Intuitionistic and uniform provability in reverse mathematics

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博　士　論　文

Intuitionistic and uniform provability in reverse mathematics

(逆数学における直観主義証明可能性と一様存在証明可能性)

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Summary

The main theme of this thesis is the relationship between intuitionistic and uniform provability in reverse mathematics. A subsystem RCA of second-order arithmetic, which is obtained by adding full second-order induction scheme to the most popular base system RCA_0 of reverse mathematics, corresponds to non-uniform computable mathematics. In particular, if a Π^1_2 theorem $\forall X(\varphi(X) \to \exists Y\psi(X,Y))$ is provable in RCA, for all X satisfying $\varphi(X)$, there is an algorithm Φ which computes Y satisfying $\psi(X,Y)$ with the use of X as oracle. However, they may not be a uniform algorithm Φ which computes the witness Y for any oracle X such that $\varphi(X)$. Corresponding to this difference, even if a Π^1_2 theorem $\forall X(\varphi(X) \to \exists Y\psi(X,Y))$ is provable in RCA, its uniform version $\exists\Phi\forall X(\varphi(X) \to \psi(X,\Phi(X)))$ or its sequential version $\forall\langle X_n\rangle_{n\in\mathbb{N}}\,(\forall n\,\varphi(X_n) \to \exists\langle Y_n\rangle_{n\in\mathbb{N}}\forall n\,\psi(X_n,Y_n))$ may not be provable in RCA^ω. On the other hand, the notion of uniform computability is closely related to constructive mathematics, which is formalized as a system of many-sorted arithmetic based on intuitionistic logic. Historically constructive mathematics has been developed informally in contrast to the formalist foundation of mathematics. Along with the development of reverse mathematics and the discovery of the arithmetical hierarchy of the law-of-excluded-middle principles, however, so-called constructive reverse mathematics, which investigates the interrelations between mathematical statements and logical principles over intuitionistic arithmetic, has been carried out in this decade.

In fact, there are several corresponding results between constructive reverse mathematics and classical reverse mathematics of sequential versions. For example, the principle of trichotomy for reals is intuitionistically equivalent to Σ^0_1-LEM whereas its sequential version is equivalent to ACA. On the other hand, the principle of dichotomy for reals is intuitionistically equivalent to Σ^0_1-DML whereas its sequential version is equivalent to WKL. More directly, ACA and WKL are intuitionistically equivalent to Σ^0_1-LEM and Σ^0_1-DML respectively in the presence of a choice scheme. Based on these facts, we provide a comprehensive analysis of the connection between intuitionistic provability and classical uniform provability in reverse mathematics. In Chapter 3, we provide a definitive connection between the aforementioned two notions. In particular, we first give an exact formulation to represent uniform provability in RCA and show that for any Π^1_2 formula of some syntactical form (rich enough), it is intuitionistically provable if and only if it is uniformly provable in RCA. The primary tool for the direction from left to right is formalized realizability with functions. The converse direction is shown by using a form of negative translation. In Chapter 4, along the line of the previous, we study the metatheorems which enable us to apply reverse mathematics to show intuitionistic unprovability. Whereas all of the previous metatheorems are now concerned with sequential versions, our metatheorems are concerned with uniform versions. Applying our metatheorems to the investigation of uniform versions in higher-order reverse mathematics, one can obtain stronger intuitionistic unprovability results

than the former case. We use several proof interpretations for the proofs. In Chapter 5, we observe that one has to pay careful attention to the formalization when one considers sequential or uniform versions. Using these results, we show that Dorais' results from [13] are optimal. In addition, we develop the reverse mathematics of concrete theorems like variants of marriage theorem and symmetric marriage theorem from the perspective of uniformity. Finally, we investigate (over the weak extensional variant of RCA_0^ω) the uniform versions of the existence of Jordan decomposition, the principle of trichotomy for reals and Π^0_1 least number principle, which demonstrates that our metatheorems in Chapter 4 are widely applicable to Π^1_2 statements whose sequential versions imply ACA. In Chapter 6, we investigate logical principles weaker than Markov's principle in the context of constructive reverse mathematics. In particular, we provide the complete classification of Π^0_1-DML, Δ^0_1-LEM, Δ^0_1-CA and WMP. However, the corresponding uniform provability in classical reverse mathematics is still missing.

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The heart of man plans his way, but the Lord establishes his steps.

Proverbs 16:9

Contents

1 Introduction

This thesis is a contribution to the foundation of mathematics. In this first chapter, we mention some background of this study.

1.1 Methodology

In the history of foundation of mathematics, there were three major schools (or four schools including predicativism [39]): logicism, intuitionism and formalism. In this thesis, we follow Hilbert's formalism which is the origin of proof theory and is the most popular in modern foundation of mathematics, and thus study formalized mathematics. Here we briefly compare the three schools. See e.g. [61] for more information.

Logicism: The basic idea of this school is to reduce all of mathematics to logic. The first systematic study was done mainly by G. Frege in the late 19th century. However, his attempt failed due to the discovery of well-known Russel's paradox at the beginning of 20th century. Nowadays there are some more attempts at accomplishing the goal of logicism.

Intuitionism: Intuitionism originates from the work of L. E. J. Brouwer at the beginning of 20th century. According to him, mathematics deals with mental constructions, which are immediately graspable by the mind. He tried to reformulate mathematics based on this philosophy. His work is the origin of constructive mathematics, which is a key subject in this thesis and will be introduced in Section 1.4. It should be mentioned that Brouwer rejected Hilbert's early formalism (1905) in his dissertation (1907).

Formalism: The origin of this school is the work of D. Hilbert in the early 20th century. The original concept of formalism is to consider mathematics as a formal game. That is, the statements of mathematics are uninterpreted strings of symbols, and proving such statements is a game in which symbols are manipulated according to fixed rules. In particular, Hilbert thought of a mathematical theory as an axiomatized formal system and a theorem in that theory as the corresponding sentence in the system respectively. In fact, he tried to find a consistent axiom system which was rich enough to formalize mathematics and the consistency proof was carried out using only ‘finitistic’ methods.[1] This is known as **Hilbert’s program**. While Hilbert’s turns out to be impossible due to Gödel’s second incompleteness theorem, it is coherent to maintain that mathematics is the science of formal systems. In fact, reverse mathematics, which is the main subject in this thesis and will be introduced in Section 1.3, is one of the research programs in the extended line of Hilbert’s work (See [67] and [10] on this issue).

1.2 Formalized Existence Theorems

Along the lines of Hilbert’s formalism, we investigate formalized mathematical theories, in particular, formal systems of many-sorted arithmetic. Then our attention is focused on existence theorems, namely, theorems of type that “for all ... (which is implicit in some case) there exists ...”. In fact, an enormous number of mathematical statements are of this type. Under the identification between theorems and formulas, we treat Π^1_2 sentences (i.e. closed formulas) having the form

$$\forall f\, (\varphi(f) \rightarrow \exists g \psi(f, g)),$$

where f and g are (possibly tuples of) sets of (or functions on) natural numbers. In fact, one can encode a real number, a complete separable metric space or a continuous function over a compact metric space as a function over natural numbers (See e.g. [55, 68]). Consequently, a large amount of existence theorems in ordinary mathematics (not only discrete mathematics) can be represented as Π^1_2 sentences in second-order arithmetic, which is the most popular framework for reverse mathematics (See Section 1.3). In many cases, the idea of such representations comes from constructive analysis [5, 6] or computable analysis [76].

1.3 Reverse Mathematics and Uniformity

Reverse mathematics is a research program, which was initiated by H. Friedman [19, 20] in the 1970’s and extensively developed by S. G. Simpson [68] and others in the 1980’s. The aim of reverse mathematics is classifying mathematical theorems from a perspective of logical complexity. The methodology of Reverse mathematics is based on Hilbert’s formalism (See Section 1.1). Thus we formalize mathematical statements in some formal system, and investigate the interrelations between the sentences and logical axioms. The most popular framework for reverse mathematics is (set based) second-order arithmetic Z_2, in which one can treat natural numbers and sets of natural numbers. For each (formalized) mathematical theorem, we look for

[1] Nowadays the system PRA of primitive recursive arithmetic is widely-accepted as the system capturing the finitism (See [71] for details).

the equivalent set existence axiom (e.g. WKL, ACA) over a weak subsystem RCA_0 of Z_2. Note that second-order arithmetic Z_2 consists of basic axioms of arithmetic, the comprehension (set existence) scheme CA:

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

and the (second-order) induction scheme IND:

$$\varphi(0) \land \forall y(\varphi(y) \to \varphi(y+1)) \to \forall y \varphi(y).$$

Then RCA_0 is the subsystem of Z_2 consisting of basic axioms of arithmetic, Δ^0_1 comprehension scheme Δ^0_1-CA (:≡ CA only for Δ^0_1 formulas) and Σ^0_1 induction scheme Σ^0_1-IND (:≡ IND only for Σ^0_1 formulas). In addition, RCA consists of basic axioms of arithmetic, Δ^0_1-CA and IND. The acronym RCA stands for "recursive comprehension axiom" and the subscript 0 in RCA_0 denotes the restriction of induction scheme to Σ^0_1 formulas. The following empirical phenomenon has been confirmed from the early age of reverse mathematics:

Fact 1.3.1. *A large number of ordinary*[2] *mathematical theorems are classified into the following three*[3]*:*

1. *provable in* RCA_0;
2. *equivalent to weak Kőnig's lemma* WKL *(expressing that every infinite subtree of* $2^{<\mathbb{N}}$ *has an infinite path) over* RCA_0;
3. *equivalent to arithmetical comprehension scheme* ACA (:≡ CA *only for arithmetical formulas) over* RCA_0.

Importantly, the corresponding subsystems form a strict hierarchy, namely,

1. $\mathsf{RCA} \nvdash$ WKL (and hence, $\mathsf{RCA}_0 \nvdash$ WKL);
2. $\mathsf{WKL}_0 :\equiv \mathsf{RCA}_0 +$ WKL $\nvdash$ ACA;
3. $\mathsf{ACA}_0 :\equiv \mathsf{RCA}_0 +$ ACA $\vdash$ WKL.

It is remarkable that RCA or RCA_0 roughly corresponds to **computable** (or recursive) mathematics [16] via well-known Post's theorem (See Section 6.1) and the equivalence to WKL or ACA corresponds to degree of non-computability. In fact, there are many results connecting between reverse and computable mathematics.

[2] Simpson [68] calls the body of mathematics which is prior to or independent of the introduction of abstract set-theoretic concepts, "ordinary" mathematics.

[3] In fact, the main observation of reverse mathematics is that most of the ordinary mathematical theorems are classified into the big five: RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0 (See [68]).

In addition, it is well-known that there is also a first-order hierarchy (i.e., hierarchy of induction schemes). An important first-order scheme is the following $\mathrm{B}\Sigma^0_{\mathrm{n}}$ (called 'bounding' or 'collection' principle):

$$\forall b(\forall a \leq b \exists c A(a,b,c) \to \exists d \forall a \leq b \exists c \leq d A(a,b,c)),$$

where A is a Σ^0_{n} formula. $\mathrm{B}\Pi^0_{\mathrm{n}}$ is defined in the same manner.

Fact 1.3.2 (First-Order Hierarchy)**.**

$$\Sigma^0_1\text{-IND} < \mathrm{B}\Pi^0_1 \equiv \mathrm{B}\Sigma^0_2 < \Sigma^0_2\text{-IND} < \ldots < \Sigma^0_{\mathrm{n}}\text{-IND} < \mathrm{B}\Pi^0_{\mathrm{n}} \equiv \mathrm{B}\Sigma^0_{\mathrm{n+1}} < \Sigma^0_{\mathrm{n+1}}\text{-IND} < \ldots.$$

In particular, the following holds:

- $\mathsf{RCA}_0 \nvdash \mathrm{B}\Pi^0_1$;
- $\mathsf{RCA}_0 + \mathrm{B}\Pi^0_1 \nvdash \Sigma^0_2$-IND ($:\equiv$ IND only for Σ^0_2 formulas);
- $\mathsf{RCA}_0 + \Sigma^0_{\mathrm{n}}\text{-IND} \subsetneq \mathsf{RCA}$.

Most results in reverse mathematics are formulated in classical logic[4]. To distinguish this mainstream from constructive reverse mathematics (See Section 1.4) based on intuitionistic logic, we call this "classical" reverse mathematics. For a comprehensive treatment of classical reverse mathematics, see Simpson's monograph [68].

