx^1 \wedge \cdots \wedge dx^n. \tag{6.14}$$

As all indices in the quantity $\Xi = \Pi - 1/12R^2$ are contracted, it is apparently a scalar. Hence the analogue of the Einstein-Hilbert action would be

$$\mathcal{L} = \sqrt{|G|}\left(\Pi - \frac{1}{12}R^2\right), \tag{6.15}$$

where we rescaled the R-flux and introduced $G = \det G_{ij}$. Since the R-flux $R = d_\theta\beta$ is an invariant quantity under gauge transformations induced by β-transformations

$$\beta \;\rightarrow\; \beta + d_\theta\Lambda, \tag{6.16}$$

and both the invariant measure and the Ricci scalar Ξ are invariant under diffeomorphisms, including β-diffoemorphisms in particular, the Einstein-Hilbert action defined above is manifestly invariant under both β-transformations and β-diffeomorphisms.

In this sense, our construction of gravity theory is closely related to the construction done by Andriot et.al. [36–42] and Blumenhagen et.al. [43–46], since the construction of the gravity theory in [36–46] is also based on β-diffeomorphisms. Their resulting gravity theory is shown to be physically equivalent to the original low-energy supergravity theory of NSNS sector

$$\mathcal{L}_{\text{NSNS}} = \sqrt{|g|}\bigg(R - \frac{1}{12}H_{ijk}H^{ijk}\bigg). \tag{6.17}$$

They also introduce a covariant derivative with upper indices

$$\tilde{\nabla}^i V^j = \tilde{D}^i V^j - \breve{\Gamma}^{ij}_{k} V^k, \quad \tilde{\nabla}^i V_j = \tilde{D}^i V_j + \breve{\Gamma}^{ik}_{j} V_k. \tag{6.18}$$

The connection coefficients are given by

$$\breve{\Gamma}^{(ij)}_{k} = \breve{\Gamma}^{ij}_{k} - G_{kl}(G^{mi}\breve{\Gamma}^{[jl]}_{m} + G^{mj}\breve{\Gamma}^{[il]}_{m}), \tag{6.19}$$

where

$$\breve{\Gamma}^{(ij)}_{k} = \widetilde{\Gamma}^{ij}_{k} - \frac{1}{2}G_{kl}(\beta^{im}\partial_m G^{jl} + \beta^{jm}\partial_m G^{il} - \beta^{lm}\partial_m G^{ij} - G^{mi}\partial_m\beta^{jl} - G^{mj}\partial_m\beta^{il}), \tag{6.20}$$

$$\breve{\Gamma}^{[ij]}_{k} = -\frac{1}{2}\partial_k\beta^{ij}, \tag{6.21}$$

$$\widetilde{\Gamma}^{ij}_{k} = \frac{1}{2}G_{kl}(\tilde{\partial}^i G^{jl} + \tilde{\partial}^j G^{il} - \tilde{\partial}^l G^{ij}). \tag{6.22}$$

Comparing with our connection

$$\Omega^{ij}_{k} = \frac{1}{2}G_{kn}(\theta^{im}\partial_m G^{jn} + \theta^{jm}\partial_m G^{in} - \theta^{nm}\partial_m G^{ij} - G^{mi}\partial_m\theta^{jn} - G^{mj}\partial_m\theta^{in}), \tag{6.23}$$

$$\Theta^{ij}_{k} = \frac{1}{2}\partial_k\theta^{ij}, \tag{6.24}$$

we find that the connection $\breve{\Gamma}^{ij}_{k}$ exactly coincides with ours, $(\Omega+\Theta)^{ij}_{k}$, if $\theta = \beta$, except the existence of the term $\tilde{\Gamma}^{ij}_{k}$ containing derivatives with respect to the dual coordinate $\tilde{\partial}$. Since in our formalism we do not refer to the dual coordinate $\tilde{x}_i$, we can naturally drop off the term $\tilde{\Gamma}^{ij}_{k}$ for comparison. Hence, the connections exactly coincide.

However, there is a significant difference between the formulation of [36–46] and our formulation as well. In [36–46] the bi-vector β is allowed to be any bi-vector i.e. the Poisson condition is no longer imposed. There the violation of the Jacobi identity itself is identified with the R-flux

$$\breve{R} = [\beta, \beta]_{SN} \neq 0. \tag{6.25}$$

On the other hand, our formulation assumes a Poisson bi-vector θ which satisfies the Poisson condition

$$[\theta, \theta]_{SN} = 0. \tag{6.26}$$

Our R-flux is globally well-defined by

$$R = [\theta, \Lambda]_{SN} = d_\theta \Lambda, \tag{6.27}$$

with a use of gauge potential Λ of bi-vector field. This formulation of the R-flux realized to introduce the notion of (Abelian) gauge transformation $\Lambda \to \Lambda + d_\theta \lambda$.
In refs. [36–46] the Q-flux

$$Q_k{}^{ij} = \partial_k \beta^{ij} \tag{6.28}$$

is introduced heuristically, in order to make the curvature tensor covariant, and it confuses ones due to its non-covariant expression, whereas in our construction it inevitably appeared as the anti-symmetric part of the connection

$$\Theta_k^{ij} = \frac{1}{2} \partial_k \theta^{ij} \tag{6.29}$$

by the first principle. It provides us with a definite interpretation on our geometry: our connection enables the corresponding covariant derivative to be compatible with both the metic G_{ij} and the Poisson tensor θ^{ij}. Besides, the anti-symmetric part of the connection guarantees that the torsion tensor vanishes in the absence of the R-flux. This notion on geometry has hardly obtained without imposing the Poisson condition.

In this dissertation we used the new Courant algebroid as a fundamental structure, however, there are many open questions related to the formulation. Along the approach of refs. [63, 64], we would like to define another non-geometric flux, Q-flux. It will be important to understand the T-duality chain in fully geometric way. It also needs more detailed study on the Poisson generalized geometry, such as a variant of generalized complex structures. The most important question is how the R-flux is realized in string theory or supergravity. In the case of H-fluxes, H should be quantized, since it appears in the Wess-Zumino-Witten term in the string world-sheet theory. Similarly, R-flux should also be quantized when it is realized as a background flux in the string world-sheet theory [60] or membrane world-volume theory [65]. It is interesting to see whether our R-flux is consistent with these formulations. There, the $U(1)$ gerbe analogue of R-flux would play a role.
It was observed that the resulting gravity theory is compatible with the Poisson structure. A Poisson structure potentially provides the non-commutative nature, since a space equipped with it can be regarded as the semi-classical approximation of a non-commutative space. Hence, by applying Kontsevich's deformation quantization formula to the Poisson structure [66, 67], the gravity theory constructed from Poisson generalized geometry is expected to be lifted to a gravity theory on a non-commutative space. It might be related to matrix models [68–70], since both the non-commutativity and the gravity must be taken into account in formulating those models.

Acknowledgements

This dissertation could never have been accomplished without help of many people. First and foremost, I would like to express my deepest appreciation to my supervisor, Dr. Satoshi Watamura, for his patience and guidance throughout the years of my doctoral course. I will never forget his encouragement with warmth. Secondly, I would like to express my gratitude to Dr. Tsuguhiko Asakawa, for his enduring support on my research. He always gave me insightful comments and suggestions. Gratitude is also due to the members of my dissertation committee, Dr. Ken-ichi Hikasa, Dr. Masahiro Yamaguchi, Dr. Hiroshi Ishikawa and Dr. Kouichi Hagino for generously sparing their time and giving invaluable advice to my work. In particular, I appreciate the feedback offered by Dr. Ishikawa. I would like to offer my special thanks to Dr. Ursula Carow-Watamura for useful comments. Special thanks also to each and every member of Particle Theory and Cosmology Group. Finally, I would like to thank my understanding parents, Yukiyo and Yukie, for their constant encouragement and support.

Appendix A

Notations for differential geometry

We use the symbols listed here without mention throughout this dissertation.

Cartan differential calculus

We list here the mathematical notations used throughout this dissertation.

$$C^\infty(M): \text{ set of smooth functions on manifold } M, \tag{A.1}$$

$$\mathcal{X}(M): \text{ set of vector fields on manifold } M, \tag{A.2}$$

$$\Omega^p(M): \text{ set of } p\text{-forms on manifold } M, \tag{A.3}$$

as usual. We use the standard notations used in differential geometry: i and d denote the interior product and exterior derivative, defined in some local coordinate $\{x^\mu\}$ as

$$i_X\omega = \frac{1}{(p-1)!}X^\nu \omega_{\nu[\mu_2\cdots\mu_p]}dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p}, \tag{A.4}$$

$$df = \frac{\partial f}{\partial x^\mu}dx^\mu, \tag{A.5}$$

$$d\omega = \frac{1}{(p+1)!}\partial_{[\mu_0}\omega_{\mu_1\cdots\mu_p]}dx^{\mu_0}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}, \tag{A.6}$$

respectively, for vector field $X \in \mathcal{X}(M)$, p-form $\omega \in \Omega^p(M)$ and $f \in C^\infty(M)$. The indices under the square bracket [ ] are anti-symmetrized, e.g. $f_{[\mu\nu]} = f_{\mu\nu} - f_{\nu\mu}$. The exterior derivative is nilpotent

$$d^2 = 0. \tag{A.7}$$

The Lie derivative is defined as

$$\mathcal{L}_X Y = [X, Y], \tag{A.8}$$

$$\mathcal{L}_X f = i_X df, \tag{A.9}$$

$$\mathcal{L}_X\omega = (di_X + i_X d)\omega. \tag{A.10}$$

In terms of local coordinates, they are rewritten as

$$\mathcal{L}_X Y = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)\partial_\nu, \tag{A.11}$$

$$\mathcal{L}_X f = X^\mu \frac{\partial f}{\partial x^\mu}, \tag{A.12}$$

$$\mathcal{L}_X \omega = \frac{1}{p!}(\partial_{[\mu_1} X^\nu \omega_{\nu\mu_2\cdots\mu_p]} + X^\nu \partial_\nu \omega_{[\mu_1\mu_2\cdots\mu_p]})dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \tag{A.13}$$

by definition. For example, for 1-form η, the Lie derivative generated by vector field X can be written as

$$\mathcal{L}_X \eta = (\partial_\mu X^\nu \eta_\nu + X^\nu \partial_\nu \eta_\mu)dx^\mu, \tag{A.14}$$

and for 2-form ξ

$$\mathcal{L}_X \xi = \frac{1}{2}(\partial_{[\mu_1} X^\nu \xi_{\nu\mu_2]} + X^\nu \partial_\nu \xi_{[\mu_1\mu_2]})dx^{\mu_1} \wedge dx^{\mu_2} \tag{A.15}$$

They satisfy following relations

$$\{i_X, i_Y\} = 0, \quad \{d, i_X\} = \mathcal{L}_X, \quad [d, \mathcal{L}_X] = 0, \quad [\mathcal{L}_X, i_Y] = i_{[X,Y]}, \quad [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}. \tag{A.16}$$