As we already mentioned in Section 1.2, a large number of existence theorems are formalized as Π^1_2 sentences of the form

$$\forall f\,(\varphi(f) \to \exists g \psi(f,g))\,,$$

and known to be provable in RCA_0. However, in some cases, the construction of the witness g from a problem f is not uniform. Let us consider the case of well-known intermediate value theorem: if f is a continuous function on the unit interval $0 \leq x \leq 1$ such that $f(0) < 0 < f(1)$, then there exists $x \in (0,1)$ such that $f(x) = 0$. In fact, it is provable in RCA_0 as follows (See [68] for details). If there exists $x \in \mathbb{Q}$ such that $0 < x < 1$ and $f(x) = 0$, we are done. Otherwise, one can construct $x \in (0,1)$ by the method of nested intervals (and it is verified) in RCA_0. In this proof, the construction of the intermediate point x from f depends on whether there is a rational intermediate point (although the construction is trivial if there is). To reveal such a non-uniformity, sequential versions of Π^1_2 statements, which assert to solve infinitely many instances of a particular problem simultaneously (See Definition 3.1.1 precisely), have been investigated. In fact, the sequential version of intermediate value theorem is equivalent to WKL over RCA_0

[4]This is the logic on which standard mathematics is based.

([9]), and hence it is not provable even in RCA. Consequently, it follows that there is no uniform algorithm to construct an intermediate point x from an arbitrary given continuous function f on the unit interval $0 \leq x \leq 1$ such that $f(0) < 0 < f(1)$. The reason why the above proof in RCA_0 does not work is that one needs to **decide** in RCA whether there exists a rational intermediate point or not for each problem **simultaneously** (rather than just an application of the law-of-excluded-middle) in the sequentialized case. However, this is not possible in RCA having only the Δ^0_1 ($\approx$ computable) set existence scheme. As suggested by this example, uniform provability in RCA (in the presented sense) is closely related to the notions of uniform computability and constructivity. In Chapter 3, we pay strict attention to them and analyze their connections. In addition, as we will mention below (See Section 3.1 and Chapter 4), interesting applications of the investigation of sequential versions to constructive mathematics has been recently formulated ([37, 14, 13]). Furthermore, along with the recent development of uniform computability in computable analysis (e.g. [7, 8]), the uniform relationships between Π^1_2 statements have been interested in the context of reverse mathematics (e.g. [15, 28]). See [15] (and also Section 3.2 below) for the exhaustive comparison between uniform computability and reverse mathematical provability.

1.4 Constructive Mathematics

As mentioned earlier, the origin of constructive mathematics is Brouwer's work in the early 20th century. The basic idea of constructive mathematics is interpreting the phrase "there exists" as "we can construct" in all discussions. In order to work fully constructively, we need to reinterpret not only the existential quantifiers but all the logical connectives as instructions on how to construct a proof of the statement involving these logical expressions. Such interpretation of logical connectives is known as the BHK(Brouwer-Heyting-Kolmogorov)-interpretation (See [75] for details).[5] While there are a number of schools of constructive mathematics, the common thing to all these schools is that they are based on intuitionistic logic, namely, all of them reject the use of classical reasoning like $A \vee \neg A$ or $\neg\neg A \to A$. Importantly, intuitionistic analysis EL (See Subsection 2.2.1) or intuitionistic finite type arithmetic $\mathsf{E\text{-}HA}^\omega$ (See Subsection 2.2.2) serves as a system for formalizing (Bishop's) constructive mathematics, while it was originally carried out in an informal manner [5, 6]. Recently, the constructive variant of reverse mathematics over such an intuitionistic system, which is called **constructive reverse mathematics** [43, 3], has been developed. We refer the reader to e.g. [12] for basic knowledge of intuitionistic predicate logic and see e.g. [75] for more information on constructive mathematics.

[5] Brouwer himself never dealt with the formal interpretation of the logical connectives. That step was made by Heyting and Kolmogorov.

1.5 Arithmetical Hierarchy of the Law-of-Excluded-Middle

Akama et al. [1] considered the following restricted variants of classical principles (under the name of Σ^0_1-LLPO for Σ^0_1-DML) and showed that there is a strict hierarchy as in Figure 1.1. Note that all of the implications in Figure 1.1 are proper, namely, each of the converse directions does not hold. In addition, it is also known that Σ^0_1-DNE and Π^0_1-LEM (or Σ^0_1-DML) are incomparable.

- Σ^0_1-LEM: $A \vee \neg A$, where A is purely existential.
- Π^0_1-LEM: $A \vee \neg A$, where A is purely universal.
- Δ^0_1-LEM: $(A \leftrightarrow A') \to (A \vee \neg A)$, where A is purely existential and A' is purely universal.
- Σ^0_1-DNE: $\neg\neg A \to A$, where A is purely existential.[6]
- Σ^0_1-DML: $\neg(A \wedge B) \to (\neg A \vee \neg B)$, where A and B are purely existential.

$$
\begin{tikzcd}
 & \Sigma^0_1\text{-LEM} \arrow[dl] \arrow[dr] & \\
\Sigma^0_1\text{-DNE} \arrow[ddr] & & \Pi^0_1\text{-LEM} \arrow[d] \\
 & & \Sigma^0_1\text{-DML} \arrow[dl] \\
 & \Delta^0_1\text{-LEM} &
\end{tikzcd}
$$

Figure 1.1: Arithmetical hierarchy of the law-of-excluded-middle

While their base theory is intuitionistic first-order arithmetic HA, inspecting their proofs reveals that the corresponding hierarchy also exists over intuitionistic second-order arithmetic EL[7] (or intuitionistic finite type arithmetic $\mathsf{E\text{-}HA}^\omega$). In Chapter 6 below, we will investigate Π^0_1-DML and other principles weaker than Markov's principle MP (equivalent to Σ^0_1-DNE) in the spirit of constructive reverse mathematics.

In addition, it is remarkable that there is a beautiful correspondence between the hierarchy of law-of-excluded-middle and the hierarchy of (classical) reverse mathematics. At first, one can easily see the following.

- Σ^0_1-LEM is intuitionistically equivalent to Σ^0_1-DNE + Π^0_1-LEM.

[6] In the following chapters, we employ the notation $\mathrm{MP}_{\mathrm{PR}}$ (or MP in second-order setting) instead of Σ^0_1-DNE as in [72].

[7] In this case, each principle may contain function parameters in addition to number parameters.

- Arithmetical comprehension axiom ACA is (intuitionistically) equivalent to Σ^0_1-LEM + Π^0_1-$\mathrm{AC}^{0,0}$(numbers-numbers choice scheme for Π^0_1 formulas) over EL_0.

On the other hand, Ishihara [43] showed that weak Kőnig's lemma WKL is (intuitionistically) equivalent to Σ^0_1-DML + Π^0_1-$\mathrm{AC}^{\vee}_0$(weaker form of Π^0_1-$\mathrm{AC}^{0,0}$) over EL_0. Thus Σ^0_1-LEM (equivalently, Π^0_1-LEM in the presence of Σ^0_1-DNE) corresponds to ACA and Σ^0_1-DML corresponds to WKL. The same correspondence also exists in constructive reverse mathematics and reverse mathematics of sequential versions. For example, Dorais, Hirst and Shafer [14] verified the following.

- The principle of trichotomy of reals, namely $\forall \alpha \in \mathbb{R}(\alpha < 0 \vee \alpha = 0 \vee \alpha > 0)$, is equivalent to Σ^0_1-LEM over EL_0 and its sequential version is (classically) equivalent to ACA over RCA_0.
- The principle of dichotomy of reals, namely $\forall \alpha \in \mathbb{R}(\alpha \leq 0 \vee \alpha \geq 0)$, is equivalent to Σ^0_1-DML over EL_0 and its sequential version is (classically) equivalent to WKL over RCA_0.

Furthermore, as we introduce in Section 3.1, Dorais [13] has recently established that for any Π^1_2 statements of some syntactical form,

1. if it is provable in $\mathsf{EL}_0 + \Sigma^0_1$-DNE, then its sequential version is provable in RCA_0;
2. if it is provable in $\mathsf{EL}_0 + \mathrm{WKL} + \Sigma^0_1$-DNE, then its sequential version is provable in WKL_0.

All of these results suggest the strong connection between the hierarchy of law-of-excluded-middle and uniform investigations in classical reverse mathematics.

2 Preliminaries

2.1 Notations and Conventions

Throughout this paper, we use the following notations and conventions.

- The superscript of each variable denotes the type of the variable. For example, we write like $\forall x^0$ or $\exists \alpha^1$.
- $\underline{x}$ denotes a finite tuple of terms $x_0, x_1, \dots, x_k$.
- A formula containing terms $\underline{x}$ is denoted like $A(\underline{x})$ or $A[\underline{x}]$ with $\underline{x}$ in brackets.
- A notation $t[z]$ expresses that z is contained in the term t.
- $A[s/x]$ is the formula obtained by replacing every free occurrence of x in the formula $A(x)$ by s. This notation is also used for terms like $t[s/z]$.
- $f \cdot g$ stands for the composite function of f and g, namely, $\lambda x.f(g(x))$.
- j denotes a pairing function and j_1, j_2 are its inverses, namely, they satisfy $j_1 \cdot j(x, y) = x$, $j_2 \cdot j(x, y) = y$ and $j(j_1(z), j_2(z)) = z$. In fact, one can choose a surjective pairing function j as $2j(x + y) = (x + y)(x + y + 1) + 2x$ (See [72, 1.3.9.A]).
- $\alpha(x, y)$ is the abbreviation for $\alpha(j(x, y))$ for a function term α and number terms x, y.
- We mean by $\langle \cdot \rangle$ a (code for a) finite sequence.
- For a (code for a) finite sequence s, $lh(s)$ denotes the length of s, s_i denotes the i-th element of s for $i < lh(s)$, and $s^\frown\langle t \rangle$ denotes the concatenation of s and $\langle t \rangle$.
- Seq is an auxiliary symbol used like $s \in \mathrm{Seq}$ to denote that the number term s is manipulated as a code for a finite sequence in the context.
- $\mathrm{sg} : \mathbb{N} \to \mathbb{N}$ is a primitive recursive function such that $\mathrm{sg}(0) = 1$ and $\mathrm{sg}(y + 1) = 0$.
- $\dot{-} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a primitive recursive function such that $x \dot{-} 0 = 0$ and $x \dot{-} (y + 1) = x - (y + 1)$.
- $\forall \underline{i} < \underline{m} A(\underline{m})$ stands for $\forall i_0 < m_0 \forall i_1 < m_1 \dots \forall i_k < m_k A(m_0, m_1, \dots, m_k)$ for some k.

- For function (type 1) term β, $\overline{\beta}n$ denotes the (code of) finite sequence $\langle \beta(0), \beta(1), \ldots, \beta(n-1)\rangle$.
- $\mathrm{FV}(A)$ denotes the set of free variables in A.
- We sometimes use the notation $\Lambda x.$ or $\Lambda \alpha.$ in a suitable setting (See Notation 2.2.23).
- The subscripts "p", "qf" and "b" are used for prime formulas, quantifier-free formulas, and bounded formulas respectively (See Notation 2.2.5).
- In the discussion of finite type arithmetic, following [55], we make use of the notation $\underline{y}\underline{x} := y_1\underline{x}, \ldots, y_n\underline{x}$ where $\underline{y} = y_1, \ldots, y_n$ and $\underline{x} = x_1, \ldots, x_k$ are tuples of functionals of suitable types and $y_i\underline{x} := y_i x_1 \ldots x_k$.
- For a given set B, $X \subset_{\mathrm{fin}} B$ denotes that X is a (code of) finite subset of B.
- $\mathbb{Q}^+$ denotes the set of positive rational numbers.

2.2 Formal Systems and Principles

In this section, we present the precise definitions of the systems and principles which we use throughout this thesis. In addition, we show some basic properties for the systems, which is used for our investigation in the next chapters.

2.2.1 Intuitionistic Analysis

We start with the description of an intuitionistic two-sorted arithmetic (usually called "intuitionistic analysis") EL in a Hilbert-type axiomatization. Our axiomatization of intuitionistic first-order predicate logic is due to Gödel [27]. One can find in [72, 1.1.5 & 1.1.11] the equivalence between the latter and axiomatization using natural deduction. Our description of EL is based on [72, 1.9.10], [13, Section 2] and [55, Chapter 3]. It is known that EL is a conservative extension of first-order (Heyting) arithmetic HA. We refer the reader to [59] for a comprehensive treatment on EL.

Language $\mathcal{L}(\mathsf{EL})$ of EL $\mathcal{L}(\mathsf{EL})$ consists of the following:

- number variables[1], which are usually denoted by Roman small letters x, y, z and so on;
- function variables[2], which are denoted by Greek lower case letters α, β, γ and so on or sometimes by Roman small letters f, g, h and so on (the type of a variable is explicitly denoted by superscript when it is not clear from the context);

[1] A number variable is also called a variable of type 0 in the following.
[2] A function variable is also called a variable of type 1 in the following.

- logical constants: $\wedge, \vee, \rightarrow, \exists x^\rho, \forall x^\rho$ ($\rho \in \{0, 1\}$);
- function constants: a zero constant 0 (with arity 0), the Lambda abstraction operators $\lambda x.$ for each number variable x, a recursor R_0, a constant function symbol S for the successor, constant function symbols for all of other primitive recursive functions and application operators Ap;
- a binary predicate constant "=" (equality for numbers).