They are easily shown as

$$[d, \mathcal{L}_X] = d\mathcal{L}_X - \mathcal{L}_X d = d(i_X d + d i_X) - (d i_X + i_X d)d = 0, \tag{A.17}$$

and for any p-form ω,

$$\begin{aligned}
&[\mathcal{L}_X, i_Y]\omega \\
&= \frac{1}{(p-1)!}\mathcal{L}_X(Y^\nu \omega_{\nu[\mu_1\cdots\mu_{p-1}]} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p-1}}) \\
&\qquad - \frac{1}{p!} i_Y [(\partial_{[\mu_0} X^\nu \omega_{\nu\mu_1\cdots\mu_{p-1}]} + X^\nu \partial_\nu \omega_{[\mu_0\mu_1\cdots\mu_{p-1}]})dx^{\mu_0} \wedge \cdots \wedge dx^{\mu_{p-1}}] \\
&= \frac{1}{(p-1)!}\big[\partial_{[\mu_1} X^\rho Y^\nu \omega_{\nu\rho\mu_2\cdots\mu_{p-1}]} + X^\rho \partial_\rho (Y^\nu \omega_{\nu[\mu_1\cdots\mu_{p-1}]})) \\
&\qquad - Y^\rho \partial_\rho X^\nu \omega_{\nu[\mu_1\cdots\mu_{p-1}]} + Y^\rho \partial_{[\mu_1} X^\nu \omega_{\nu\rho\mu_2\cdots\mu_{p-1}]} - Y^\rho X^\nu \partial_\nu \omega_{\rho[\mu_1\cdots\mu_{p-1}]}\big] dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p-1}} \\
&= \frac{1}{(p-1)!}\big[X^\rho \partial_\rho Y^\nu \omega_{\nu[\mu_1\cdots\mu_{p-1}]}) - Y^\rho \partial_\rho X^\nu \omega_{\nu[\mu_1\cdots\mu_{p-1}]}\big] dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p-1}} = i_{[X,Y]}\omega,
\end{aligned} \tag{A.18}$$

and finally,

$$\begin{aligned}
[\mathcal{L}_X, \mathcal{L}_Y] &= [\mathcal{L}_X, d i_Y] + [\mathcal{L}_X, i_Y d] \\
&= [\mathcal{L}_X, d] i_Y + d[\mathcal{L}_X, i_Y] + [\mathcal{L}_X, i_Y]d + i_Y[\mathcal{L}_X, d] = d i_{[X,Y]} + i_{[X,Y]} d = \mathcal{L}_{[X,Y]}.
\end{aligned} \tag{A.19}$$

Schouten-Nijenhuis bracket

The Schouten-Nijenhuis bracket is an extension of the Lie bracket of the vector fields to that of the poly-vector fields. For two poly-vectors of the form $V = X_1 \wedge \cdots \wedge X_k$ and $W = Y_1 \wedge \cdots \wedge Y_l$, with $X_i, Y_j \in \mathcal{X}(M)$, the Schouten-Nijenhuis bracket is defined as

$$[V, W]_S = \sum_{i=1,j=1}^{k,l} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots X_k \wedge Y_1 \wedge \cdots \hat{Y}_j \wedge \cdots Y_l. \tag{A.20}$$

Here $[X_i, Y_j]$ denotes the usual Lie bracket mentioned above and the vector fields with the symbol hat $\hat{}$ are understood to be removed. Hence, for 1-vectors the Shouten-Nijenhuis bracket reduces to the usual Lie bracket.
For example, for $X, Y, Z, W \in \mathcal{X}(M)$

$$[X \wedge Y, Z \wedge W] = [X, Z] \wedge Y \wedge W - [X, W] \wedge Y \wedge Z - [Y, Z] \wedge X \wedge W + [Y, W] \wedge X \wedge Z. \tag{A.21}$$

Vector bundles

$$TM: \text{ tangent bundle over manifold } M, \tag{A.22}$$

$$\Gamma(TM): \text{ set of sections of tangent bundle } TM \text{ i.e. set of vector fields}, \tag{A.23}$$

$$T^*M: \text{ cotangent bundle over manifold } M, \tag{A.24}$$

$$\Gamma(T^*M): \text{ set of sections of cotangent bundle } T^*M \text{ i.e. set of 1-forms}. \tag{A.25}$$

Appendix B

Computational details

B.1 Generalized geometry

Proof of (3.75) We show the following equation

$$R(fX, gY)hu = fghR(X, Y)u. \tag{B.1}$$

Indeed, using the Leibniz rule of the connection, we have

$$\begin{aligned}
&R(fX, gY)hu \\
&= (\nabla_{fX}\nabla_{gY} - \nabla_{gY}\nabla_{fX} - \nabla_{[fX,gY]})hu \\
&= f\nabla_X(gh\nabla_Y u + g(Yh)u) - g\nabla_Y(fh\nabla_X u + f(Xh)u) - \nabla_{fg[X,Y]+f(Xg)Y-g(Yf)X}hu = \\
&= fgh\nabla_X\nabla_Y u + fX(gh)\nabla_Y u + fg(Yh)\nabla_X u + fX(g(Yh))u- \\
&\quad - gfh\nabla_Y\nabla_X u - gY(fh)\nabla_X u - gf(Xh)\nabla_Y u - gX(f(Xh))u- \\
&\quad - fg\nabla_{[X,Y]}hu - f(Xg)\nabla_Y hu + g(Yf)\nabla_X hu = \\
&= fgh(\nabla_X\nabla_Y u - \nabla_Y\nabla_X u - \nabla_{[X,Y]}u) + (fX(gh) - gf(Xh) - fh(Xg))\nabla_Y u+ \\
&\quad + (fg(Yh) - gY(fh) + gh(Yf))\nabla_X u+ \\
&\quad + (fX(g(Yh)) - gY(f(Xh)) - fg([X,Y]h) - f(Xg)(Yh) + g(Yf)(Xh))u = \\
&= fgh(\nabla_X\nabla_Y u - \nabla_Y\nabla_X u - \nabla_{[X,Y]}u)+ \\
&\quad + [(f(Xg)(Yh)) + fgX(Yh) - g(Yf)(Xh) - gfY(Xh)- \\
&\qquad - fg([X,Y]h) - f(Xg)(Yh) + g(Yf)(Xh)]u \\
&= fgh(\nabla_X\nabla_Y u - \nabla_Y\nabla_X u - \nabla_{[X,Y]}u).
\end{aligned} \tag{B.2}$$

Details of the curvature in generalized geometry

$$\begin{aligned}
\Upsilon^m_{kij} &= \partial_i \Xi^m_{jk} + \Xi^l_{jk}\Xi^m_{il} - (i \leftrightarrow j) \\
&= \partial_i \left(\Gamma^m_{jk} + \frac{1}{2} g^{ml} H_{ljk} \right) + \left(\Gamma^l_{jk} + \frac{1}{2} g^{ln} H_{njk} \right) \left(\Gamma^m_{il} + \frac{1}{2} g^{mp} H_{pil} \right) - (i \leftrightarrow j) \\
&= R^m_{kij} + \frac{1}{2}\partial_i (g^{ml} H_{ljk}) + \frac{1}{2}\Gamma^l_{jk} g^{mp} H_{pil} + \frac{1}{2} g^{ln} H_{njk} \Gamma^m_{il} + \frac{1}{4} g^{ln} H_{njk} g^{mp} H_{pil} - \\
&\quad - \frac{1}{2}\partial_j (g^{ml} H_{lik}) - \frac{1}{2}\Gamma^l_{ik} g^{mp} H_{pjl} - \frac{1}{2} g^{ln} H_{nik} \Gamma^m_{jl} - \frac{1}{4} g^{ln} H_{nik} g^{mp} H_{pjl}.
\end{aligned} \tag{B.3}$$

At a first glance this object Υ^m_{kij} is not a covariant quantity because it has terms involving the partial derivatives, not the covariant derivatives. However, in fact they form covariant tensors:

$$\begin{aligned}
&(\text{terms seeming not to be covariant}) \\
&= \frac{1}{2}\partial_i (g^{ml} H_{ljk}) - \frac{1}{2}\partial_j (g^{ml} H_{lik}) + \frac{1}{2}\Gamma^l_{jk} g^{mp} H_{pil} + \frac{1}{2} g^{ln} H_{njk} \Gamma^m_{il} - \frac{1}{2}\Gamma^l_{ik} g^{mp} H_{pjl} - \frac{1}{2} g^{ln} H_{nik} \Gamma^m_{jl} \\
&= \frac{1}{2}\partial_i g^{ml} H_{ljk} + \frac{1}{2} g^{ml} \partial_i H_{ljk} - \frac{1}{2}\partial_j g^{ml} H_{lik} - \frac{1}{2} g^{ml} \partial_j H_{lik} + \\
&\quad + \frac{1}{2} g^{mp} \Gamma^l_{jk} H_{pil} + \frac{1}{2} g^{ln} \Gamma^m_{il} H_{njk} - \frac{1}{2} g^{mp} \Gamma^l_{ik} H_{pjl} - \frac{1}{2} g^{ln} \Gamma^m_{jl} H_{nik}.
\end{aligned} \tag{B.4}$$

Note that $0 = \nabla_i g^{ml} = \partial_i g^{ml} + \Gamma^m_{ip} g^{pl} + \Gamma^l_{ip} g^{mp}$, so that $\partial_i g^{ml} = -g^{pl}\Gamma^m_{ip} - g^{mp}\Gamma^l_{ip}$. With making a use of this relation, we find

$$\begin{aligned}
&(\text{terms seeming not to be covariant}) \\
&= \frac{1}{2}(-g^{pl}\Gamma^m_{ip} - g^{mp}\Gamma^l_{ip}) H_{ljk} + \frac{1}{2} g^{ml}\partial_i H_{ljk} - \frac{1}{2}(-g^{pl}\Gamma^m_{jp} - g^{mp}\Gamma^l_{jp}) H_{lik} - \frac{1}{2} g^{ml}\partial_j H_{lik} + \\
&\quad + \frac{1}{2} g^{mp}\Gamma^l_{jk} H_{pil} + \frac{1}{2} g^{ln}\Gamma^m_{il} H_{njk} - \frac{1}{2} g^{mp}\Gamma^l_{ik} H_{pjl} - \frac{1}{2} g^{ln}\Gamma^m_{jl} H_{nik} \\
&= \frac{1}{2} g^{ml}\partial_i H_{ljk} - \frac{1}{2} g^{mp}\Gamma^l_{ip} H_{ljk} - \frac{1}{2} g^{mp}\Gamma^l_{ik} H_{pjl} - \frac{1}{2} g^{ml}\partial_j H_{lik} + \frac{1}{2} g^{mp}\Gamma^l_{jp} H_{lik} + \frac{1}{2} g^{mp}\Gamma^l_{jk} H_{pil} \\
&= \frac{1}{2} g^{ml}[\nabla_i H_{ljk} + \Gamma^p_{il} H_{pjk} + \Gamma^p_{ij} H_{lpk} + \Gamma^p_{ik} H_{ljp}] - \frac{1}{2} g^{mp}\Gamma^l_{ip} H_{ljk} - \frac{1}{2} g^{mp}\Gamma^l_{ik} H_{pjl} - \\
&\quad - \frac{1}{2} g^{ml}[\nabla_j H_{lik} + \Gamma^p_{jl} H_{pik} + \Gamma^p_{ji} H_{lpk} + \Gamma^p_{jk} H_{lip}] + \frac{1}{2} g^{mp}\Gamma^l_{jp} H_{lik} + \frac{1}{2} g^{mp}\Gamma^l_{jk} H_{pil} \\
&= \frac{1}{2} g^{ml}\nabla_i H_{ljk} - \frac{1}{2} g^{ml}\nabla_j H_{lik}.
\end{aligned} \tag{B.5}$$