Terms of EL The number terms and function terms (called functors in [72]) of EL are built as follows:

- number variables are number terms;
- function variables are function terms;
- the zero constant 0 is a number term;
- the successor constant S is a function term;
- if $t_1, \ldots, t_k$ are number terms and f is a constant function symbol for a k-ary primitive recursive function, then $Ap(f, t_1, \ldots, t_k)$ is a number term (As usual, however, we suppress this cumbersome description and simply write $f(t_1, \ldots, t_k)$ throughout this thesis as well as for finite type arithmetic);
- if t is a number term and τ is a function term, then $Ap(\tau, t)$ (abbreviated as $\tau(t)$) is a number term;
- if t and t' are number terms and τ is a function term, then $R_0(t, \tau, t')$ is a number term[3];
- if t is a number term and x is a number variable, $\lambda x.t$ is a function term.

Formulas of EL The formulas of EL are built as follows:

- if t and t' are number terms, then $t = t'$ is a "prime" (also called "atomic") formula;
- if A and B are formulas, then $(A \wedge B)$, $(A \vee B)$, $A \rightarrow B$ are also formulas;
- if A is a formula and x is a (number or function) variable, then $(\forall x A)$ and $(\exists x A)$ are formulas.

Abbreviations

- As usual, we drop many parentheses around formulas under the agreement that negation and quantifiers bind stronger than $\wedge$ and $\vee$ which bind stronger than $\rightarrow$ and $\leftrightarrow$. In addition, $A \leftrightarrow B :\equiv ((A \rightarrow B) \wedge (B \rightarrow A))$, $\perp :\equiv 0 = 1$, $\neg A :\equiv (A \rightarrow \perp)$, and $t \neq t' :\equiv \neg(t = t')$.

[3] For notational simplicity, we sometimes use the abbreviation $R_0 t\tau t'$ for $R_0(t, \tau, t')$.

- It is well-known that the relation $<$ is primitive recursive, and the symbol $f_<$ for the characteristic function of $<$ is contained in the language of EL. We use the notation $x < y$ for $f_<(x, y) = 0$ as usual. In addition, $x \leq y$ stands for $x < y \vee x = y$.

Remark 2.2.1. *One can assume that $\mathcal{L}(\mathsf{EL})$ has $\perp$ as a logical constant and that $\perp$ is a prime formula of EL since $\perp \leftrightarrow 0 = 1$ is provable in the extended EL (See [55, Remark 3.3]).*

Axioms and Rules of EL

- The logical axioms and rules (the names are taken from [72]) of EL are the following:

  (PL2) $A, A \to B \Rightarrow B$;

  (PL3) $A \to B, B \to C \Rightarrow A \to C$;

  (PL7) $A \wedge B \to C \Rightarrow A \to (B \to C)$;

  (PL8) $A \to (B \to C) \Rightarrow A \wedge B \to C$;

  (PL9) $\perp \to A$;

  (PL10) $A \vee A \to A, A \to A \wedge A$;

  (PL11) $A \to A \vee B, A \wedge B \to A$;

  (PL12) $A \vee B \to B \vee A, A \wedge B \to B \wedge A$;

  (PL13) $A \to B \Rightarrow C \vee A \to C \vee B$;

  ($Q1^i$) $B \to A(x) \Rightarrow B \to \forall x^i A(x)$, where $i \in \{0, 1\}$;

  ($Q2^i$) $\forall x^i A(x) \to A[t/x]$, where $i \in \{0, 1\}$;

  ($Q3^i$) $A[t/x] \to \exists x^i A(x)$, where $i \in \{0, 1\}$;

  ($Q4^i$) $A(x) \to B \Rightarrow \exists x^i A(x) \to B$, where $i \in \{0, 1\}$.

  In ($Q1^i$) and ($Q4^i$), x is not free in B. In ($Q2^i$) and ($Q3^i$), x is a variable of type i and t is a term of type i such that t is free for x in A.

- The equality axioms are the following:

  (i) $x = x$;

  (ii) $x = y \wedge z = y \to x = z$;

  (iii) $x_1 = y_1 \wedge \cdots \wedge x_n = y_n \to f(x_i, \ldots, x_n) = f(y_1, \ldots, y_n)$.

- The successor axioms are the following:

  (iv) $S(x) \neq 0$;

  (v) $S(x) = S(y) \to x = y$.

- The defining axioms of the symbols for primitive recursive functions.
- The defining axioms of λ-operators, namely,

$$\lambda x.t(t') = t[t'/x].$$

- The defining axiom of the recursor R_0, namely,

$$\begin{cases} R_0(t, \tau, 0) = t, \\ R_0(t, \tau, S(t')) = \tau(j(R_0(t, \tau, t'), t')), \end{cases}$$

  where j denotes the symbol for the (primitive recursive) pairing function.
- The axiom scheme of induction IND:

$$A(0) \wedge \forall y(A(y) \to A(S(y))) \to \forall y A(y),$$

  where $A(y)$ may contain parameters.
- The axiom scheme of quantifier-free axiom of choice QF-AC^{0,0}:

$$\forall x^0 \exists y^0 A_{qf}(x, y) \to \exists g^1 \forall x A_{qf}(x, g(x)),$$

  where $A_{qf}(x, y)$ is quantifier-free (See Definition2.2.3) and may contain parameters.

Remark 2.2.2. *In fact, the successor axiom $S(x) \neq 0$ is redundant as mentioned in [55, Remark 3.3].*

Definition 2.2.3.
- A formula A is said to be "quantifier-free" if no quantifiers occur in A.
- A formula A is said to be "bounded" if all of the quantifiers in A are bounded, namely, the quantifiers occurs only in the form $\exists x^0(x < t \wedge B(x))$ or $\forall x^0(x < t \to B(x))$ with some number term t which does not contain x freely.

The same notations are also used for formulas of finite type arithmetic.

Notation 2.2.4. *For a number variable x and a number term t which does not contain x freely, $\exists x < t\, A(x)$ and $\forall x < t\, A(x)$ is the abbreviation for $\exists x(x < t \wedge A(x))$ and $\forall x(x < t \to A(x))$ respectively. We also use the abbreviations $\exists x \leq t\, A(x)$ and $\forall x \leq t\, A(x)$ as well. The same notations are also used for formulas of finite type arithmetic.*

Notation 2.2.5.
- *The subscript "p" (like A_p) denotes that the formula in question is prime.*
- *The subscript "qf" (like A_{qf}) denotes that the formula in question is quantifier-free.*
- *The subscript "b" (like A_b) denotes that the formula in question is bounded.*

Description of EL_0 ([13]) The system EL_0 is the fragment of EL where the axiom scheme of induction IND is replaced by QF-IND:

$$A_{qf}(0) \wedge \forall y^0(A_{qf}(y) \to A_{qf}(S(y))) \to \forall y^0 A_{qf}(y),$$

where $A_{qf}(y)$ is quantifier-free and may contain parameters.

2.2.2 Intuitionistic Arithmetic in All Finite Types

Here we describe some systems of intuitionistic arithmetic in all finite types. Our description follows [55, Chapter 3], which is based on [72]. However, note that our $\mathsf{WE\text{-}HA}^\omega$ is defined in the sense of [55], which is essentially different from that in [72, 1.6.12] (See [51] for details).

Types The set $\mathbf{T}$ of all finite types is generated inductively by the clauses $0 \in \mathbf{T}$ and $\rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}$. Note that type 0 is assigned to natural numbers and type $\tau(\rho)$ is assigned to functions which map type ρ objects to type τ objects. The degrees of types (denoted as '$deg(\rho)$') are also defined inductively as follows:

$$deg(0) := 0, deg(\tau(\rho)) := \max\{deg(\rho) + 1, deg(\tau)\}.$$

The set $\mathbf{P} \subset \mathbf{T}$ of pure types is defined by the clauses $0 \in \mathbf{P}$ and $\rho \in \mathbf{P} \Rightarrow 0(\rho) \in \mathbf{P}$. The pure types are often denoted by natural numbers:

$$0(n) := n + 1 \text{ (e.g. } 0(0) = 1, 0(0(0)) = 2).$$

Notation 2.2.6. *We often omit brackets which are uniquely determined and write e.g. 0(00) instead of* $0(0(0))$. *Every type $\rho \neq 0$ can uniquely be written as $\rho = 0(\rho_k)\dots(\rho_1)$, which is denoted just as $0\rho_k\dots\rho_1$ when there is no danger of confusion. Note that the superscripts on quantified variables or constants indicate their type (and $\underline{x}^{\underline{\rho}}$ denotes the tuple of variables with corresponding types). In addition, we sometimes use the notation $\rho \to \tau$ instead of $\tau(\rho)$ to indicate directly the formation of a function space.*

language $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ of $\mathsf{E\text{-}HA}^\omega$ $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ consists of the following:

- variables for all finite types;
- logical constants: $\wedge, \vee, \to, \exists x^\rho, \forall x^\rho$ $(\rho \in \mathbf{T})$;
- function constants: a zero constant 0^0 (with arity 0), a constant function symbol S^1 for the successor, projectors $\Pi^{\rho\tau\rho}_{\rho,\tau}$ for $\rho, \tau \in \mathbf{T}$, combinators $\Sigma_{\delta,\rho,\tau}$ of type $\tau\delta(\rho\delta)(\tau\rho\delta)$ for

$\delta, \rho, \tau \in \mathbf{T}$, recursors R_ρ of type $\rho 0(\rho 0 \rho)\rho$;

- a binary predicate constant "$=_0$" (equality for numbers)[4].

Remark 2.2.7. *Kohlenbach's formulation of* $\mathsf{E\text{-}HA}^\omega$ *[55], which has the simultaneous primitive recursors* $\underline{R}_{\underline{\rho}}$*, is essentially the same as our formulation as explained in [55, Remark 3.14(1)].*

Terms of $\mathsf{E\text{-}HA}^\omega$ The terms of $\mathsf{E\text{-}HA}^\omega$ are built as follows:

- constants c^ρ and variables x^ρ of type ρ are terms of type ρ;
- if $t^{\tau\rho}$ is a term of type $\tau\rho$ and s^ρ is a term of type ρ, then $t(s)$ is a term of type τ.

Notation 2.2.8. *We sometimes write simply* $ts_1 \dots s_k$ *instead of* $t(s_1)\dots(s_k)$.

Formulas of $\mathsf{E\text{-}HA}^\omega$ The formulas of $\mathsf{E\text{-}HA}^\omega$ are built as follows:

- if t and t' are terms of type 0, then $t =_0 t'$ is a "prime" (also called "atomic") formula;
- if A and B are formulas, then $(A \wedge B)$, $(A \vee B)$, $A \to B$ are also formulas;
- if A is a formula and x is a variable (or any type), then $(\forall x A)$ and $(\exists x A)$ are formulas.

Abbreviations

- Higher type equations $s =_\rho t$ between terms s, t of type $\rho = 0\rho_k \dots \rho_1$ $(k \geq 1)$ are abbreviations for
$$\forall {y_1}^{\rho_1}, \dots, {y_k}^{\rho_k}(sy_1 \dots y_k =_0 ty_1 \dots y_k),$$
where $y_1, \dots, y_k$ are variables which do not occur in s, t.
- We use the standard abbreviations $A \leftrightarrow B, \perp, \neg A$ etc. as in the case of EL (See Section 2.2.1).

Axioms and Rules of $\mathsf{E\text{-}HA}^\omega$

- The logical axioms and rules of $\mathsf{E\text{-}HA}^\omega$ are the same as those for EL except that it has the quantifier axioms and rules for all finite types.
- The equality axioms for $=_0$: $x =_0 x$, $x =_0 y \to y =_0 x$ and $x =_0 y \wedge y =_0 z \to x =_0 z$.
- The extensionality axiom EA:
$$\bigcup_{\rho = 0\rho_k \dots \rho_1,\ \rho_i \in \mathbf{T}} \forall z^\rho, x_1^{\rho_1}, y_1^{\rho_1}, \dots, x_k^{\rho_k}, y_k^{\rho_k} \left(\bigwedge_{i=1}^{k} (x_i =_{\rho_i} y_i) \to z\underline{x} =_0 z\underline{y} \right).$$

[4] We often omit the subscript 0 when it is clear from context.

- The successor axioms as in EL.
- The axiom scheme of induction IND:

$$A(0) \wedge \forall y^0 (A(y) \to A(S(y))) \to \forall y^0 A(y),$$

where $A(y)$ may contain parameters of any type.

- The defining axiom of $\Pi_{\rho,\tau}$, $\Sigma_{\delta,\rho,\tau}$ and R_ρ:

(Π): $\Pi_{\rho,\tau} x^\rho y^\tau =_\rho x^\rho$;

(Σ): $\Sigma_{\delta,\rho,\tau} xyz =_\tau xz(yz)$, where $x^{\tau\rho\delta}, y^{\rho\delta}, z^\delta$;

(R): $\begin{cases} R_\rho xy0 =_\rho x, \\ R_\rho xy(Sz) =_\rho y(R_\rho xyz)z, \end{cases}$ where $x^\rho, y^{\rho 0 \rho}, z^0$.