Thus we obtain a tensor expression

$$\Upsilon^m_{kij} = R^m_{kij} + \frac{1}{4} g^{ln} H_{njk} g^{mp} H_{pil} - \frac{1}{4} g^{ln} H_{nik} g^{mp} H_{pjl} + \frac{1}{2} g^{ml}\nabla_i H_{ljk} - \frac{1}{2} g^{ml}\nabla_j H_{lik}. \tag{B.6}$$

B.2 Poisson generalized geometry

Proof of the third equation of (4.15) We will prove the third equation of (4.15),

$$\mathcal{L}_\zeta X = \mathcal{L}_{\theta(\zeta)} X + \theta(i_X d\zeta), \tag{B.7}$$

in the the components calculation. Because of

$$\begin{aligned}[\theta, X]_S &= \left[\tfrac{1}{2}\theta^{\mu\nu}\partial_\mu \wedge \partial_\nu, X^\alpha\partial_\alpha\right]_S \\ &= \tfrac{1}{2}\left[\theta^{\mu\nu}\partial_\mu, X^\alpha\partial_\alpha\right]_S \wedge \partial_\nu - \tfrac{1}{2}\left[\partial_\nu, X^\alpha\partial_\alpha\right]_S \wedge \theta^{\mu\nu}\partial_\mu \\ &= \tfrac{1}{2}\theta^{\mu\nu}\partial_\mu X^\alpha\partial_\alpha \wedge \partial_\nu - \tfrac{1}{2}\theta^{\mu\nu}\partial_\nu X^\alpha\partial_\alpha \wedge \partial_\mu - \tfrac{1}{2}X^\alpha\partial_\alpha\theta^{\mu\nu}\partial_\mu \wedge \partial_\nu \\ &= \theta^{\mu\nu}\partial_\mu X^\alpha\partial_\alpha \wedge \partial_\nu - \tfrac{1}{2}X^\alpha\partial_\alpha\theta^{\mu\nu}\partial_\mu \wedge \partial_\nu, \end{aligned} \tag{B.8}$$

so we have

$$\begin{aligned} i_\zeta d_\theta X &= i_\zeta[\theta, X]_S \\ &= \theta^{\mu\nu}\partial_\mu X^\alpha\zeta_\alpha\partial_\nu - \theta^{\mu\nu}\partial_\mu X^\alpha\zeta_\nu\partial_\alpha - X^\alpha\partial_\alpha\theta^{\mu\nu}\zeta_\mu\partial_\nu \\ &= \left(\theta^{\mu\rho}\partial_\mu X^\alpha\zeta_\alpha + \theta^{\mu\nu}\partial_\nu X^\rho\zeta_\mu - X^\alpha\partial_\alpha\theta^{\mu\rho}\zeta_\mu\right)\partial_\rho. \end{aligned} \tag{B.9}$$

Next, we compute

$$\begin{aligned} d_\theta i_\zeta X &= -\theta(d(i_\zeta X)) \\ &= -\theta^{\mu\nu}\partial_\mu(\zeta_\alpha X^\alpha)\partial_\nu \\ &= -\theta^{\mu\rho}(\partial_\mu\zeta_\alpha X^\alpha + \zeta_\alpha\partial_\mu X^\alpha)\partial_\rho. \end{aligned} \tag{B.10}$$

Therefore, the l.h.s. is written as

$$\begin{aligned} \mathcal{L}_\zeta X &= i_\zeta d_\theta X + d_\theta i_\zeta X \\ &= \left(\theta^{\mu\rho}\partial_\mu X^\alpha\zeta_\alpha + \theta^{\mu\nu}\partial_\nu X^\rho\zeta_\mu - X^\alpha\partial_\alpha\theta^{\mu\rho}\zeta_\mu\right)\partial_\rho - \theta^{\mu\rho}(\partial_\mu\zeta_\alpha X^\alpha + \zeta_\alpha\partial_\mu X^\alpha)\partial_\rho \\ &= \left(\theta^{\mu\nu}\partial_\nu X^\rho\zeta_\mu - \theta^{\mu\rho}\partial_\mu\zeta_\alpha X^\alpha - X^\alpha\partial_\alpha\theta^{\mu\rho}\zeta_\mu\right)\partial_\rho. \end{aligned} \tag{B.11}$$

On the other hand, the r.h.s. is computed as follows. The first term is written as

$$\begin{aligned} \mathcal{L}_{\theta(\zeta)} X &= [\theta(\zeta), X]_S = [\theta^{\mu\nu}\zeta_\mu\partial_\nu, X^\alpha\partial_\alpha]_S \\ &= \theta^{\mu\nu}\zeta_\mu\partial_\nu X^\alpha\partial_\alpha - X^\alpha\partial_\alpha(\theta^{\mu\nu}\zeta_\mu)\partial_\nu \\ &= \left(\theta^{\mu\nu}\zeta_\mu\partial_\nu X^\rho - X^\alpha\partial_\alpha\theta^{\mu\rho}\zeta_\mu - X^\alpha\theta^{\mu\rho}\partial_\alpha\zeta_\mu\right)\partial_\rho, \end{aligned} \tag{B.12}$$

and the second term is

$$\begin{aligned} \theta(i_X d\zeta) &= \theta\left(i_X\left(\tfrac{1}{2}(\partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu)dx^\mu \wedge dx^\nu\right)\right) \\ &= \theta\left(X^\mu(\partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu)dx^\nu\right) = \theta\left(X^\nu(\partial_\nu\zeta_\mu - \partial_\mu\zeta_\nu)dx^\mu\right) \\ &= \theta^{\mu\rho}X^\nu(\partial_\nu\zeta_\mu - \partial_\mu\zeta_\nu)\partial_\rho. \end{aligned} \tag{B.13}$$

Summing up, we obtain

$$\begin{aligned} \mathcal{L}_{\theta(\zeta)} X + \theta(i_X d\zeta) &= \left(\theta^{\mu\nu}\zeta_\mu\partial_\nu X^\rho - X^\alpha\partial_\alpha\theta^{\mu\rho}\zeta_\mu - X^\alpha\theta^{\mu\rho}\partial_\alpha\zeta_\mu + \theta^{\mu\rho}X^\nu(\partial_\nu\zeta_\mu - \partial_\mu\zeta_\nu)\right)\partial_\rho \\ &= \left(\theta^{\mu\nu}\zeta_\mu\partial_\nu X^\rho - X^\alpha\partial_\alpha\theta^{\mu\rho}\zeta_\mu - \theta^{\mu\rho}X^\nu\partial_\mu\zeta_\nu\right)\partial_\rho \\ &= \left(\theta^{\mu\nu}\partial_\nu X^\rho\zeta_\mu - \theta^{\mu\rho}\partial_\mu\zeta_\alpha X^\alpha - X^\alpha\partial_\alpha\theta^{\mu\rho}\zeta_\mu\right)\partial_\rho. \end{aligned} \tag{B.14}$$

Thus, the equation is proved.

Proofs of (4.17) The proof of the first equation is shown as follows.

$$\begin{aligned}\langle\mathcal{L}_\zeta(X+\xi),Y+\eta\rangle+\langle X+\xi,\mathcal{L}_\zeta(Y+\eta)\rangle =&\tfrac{1}{2}(i_\eta\mathcal{L}_\zeta X+i_{\mathcal{L}_\zeta\xi}Y+i_\xi\mathcal{L}_\zeta Y+i_{\mathcal{L}_\zeta\eta}X)\\ =&\tfrac{1}{2}\mathcal{L}_\zeta(i_\eta X+i_\xi Y)\\ =&\mathcal{L}_\zeta\langle X+\xi,Y+\eta\rangle,\end{aligned} \tag{B.15}$$

where we used $i_{\mathcal{L}_\zeta\xi}=i_{[\zeta,\xi]_\theta}=[\mathcal{L}_\zeta,i_\xi]$ given in (4.4). Next, let us prove the third equation. The r.h.s. is

$$\mathcal{L}_\zeta[X+\xi,Y+\eta]=\mathcal{L}_\zeta[\xi,\eta]_\theta+\mathcal{L}_\zeta\mathcal{L}_\xi Y-\mathcal{L}_\zeta\mathcal{L}_\eta X+\tfrac{1}{2}\mathcal{L}_\zeta d_\theta(i_X\eta-i_Y\xi). \tag{B.16}$$

By using the relations following from (4.4),

$$\begin{aligned}&\mathcal{L}_\zeta[\xi,\eta]_\theta=[\zeta,[\xi,\eta]_\theta]_\theta=[\mathcal{L}_\zeta\xi,\eta]_\theta+[\xi,\mathcal{L}_\zeta\eta]_\theta,\\ &\mathcal{L}_\zeta\mathcal{L}_\xi Y=\mathcal{L}_\xi\mathcal{L}_\zeta Y+\mathcal{L}_{[\zeta,\xi]_\theta}Y,\end{aligned} \tag{B.17}$$

the r.h.s. is further rewritten as

$$\begin{aligned}\mathcal{L}_\zeta[X+\xi,Y+\eta]=&[\mathcal{L}_\zeta\xi,\eta]_\theta+[\xi,\mathcal{L}_\zeta\eta]_\theta+\mathcal{L}_\xi\mathcal{L}_\zeta Y+\mathcal{L}_{[\zeta,\xi]_\theta}Y-\mathcal{L}_\eta\mathcal{L}_\zeta X-\mathcal{L}_{[\zeta,\eta]_\theta}X\\ &+\tfrac{1}{2}d_\theta i_\zeta d_\theta(i_X\eta-i_Y\xi).\end{aligned} \tag{B.18}$$

On the other hand, the first term in the l.h.s. is

$$[\mathcal{L}_\zeta(X+\xi),Y+\eta]=[\mathcal{L}_\zeta\xi,\eta]_\theta+\mathcal{L}_{[\zeta,\xi]_\theta}Y-\mathcal{L}_\eta(\mathcal{L}_\zeta X)+\tfrac{1}{2}d_\theta(i_{\mathcal{L}_\zeta X}\eta-i_Y(\mathcal{L}_\zeta\xi)), \tag{B.19}$$

and similar for the second term. Thus, the l.h.s gives

$$\begin{aligned}&[\mathcal{L}_\zeta(X+\xi),Y+\eta]+[X+\xi,\mathcal{L}_\zeta(Y+\eta)]\\ =&[\mathcal{L}_\zeta\xi,\eta]_\theta+\mathcal{L}_{[\zeta,\xi]_\theta}Y-\mathcal{L}_\eta(\mathcal{L}_\zeta X)+\tfrac{1}{2}d_\theta(i_{\mathcal{L}_\zeta X}\eta-i_Y(\mathcal{L}_\zeta\xi))\\ +&[\xi,\mathcal{L}_\zeta\eta]_\theta+\mathcal{L}_\xi(\mathcal{L}_\zeta Y)-\mathcal{L}_{[\zeta,\eta]_\theta}X+\tfrac{1}{2}d_\theta(i_X(\mathcal{L}_\zeta\eta)-i_{\mathcal{L}_\zeta Y}\xi).\end{aligned} \tag{B.20}$$

Then except for the d_θ-exact terms, it is apparent that (B.18) and (B.20) coincide. Moreover, the d_θ-exact terms are also the same, since

$$i_{\mathcal{L}_\zeta X}\eta+i_X(\mathcal{L}_\zeta\eta)=\mathcal{L}_\zeta(i_X\eta)=i_\zeta d_\theta(i_X\eta). \tag{B.21}$$

Here we used the formula of the action of the Lie derivative on a function, $\mathcal{L}_\zeta f=i_\zeta d_\theta f$.

Finally, we check the second equation. The l.h.s is given as $\rho(\mathcal{L}_\zeta(X+\xi))=\theta(\mathcal{L}_\zeta\xi)$, while the r.h.s is $\mathcal{L}_\zeta(\rho(X+\xi))=(\mathcal{L}_\zeta\theta)(\xi)+\theta(\mathcal{L}_\zeta\xi)$, so that the equation is satisfied if

$$\mathcal{L}_\zeta\theta=d_\theta i_\zeta\theta=d_\theta\theta(\zeta)=0. \tag{B.22}$$

Proof of the third equation of (4.18) To this end we will show that

$$e^{\beta}[X+\xi, Y+\eta] = [e^{\beta}(X+\xi), e^{\beta}(Y+\eta)] + [\theta,\beta]_S(\xi,\eta) \tag{B.23}$$

then, a β-transformation is a symmetry if $d_\theta\beta = [\theta,\beta]_S = 0$. The l.h.s. is written as

$$e^{\beta}[X+\xi, Y+\eta] = [X+\xi, Y+\eta] + \beta([\xi,\eta]_\theta). \tag{B.24}$$

while the r.h.s. is

$$\begin{aligned}[e^{\beta}(X+\xi), e^{\beta}(Y+\eta)] &= [X+\xi+\beta(\xi), Y+\eta+\beta(\eta)] \\ &= [X+\xi, Y+\eta] + \mathcal{L}_\xi\beta(\eta) - \mathcal{L}_\eta\beta(\xi) + \tfrac{1}{2}d_\theta(i_{\beta(\xi)}\eta - i_{\beta(\eta)}\xi).\end{aligned} \tag{B.25}$$

By using the formula $\mathcal{L}_\zeta X = \mathcal{L}_{\theta(\zeta)}X + \theta(i_X d\zeta)$, we have

$$\begin{aligned}\mathcal{L}_\xi\beta(\eta) &= \mathcal{L}_{\theta(\xi)}\beta(\eta) + \theta(i_{\beta(\eta)}d\xi) \\ &= [\theta(\xi), \beta(\eta)]_\theta + \theta(i_{\beta(\eta)}d\xi),\end{aligned} \tag{B.26}$$

By using $d_\theta f = -\theta(df)$, we have

$$d_\theta i_{\beta(\xi)}\eta = d_\theta(\beta(\xi,\eta)) = -\theta(d(\beta(\xi,\eta))) \tag{B.27}$$

Substituting these, the r.h.s. becomes

$$\begin{aligned}&[e^{\beta}(X+\xi), e^{\beta}(Y+\eta)] \\ =&[X+\xi, Y+\eta] + [\theta(\xi),\beta(\eta)]_\theta - [\theta(\eta),\beta(\xi)]_\theta + \theta(i_{\beta(\eta)}d\xi - i_{\beta(\xi)}d\eta - d(\beta(\xi,\eta))) \\ =&[X+\xi, Y+\eta] + [\theta(\xi),\beta(\eta)]_\theta + [\beta(\xi),\theta(\eta)]_\theta - \theta([\xi,\eta]_\beta),\end{aligned} \tag{B.28}$$

where in the last line we define $[\xi,\eta]_\beta$ by the same formula as the Koszul bracket for an arbitrary bivector β (It is not a Lie bracket but we do not use this property.). Then, by using

$$[(\theta+\beta)(\xi), (\theta+\beta)(\eta)]_S = [\theta(\xi),\theta(\eta)]_S + [\theta(\xi),\beta(\eta)]_S + [\beta(\xi),\theta(\eta)]_S + [\beta(\xi),\beta(\eta)]_S, \tag{B.29}$$

it is further rewritten as

$$\begin{aligned}&[e^{\beta}(X+\xi), e^{\beta}(Y+\eta)] \\ =&[X+\xi, Y+\eta] + [(\theta+\beta)(\xi), (\theta+\beta)(\eta)]_S - [\theta(\xi),\theta(\eta)]_S - [\beta(\xi),\beta(\eta)]_S - \theta([\xi,\eta]_\beta),\end{aligned} \tag{B.30}$$

To rewrite it further, we use a formula

$$[\beta(\xi),\beta(\eta)]_S = \beta([\xi,\eta]_\beta) + \frac{1}{2}[\beta,\beta]_S(\xi,\eta) \tag{B.31}$$

which is valid for any bivector β. In particular,

$$\begin{aligned}[(\theta+\beta)(\xi), (\theta+\beta)(\eta)]_S &= (\theta+\beta)([\xi,\eta]_{\theta+\beta}) + \frac{1}{2}[\theta+\beta,\theta+\beta]_S(\xi,\eta) \\ &= (\theta+\beta)([\xi,\eta]_\theta + [\xi,\eta]_\beta) + \frac{1}{2}[\theta+\beta,\theta+\beta]_S(\xi,\eta).\end{aligned} \tag{B.32}$$

Then, we finally obtain

$$\begin{aligned}
&[e^{\beta}(X+\xi), e^{\beta}(Y+\eta)]\\
=&[X+\xi, Y+\eta] + (\theta+\beta)([\xi,\eta]_{\theta} + [\xi,\eta]_{\beta}) - \theta([\xi,\eta]_{\theta}) - \beta([\xi,\eta]_{\beta}) - \theta([\xi,\eta]_{\beta})\\
&+\frac{1}{2}[\theta+\beta,\theta+\beta]_S(\xi,\eta) - \frac{1}{2}[\theta,\theta]_S(\xi,\eta) - \frac{1}{2}[\beta,\beta]_S(\xi,\eta)\\
=&[X+\xi, Y+\eta] + \beta([\xi,\eta]_{\theta}) + [\theta,\beta]_S(\xi,\eta).
\end{aligned} \tag{B.33}$$

Review on twisting of $TM \oplus T^*M$ with H-flux When there is a H-flux, one can define the corresponding Courant algebroid $(E, \rho, [\cdot,\cdot])$ from $TM \oplus T^*M$ by twist as follows [24, 61, 62]:

1) Take a good cover $\{U_i\}$ of M. Before twisting, a global section of $TM \oplus T^*M$ satisfies $X_i + \xi_i = X_j + \xi_j$ on a overlap $U_{ij} = U_i \cap U_j$.

2) Modify the gluing condition to $X_i + \xi_i = X_j + \xi_j - dA_{ij}(X_j)$ for a set of 1-forms $A_{ij} \in T^*U_{ij}$. Note that T^*M is twisted by local B-gauge transformations.

3) Define a bundle $E = \amalg_i (TU_i \oplus T^*U_i)/\sim$ by a standard clutching construction. Then, $(E, \rho, [\cdot,\cdot])$ is a Courant algebroid, because the B-gauge transformation preserves both the anchor ρ and the bracket $[\cdot,\cdot]$.

This twisting defines an exact Courant algebroid

$$0 \to T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \to 0. \tag{B.34}$$

with an isotropic splitting $s: TM \to E$. That is $E = s(TM) \otimes \rho^*(T^*M)$. Locally, the splitting is given by a local B-transform as

$$s_i(X) = e^{B_i}(X) = X + B_i(X), \tag{B.35}$$

where $B_i \in \wedge^2 T^*U_i$. In order that it is globally defined, it should satisfy $s_i(X) = s_j(X)$ on U_{ij}. Taking into account the gluing condition 2), it leads to conditions $B_j = B_i + dA_{ij}$ for local 2-forms. It also implies that $H := dB_i$ on M is a global closed 3-form.

Thus, we need a data (H, B_i, A_{ij}) to construct E. More specifically, it is known that the geometric object corresponding to a closed 3-form H flux is a $U(1)$-gerbe with connection, when its cohomology class $[H]$ is in the integer cohomology $H^3(M;\mathbb{Z})$. It is defined by a set $(H, B_i, A_{ij}, \Lambda_{ijk})$ in the Čech-de Rham double complex, with a set of equations

$$\begin{aligned}
U_i:&\quad H = dB_i,\\
U_{ij}:&\quad B_j - B_i = dA_{ij},\\
U_{ijk}:&\quad A_{ij} + A_{jk} + A_{ki} = d\Lambda_{ijk},\\
U_{ijkl}:&\quad \Lambda_{jkl} - \Lambda_{ikl} + \Lambda_{ijl} - \Lambda_{ijk} = n_{ijkl}.
\end{aligned} \tag{B.36}$$

This H-twisting is also regarded as a change of the Courant bracket of $TM \oplus T^*M$ to the H-twisted Courant bracket. To see this recall that the relation

$$[e^{B_i}(X+\xi), e^{B_i}(Y+\eta)] = e^{B_i}[X+\xi, Y+\eta] + i_X i_Y dB_i \tag{B.37}$$

is still true for local B-transformations. Therefore, if we define an H-twisted Courant bracket

$$[X+\xi, Y+\eta]_H = [X+\xi, Y+\eta] + i_X i_Y H, \tag{B.38}$$

then we have locally

$$e^{B_i}[X+\xi, Y+\eta]_H = [e^{B_i}(X+\xi), e^{B_i}(Y+\eta)], \tag{B.39}$$

and globally

$$[X+s(\xi), Y+s(\eta)] = (\rho^* \oplus s)([X+\xi, Y+\eta]_H). \tag{B.40}$$

This defines an isomorphism of Courant algebroids

$$(TM \oplus T^*M, \rho, [\cdot,\cdot]_H) \to (E, \rho, [\cdot,\cdot]). \tag{B.41}$$

We end this section with a remark about global B-transformations. The another choice of the splitting s' should differ from s by a B-transformation with a global 2-form b and change $E = s'(TM) \otimes \rho^*(T^*M)$, where $s'_i(X) = X + (B_i + b)(X)$. It leads to the twisted bracket $[\cdot,\cdot]_{H+db}$ but does not change the cohomology class in $H^3_{\mathrm{dR}}(M)$.