Description of other finite type systems

- The system $\mathsf{WE\text{-}HA}^\omega$ is obtained from $\mathsf{E\text{-}HA}^\omega$ by replacing the full extensionality axiom EA with a quantifier-free rule of extensionality QF-ER:

$$\frac{A_{qf} \to s =_\rho t}{A_{qf} \to r^\tau[s/x^\rho] =_\tau r[t/x^\rho]} .[5]$$

- The systems $\mathsf{E\text{-}PA}^\omega$ and $\mathsf{WE\text{-}PA}^\omega$ are defined respectively from $\mathsf{E\text{-}HA}^\omega$ and $\mathsf{WE\text{-}HA}^\omega$ by adding the law-of-excluded-middle scheme LEM:

$$A \vee \neg A \text{ (for arbitrary formula } A).$$

- The system $\widehat{\mathsf{E\text{-}HA}}^\omega\!\restriction$ is the fragment of $\mathsf{E\text{-}HA}^\omega$ where there is only the recursor R_0 for type 0 objects and the axiom scheme of induction IND is replaced by QF-IND:

$$A_{qf}(0) \wedge \forall y^0 (A_{qf}(y) \to A_{qf}(S(y))) \to \forall y^0 A_{qf}(y),$$

where $A_{qf}(y)$ is quantifier-free and may contain parameters of any type. In addition, $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$, $\widehat{\mathsf{E\text{-}PA}}^\omega\!\restriction$, $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction$ are the corresponding fragments of $\mathsf{WE\text{-}HA}^\omega$, $\mathsf{E\text{-}PA}^\omega$, $\mathsf{WE\text{-}PA}^\omega$ respectively.

- $\mathsf{RCA}_0^\omega := \widehat{\mathsf{E\text{-}PA}}^\omega\!\restriction + \text{QF-AC}^{1,0}$, where $\text{QF-AC}^{1,0}$ is the following axiom scheme:

$$\forall x^1 \exists y^0 A_{qf}(x,y) \to \exists Y^{1\to 0} \forall x A_{qf}(x, Yx),$$

[5] Note that $\mathsf{WE\text{-}HA}^\omega$ is a (proper) subsystem of $\mathsf{E\text{-}HA}^\omega$.

where $A_{qf}(x, y)$ may contain parameters of any type.

- RCA^ω := $\mathsf{E\text{-}PA}^\omega$ + QF-AC^{1,0}.
- WRCA_0^ω := $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\upharpoonright$ + QF-AC^{1,0}.
- WRCA^ω := $\mathsf{WE\text{-}PA}^\omega$ + QF-AC^{1,0}.

Remark 2.2.9. *The fragments of finite type systems were first introduced by Feferman [17]. Later the system* RCA_0^ω *was formulated by Kohlenbach [54] as a candidate of base system for reverse mathematics. In addition,* RCA^ω *is defined in [37]. Here we introduce* WRCA_0^ω *and* WRCA^ω *with respect to the application of our metatheorems in Chapter 4 (See Section 4.4 and Section 5.4 below).*

Warning. The weakly extensional systems do not satisfy the deduction theorem (See [51]).

2.2.3 Principles

Throughout this thesis, we treat a large number of principles over many-sorted intuitionistic arithmetic. Here we list the primary principles in this thesis.

We consider the following principles in second-order arithmetic. GC may have parameters.

- MP (Markov's principle): $\forall \alpha(\neg\neg\exists x(\alpha(x) = 0) \to \exists x(\alpha(x) = 0))$.
- GC (generalized continuity principle):

$$(\forall \xi A(\xi) \to \exists \zeta B(\xi, \zeta)) \to \exists \gamma(\forall \xi A(\xi) \to \gamma \,|\, \xi \downarrow \wedge B(\xi, \gamma \,|\, \xi)),$$

  where $A(\xi)$ is almost negative (See Definition 2.3.6).

In addition, we consider the following axiom schemes in finite type arithmetic. They may have parameters of arbitrary type.

- AC$^{\rho,\tau}$: $\forall x^\rho \exists y^\tau A(x, y) \to \exists Y^{\tau\rho} \forall x^\rho A(x, Yx)$.

  AC := $\bigcup_{\rho,\tau \in \mathbf{T}}\{\mathrm{AC}^{\rho,\tau}\}$ (axiom scheme of choice).

  AC^0 := $\bigcup_{\tau \in \mathbf{T}}\{\mathrm{AC}^{0,\tau}\}$ (axiom scheme of countable choice).
- AC!$^{\rho,\tau}$: $\forall x^\rho \exists y^\tau (A(x, y) \wedge \forall z^\tau (A(x, z) \to y = z)) \to \exists Y^{\tau\rho} \forall x^\rho A(x, Yx)$.

  AC!^{1} := $\bigcup_{\rho,\tau \in \mathbf{T},\ deg(\rho) \leq 1}\{\mathrm{AC!}^{\rho,\tau}\}$ (axiom scheme of unique choice for functions).
- QF-AC$^{\rho,\tau}$ is the restriction of AC$^{\rho,\tau}$ to quantifier-free A.

  QF-AC := $\bigcup_{\rho,\tau \in \mathbf{T}}\{\text{QF-AC}^{\rho,\tau}\}$ (quantifier-free axiom of choice).

- $\mathrm{IP}^{\rho}_{\mathrm{ef}} : (A_{\mathrm{ef}} \to \exists x^{\rho} B(x)) \to \exists x^{\rho}(A_{\mathrm{ef}} \to B(x^{\rho}))$, where A_{ef} is $\exists$-free (i.e. A_{ef} does not contain $\exists, \vee$) and does not contain x free.

  $\mathrm{IP}^{\omega}_{\mathrm{ef}} := \bigcup_{\rho \in \mathbf{T}} \{\mathrm{IP}^{\rho}_{\mathrm{ef}}\}$ (independence-of-premise schema for $\exists$-free premises).

- $\mathrm{IP}^{\rho}_{\neg} : (\neg A \to \exists x^{\rho} B(x)) \to \exists x^{\rho}(\neg A \to B(x^{\rho}))$, where A does not contain x free.

  $\mathrm{IP}^{\omega}_{\neg} := \bigcup_{\rho \in \mathbf{T}} \{\mathrm{IP}^{\rho}_{\neg}\}$ (independence-of-premise schema for negated premises).

- $\mathrm{IP}^{\rho,\tau}_{\forall} : \left(\forall u^{\tau} A_{qf}(u) \to \exists x^{\rho} B(x)\right) \to \exists x^{\rho} \left(\forall u^{\tau} A_{qf}(u) \to B(x)\right)$, where A_{qf} does not contain x free.

  $\mathrm{IP}^{\omega}_{\forall} := \bigcup_{\rho,\tau \in \mathbf{T}} \{\mathrm{IP}^{\rho,\tau}_{\forall}\}$ (independence-of-premise schema for universal premises).

  $\mathrm{IP}^{\leq 1, \leq 1}_{\forall}$ is the restriction of $\mathrm{IP}^{\omega}_{\forall}$ types ρ and τ of degree ≤ 1 (note that B may contain other variables of arbitrary type).

- $\mathrm{M}^{\rho} := \neg\neg\exists x^{\rho} A_{qf}(x) \to \exists x^{\rho} A_{qf}(x)$

  $\mathrm{M}^{\omega} := \bigcup_{\rho \in \mathbf{T}} \mathrm{M}^{\rho}$ (Markov's principle for finite types).

  $\mathrm{M}^{\leq 1}$ is the restriction of M^{ω} to types ρ of degree ≤ 1.

Remark 2.2.10. *In this thesis, we formulate the axioms with single variables for simplicity. Note that the version with single variables (x^{ρ}, y^{τ} etc.) implies the one with tuples ($\underline{x}^{\underline{\rho}}, \underline{y}^{\underline{\tau}}$ etc.) since one can show in* $\mathsf{WE\text{-}HA}^{\omega}$ *that finite tuples of variables of different types can be coded together into a single variable (See e.g.* [72] *for details).*

2.2.4 Basic Properties

Lemma 2.2.11. $\mathsf{EL}_0 \vdash x = y \to (A[x/u] \leftrightarrow A[y/u])$.

Proof. If the assertion holds for quantifier-free A_{qf}, the general case is verified by induction on the structure of A. Then it suffices to show only the case for quantifier-free A_{qf}. We reason in EL_0. It is well-known that there is a term t^0 such that $t = 0 \leftrightarrow A_{qf}[u]$ (See e.g. [55, Proposition 3.8]). Consider a function term $\lambda u.t$. Then $x = y$ implies $(\lambda u.t)(x) = (\lambda u.t)(y)$ by the equality axiom. Therefore $A_{qf}[x/u] \leftrightarrow t[x/u] = 0 \leftrightarrow (\lambda u.t)(x) = 0 \leftrightarrow (\lambda u.t)(y) = 0 \leftrightarrow t[y/u] = 0 \leftrightarrow A_{qf}[y/u]$. □

Lemma 2.2.12. *For each bounded $\mathcal{L}(\mathsf{EL})$-formulas A_b (which may contain parameters), there exists a term t^0 of* EL *such that* $\mathsf{EL}_0 \vdash t = 0 \leftrightarrow A_b$.

Proof. Let $A_{qf}(i, m)$ be a quantifier-free formula (which may contain parameters) where m does not contain i freely. By [55, Proposition 3.8], there is a term $s^0[i, m]$ such that $s = 0 \leftrightarrow A_{qf}(i, m)$. Let $f^1[m]$ be a function such that $f(0) = 1$ and $f(j + 1) = f(j) \cdot \mathrm{sg}(s[j, m])$. Then it is straightforward to see in EL_0 that $\forall i < m A_{qf}(i, m) \leftrightarrow \mathrm{sg}(f(m)) = 0$. On the other hand, let $g^1[m]$ be a function such that $g(0) = 0$ and

$$g(j+1) = \begin{cases} g(j) & \text{if } s[g(j), m] = 0, \\ j+1 & \textit{otherwise}. \end{cases}$$

Then one can show in EL_0 that $\exists i < m A_{qf}(i, m) \leftrightarrow g(m) < m$ as follows. The direction from the right to the left is obvious from the definition of g. For the converse direction, it suffices to show $\neg g(m) < m \to \neg \exists i < m A_{qf}(i, m)$. Assume that $\neg g(m) < m$, $i < m$ and $A_{qf}(i, m)$. By (quantifier-free) induction, it follows that for all $j > g(i)$, $g(j) = g(i)$. Since $m > i \geq g(i)$, we have $g(m) = g(i) < m$, which is a contradiction. Therefore the induction step has been established. By the observation that f and g are constructed from s as function terms of EL (using recursors and λ-operators), we complete the proof of our lemma. □

Lemma 2.2.13. *For any bounded $\mathcal{L}(\mathsf{EL})$-formulas A_b (which may contain parameters),* EL_0 *proves $A_b \vee \neg A_b$.*

Proof. As in the proof of [55, Lemma 3.4], we have $\mathsf{EL}_0 \vdash \forall x(x = 0 \vee x \neq 0)$. Then our lemma follows from Lemma 2.2.12. □

Lemma 2.2.14. EL_0 *proves*

$$\Sigma^0_1\text{-IND}: \ \exists z A_b(0, z) \wedge \forall y(\exists z A_b(y, z) \to \exists z A_b(S(y), z)) \to \forall y \exists z A_b(y, z),$$

where $A_b(y, z)$ is a bounded formula which may contain parameters.

Proof. This follows from the proof of [55, Proposition 3.21] with Lemma 2.2.12. □

The following lemmas are used in Chapter 6.

Lemma 2.2.15. EL_0 *proves*

$$\mathrm{B}\Sigma^0_0: \ \forall b(\forall a \leq b \exists c A_{qf}(a, b, c) \to \exists d \forall a \leq b \exists c \leq d A_{qf}(a, b, c)),$$

where $A_{qf}(a, b, c)$ is a quantifier-free formula which may contain parameters.

Proof. Fix $b \in \mathbb{N}$ and assume

$$\forall a \leq b \exists c A_{qf}(a, b, c). \tag{2.1}$$

We show $\forall n \leq b \exists d \forall a \leq n \exists c \leq d A_{qf}$ by Σ^0_1 induction (See Lemma 2.2.14) on n. In the case of $n = 0$, by (2.1), there exists c_0 such that $A_{qf}(0, b, c_0)$. Then one can take d as c_0. For induction step, suppose $\forall a \leq n \exists c \leq d_n A_{qf}(a, b, c)$. If $n + 1 > b$, take d_{n+1} as d_n. If $n + 1 \leq b$, then by

(2.1), there exists c_{n+1} such that $A_{qf}(n+1, b, c)$, and take d_{n+1} as $\max(d_n, c_{n+1})$. Then clearly $\forall a \le n+1 \exists c \le d_{n+1} A_{qf}(a, b, c)$ holds. This completes the proof. □

Lemma 2.2.16.

$$\mathsf{EL}_0 \vdash \forall \alpha, x, m, k \left(\begin{array}{l} R_0 x\alpha m = k \leftrightarrow \\ \exists s \in \mathrm{Seq}(\mathrm{lh}(s) = m+1 \wedge s_0 = x \wedge \forall i < m(s_{i+1} = \alpha(s_i, i)) \wedge s_m = k) \end{array} \right).$$

Proof. Fix α, x, m, k. If there exists a finite sequence s satisfying the condition, then one can easily show $\forall n \le m(R_0 x\alpha n = s_n)$ by quantifier-free induction, and hence $R_0 x\alpha m = s_m = k$. For the converse direction, assume $R_0 x\alpha m = k$. We show that $\forall n \le m \exists s \in \mathrm{Seq}(\mathrm{lh}(s) = n+1 \wedge s_0 = x \wedge \forall i < n(s_{i+1} = \alpha(s_i, i)) \wedge s_n = k)$ by Σ^0_1 induction on n. The initial step is verified by using the axiom $R_0 x\alpha 0 = x$. Assume that there exists $s \in \mathrm{Seq}$ such that $\mathrm{lh}(s) = n+1 \wedge s_0 = x \wedge \forall i < n(s_{i+1} = \alpha(s_i, i)) \wedge s_n = k$. Let $s' := s^\frown\langle \alpha(R_0 x\alpha n, n)\rangle$. Then $s_{i+1} = \alpha(R_0 x\alpha n, n) = R_0 x\alpha(n+1)$ by the defining axiom for R_0. On the other hand, $\forall i < n+1(s_{i+1} = \alpha(s_i, i))$ holds from our induction hypothesis. Thus s' satisfies the condition for $n+1$. This completes the proof. □

Lemma 2.2.17. *For any $\mathcal{L}(\mathsf{EL})$-formula $A[R_0 x\alpha m]$,*

$$\mathsf{EL}_0 \vdash A[R_0 x\alpha m] \leftrightarrow \exists s \in \mathrm{Seq}(\mathrm{lh}(s) = m+1 \wedge s_0 = x \wedge \forall i < m(s_{i+1} = \alpha(s_i, i)) \wedge A[s_m/R_0 x\alpha m]),$$

where $A[s_m/R_0 x\alpha m]$ is the formula obtained from $A[R_0 x\alpha m]$ by replacing each occurrence of $R_0 x\alpha m$ with s_m and may have parameters.

Proof. It is easy to see that $A[R_0 x\alpha m]$ is equivalent to $\exists k(A[k/R_0 x\alpha m] \wedge R_0 x\alpha m = k)$ via Lemma 2.2.11. By Lemma 2.2.16, this is equivalent to $\exists k \exists s \in \mathrm{Seq}(\mathrm{lh}(s) = m+1 \wedge s_0 = x \wedge \forall i < m(s_{i+1} = \alpha(s_i, i)) \wedge s_m = k \wedge A[k/R_0 x\alpha m])$. Again by Lemma 2.2.11, this is equivalent to $\exists s \in \mathrm{Seq}(\mathrm{lh}(s) = m+1 \wedge s_0 = x \wedge \forall i < m(s_{i+1} = \alpha(s_i, i)) \wedge A[s_m/R_0 x\alpha m])$. □

Lemma 2.2.18. *For any prime formula $t = 0$ of* EL*, there exists an equivalent (provably in* EL_0*) formula of the form $\exists n \forall \underline{i} < \underline{m}(t' = 0)$ where $\underline{m}$ and t' are terms not containing recursors, λ-operators and free variables not in t.*

Proof (communicated with Takeshi Yamazaki). By the defining axiom for λ-operator and Lemma 2.2.11, one can assume that t contains λ-operators only in the form of $R_0 x(\lambda l.u)m$. We show our assertion by induction on the occurrence of recursors in t. If t contains no recursors, then we are done. For the induction step, let t_1 be a term containing a recursor R_0^* in the form of $R_0^* x\beta m$ with free variables n', $\underline{z}$ and $\underline{\alpha}$. Without loss of generality, one can assume that x and m contain neither recursors nor λ-operators and β contains no recursors. Then by Lemma 2.2.17, we have that $\exists n' \forall \underline{i'} < \underline{m'}(t_1[n', \underline{z}, \underline{\alpha}, R_0^* x\beta m] = 0)$ is equivalent to

$$\exists s \exists n' \forall \underline{i'} < \underline{m'} \forall i < m(\mathrm{lh}(s) = m+1 \wedge s_0 = x \wedge s_{i+1} = \beta(s_i, i) \wedge t_1[n', \underline{z}, \underline{\alpha}, s_m/R_0^* x\beta m] = 0). \quad (2.2)$$

If β has the form of $(\lambda l^*.u)$, $s_{i+1} = \beta(s_i, i)$ in (2.2) is equivalent to $s_{i+1} = u[(s_i, i)/l^*]$. Then (2.2) is equivalent to some formula of the form $\exists z \forall \underline{i'} < \underline{m'} \forall i < m(t_2 = 0)$ where t_2 is a term containing recursors and λ-operators only in t_1 except R^*_0 (and λl^*. if $\beta \equiv \lambda l^*.u$). Thus the induction step has been established. □

The following fact is (implicitly in many cases) used throughout this thesis.

Lemma 2.2.19. $\mathsf{EL}_0 \vdash \forall f^1, g^1(\forall x^0(f(x) = g(x)) \to (A[f] \leftrightarrow A[g/f]))$.

Proof. If the assertion holds for quantifier-free A_{qf}, the general case is verified by induction on the structure of A. Then it suffices to show only the case for quantifier-free A_{qf}. We reason in EL_0, assuming $\forall x^0(f(x) = g(x))$. It is well-known that there is a term $t^0[f]$ such that $t[f] = 0 \leftrightarrow A_{qf}[f]$ (See e.g. [55, Proposition 3.8]). By Lemma 2.2.18, there exists an equivalent formula of the form $\exists n \forall \underline{i} < \underline{m}(t'[f] = 0)$ where $\underline{m}$ and t' are terms containing no recursors, λ-operators or free variables not in t. Since f appears only in the form of $f(\cdot)$ in t', one can show $t'[f] = t'[g/f]$ by the multiple use of Lemma 2.2.11. Then we have

$$A_{qf}[f] \leftrightarrow t[f] = 0 \leftrightarrow \exists n \forall \underline{i} < \underline{m}(t'[f] = 0) \leftrightarrow \exists n \forall \underline{i} < \underline{m}(t'[g/f] = 0) \leftrightarrow t[g/f] = 0 \leftrightarrow A_{qf}[g/f].$$

□

Definition 2.2.20 (Partially defined application operations)**.** For $\alpha, \beta : \mathbb{N} \to \mathbb{N}$,

$$\alpha(\beta) := \begin{cases} \alpha(\bar{\beta}n) - 1 & \text{where n is the least } n' \text{ such that } \alpha(\bar{\beta}n') \neq 0. \\ \uparrow & \text{if there is no such } n'. \end{cases}$$

Then

$$\alpha \,|\, \beta := \lambda n.\alpha(\langle n \rangle ^\frown \beta),$$

where $\langle n \rangle ^\frown \beta$ is the function such that $\langle n \rangle ^\frown \beta(0) = n$ and $\langle n \rangle ^\frown \beta(y + 1) = \beta(y)$.

Definition 2.2.21 (P-functors, See Subsection 1.9.12 in [72])**.** The partially defined expressions constructed from terms of EL and the partially defined application operations $\cdot(\cdot)$ and $\cdot \,|\, \cdot$ are called *p-terms* of EL. In particular, type 1 p-terms are called *p-functors*.

Proposition 2.2.22.

- *For all p-functor* $\varphi[\alpha, \underline{v}]$ *of* EL_0 *with free variables* α^1 *and* $\underline{v}$ *(of type* 0 *or* 1*), there exists a term* t^1 *of* EL_0 *with* $\mathrm{FV}(t) = \{\underline{v}\}$ *such that*

$$\mathsf{EL}_0 \vdash t \,|\, \alpha \simeq \varphi.$$

- *For all p-functor $\varphi[z, \underline{v}]$ of EL_0 with free variables z^0 and $\underline{v}$ (of type 0 or 1), there exists a term t^1 of EL_0 with $\mathrm{FV}(t) = \{\underline{v}\}$ such that*

$$\mathsf{EL}_0 \vdash t \,|\, \lambda w.z \simeq \varphi.$$

Proof. By inspecting the proof of [72, Theorem 1.9.14]. □

The previous proposition allows us to use the following notation ([72, 1.9.17]).

Notation 2.2.23.

- *For a p-functor φ of EL_0, $\Lambda\alpha.\varphi$ denotes a term t^1 of EL_0 with $\mathrm{FV}(t) = \mathrm{FV}(\varphi) \setminus \{\alpha\}$ such that $\mathsf{EL}_0 \vdash t \,|\, \alpha \simeq \varphi$.*
- *For a p-functor φ of EL_0, $\Lambda z.\varphi$ denotes a term t^1 of EL_0 with $\mathrm{FV}(t) = \mathrm{FV}(\varphi) \setminus \{z\}$ such that $\mathsf{EL}_0 \vdash t \,|\, \lambda w.z \simeq \varphi$.*

2.2.5 Conservation Results

Proposition 2.2.24. *$\mathsf{E\text{-}HA}^\omega$ is a conservative extension of EL under the identification of EL with its canonical embedding into $\mathsf{E\text{-}HA}^\omega$. This also holds for $\widehat{\mathsf{E\text{-}HA}}^\omega\!\upharpoonright$ and EL_0 instead of $\mathsf{E\text{-}HA}^\omega$ and EL.*

Proof. Straightforward from the discussion in [72, Section 2.6]. In addition, a careful inspection shows that this is also the case for the fragments (See also [54, Proposition 3.1]). □

In particular, we have the following classical counterpart to the previous proposition.

Proposition 2.2.25.

1. *RCA_0^ω and WRCA_0^ω are conservative extensions of RCA_0($:= \mathsf{EL}_0 + \mathrm{LEM}$).*
2. *RCA^ω and WRCA^ω are conservative extensions of RCA($:= \mathsf{EL} + \mathrm{LEM}$).*

Proof. See [54, Proposition 3.1] and [37, Theorem 2.8]. □

2.3 Proof Interpretations

In this section, we present several proof interpretations which are crucial tools for our investigation in the next chapters.

2.3.1 Negative Translation

It is well-known that many theories based on classical logic can be embedded into their intuitionistic variants by means of various syntactic translations. Such translations are called "negative translation", for such translations $A \mapsto A'$ have in common that A' is (or is intuitionistically equivalent to) a negative formula. In fact, the well-known negative translations are intuitionistically equivalent each other. What we chose in this thesis is originally due to Kuroda [60], and the results presented here is based on [55, Section 10.1]. For more general information on negative translations, see e.g. [72, Section 1.10] or [75, Section 2.3].

Negative translation can be a strong tool to show the conservativity of a classical theory over its intuitionistic variant. In Chapter 3 below, we use the negative translation to show that classical uniform provability ensures intuitionistic provability.

Definition 2.3.1 (Kuroda's negative translation [60]). A^q is defined as $A^q :\equiv \neg\neg A^*$, where A^* is defined by induction on the logical structure of A:

- $A^* :\equiv A$, if A is a prime formula,
- $(A \Box B)^* :\equiv (A^* \Box B^*)$, where $\Box \in \{\wedge, \vee, \to\}$,
- $(\exists x^\rho A)^* :\equiv \exists x^\rho A^*$,
- $(\forall x^\rho A)^* :\equiv \forall x^\rho \neg\neg A^*$.

Lemma 2.3.2. *If* $\mathsf{RCA} \vdash A$*, then* $\mathsf{EL} + \mathrm{MP} \vdash A^q$.

Proof. The proof is straightforward by induction on the length of the derivation as for [55, Proposition 10.3 (ii)]. Note that MP is used only to derive $\left(\text{QF-AC}^{0,0}\right)^q$ intuitionistically from QF-AC^{0,0} (See [55, Proposition 10.6]). □

Lemma 2.3.3. *For every formula A and B of $\mathcal{L}(\mathsf{WE\text{-}PA}^\omega)$, if*

$$\mathsf{WE\text{-}PA}^\omega \pm \text{QF-AC} \pm \Delta \pm A \vdash B,$$

then

$$\mathsf{WE\text{-}HA}^\omega \pm \{\text{QF-AC} + \mathrm{M}^\omega\} \pm \Delta \pm A^q \vdash B^q,$$

where $\mathcal{T} \pm \mathrm{T}$ indicates that we consider both case: $\mathcal{T}$ and $\mathcal{T} + \mathrm{T}$.

Proof. Induction on the length of the derivation as for [55, Proposition 10.3] (See also [55, Proposition 10.6& Proposition 10.19]). Note that M^ω is only used to verify the negative translation of QF-AC. □

Remark 2.3.4. *The corresponding results to Lemma 2.3.2 and Lemma 2.3.3 for the fragments also hold.*

2.3.2 Realizability with Functions

Realizability with functions was first introduced by Kleene and Vesley [47]. We employ its formalized variant. Our treatment is based on [72, Section Ⅲ. 3], but most of the proofs are in fact based on Kleene's work [46]. This is the crucial tool for our investigation in Chapter 3. See e.g. [74] for a comprehensive treatment of formalized realizability.