Proofs of (4.8) and (4.9) We give the proofs of (4.8) and (4.9). To show them we preliminarily examine the following stuffs. In this section $f, g \in C^\infty(M)$ and $\xi, \zeta \in T^*M$ and $X, Y \in TM$. Furthermore, we may use $u = X+\xi$, $v = Y+\eta$, $w = Z+\zeta \in (TM)_0 \oplus (T^*M)_\theta$. By definition, we trivially see

$$\mathcal{L}_{(f\xi)} g = i_{f\xi} d_\theta g = f \mathcal{L}_\xi g. \tag{B.42}$$

Similarly, by definition we find

$$\begin{aligned}
\mathcal{L}_{(f\xi)}\zeta &= [f\xi, \zeta]_\theta \\
&= \mathcal{L}_{\theta(f\xi)}\zeta - i_{\theta(\zeta)} d(f\xi) \\
&= i_{\theta(f\xi)} d\zeta + d i_{\theta(f\xi)}\zeta - i_{\theta(\zeta)}(df \wedge \xi + f d\xi) \\
&= f i_{\theta(\xi)} d\zeta + d(f i_{\theta(\xi)}\zeta) - f i_{\theta(\zeta)} d\xi - i_{\theta(\zeta)}(df \wedge \xi) \\
&= f i_{\theta(\xi)} d\zeta + f d i_{\theta(\xi)}\zeta + (i_{\theta(\xi)}\zeta) df - f i_{\theta(\zeta)} d\xi - \xi(i_{\theta(\zeta)} df) + (i_{\theta(\zeta)}\xi) df \\
&= f \mathcal{L}_\xi \zeta - \xi(i_{\theta(\zeta)} df) + (i_{\theta(\xi)}\zeta) df + (i_{\theta(\zeta)}\xi) df.
\end{aligned}$$

Noticing that

$$\begin{aligned}
(i_{\theta(\xi)}\zeta) df + (i_{\theta(\zeta)}\xi) df = (i_{\theta(\xi)}\zeta) df - (i_{\theta(\xi)}\zeta) df = 0, \\
-\xi(i_{\theta(\zeta)} df) = -\xi \mathcal{L}_{\theta(\zeta)} f = -\xi \mathcal{L}_\zeta f,
\end{aligned}$$

we obtain

$$\mathcal{L}_{(f\xi)}\zeta = f \mathcal{L}_\xi \zeta - (\mathcal{L}_\zeta f)\xi. \tag{B.43}$$

Next, we have

$$\begin{aligned}\mathcal{L}_{(f\xi)}X &= (d_\theta i_{f\xi} + i_{f\xi} d_\theta)X \\ &= d_\theta(f i_\xi X) + f i_\xi d_\theta X \\ &= (i_\xi X) d_\theta f + f d_\theta i_\xi X + f i_\xi d_\theta X = f\mathcal{L}_\xi X + (i_\xi X) d_\theta f,\end{aligned} \tag{B.44}$$

and,

$$\begin{aligned}\mathcal{L}_\xi(fg) &= i_\xi d_\theta(fg) \\ &= -i_\xi \theta(d(fg)) \\ &= -i_\xi \theta((df)g + f dg) \\ &= i_\xi(d_\theta f)g + i_\xi(f d_\theta g) = (\mathcal{L}_\xi f)g + f\mathcal{L}_\xi g,\end{aligned} \tag{B.45}$$

furthermore,

$$\begin{aligned}\mathcal{L}_\xi(f\zeta) &= [\xi, f\zeta]_\theta \\ &= -[f\zeta, \xi]_\theta \\ &= -\mathcal{L}_{(f\zeta)}\xi \\ &= -f\mathcal{L}_\zeta \xi + (\mathcal{L}_\xi f)\zeta = f\mathcal{L}_\xi \zeta + (\mathcal{L}_\xi f)\zeta,\end{aligned} \tag{B.46}$$

where we used (A.2). Finally, we see

$$\begin{aligned}\mathcal{L}_\xi(fX) &= (d_\theta i_\xi + i_\xi d_\theta)(fX) \\ &= d_\theta(f i_\xi X) + i_\xi(d_\theta \wedge X + f d_\theta X) \\ &= (i_\xi X) d_\theta f + f d_\theta i_\xi X + f i_\xi d_\theta X + i_\xi(d_\theta f \wedge X) \\ &= f\mathcal{L}_\xi X + (i_\xi X) d_\theta f + (i_\xi d_\theta f)X - d_\theta f(i_\xi X) = f\mathcal{L}_\xi X + (\mathcal{L}_\xi f)X,\end{aligned} \tag{B.47}$$

where we used $d_\theta(fX) = d_\theta f \wedge X + f d_\theta X$. This is shown as

$$\begin{aligned}d_\theta(fX) &= [\theta, fX]_S \\ &= \frac{1}{2}[\theta^{ij}\partial_i \wedge \partial_j, fX^k\partial_k]_S \\ &= \frac{1}{2}\big([\theta^{ij}\partial_i, fX^k\partial_k] \wedge \partial_j - \theta^{ij}[\partial_j, fX^k\partial_k] \wedge \partial_i\big) \\ &= \frac{1}{2}\big(\theta^{ij}\partial_i(fX^k)\partial_k \wedge \partial_j - fX^k(\partial_k\theta^{ij})\partial_i \wedge \partial_j - \theta^{ij}\partial_j(fX^k)\partial_k \wedge \partial_i\big) \\ &= \theta^{ij}(\partial_i f)X^k\partial_k \wedge \partial_j + f[\theta, X]_S \\ &= -\theta(df) \wedge X + f[\theta, X]_S = d_\theta f \wedge X + f d_\theta X,\end{aligned} \tag{B.48}$$

where $\theta(df) = \theta^{ij}(\partial_i f)\partial_j$. With these preliminaries, we address to showing equations (4.8) and (4.9):

$$\begin{aligned}
&[X+\xi, f(Y+\eta)] \\
&= \mathcal{L}_\xi(fY + f\eta) - \mathcal{L}_{(f\eta)}X + \frac{1}{2}d_\theta(i_X(f\eta) - i_{fY}\xi) \\
&= f\mathcal{L}_\xi(Y+\eta) + (\mathcal{L}_\xi f)(Y+\eta) - f\mathcal{L}_\eta X - (i_\eta X)d_\theta f + \frac{1}{2}f d_\theta(i_X\eta - i_Y\xi) + \frac{1}{2}(d_\theta f)(i_X\eta - i_Y\xi) \\
&= f[X+\xi, Y+\eta] + (\mathcal{L}_\xi f)(Y+\eta) - \frac{1}{2}(d_\theta f)(i_X\eta + i_Y\xi).
\end{aligned} \tag{B.49}$$

This is (4.8). The equation (4.9) is given by

$$\begin{aligned}
&\langle [u,v] + d_\theta\langle u,v\rangle, w\rangle + \langle v, [u,w] + d_\theta\langle u,w\rangle\rangle \\
&= \langle \mathcal{L}_\xi v - \mathcal{L}_\eta X + \frac{1}{2}d_\theta(i_X\eta - i_Y\xi) + \frac{1}{2}d_\theta(i_X\eta + i_Y\xi), w\rangle + \langle v, [u,w] + d_\theta\langle u,w\rangle\rangle \\
&= \langle \mathcal{L}_\xi v - i_\eta d_\theta X - d_\theta i_\eta X + d_\theta i_X\eta, w\rangle + \langle v, [u,w] + d_\theta\langle u,w\rangle\rangle \\
&= \langle \mathcal{L}_\xi(Y+\eta) - i_\eta d_\theta X, w\rangle + \langle v, [u,w] + d_\theta\langle u,w\rangle\rangle \\
&= \frac{1}{2}(i_\zeta\mathcal{L}_\xi Y + i_Z\mathcal{L}_\xi\eta - i_\zeta i_\eta d_\theta X) + \langle v, [u,w] + d_\theta\langle u,w\rangle\rangle \\
&= \frac{1}{2}(i_\zeta\mathcal{L}_\xi Y + i_Z\mathcal{L}_\xi\eta - i_\zeta i_\eta d_\theta X) + \frac{1}{2}(i_\eta\mathcal{L}_\xi Z + i_Y\mathcal{L}_\xi\zeta - i_\eta i_\zeta d_\theta X) = \frac{1}{2}\mathcal{L}_\xi(i_\zeta Y + i_Z\eta) = \mathcal{L}_\xi\langle v,w\rangle.
\end{aligned} \tag{B.50}$$

Here we used

$$\begin{aligned}
i_\zeta\mathcal{L}_\xi Y + i_Y\mathcal{L}_\xi\zeta &= i_\zeta\{\mathcal{L}_{\theta(\xi)}Y + \theta(i_Y d\xi)\} + i_Y\{\mathcal{L}_{\theta(\xi)}\zeta - i_{\theta(\zeta)}d\xi\} \\
&= \mathcal{L}_{\theta(\xi)}(i_\zeta Y) + i_\zeta\theta(i_Y d\xi) - i_Y i_{\theta(\zeta)}d\xi \\
&= \mathcal{L}_{\theta(\xi)}(i_\zeta Y) + i_\zeta\theta(i_Y d\xi) + i_{\theta(\zeta)}i_Y d\xi \\
&= \mathcal{L}_{\theta(\xi)}(i_\zeta Y) + i_\zeta\theta(i_Y d\xi) - i_{\theta(i_Y d\xi)}\zeta = \mathcal{L}_{\theta(\xi)}(i_\zeta Y) = \mathcal{L}_\xi(i_\zeta Y).
\end{aligned} \tag{B.51}$$

Proofs of (5.8) and (5.9) In this section we denote $\xi \in T^*M$, $\xi^- = \xi - (G+\beta)(\xi) \in C_-$, $u = X + \zeta \in C_+$. By definition and (4.8),

$$\begin{aligned}
\nabla_{(f\xi)}u &= \pi_+([f\xi^-, u]) \\
&= \pi_+(f[\xi^-, u] - (\mathcal{L}_\zeta f)\xi^- + \frac{1}{2}d_\theta f\langle\xi^-, u\rangle) = \pi_+(f[\xi^-, u]) = f\nabla_\xi u,
\end{aligned} \tag{B.52}$$

where we used $\pi_+(\xi^-) = 0$ and $\langle\xi^-, u\rangle = 0$. Similarly,

$$\begin{aligned}
\nabla_\xi(fu) &= \pi_+([\xi^-, fu]) \\
&= \pi_+(f[\xi^-, u] + (\mathcal{L}_\xi f)u - \frac{1}{2}d_\theta f\langle\xi^-, u\rangle) \\
&= \pi_+(f[\xi^-, u]) + (\mathcal{L}_\xi f)u = f\nabla_\xi u + (\mathcal{L}_\xi f)u,
\end{aligned} \tag{B.53}$$

where $\pi_+(u) = u$. These are what we wanted to prove.