Definition 2.3.5. For each formula A of $\mathcal{L}(\mathsf{EL})$, we define a formula α rf A of $\mathcal{L}(\mathsf{EL})$, where α is a new (possibly empty) function variable, namely, free variables of α rf A are contained in that of A and α. The definition is by induction over the logical structure of A.

- $\alpha \text{ rf } A$ is A for prime A,
- $\alpha \text{ rf } (A \wedge B)$ is $j_1 \cdot \alpha \text{ rf } A \wedge j_2 \cdot \alpha \text{ rf } B$,
- $\alpha \text{ rf } (A \vee B)$ is $(j_1 \cdot \alpha(0) = 0 \to j_2 \cdot \alpha \text{ rf } A) \wedge (j_1 \cdot \alpha(0) \neq 0 \to j_2 \cdot \alpha \text{ rf } B)$,
- $\alpha \text{ rf } (A \to B)$ is $\forall \beta (\beta \text{ rf } A \to \alpha \,|\, \beta \downarrow \wedge\, \alpha \,|\, \beta \text{ rf } B)$,
- $\alpha \text{ rf } \forall x A(x)$ is $\forall x (\alpha \,|\, \lambda w.x \downarrow \wedge\, \alpha \,|\, \lambda w.x \text{ rf } A(x))$,
- $\alpha \text{ rf } \exists x A(x)$ is $j_2 \cdot \alpha \text{ rf } A(j_1 \cdot \alpha(0))$,
- $\alpha \text{ rf } \forall \beta A(\beta)$ is $\forall \beta (\alpha \,|\, \beta \downarrow \wedge \alpha \,|\, \beta \text{ rf } A(\beta))$,
- $\alpha \text{ rf } \exists \beta A(\beta)$ is $j_2 \cdot \alpha \text{ rf } A(j_1 \cdot \alpha)$,

where $j_i \cdot \alpha$ is the composition of j_i and α $(i \in \{1, 2\})$, $|$ is the continuous operation in Definition 2.2.20 and $(\alpha \,|\, \beta \downarrow \wedge\, \alpha \,|\, \beta \text{ rf } B)$ is the abbreviation of

$$\Big(\forall n \exists m (\alpha(\overline{\beta} m) > 0) \wedge \forall \gamma^1 \Big(\forall n \exists m \Big(\alpha(\overline{\beta} m) = \gamma(n) + 1 \wedge \forall m' < m (\alpha(\overline{\beta} m') = 0)\Big) \to \gamma \text{ rf } B\Big)\Big).$$

Definition 2.3.6. A formula $A \in \mathcal{L}(\mathsf{EL})$ is said to be *almost negative* if it built up from purely existential formulas (i.e. $\exists x^\rho A_{qf}(x)$, $\rho \in \{0, 1\}$) by means of $\wedge$, $\to$ and $\forall$ only.

Lemma 2.3.7 (Lemma 3.3 in [13]). *For any almost negative formula $A(\xi^1) \in \mathcal{L}(\mathsf{EL})$, there exists a function term t_A such that*

$$\mathsf{EL}_0 \vdash A(\xi) \leftrightarrow t_A \,|\, \xi \downarrow \wedge\, t_A \,|\, \xi \text{ rf } A(\xi).$$

Proof. See [72, Lemma 3.3.8]. □

In the following, we show the formalized soundness theorem for realizability with functions. The original proof of the formalized soundness theorem in a somewhat different formulation can be found in [46, Theorem 50]. However, the author does not find its proof in more recent literature. Thus we now present a sketch of the proof in our formulation (cf. Subsection 2.2.1).

Theorem 2.3.8 (Soundness theorem for realizability with functions [72])**.** *Let Γ, Γ' be sets of closed $\mathcal{L}(\mathsf{EL})$-formulas such that $\Gamma \subset \Gamma'$ and for each $\varphi \in \Gamma'$, there is a closed p-functor ψ_φ such that* $\mathsf{EL} + \Gamma \vdash \psi_\varphi \downarrow \wedge \psi_\varphi \text{ rf } \varphi$. *For any $\mathcal{L}(\mathsf{EL})$-formula A, if* $\mathsf{EL} + \mathrm{GC} + \Gamma' \vdash A$, *then there exists a p-functor ψ with* $\mathrm{FV}(\psi) \subset \mathrm{FV}(A)$ *such that* $\mathsf{EL} + \Gamma \vdash \psi \downarrow \wedge \psi \text{ rf } A$.

Proof. The proof is by induction on the length of deductions. Then it suffices to show the following;

(A) For each axiom Ax of EL, there exists a p-functor ψ^1 with $\mathrm{FV}(\psi_1) \subset \mathrm{FV}(Ax)$ such that $\mathsf{EL} \vdash \psi_1 \text{ rf } Ax$.

(B) For each rule $D \Rightarrow D'$ of EL, if $\mathsf{EL} + \Gamma \vdash \psi_1 \downarrow \wedge \psi_1 \text{ rf } D$ with a p-functor ψ_1 such that $\mathrm{FV}(\psi_1) \subset \mathrm{FV}(D)$, then there exists a p-functor ψ_2 with $\mathrm{FV}(\psi_2) \subset \mathrm{FV}(D')$ such that $\mathsf{EL} + \Gamma \vdash \psi_2 \downarrow \wedge \psi_2 \text{ rf } D'$.

(C) [72, Lemma 3.3.10]: For any universal closure A of an instance of GC, there exists a closed p-functor ψ_A such that $\mathsf{EL} \vdash \psi_A \downarrow \wedge \psi_A \text{ rf } A$.

We now prove (A) and (B) while freely using the notation Λ. (See Proposition 2.2.22 and Notation 2.2.23). The Greek lower-case letter ψ_i is used for a p-funcor.

(PL2). Assume $\mathsf{EL} + \Gamma \vdash \psi_0[\underline{z}]\downarrow \wedge \psi_0[\underline{z}] \text{ rf } A[\underline{z}]$ and $\mathsf{EL} + \Gamma \vdash \psi_1[\underline{z}, \underline{z'}]\downarrow \wedge \psi_1[\underline{z}, \underline{z'}] \text{ rf } (A[\underline{z}] \to B[\underline{z'}])$ where $\underline{z}$ contains all of the free variables occurring in A and $\underline{z'}$ contains all of the free variables occurring in B. Note that $\psi_1[\underline{z}, \underline{z'}] \text{ rf } (A[\underline{z}] \to B[\underline{z'}])$ means that $\forall g(g \text{ rf } A[\underline{z}] \to \psi_1[\underline{z}, \underline{z'}] \,|\, g \downarrow \wedge \psi_1[\underline{z}, \underline{z'}] \,|\, g \text{ rf } B[\underline{z'}])$. Let $\psi_2[\underline{z'}]$ be a p-functor with free variable $\underline{z'}$ obtained from $\psi_1 \,|\, \psi_0$ by replacing all the variables occurring in A but not occurring in B by 0 or $\lambda w.0$ (depending on their types). Then it is straightforward (using (PL2)) to show $\mathsf{EL} + \Gamma \vdash \psi_2[\underline{z'}]\downarrow \wedge \psi_2[\underline{z'}] \text{ rf } B[\underline{z'}]$.

For notational simplicity, we suppress parameters in the following.

(PL3). Assume $\mathsf{EL} + \Gamma \vdash \psi_0 \downarrow \wedge \psi_0 \text{ rf } (A \to B)$ and $\mathsf{EL} + \Gamma \vdash \psi_1 \downarrow \wedge \psi_1 \text{ rf } (B \to C)$. Let ψ_2 be a p-functor obtained from $\Lambda\gamma.\ \psi_1 \,|\, (\psi_0 \,|\, \gamma)$ by taking zeros for parameters occurring in B but occurring neither in A nor in C (as for (PL2)). Then $\mathrm{FV}(\psi_2) \subset \mathrm{FV}(\mathrm{A} \to \mathrm{C})$ and one can straightforwardly show that $\mathsf{EL} + \Gamma \vdash \psi_2\downarrow \wedge \psi_2 \text{ rf } (B \to C)$.

(PL7). Assume $\mathsf{EL} + \Gamma \vdash \psi_0\downarrow \wedge \psi_0 \text{ rf } ((A \wedge B) \to C)$. Defining ψ_1 as $\Lambda\gamma_A.\Lambda\gamma_B.\ \psi_0 \,|\, \lambda x.\, j(\gamma_A(x), \gamma_B(x))$, we have that $\mathsf{EL} + \Gamma \vdash \psi_1\downarrow \wedge \psi_1 \text{ rf } (A \to (B \to C))$.

(PL8). Assume $\mathsf{EL} + \Gamma \vdash \psi_0\downarrow \wedge \psi_0 \text{ rf } (A \to (B \to C))$. Defining ψ_1 as $\Lambda\gamma.(\psi_0 \,|\, (j_1 \cdot \gamma))\Big|(j_2 \cdot \gamma)$, we have that $\mathsf{EL} + \Gamma \vdash \psi_1\downarrow \wedge \psi_1 \text{ rf } ((A \wedge B) \to C)$.

(PL9). It follows from (PL9) that $\mathsf{EL} \vdash \Lambda\gamma.(\lambda w.0) \text{ rf } (\bot \to A)$.

(PL10). $\mathsf{EL} \vdash \Lambda\gamma.(j_2 \cdot \gamma)$ rf $(A \vee A \to A)$ and $\mathsf{EL} \vdash \Lambda\gamma.(\lambda x.\, j(\gamma(x), \gamma(x)))$ rf $(A \to A \wedge A)$.

(PL11). $\mathsf{EL} \vdash \Lambda\gamma.(\lambda w.\, j(0, \gamma(x)))$ rf $(A \to A \vee B)$ and $\mathsf{EL} \vdash \Lambda\gamma.(j_1 \cdot \gamma)$ rf $(A \wedge B \to A)$.

(PL12). $\mathsf{EL} \vdash \Lambda\gamma.(\lambda x.\, j(1 \dot{-} j_1 \cdot \gamma(0), j_2 \cdot \gamma(x)))$ rf $(A \vee B \to B \vee A)$ and $\mathsf{EL} \vdash \Lambda\gamma.(\lambda x.\, j(j_2 \cdot \gamma(x), j_1 \cdot \gamma(x)))$ rf $(A \wedge B \to B \wedge A)$.

(PL13). Assume $\mathsf{EL} + \Gamma \vdash \psi_0{\downarrow} \wedge \psi_0$ rf $(A \to B)$. Define ψ_1 as $\Lambda\gamma.\lambda x.\, j(j_1 \cdot \gamma(x), (1 \dot{-} j_1 \cdot \gamma(0)) \cdot (j_2 \cdot \gamma(x)) + \mathrm{sg}(j_1 \cdot \gamma(0)) \cdot (\psi \mid (j_2 \cdot \gamma)\ (x)))$. Then we have $\mathsf{EL} + \Gamma \vdash \psi_1$ rf $(C \vee A \to C \vee B)$ for any C.

(Q1^0). Assume $\mathsf{EL} + \Gamma \vdash \psi_0{\downarrow} \wedge \psi_0$ rf $(B \to A(x^0))$ where $\mathrm{FV}(\psi_0) \subset \mathrm{FV}(B \to A(x))$. Define ψ_1 as $\Lambda\gamma.\Lambda x.\psi_0 \mid \gamma$. Then $\mathrm{FV}(\psi_1) \subset \mathrm{FV}(B \to \forall x A(x))$ and we have $\mathsf{EL} + \Gamma \vdash \psi_1$ rf $(B \to \forall x A(x))$.

(Q1^1). Same as for (Q1^0).

(Q2^0). $\mathsf{EL} \vdash \Lambda\gamma.(\gamma \mid \lambda w.t)$ rf $(\forall x^0 A(x) \to A(t))$.

(Q2^1). $\mathsf{EL} \vdash \Lambda\gamma.(\gamma \mid \xi)$ rf $(\forall \beta^1 A(\beta) \to A(\xi))$.

(Q3^0). $\mathsf{EL} \vdash \Lambda\gamma.(\lambda x.\, j(t, \gamma(x)))$ rf $(A(t) \to \exists x^0 A(x))$.

(Q3^0). $\mathsf{EL} \vdash \Lambda\gamma.(\lambda x.\, j(\xi(x), \gamma(x)))$ rf $(A(\xi) \to \exists \beta^1 A(\beta))$.

(Q4^0). Assume $\mathsf{EL} + \Gamma \vdash \psi_0{\downarrow} \wedge \psi_0$ rf $(A(x^0) \to B)$ where $\mathrm{FV}(\psi_0) \subset \mathrm{FV}(A(x) \to B)$. Define ψ_1 as $\Lambda\gamma.\psi_0[j_1 \cdot \gamma\,(0)/x] \big| (j_2 \cdot \gamma)$. Then $\mathrm{FV}(\psi_1) \subset \mathrm{FV}(\exists x A(x) \to B)$ and it is not hard to see $\mathsf{EL} \vdash \psi_1$ rf $(\exists x A(x) \to B)$ (using (Q4^i) if ψ_0 does not contain x as free variable).