Proof of the compatibility (5.10) Using the equation (4.9) and the fact mentioned after (4.9),

$$\begin{aligned}\mathcal{L}_\xi\langle u,v\rangle &= \langle[\xi,u]+d_\theta\langle\xi,u\rangle,v\rangle+\langle u,[\xi,v]+d_\theta\langle\xi,v\rangle\rangle\\ &= \langle[\xi^-,u]+d_\theta\langle\xi^-,u\rangle,v\rangle+\langle u,[\xi^-,v]+d_\theta\langle\xi^-,v\rangle\rangle\\ &= \langle[\xi^-,u],v\rangle+\langle u,[\xi^-,v]\rangle\\ &= \langle\pi_+[\xi^-,u],v\rangle+\langle u,\pi_+[\xi^-,v]\rangle=\langle\nabla_\xi u,v\rangle+\langle u,\nabla_\xi v\rangle.\end{aligned} \tag{B.54}$$

Proof of (5.14) Firstly notice that by combining the results (B.42), (B.43) and (B.46), we find

$$\begin{aligned}[f\xi,g\eta] &= [f\xi,g\eta]_\theta\\ &= f\mathcal{L}_\xi(g\eta)-(\mathcal{L}_{(g\eta)}f)\xi\\ &= fg\mathcal{L}_\xi\eta+f(\mathcal{L}_\xi g)\eta-g(\mathcal{L}_\eta f)\xi=fg[\xi,\eta]+f(\mathcal{L}_\xi g)\eta-g(\mathcal{L}_\eta f)\xi.\end{aligned} \tag{B.55}$$

Secondly, from (5.8) and (5.9) we obtain

$$\begin{aligned}\nabla_{f\xi}\nabla_{g\eta}(hu) &= f\nabla_\xi(gh\nabla_\eta u+(g\mathcal{L}_\eta h)u)\\ &= f(\mathcal{L}_\xi(gh)\nabla_\eta u+gh\nabla_\xi\nabla_\eta u+\mathcal{L}_\xi(g\mathcal{L}_\eta h)u+(g\mathcal{L}_\eta h)\nabla_\xi u).\end{aligned} \tag{B.56}$$

Hence, on one hand

$$\begin{aligned}&(\nabla_{f\xi}\nabla_{g\eta}-\nabla_{g\eta}\nabla_{f\xi})(hu)\\ &= f(\mathcal{L}_\xi(gh)\nabla_\eta u+gh\nabla_\xi\nabla_\eta u+\mathcal{L}_\xi(g\mathcal{L}_\eta h)u+(g\mathcal{L}_\eta h)\nabla_\xi u)\\ &\quad -g(\mathcal{L}_\eta(fh)\nabla_\xi u+fh\nabla_\eta\nabla_\xi u+\mathcal{L}_\eta(f\mathcal{L}_\xi h)u+(f\mathcal{L}_\xi h)\nabla_\eta u)\\ &= fgh(\nabla_\xi\nabla_\eta-\nabla_\eta\nabla_\xi)u+fh(\mathcal{L}_\xi g)\nabla_\eta u-gh(\mathcal{L}_\eta f)\nabla_\xi u+f\mathcal{L}_\xi(g\mathcal{L}_\eta h)u-g\mathcal{L}_\eta(f\mathcal{L}_\xi h)u.\end{aligned} \tag{B.57}$$

And on the other hand

$$\begin{aligned}&\nabla_{[f\xi,g\eta]}hu\\ &= \nabla_{fg[\xi,\eta]+f(\mathcal{L}_\xi g)\eta-g(\mathcal{L}_\eta f)\xi}hu\\ &= (\nabla_{fg[\xi,\eta]}+\nabla_{f(\mathcal{L}_\xi g)\eta}-\nabla_{g(\mathcal{L}_\eta f)\xi})hu\\ &= (fg\nabla_{[\xi,\eta]}+f(\mathcal{L}_\xi g)\nabla_\eta-g(\mathcal{L}_\eta f)\nabla_\xi)hu\\ &= fgh\nabla_{[\xi,\eta]}u+fg(\mathcal{L}_{[\xi,\eta]}h)u+fh(\mathcal{L}_\xi g)\nabla_\eta u+f(\mathcal{L}_\xi g)(\mathcal{L}_\eta f)u-gh(\mathcal{L}_\eta f)\nabla_\xi u-g(\mathcal{L}_\eta f)(\mathcal{L}_\xi h)u\\ &= fgh\nabla_{[\xi,\eta]}u+fh(\mathcal{L}_\xi g)\nabla_\eta u-gh(\mathcal{L}_\eta f)\nabla_\xi u+f(\mathcal{L}_\xi g)(\mathcal{L}_\eta f)u-g(\mathcal{L}_\eta f)(\mathcal{L}_\xi h)u+fg(\mathcal{L}_{[\xi,\eta]}h)u.\end{aligned} \tag{B.58}$$

Thus

$$(\nabla_{f\xi}\nabla_{g\eta}-\nabla_{g\eta}\nabla_{f\xi}-\nabla_{[f\xi,g\eta]})(hu)=fgh(\nabla_\xi\nabla_\eta-\nabla_\eta\nabla_\xi-\nabla_{[\xi,\eta]})u. \tag{B.59}$$

Proof of (5.16) The bracket under the projection operator is computed as follows:

$$\begin{aligned}
&[dx^i - g^{ik}\partial_k, dx^j + g^{jl}\partial_l] \\
&= [dx^i, dx^j]_\theta + \mathcal{L}_{dx^i}(g^{jl}\partial_l) - \mathcal{L}_{dx^j}(-g^{jl}\partial_l) + \frac{1}{2}d_\theta(i_{-g^{jl}\partial_l}dx^j - i_{g^{jl}\partial_l}dx^i) \\
&= [dx^i, dx^j]_\theta + (d_\theta i_{dx^i} + i_{dx^i}d_\theta)(g^{jl}\partial_l) + (d_\theta i_{dx^j} + i_{dx^j}d_\theta)(g^{jl}\partial_l) - d_\theta g^{ij} \\
&= [dx^i, dx^j]_\theta + i_{dx^i}d_\theta(g^{jl}\partial_l) + i_{dx^j}d_\theta(g^{jl}\partial_l) + d_\theta g^{ij}.
\end{aligned} \tag{B.60}$$

Each terms results in

$$\begin{aligned}
[dx^i, dx^j] &= [dx^i, dx^j]_\theta = \mathcal{L}_{\theta(dx^i)}dx^j - i_{\theta(dx^j)}d(dx^i) \\
&= di_{\theta(dx^i)}dx^j = d\theta^{ij} = \partial_k\theta^{ij}dx^k,
\end{aligned} \tag{B.61}$$

$$\begin{aligned}
i_{dx^i}d_\theta(g^{jl}\partial_l) &= i_{dx^i}[\frac{1}{2}\theta^{mn}\partial_m \wedge \partial_n, g^{jl}\partial_l]_S \\
&= i_{dx^i}\left([\frac{1}{2}\theta^{mn}\partial_m, g^{jl}\partial_l]_S \wedge \partial_n - [\partial_n, g^{jl}\partial_l]_S \wedge \frac{1}{2}\theta^{mn}\partial_m\right) \\
&= i_{dx^i}\left(\frac{1}{2}\theta^{mn}(\partial_m g^{jl})\partial_l \wedge \partial_n - \frac{1}{2}g^{jl}(\partial_l\theta^{mn})\partial_m \wedge \partial_n - \frac{1}{2}\theta^{mn}(\partial_n g^{jl})\partial_l \wedge \partial_m\right) \\
&= i_{dx^i}\left(\theta^{mn}(\partial_m g^{jl})\partial_l \wedge \partial_n - \frac{1}{2}g^{jl}(\partial_l\theta^{mn})\partial_m \wedge \partial_n\right) \\
&= \theta^{mn}(\partial_m g^{ji})\partial_n - \theta^{mi}(\partial_m g^{jl})\partial_l - g^{jl}(\partial_l\theta^{in})\partial_n,
\end{aligned} \tag{B.62}$$

$$i_{dx^j}d_\theta(g^{il}\partial_l) = \theta^{mn}(\partial_m g^{ij})\partial_n - \theta^{mj}(\partial_m g^{il})\partial_l - g^{il}(\partial_l\theta^{jn})\partial_n, \tag{B.63}$$

$$d_\theta g^{ij} = -\theta(dg^{ij}) = -\theta(\partial_k g^{ij}dx^k) = -\theta^{kl}(\partial_k g^{ij})\partial_l. \tag{B.64}$$

Hence the bracket reads

$$\begin{aligned}
&[dx^i - g^{ik}\partial_k, dx^j + g^{jl}\partial_l] \\
&= [dx^i, dx^j]_\theta + i_{dx^i}d_\theta(g^{jl}\partial_l) + i_{dx^j}d_\theta(g^{jl}\partial_l) + d_\theta g^{ij} \\
&= \partial_k\theta^{ij}dx^k + \theta^{mn}(\partial_m g^{ji})\partial_n - \theta^{mi}(\partial_m g^{jn})\partial_n - g^{jl}(\partial_l\theta^{in})\partial_n \\
&\quad + \theta^{mn}(\partial_m g^{ij})\partial_n - \theta^{mj}(\partial_m g^{in})\partial_n - g^{il}(\partial_l\theta^{jn})\partial_n - \theta^{mn}(\partial_m g^{ij})\partial_n \\
&= \partial_k\theta^{ij}dx^k + [\theta^{mn}(\partial_m g^{ji}) - \theta^{mi}(\partial_m g^{jn}) - \theta^{mj}(\partial_m g^{in}) - g^{jl}(\partial_l\theta^{in}) - g^{il}(\partial_l\theta^{jn})]\partial_n.
\end{aligned} \tag{B.65}$$

Proof of (5.38) In order to write down its explicit formula in terms of g and θ, we compute preliminarily

$$\begin{aligned}
\theta^{im}\partial_m(\Omega + \Theta)^{jk}_l &= \theta^{im}\partial_m\left(\Gamma^k_{nl}\theta^{nj} + \frac{1}{2}T^{jk}_l\right) \\
&= \theta^{im}\left((\partial_m\Gamma^k_{nl})\theta^{nj} + \Gamma^k_{nl}\partial_m\theta^{nj} + \frac{1}{2}\partial_m T^{jk}_l\right),
\end{aligned} \tag{B.66}$$

$$(\partial_n\theta^{ij})(\Omega + \Theta)^{nk}_l = (\partial_n\theta^{ij})\left(\Gamma^k_{pl}\theta^{pn} + \frac{1}{2}T^{nk}_l\right), \tag{B.67}$$

$$(\Omega + \Theta)^{jk}_m(\Omega + \Theta)^{im}_l = \left(\Gamma^k_{nm}\theta^{nj} + \frac{1}{2}T^{jk}_m\right)\left(\Gamma^m_{pl}\theta^{pi} + \frac{1}{2}T^{im}_l\right) \tag{B.68}$$

Gathering these results we find that the curvature's explicit form becomes

$$\begin{aligned}\Pi_l^{kij} &= \theta^{im}\left((\partial_m\Gamma^k_{nl})\theta^{nj} + \Gamma^k_{nl}\partial_m\theta^{nj} + \frac{1}{2}\partial_m T_l^{jk}\right) - \theta^{jm}\left((\partial_m\Gamma^k_{nl})\theta^{ni} + \Gamma^k_{nl}\partial_m\theta^{ni} + \frac{1}{2}\partial_m T_l^{ik}\right) - \\ &\quad - (\partial_n\theta^{ij})\left(\Gamma^k_{pl}\theta^{pn} + \frac{1}{2}T_l^{nk}\right) + \\ &\quad + \left(\Gamma^k_{nm}\theta^{nj} + \frac{1}{2}T_m^{jk}\right)\left(\Gamma^m_{pl}\theta^{pi} + \frac{1}{2}T_l^{im}\right) - \left(\Gamma^k_{nm}\theta^{ni} + \frac{1}{2}T_m^{ik}\right)\left(\Gamma^m_{pl}\theta^{pj} + \frac{1}{2}T_l^{jm}\right) \\ &= \theta^{im}(\partial_m\Gamma^k_{nl})\theta^{nj} - \theta^{jm}(\partial_m\Gamma^k_{nl})\theta^{ni} + \Gamma^k_{nm}\theta^{nj}\Gamma^m_{pl}\theta^{pi} - \Gamma^k_{nm}\theta^{ni}\Gamma^m_{pl}\theta^{pj} + \\ &\quad + \theta^{im}\Gamma^k_{nl}\partial_m\theta^{nj} - \theta^{jm}\Gamma^k_{nl}\partial_m\theta^{ni} + \frac{1}{2}\theta^{im}\partial_m T_l^{jk} - \frac{1}{2}\theta^{jm}\partial_m T_l^{ik} - (\partial_n\theta^{ij})\left(\Gamma^k_{pl}\theta^{pn} + \frac{1}{2}T_l^{nk}\right) + \\ &\quad + \frac{1}{2}\Gamma^k_{nm}\theta^{nj}T_l^{im} - \frac{1}{2}T_m^{ik}\Gamma^m_{pl}\theta^{pj} + \frac{1}{2}T_m^{jk}\Gamma^m_{pl}\theta^{pi} - \frac{1}{2}\Gamma^k_{nm}\theta^{ni}T_l^{jm} + \frac{1}{4}T_m^{jk}T_l^{im} - \frac{1}{4}T_m^{ik}T_l^{jm}.\end{aligned} \tag{B.69}$$

The first line in (B.69) yields the celebrated Riemann curvature tensor:

$$\begin{aligned}&\theta^{im}(\partial_m\Gamma^k_{nl})\theta^{nj} - \theta^{jm}(\partial_m\Gamma^k_{nl})\theta^{ni} + \Gamma^k_{nm}\theta^{nj}\Gamma^m_{pl}\theta^{pi} - \Gamma^k_{nm}\theta^{ni}\Gamma^m_{pl}\theta^{pj} \\ &= \theta^{im}\theta^{nj}[\partial_m\Gamma^k_{ln} - \partial_n\Gamma^k_{lm} + \Gamma^p_{ln}\Gamma^k_{pm} - \Gamma^p_{lm}\Gamma^k_{pn}] = \theta^{im}\theta^{nj}R^k_{lmn}.\end{aligned} \tag{B.70}$$

Here in our notation the Riemann curvature tensor is defined by

$$R^k_{lmn} = \partial_m\Gamma^k_{ln} - \partial_n\Gamma^k_{lm} + \Gamma^p_{ln}\Gamma^k_{pm} - \Gamma^p_{lm}\Gamma^k_{pn}, \tag{B.71}$$

as usual. The third and fourth terms in the second line in (B.69) and the first four terms in the last line in (B.69) is

$$\begin{aligned}&\frac{1}{2}\theta^{im}\partial_m T_l^{jk} - \frac{1}{2}\theta^{jm}\partial_m T_l^{ik} + \frac{1}{2}\Gamma^k_{nm}\theta^{nj}T_l^{im} - \frac{1}{2}T_m^{ik}\Gamma^m_{pl}\theta^{pj} + \frac{1}{2}T_m^{jk}\Gamma^m_{pl}\theta^{pi} - \frac{1}{2}\Gamma^k_{nm}\theta^{ni}T_l^{jm} \\ &= -\frac{1}{2}\theta^{ni}\partial_n T_l^{jk} + \frac{1}{2}\theta^{nj}\partial_n T_l^{ik} + \frac{1}{2}\theta^{nj}\Gamma^k_{nm}T_l^{im} - \frac{1}{2}\theta^{pj}\Gamma^m_{pl}T_m^{ik} + \frac{1}{2}\theta^{pi}\Gamma^m_{pl}T_m^{jk} - \frac{1}{2}\theta^{ni}\Gamma^k_{nm}T_l^{jm} \\ &= \frac{1}{2}\theta^{nj}\partial_n T_l^{ik} + \frac{1}{2}\theta^{nj}\Gamma^k_{nm}T_l^{im} - \frac{1}{2}\theta^{nj}\Gamma^m_{nl}T_m^{ik} - \frac{1}{2}\theta^{ni}\partial_n T_l^{jk} + \frac{1}{2}\theta^{ni}\Gamma^m_{nl}T_m^{jk} - \frac{1}{2}\theta^{ni}\Gamma^k_{nm}T_l^{jm} \\ &= \frac{1}{2}\theta^{nj}\nabla_n T_l^{ik} - \frac{1}{2}\theta^{nj}\Gamma^i_{nm}T_l^{mk} - \frac{1}{2}\theta^{ni}\nabla_n T_l^{jk} + \frac{1}{2}\theta^{ni}\Gamma^j_{nm}T_l^{mk}.\end{aligned} \tag{B.72}$$

Here we use the fact that the quantity T is a tensor, that is, it is a covariant object. Hence so far the curvature is summarized as

$$\begin{aligned}\Pi_l^{kij} &= \theta^{im}\theta^{nj}R^k_{lmn} + \theta^{im}\Gamma^k_{nl}\partial_m\theta^{nj} - \theta^{jm}\Gamma^k_{nl}\partial_m\theta^{ni} - (\partial_n\theta^{ij})\left(\Gamma^k_{pl}\theta^{pn} + \frac{1}{2}T_l^{nk}\right) + \\ &\quad + \frac{1}{2}\theta^{nj}\nabla_n T_l^{ik} - \frac{1}{2}\theta^{ni}\nabla_n T_l^{jk} - \frac{1}{2}\theta^{nj}\Gamma^i_{nm}T_l^{mk} + \frac{1}{2}\theta^{ni}\Gamma^j_{nm}T_l^{mk} + \frac{1}{4}T_m^{jk}T_l^{im} - \frac{1}{4}T_m^{ik}T_l^{jm}.\end{aligned} \tag{B.73}$$

This expression still has terms involving the partial derivatives and the Christoffel symbols and seems not to be covariant. However, the non-covariant-looking part turns out to form a covariant tensor

$$\begin{aligned}
&\theta^{im}\Gamma^k_{nl}\partial_m\theta^{nj} - \theta^{jm}\Gamma^k_{nl}\partial_m\theta^{ni} - (\partial_n\theta^{ij})\left(\Gamma^k_{pl}\theta^{pn} + \frac{1}{2}T^{nk}_l\right) - \frac{1}{2}\theta^{nj}\Gamma^i_{nm}T^{mk}_l + \frac{1}{2}\theta^{ni}\Gamma^j_{nm}T^{mk}_l \\
&= -(\partial_n\theta^{ij})\Gamma^k_{pl}\theta^{pn} - \frac{1}{2}(\partial_n\theta^{ij})T^{nk}_l + \Gamma^k_{nl}[\theta^{im}\partial_m\theta^{nj} - \theta^{jm}\partial_m\theta^{ni}] - \frac{1}{2}\theta^{nj}\Gamma^i_{nm}T^{mk}_l + \frac{1}{2}\theta^{ni}\Gamma^j_{nm}T^{mk}_l \\
&= -(\partial_n\theta^{ij})\Gamma^k_{pl}\theta^{pn} - \frac{1}{2}(\partial_n\theta^{ij})T^{nk}_l + \Gamma^k_{nl}[\theta^{im}\partial_m\theta^{nj} + \theta^{jm}\partial_m\theta^{in}] - \frac{1}{2}\theta^{nj}\Gamma^i_{nm}T^{mk}_l + \frac{1}{2}\theta^{ni}\Gamma^j_{nm}T^{mk}_l \\
&= -\Gamma^k_{pl}\theta^{pn}(\partial_n\theta^{ij}) - \frac{1}{2}(\partial_n\theta^{ij})T^{nk}_l - \Gamma^k_{nl}\theta^{nm}\partial_m\theta^{ji} - \frac{1}{2}\theta^{nj}\Gamma^i_{nm}T^{mk}_l + \frac{1}{2}\theta^{ni}\Gamma^j_{nm}T^{mk}_l \\
&= -\frac{1}{2}[(\partial_n\theta^{ij})T^{nk}_l + \theta^{nj}\Gamma^i_{nm}T^{mk}_l - \theta^{ni}\Gamma^j_{nm}T^{mk}_l] \\
&= -\frac{1}{2}[(\partial_n\theta^{ij}) + \Gamma^i_{mn}\theta^{mj} - \Gamma^j_{mn}\theta^{mi}]T^{nk}_l = -\frac{1}{2}(\nabla_n\theta^{ij})T^{nk}_l. \qquad \text{(B.74)}
\end{aligned}$$

Here in the third equality we use the Poisson condition on θ. Note that the Poisson condition on the Poisson bi-vector θ

$$\theta^{im}\partial_m\theta^{jk} + \theta^{jm}\partial_m\theta^{ki} + \theta^{km}\partial_m\theta^{ij} = 0, \qquad \text{(B.75)}$$

is equivalent to the one in which the partial derivatives are replaced by the covariant derivatives:

$$\theta^{im}\nabla_m\theta^{jk} + \theta^{jm}\nabla_m\theta^{ki} + \theta^{km}\nabla_m\theta^{ij} = 0, \qquad \text{(B.76)}$$

because of the antisymmetric property of the Poisson tensor and the symmetric property of the Christoffel symbol in their upper and lower indices, respectively. Finally we obtain the covariant expression of the curvature

$$\begin{aligned}
\Pi^{kij}_l &= \theta^{im}\theta^{nj}R^k_{lmn} - \frac{1}{2}(\nabla_n\theta^{ij})T^{nk}_l + \frac{1}{2}\theta^{nj}\nabla_n T^{ik}_l - \frac{1}{2}\theta^{ni}\nabla_n T^{jk}_l + \frac{1}{4}T^{jk}_m T^{im}_l - \frac{1}{4}T^{ik}_m T^{jm}_l, \qquad \text{(B.77)} \\
T^{jk}_l &= \nabla_l\theta^{jk} - \nabla^k(\theta^{jp}g_{pl}) - \nabla^j(\theta^{kp}g_{pl}). \qquad \text{(B.78)}
\end{aligned}$$

Bibliography

[1] A. Einstein, "The Foundation of the General Theory of Relativity," *Annalen Phys.* **49** (1916) 769–822.

[2] J. Scherk and J. H. Schwarz, "Dual Models for Nonhadrons," *Nucl.Phys.* **B81** (1974) 118–144.

[3] T. Yoneya, "Connection of Dual Models to Electrodynamics and Gravidynamics," *Prog.Theor.Phys.* **51** (1974) 1907–1920.

[4] M. B. Green and J. H. Schwarz, "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory," *Phys.Lett.* **B149** (1984) 117–122.

[5] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "The Heterotic String," *Phys.Rev.Lett.* **54** (1985) 502–505.

[6] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "Heterotic String Theory. 1. The Free Heterotic String," *Nucl.Phys.* **B256** (1985) 253.

[7] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "Heterotic String Theory. 2. The Interacting Heterotic String," *Nucl.Phys.* **B267** (1986) 75.

[8] K. Kikkawa and M. Yamasaki, "Casimir Effects in Superstring Theories," *Phys.Lett.* **B149** (1984) 357.

[9] T. Buscher, "A Symmetry of the String Background Field Equations," *Phys.Lett.* **B194** (1987) 59.

[10] T. Buscher, "Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models," *Phys.Lett.* **B201** (1988) 466.

[11] L. Alvarez-Gaume, D. Z. Freedman, and S. Mukhi, "The Background Field Method and the Ultraviolet Structure of the Supersymmetric Nonlinear Sigma Model," *Annals Phys.* **134** (1981) 85.

[12] J. Callan, Curtis G., E. Martinec, M. Perry, and D. Friedan, "Strings in Background Fields," *Nucl.Phys.* **B262** (1985) 593.

[13] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, "Vacuum Configurations for Superstrings," *Nucl.Phys.* **B258** (1985) 46–74.