(Q4^1). Same as for (Q4^0) but defining ψ_1 as $\Lambda\gamma.\psi_0[j_1 \cdot \gamma/x^1] \big| (j_2 \cdot \gamma)$ in this case.

(Equality axioms and the defining axioms for primitive recursive functions, recursor R_0 and λ-operators). All of these axioms have a form of $\forall \underline{z}(x = y)$. They are realized by terms of the form $\Lambda\underline{z}.\lambda w.0$.

(QF-AC^{0,0}**).** Define ψ_0 as $\Lambda\gamma.\lambda z.\, j(j_1 \cdot (\gamma \mid \lambda w.z)\ (0), (\Lambda x.\, j_2 \cdot (\gamma \mid \lambda w.x))(z))$. Then by a careful inspection, one can see $\mathsf{EL} \vdash \psi_0$ rf $(\forall x \exists y A_{qf}(x, y) \to \exists f^1 \forall x A_{qf}(x, f(x)))$. In fact, $\mathsf{EL} \vdash \psi_0$ rf $\mathrm{AC}^{0,0}$ holds.

(IND)**.** We shall construct a p-functor (in fact, a function term) ψ_0 such that $\mathsf{EL} \vdash \psi_0$ rf $(A(0) \wedge \forall y(A(y) \to A(y+1)) \to \forall y A(y))$. Note that ψ_0 rf $(A(0) \wedge \forall y(A(y) \to A(y+1)) \to \forall y A(y))$ is the following formula:

$$\forall f \left(\begin{array}{l} f \text{ rf } (A(0) \wedge \forall y(A(y) \to A(y+1))) \\ \to \psi_0 \mid f{\downarrow} \wedge\ \forall y((\psi_0 \mid f) \mid \lambda w.y{\downarrow} \wedge\ (\psi_0 \mid f) \mid \lambda w.y \text{ rf } A(y)) \end{array} \right).$$

Once we have obtained a p-functor t such that

$$\begin{cases} (t \mid f) \mid \lambda w.0 \simeq j_1 \cdot f, \\ (t \mid f) \mid \lambda w.S(y) \simeq ((j_2 \cdot f) \mid \lambda w.y)\big|(t \mid f) \mid \lambda w.y, \end{cases} \tag{2.3}$$

for given f satisfying the condition, one can show by (analytical) induction that for all y,

$$(t \mid f) \mid \lambda w.y\downarrow \wedge\ (t \mid f) \mid \lambda w.y \text{ rf } A(y).$$

Therefore our goal is to construct such a p-functor. Now there exists a numeral $\underline{e}$ such that

$$\mathsf{EL} \vdash \{\underline{e}\}(\beta, t, f, y) \simeq \begin{cases} j_1 \cdot f(t) & \text{if } y = 0, \\ ((j_2 \cdot f) \mid \lambda w.y \dot{-} 1)\big|(\beta \mid f) \mid \lambda w.y & \text{otherwise.} \end{cases}$$

As in the proof of [72, Theorem 1.9.14], take a function term $t_{\underline{e}}$ such that $t_{\underline{e}} \simeq \{\underline{e}\}$. By the recursion theorem analogue [72, Theorem 1.9.16], we have a p-functor (in fact, a function term) t such that $\mathsf{EL} \vdash ((t_{\underline{e}} \mid t)\big|f)\big|\lambda w.y \simeq (t \mid f)\big|\lambda w.y$. It is straightforward to see that this t satisfies the condition (2.3). □

Remark 2.3.9. *The variant of Theorem 2.3.8 where* EL *is replaced by* EL_0 *is also true. In fact, by the fact that* QF-IND *is almost negative and Lemma 2.3.7, one can construct a function term* ψ_0 *of* EL_0 *such that* $\mathsf{EL}_0 \vdash \psi_0$ rf QF-IND.

Remark 2.3.10. *The proof of Theorem 2.3.8 shows that if A (possibly containing free variables) is provable in* $\mathsf{EL} + \mathrm{GC} + \Gamma'$ *without using axioms Q2 and Q3, then one can extract a "closed" p-functor ψ such that* $\mathsf{EL} + \Gamma \vdash \psi\downarrow \wedge\, \psi$ rf A.

2.3.3 Modified Realizability

Modified realizability was first introduced by Kreisel [57, 58]. Our treatment is based on [72] and [55]. We use the results presented in the next chapters. See e.g. [74] for more background information.

Definition 2.3.11. For each formula A of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$, we define a formula $\underline{x}$ mr A of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ where $\underline{x}$ is a (possibly empty) tuple of variables (of suitable types) which do not occur freely in A. The definition is by induction over the logical structure of A.

- $\underline{x}$ mr A is A for prime A,
- $\underline{x}, \underline{y}$ mr $(A \wedge B)$ is $\underline{x}$ mr $A \wedge \underline{y}$ mr B,
- $z^0, \underline{x}, \underline{y}$ mr $(A \vee B)$ is $(z =_0 0 \to \underline{x}$ mr $A) \wedge (z \neq_0 0 \to \underline{y}$ mr $B)$,

- $\underline{y}$ mr $(A \to B)$ is $\forall \underline{x}(\underline{x}$ mr $A \to \underline{y}\underline{x}$ mr $B)$,
- $\underline{x}$ mr $(\forall y^\rho A(y))$ is $\forall y^\rho(\underline{x}y$ mr $A(y))$,
- $z^\rho, \underline{x}$ mr $(\exists y^\rho A(y))$ is $\underline{x}$ mr $A(z)$,

where $y_i\underline{x} := y_i x_1 \dots x_k$ and $\underline{y}\underline{x} := y_1\underline{x}, \dots, y_n\underline{x}$ of suitable types.

Theorem 2.3.12 (Soundness theorem of modified realizability, Theorem 5.8 in [55])**.** *Let A be an arbitrary $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$-formula and Δ_{ef} be an arbitrary set of $\exists$-free sentences. If* $\mathsf{E\text{-}HA}^\omega$ + AC + $\mathrm{IP}^\omega_{\mathrm{ef}}$ + $\Delta_{\mathrm{ef}} \vdash A$*, then there exists a tuple $\underline{t}$ of terms of* $\mathsf{E\text{-}HA}^\omega$ *of suitable types such that* $\mathrm{FV}(\underline{t}) \subset \mathrm{FV}(A)$ *and* $\mathsf{E\text{-}HA}^\omega + \Delta_{\mathrm{ef}} \vdash \underline{t}$ mr A

Proof. See [55, Theorem 5.8]. □

Definition 2.3.13 ([72, 55, 37])**.**

1. A formula of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ is $\exists$-free if it is built up from prime formulas by means of $\wedge$, $\to$ and $\forall$ only.
2. Γ_1 is the syntactic class of formulas of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ defined inductively as follows.
   a) Prime formulas are in Γ_1.
   b) If A, B are in Γ_1, then $A \wedge B, A \vee B, \forall x A(x), \exists x A(x)$ are in Γ_1.
   c) If A is $\exists$-free and $B \in \Gamma_1$, then $(\exists \underline{x} A \to B) \in \Gamma_1$.

Lemma 2.3.14 (Lemma 5.20 in [55])**.** *For $A \in \Gamma_1$,* $\mathsf{E\text{-}HA}^\omega \vdash (\underline{x}$ mr $A) \to A$ *holds.*

Remark 2.3.15. *The corresponding results to Theorem 2.3.12 and Lemma 2.3.14 for the fragments also hold.*

2.3.4 Dialectica Interpretation

The Dialectica interpretation was introduced by Gödel [27]. Our treatment is based on Kohlenbach [55], where a detailed exposition is given. Here we present the basic results on the Dialectica interpretation, which are used in the next chapters as crucial tools. See e.g. [2] for a comprehensive treatment of the Dialectica interpretation.

Definition 2.3.16. For each formula A of $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$, we define a formula $A^D \equiv \exists \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y})$ of $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$ where $\underline{x}, \underline{y}$ are (possibly empty) tuples of variables (of suitable types), A_D is quantifier-free, and the free variables of A are that of A^D. The definition is by induction over the logical structure of A.

- $A^D (\equiv A_D)$ is A for prime A,

- $(A \wedge B)^D$ is $\exists \underline{x}, \underline{u} \forall \underline{y}, \underline{v}(A_D(\underline{x}, \underline{y}) \wedge B_D(\underline{u}, \underline{v}))$ (namely, $(A \wedge B)_D$ is $(A_D(\underline{x}, \underline{y}) \wedge B_D(\underline{u}, \underline{v}))$),
- $(A \vee B)^D$ is $\exists z^0 \underline{x}, \underline{u} \forall \underline{y}, \underline{v}(z = 0 \to A_D(\underline{x}, \underline{y}) \wedge z \neq 0 \to B_D(\underline{u}, \underline{v}))$),
- $(A \to B)^D$ is $\exists \underline{U}, \underline{Y} \forall \underline{x}, \underline{v}(A_D(\underline{x}, \underline{Y}\,\underline{x}\,\underline{v}) \to B_D(\underline{U}\,\underline{x}, \underline{v}))$,
- $(\forall z^\rho A(z))^D$ is $\exists \underline{X} \forall z, \underline{y} A_D(\underline{X} z, \underline{y}, z)$,
- $(\exists z^\rho A(z))^D$ is $\exists z, \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y}, z)$,

where A^D is $\exists \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y})$ and B^D is $\exists \underline{u} \forall \underline{v} B_D(\underline{u}, \underline{v})$.

Theorem 2.3.17 (Soundness theorem of the Dialectica interpretation, Theorem 8.6 in [55])**.** *Let $A(\underline{a})$ be a formula of $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$ containing only $\underline{a}$ free. If $\mathsf{WE\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega$ proves $A(\underline{a})$, then there exists a tuple $\underline{t}$ of closed terms of $\mathsf{WE\text{-}HA}^\omega$ such that $\mathsf{WE\text{-}HA}^\omega$ proves $\forall \underline{y} A_D(\underline{t}\,\underline{a}, \underline{y}, \underline{a})$.*

Proof. See [55, Theorem 8.6]. □

Definition 2.3.18 ([72, 55, 37])**.** Γ_2 ($\subseteq \Gamma_1$) is the syntactic class of formulas of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ defined inductively as follows.

1. Prime formulas are in Γ_2.
2. If A, B are in Γ_2, then $A \wedge B, A \vee B, \forall x A(x), \exists x A(x)$ are in Γ_2.
3. If A is purely universal $\forall \underline{y}^{\underline{\rho}} A_{qf}(\underline{y})$ and $B \in \Gamma_2$, then $(\exists \underline{x} A \to B) \in \Gamma_2$.

Lemma 2.3.19 (Lemma 8.11 in [55])**.** *For $A \in \Gamma_2$, $\mathsf{WE\text{-}HA}^\omega \vdash A^D \to A$ holds.*

Remark 2.3.20. *The corresponding results to Theorem 2.3.17 and Lemma 2.3.19 for the fragments also hold.*

2.3.5 Elimination of Extensionality

We discuss a syntactic method for elimination of the extensionality axiom from proofs in $\mathsf{E\text{-}HA}^\omega$ (or $\widehat{\mathsf{E\text{-}HA}}^\omega\!\upharpoonright$), which enables us to apply the soundness theorem of the Dialectica interpretation. Our treatment is based on [55, Section 10.4], which is a simplification of Luckhard's original work [63].

Definition 2.3.21 (Translation for elimination of extensionality, [55])**.** Let A be a formula of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$. A_e is defined by induction on the logical structure of A:

- $A_e :\equiv A$, if A is a prime formula,
- $(A \square B)_e :\equiv (A_e \square B_e)$, where $\square \in \{\wedge, \vee, \to\}$,

- $(\exists x^\rho A)_e :\equiv \exists x^\rho A(x =^e_\rho x \wedge A_e)$,
- $(\forall x^\rho A)_e :\equiv \forall x^\rho (x =^e_\rho x \to A_e)$,

where the relation $x =^e_\rho y$ is defined by induction on ρ as follows:

- $x =^e_0 y :\equiv x =_0 y$,
- $x =^e_{\tau\rho} y :\equiv \forall u^\rho, v^\rho \left(u =^e_\rho v \to xu =^e_\tau xv \wedge xu =^e_\tau yv\right)$.

Remark 2.3.22. *As observed by Ferreira* [18]*, this definition can be shown to be equivalent to the simpler version where the first conjunct in $x =^e_{\tau\rho} y$ is dropped.*

Proposition 2.3.23 (Proposition 10.45 in [55])**.** *Let $A(\underline{a})$ be a formula of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ containing only $\underline{a}$ free. If*

$$\mathsf{E\text{-}HA}^\omega + \text{QF-AC}^{0,1} + \text{QF-AC}^{1,0} \vdash A(\underline{a}),$$

then

$$\mathsf{WE\text{-}HA}^\omega + \text{QF-AC}^{0,1} + \text{QF-AC}^{1,0} \vdash \underline{a} =^e \underline{a} \to A(\underline{a}).$$

This also holds for $\widehat{\mathsf{E\text{-}HA}}^\omega\!\upharpoonright$ and $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\upharpoonright$ instead of $\mathsf{E\text{-}HA}^\omega$ and $\mathsf{WE\text{-}HA}^\omega$.