[14] N. Kaloper and R. C. Myers, "The Odd story of massive supergravity," *JHEP* **9905** (1999) 010, arXiv:hep-th/9901045 [hep-th].

[15] J. Scherk and J. H. Schwarz, "How to Get Masses from Extra Dimensions," *Nucl.Phys.* **B153** (1979) 61–88.

[16] S. Kachru, M. B. Schulz, and S. Trivedi, "Moduli stabilization from fluxes in a simple IIB orientifold," *JHEP* **0310** (2003) 007, arXiv:hep-th/0201028 [hep-th].

[17] S. Kachru, M. B. Schulz, P. K. Tripathy, and S. P. Trivedi, "New supersymmetric string compactifications," *JHEP* **0303** (2003) 061, arXiv:hep-th/0211182 [hep-th].

[18] C. Hull, "A Geometry for non-geometric string backgrounds," *JHEP* **0510** (2005) 065, arXiv:hep-th/0406102 [hep-th].

[19] J. Shelton, W. Taylor, and B. Wecht, "Nongeometric flux compactifications," *JHEP* **0510** (2005) 085, arXiv:hep-th/0508133 [hep-th].

[20] I. T. Ellwood, "NS-NS fluxes in Hitchin's generalized geometry," *JHEP* **0712** (2007) 084, arXiv:hep-th/0612100 [hep-th].

[21] M. Grana, R. Minasian, M. Petrini, and D. Waldram, "T-duality, Generalized Geometry and Non-Geometric Backgrounds," *JHEP* **0904** (2009) 075, arXiv:0807.4527 [hep-th].

[22] A. Coimbra, C. Strickland-Constable, and D. Waldram, "Supergravity as Generalised Geometry I: Type II Theories," *JHEP* **1111** (2011) 091, arXiv:1107.1733 [hep-th].

[23] N. Hitchin, "Generalized Calabi-Yau manifolds," *Quart.J.Math.Oxford Ser.* **54** (2003) 281–308, arXiv:math/0209099 [math-dg].

[24] M. Gualtieri, "Generalized complex geometry," arXiv:math/0401221 [math-dg].

[25] H. Bursztyn, G. R. Cavalcanti, and M. Gualtieri, "Reduction of Courant algebroids and generalized complex structures," *Adv.Math.* **211** (2007) 726–765, arXiv:math/0509640 [math.DG].

[26] D. Baraglia, *Generalized geometry.* Master's thesis, The University of Adelaide, 2007.

[27] N. Hitchin, "Lectures on generalized geometry," arXiv:1008.0973 [math.DG].

[28] T. Asakawa, H. Muraki, S. Sasa, and S. Watamura, "Poisson-generalized geometry and R-flux," arXiv:1408.2649 [hep-th].

[29] M. E. Peskin, "INTRODUCTION TO STRING AND SUPERSTRING THEORY. 2.,".

[30] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory.* Cambridge University Prss, 1987.

[31] H. Ooguri and Z. Yin, "TASI lectures on perturbative string theories," arXiv:hep-th/9612254 [hep-th].

[32] J. Polchinski, *String Theory*. Cambridge University Prss, 1998.

[33] K. Becker, M. Becker, and J. H. Schwarz, *STRING THEORY AND M-THEORY*. Cambridge University Prss, 2007.

[34] E. Alvarez, L. Alvarez-Gaume, and Y. Lozano, "An Introduction to T duality in string theory," *Nucl.Phys.Proc.Suppl.* **41** (1995) 1–20, `arXiv:hep-th/9410237 [hep-th]`.

[35] A. Alekseev and T. Strobl, "Current algebras and differential geometry," *JHEP* **0503** (2005) 035, `arXiv:hep-th/0410183 [hep-th]`.

[36] D. Andriot, M. Larfors, D. Lust, and P. Patalong, "A ten-dimensional action for non-geometric fluxes," *JHEP* **1109** (2011) 134, `arXiv:1106.4015 [hep-th]`.

[37] D. Andriot, O. Hohm, M. Larfors, D. Lust, and P. Patalong, "A geometric action for non-geometric fluxes," *Phys.Rev.Lett.* **108** (2012) 261602, `arXiv:1202.3060 [hep-th]`.

[38] D. Andriot, O. Hohm, M. Larfors, D. Lust, and P. Patalong, "Non-Geometric Fluxes in Supergravity and Double Field Theory," *Fortsch.Phys.* **60** (2012) 1150–1186, `arXiv:1204.1979 [hep-th]`.

[39] D. Andriot, M. Larfors, D. Lust, and P. Patalong, "(Non-)commutative closed string on T-dual toroidal backgrounds," *JHEP* **1306** (2013) 021, `arXiv:1211.6437 [hep-th]`.

[40] D. Andriot and A. Betz, "β-supergravity: a ten-dimensional theory with non-geometric fluxes, and its geometric framework," *JHEP* **1312** (2013) 083, `arXiv:1306.4381 [hep-th]`.

[41] D. Andriot, "Non-geometric fluxes versus (non)-geometry," `arXiv:1303.0251 [hep-th]`.

[42] D. Andriot and A. Betz, "NS-branes, source corrected Bianchi identities, and more on backgrounds with non-geometric fluxes," *JHEP* **1407** (2014) 059, `arXiv:1402.5972 [hep-th]`.

[43] R. Blumenhagen, A. Deser, E. Plauschinn, and F. Rennecke, "Palatini-Lovelock-Cartan Gravity - Bianchi Identities for Stringy Fluxes," *Class.Quant.Grav.* **29** (2012) 135004, `arXiv:1202.4934 [hep-th]`.

[44] R. Blumenhagen, A. Deser, E. Plauschinn, and F. Rennecke, "Bianchi Identities for Non-Geometric Fluxes - From Quasi-Poisson Structures to Courant Algebroids," *Fortsch.Phys.* **60** (2012) 1217–1228, `arXiv:1205.1522 [hep-th]`.

[45] R. Blumenhagen, A. Deser, E. Plauschinn, and F. Rennecke, "Non-geometric strings, symplectic gravity and differential geometry of Lie algebroids," *JHEP* **1302** (2013) 122, `arXiv:1211.0030 [hep-th]`.

[46] R. Blumenhagen, A. Deser, E. Plauschinn, and F. Rennecke, "A bi-invariant Einstein-Hilbert action for the non-geometric string," *Phys.Lett.* **B720** (2013) 215–218, `arXiv:1210.1591 [hep-th]`.

[47] C. Hull and R. Reid-Edwards, “Non-geometric backgrounds, doubled geometry and generalised T-duality,” *JHEP* **0909** (2009) 014, `arXiv:0902.4032 [hep-th]`.

[48] C. Hull and B. Zwiebach, “The Gauge algebra of double field theory and Courant brackets,” *JHEP* **0909** (2009) 090, `arXiv:0908.1792 [hep-th]`.

[49] B. Zwiebach, “Double Field Theory, T-Duality, and Courant Brackets,” *Lect.Notes Phys.* **851** (2012) 265–291, `arXiv:1109.1782 [hep-th]`.

[50] G. Aldazabal, D. Marques, and C. Nunez, “Double Field Theory: A Pedagogical Review,” *Class.Quant.Grav.* **30** (2013) 163001, `arXiv:1305.1907 [hep-th]`.

[51] O. Hohm, D. Lüst, and B. Zwiebach, “The Spacetime of Double Field Theory: Review, Remarks, and Outlook,” *Fortsch.Phys.* **61** (2013) 926–966, `arXiv:1309.2977 [hep-th]`.

[52] M. Zabzine, “Lectures on Generalized Complex Geometry and Supersymmetry,” *Archivum Math.* **42** (2006) 119–146, `arXiv:hep-th/0605148 [hep-th]`.

[53] P. Koerber, “Lectures on Generalized Complex Geometry for Physicists,” *Fortsch.Phys.* **59** (2011) 169–242, `arXiv:1006.1536 [hep-th]`.

[54] Z. ju Liu, A. Weinstein, and P. Xu, “Manin triples for lie bialgebroids,” tech. rep., J. Differential Geom, 1997.

[55] D. Roytenberg, “A Note on quasi Lie bialgebroids and twisted Poisson manifolds,” *Lett.Math.Phys.* **61** (2002) 123–137, `arXiv:math/0112152 [math-qa]`.

[56] A. C. da Silva and A. Weinstein, *Geometric Models for Noncommutative Algebras.* American Mathematical Society, 1999.

[57] B. Jurco, P. Schupp, and J. Vysoky, “On the Generalized Geometry Origin of Noncommutative Gauge Theory,” *JHEP* **1307** (2013) 126, `arXiv:1303.6096 [hep-th]`.

[58] T. Asakawa, S. Sasa, and S. Watamura, “D-branes in Generalized Geometry and Dirac-Born-Infeld Action,” *JHEP* **1210** (2012) 064, `arXiv:1206.6964 [hep-th]`.

[59] T. Asakawa, H. Muraki, and S. Watamura, “D-brane on Poisson manifold and Generalized Geometry,” *Int.J.Mod.Phys.* **A29** (2014) 1450089, `arXiv:1402.0942 [hep-th]`.

[60] N. Halmagyi, “Non-geometric String Backgrounds and Worldsheet Algebras,” *JHEP* **0807** (2008) 137, `arXiv:0805.4571 [hep-th]`.

[61] P. Severa and A. Weinstein, “Poisson geometry with a 3 form background,” *Prog. Theor. Phys. Suppl.* **144** (2001) 145–154.

[62] C. L. Rogers, “2-plectic geometry, courant algebroids, and categorified prequantization,” *J. Sympl. Geom.* **11** (2013) 53–91.

[63] P. Bouwknegt, K. Hannabuss, and V. Mathai, “T duality for principal torus bundles,” *JHEP* **0403** (2004) 018, `arXiv:hep-th/0312284 [hep-th]`.

[64] P. Bouwknegt, K. Hannabuss, and V. Mathai, "T-duality for principal torus bundles and dimensionally reduced Gysin sequences," *Adv.Theor.Math.Phys.* **9** (2005) 749–773, `arXiv:hep-th/0412268 [hep-th]`.

[65] D. Mylonas, P. Schupp, and R. J. Szabo, "Membrane Sigma-Models and Quantization of Non-Geometric Flux Backgrounds," *JHEP* **1209** (2012) 012, `arXiv:1207.0926 [hep-th]`.

[66] M. Kontsevich, "Deformation quantization of Poisson manifolds. 1.," *Lett.Math.Phys.* **66** (2003) 157–216, `arXiv:q-alg/9709040 [q-alg]`.

[67] A. S. Cattaneo and G. Felder, "A Path integral approach to the Kontsevich quantization formula," *Commun.Math.Phys.* **212** (2000) 591–611, `arXiv:math/9902090 [math]`.

[68] T. Banks, W. Fischler, S. Shenker, and L. Susskind, "M theory as a matrix model: A Conjecture," *Phys.Rev.* **D55** (1997) 5112–5128, `arXiv:hep-th/9610043 [hep-th]`.

[69] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, "A Large N reduced model as superstring," *Nucl.Phys.* **B498** (1997) 467–491, `arXiv:hep-th/9612115 [hep-th]`.

[70] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa, *et al.*, "Noncommutative Yang-Mills in IIB matrix model," *Nucl.Phys.* **B565** (2000) 176–192, `arXiv:hep-th/9908141 [hep-th]`.