Proof. See [55, Proposition 10.45]. The same proof also works for the fragments as mentioned in [55, Section 10.5]. □

Lemma 2.3.24. *For x_1^ρ, x_2^ρ where $\rho = 0\rho_k \ldots \rho_1$, $x_1 =^e_\rho x_2 \leftrightarrow \left(x_1 =_\rho x_2\right)_e \wedge x_2 =^e_\rho x_2$.*

Proof. Note that by [55, Lemma 10.40.2],

$$x_1 =^e_\rho x_2 \leftrightarrow \forall y_1^{\rho_1}, \tilde{y}_1^{\rho_1}, \ldots, y_k^{\rho_k}, \tilde{y}_k^{\rho_k} \left(\bigwedge_{i=1}^{k} y_i =^e_{\rho_i} \tilde{y}_i \to x_1\underline{y} =_0 x_1\underline{\tilde{y}} \wedge x_1\underline{y} =_0 x_2\underline{\tilde{y}} \right).$$

On the other hand, $x_1 =_\rho x_2$ is the abbreviation of

$$\forall y_1^{\rho_1}, \ldots, y_k^{\rho_k} \left(x_1 y_1, \ldots, y_k =_0 x_2 y_1 \ldots y_k\right).$$

One can verify the equivalence by a careful inspection using [55, Lemma 10.37 and Lemma 10.39]. □

Lemma 2.3.25 (Elimination of extensionality)**.** *For every formula $A(\underline{a})$ of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ where $\underline{a}$ are all the free variables of A, if*

$$\mathsf{E\text{-}HA}^\omega + \text{AC!}^1 + \text{AC}^0 \vdash A(\underline{a}),$$

then

$$\mathsf{WE\text{-}HA}^{\omega} + \mathrm{AC} \vdash \underline{a} =^{e} \underline{a} \to A(\underline{a}).$$

This also holds for $\widehat{\mathsf{E\text{-}HA}}^{\omega}\!\upharpoonright$ *and* $\widehat{\mathsf{WE\text{-}HA}}^{\omega}\!\upharpoonright$ *instead of* $\mathsf{E\text{-}HA}^{\omega}$ *and* $\mathsf{WE\text{-}HA}^{\omega}$.

Proof. By induction on the derivation. In fact, the proof is the same as for [55, Proposition 10.45] (See also [55, Remark 10.46.2]) except the interpretation of $\mathrm{AC}!^1$ and AC^0. We only discuss their interpretation in $\mathsf{WE\text{-}HA}^{\omega} + \mathrm{AC}$ (See [63] for more details).

Since $x =^{e}_{\rho} x$ holds for x^{ρ} such that $deg(\rho) \leq 1$ ([55, Lemma 10.41]), $(\mathrm{AC}!^{\rho,\tau})_e$ where $deg(\rho) \leq 1$ is equivalent to the following formula (with implicit parameters $\underline{p}$);

$$\begin{aligned} & \forall x^{\rho}\exists y^{\tau}(y =^{e}_{\tau} y \wedge A(x,y)_e \wedge \forall z^{\tau}\left(z =^{e}_{\tau} z \wedge A(x,z)_e \to (y =_{\tau} z)_e\right)) \\ \to \quad & \exists Y^{\tau\rho}(Y =^{e}_{\tau\rho} Y \wedge \forall x^{\rho} A(x, Yx)_e). \end{aligned} \tag{2.4}$$

Assume (2.4). Then we have

$$\exists Y^{\tau\rho}\forall x(Yx =^{e}_{\tau} Yx \wedge A(x,Yx)_e \wedge \forall z^{\tau}\left(z =^{e}_{\tau} z \wedge (A(x,z)_e \to (Yx =_{\tau} z)_e)\right))$$

by $\mathrm{AC}^{\rho,\tau}$. What we have to show is only that such a Y is extensional, i.e.

$$\forall x_1^{\rho}, x_2^{\rho}\left(x_1 =^{e}_{\rho} x_2 \to Yx_1 =^{e}_{\tau} Yx_2\right).$$

Note that

$$Yx_1 =^{e}_{\tau} Yx_2 \;\equiv\; \forall v_1^{\tau_1}, v_2^{\tau_1}\left(v_1 =^{e}_{\tau_1} v_2 \to Yx_1v_1 =^{e}_{\tau_2} Yx_1v_2 \wedge Yx_1v_1 =^{e}_{\tau_2} Yx_2v_2\right)$$

where $\tau = \tau_2\tau_1$. On one hand, $Yx_1v_1 =^{e} Yx_1v_2$ is a direct consequence of $Yx_1 =^{e}_{\tau} Yx_1$. On the other hand, $Yx_1v_1 =^{e} Yx_2v_2$ follows form $Yx_2 =^{e} Yx_2$ and $Yx_2 =^{e} Yx_2 \wedge A(x_1, Yx_2)_e \to (Yx_1 =_{\tau} Yx_2)_e$ via Lemma 2.3.24, since $A(x_1, Yx_2)_e$ follows from $x_1 =^{e} x_2$ and $A(x_2, Yx_2)_e$ (note that each term $t[\underline{p}]$ occurring in $A(x,y)$ is extensional under the assumption $\underline{p} =^{e} \underline{p}$ [55, Lemma 10.42]).

Next we turn to the interpretation of AC^0. For any type τ, $(\mathrm{AC}^{0,\tau})_e$ is equivalent to

$$\forall x^0\exists y^{\tau}\left(y =^{e}_{\tau} y \wedge A(x,y)_e\right) \to \exists Y^{\tau(0)}\left(Y =^{e}_{\tau} Y \wedge \forall x\left(A(x, Yx)_e\right)\right).$$

This is derived from $\mathrm{AC}^{0,\tau}$ applied to $\forall x^0\exists y^{\tau}\left(y =^{e}_{\tau} y \wedge A(x,y)_e\right)$ using the fact that the full extensionality for equality of type 0 holds in $\mathsf{WE\text{-}HA}^{\omega}$ ([55, Remark 3.13.2]).

The same proof works for the analogous result for $\widehat{\mathsf{E\text{-}HA}}^{\omega}\!\upharpoonright$ and $\widehat{\mathsf{WE\text{-}HA}}^{\omega}\!\upharpoonright$. □

3 Intuitionistic Provability versus Uniform Provability in RCA

In this chapter, we give an exact formulation to represent uniform provability in RCA and show that for any Π^1_2 formula of some syntactical form (rich enough), it is intuitionistically provable if and only if it is uniformly provable in RCA.

Notation 3.0.26. *The most popular base system* RCA_0 *of reverse mathematics, presented in [68], and its extension* RCA *having full induction scheme use the set-based language (namely, has variables for numbers and sets of numbers) with the membership relation symbol.*[1] *On the other hand, the systems* EL_0 *and* EL *use the function-based language. However, as mentioned in [37] (See also [54]), one can identify* EL_0 + LEM *with* RCA_0 *and* EL + LEM *with* RCA *respectively in the sense that each is included in a canonical definitional extension (See [72, Section 1.2]) of the other. In fact, one can see sets in* RCA_0 *as their characteristic functions in* EL_0 + LEM *and conversely see functions in* EL_0 + LEM *as their graphs in* RCA_0*. Throughout this thesis, we also write* RCA_0 *and* RCA *instead of* EL_0 + LEM *and* EL + LEM *under this identification.*

3.1 Known Uniformization Results

Definition 3.1.1 (Sequential version)**.** The sequential version of a Π^1_2 statement having a form

$$(\spadesuit): \qquad \forall f\,(\varphi(f) \to \exists g \psi(f,g))$$

is the statement

$$\forall \langle f_n \rangle_{n\in\mathbb{N}}\,(\forall n\,\varphi(f_n) \to \exists \langle g_n \rangle_{n\in\mathbb{N}} \forall n\,\psi(f_n, g_n))\,,$$

where f is possibly a tuple of function (or set) variables. Throughout this thesis, we denote the sequential version of a statement T having the form ($\spadesuit$) as Seq(T).

It has been recently established in [37] and [13] that for Π^1_2-statements of some syntactical

[1] The original base system proposed by Friedman [20] had function variables.

form, the provability in certain (semi-)intuitionistic systems guarantees the sequential provability in RCA (or +WKL). Such kind of results are called "uniformization theorems". As demonstrated in [37] and [13], uniformization theorems enable us to use the investigation of sequential versions in classical reverse mathematics to show intuitionistic unprovability of some Π^1_2 statements.

In this section, we bring together the existing uniformization theorems for sequential versions. The first results of this kind was established by Hirst and Mummert [37]. The following are their (slightly refined) uniformization results.

Proposition 3.1.2 (Theorem 3.6 and Theorem 5.6 in [37])**.**

1. *For any $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$-formula* $\mathrm{T} :\equiv \forall f\,(\varphi(f) \to \exists g \psi(f,g))$ *such that* $\varphi(f)$ *is* $\exists$*-free and* $\psi(f,g)$ *is in* Γ_1 *(See Definition 2.3.13), if*

$$\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}} \vdash \mathrm{T},$$

then

$$\mathsf{RCA}^\omega \vdash \mathrm{Seq}(\mathrm{T}).$$

2. *For any $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$-formula* $\mathrm{T} :\equiv \forall f\,(\varphi(f) \to \exists g \psi(f,g))$ *such that* $\varphi(f)$ *is purely universal and* $\psi(f,g)$ *is in* Γ_2 *(See Definition 2.3.18), if*

$$\mathsf{WE\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega \vdash \mathrm{T},$$

then

$$\mathsf{RCA}^\omega \vdash \mathrm{Seq}(\mathrm{T}).$$

On the other hand, Dorais [13] showed the similar uniformization theorems in second-order setting.

Definition 3.1.3 ([13])**.**

- $\mathrm{N_K}$ is the class of almost negative formulas (Definition 2.3.6). In other words, $\mathrm{N_K}$ is defined inductively as:
  - A_{qf}, $\exists x^\rho A_{qf}$ are in $\mathrm{N_K}$, where $\rho \in \{0,1\}$.
  - If A_1, A_2 are in $\mathrm{N_K}$, then $A_1 \wedge A_2$, $A_1 \to A_2$, $\forall x^\rho A_1$ are in $\mathrm{N_K}$, where $\rho \in \{0,1\}$.
- Γ_K is the class of formulas defined inductively as:
  - A_{qf} is in Γ_K.
  - If A_1, A_2 are in Γ_K, then $A_1 \wedge A_2$, $\forall x^\rho A_1$ and $\exists x^\rho A_1$ are in Γ_K, where $\rho \in \{0,1\}$.

– If A_1 is in $\mathrm{N_K}$ and A_2 is in Γ_K, then $A_1 \to A_2$ is in Γ_K, where $\rho \in \{0, 1\}$.

- $\mathrm{N_L}$ is the class of formulas defined inductively as:
  – A_{qf}, $\exists x^\rho A_{qf}$ are in $\mathrm{N_L}$, where $\rho \in \{0, 1\}$.
  – $\exists x \le t \forall y A_{qf}$, where t is a number term in which x does not occur, is in $\mathrm{N_L}$.
  – $\exists \xi \le \tau \forall y A_{qf}$[2], where τ is a function term in which x does not occur, is in $\mathrm{N_L}$.
  – If A_1, A_2 are in $\mathrm{N_L}$, then $A_1 \wedge A_2$, $A_1 \to A_2$, $\forall x^\rho A_1$ are in $\mathrm{N_L}$, where $\rho \in \{0, 1\}$.
- Γ_L is the class of formulas defined inductively as:
  – A_{qf} is in Γ_L.
  – If A_1, A_2 are in Γ_L, then $A_1 \wedge A_2$, $\forall x^\rho A_1$ and $\exists x^\rho A_1$ are in Γ_L, where $\rho \in \{0, 1\}$.
  – If A_1 is in $\mathrm{N_L}$ and A_2 is in Γ_L, then $A_1 \to A_2$ is in Γ_L, where $\rho \in \{0, 1\}$.

Definition 3.1.4 ([13])**.**

- CN is the set of all sentences φ from $\mathrm{N_K}$ such that $\mathsf{RCA} \vdash \varphi$.
- $\mathrm{CN_L}$ is the set of all sentences φ from $\mathrm{N_L}$ such that $\mathsf{RCA} + \mathrm{WKL} \vdash \varphi$.

The following are Dorais' Uniformization results.

Proposition 3.1.5 (Corollary 3.9 and Corollary 4.9 in [13])**.**

1. *For any* $\mathrm{T} :\equiv \forall f\, (\varphi(f) \to \exists g \psi(f, g))$ *such that* $\varphi(f)$ *is in* $\mathrm{N_K}$ *and* $\psi(f, g)$ *is in* Γ_K*, if*

$$\mathsf{EL} + \mathrm{GC} + \mathrm{CN} \vdash \mathrm{T},$$

where GC *is Troelstra's generalized continuity principle (Subsection 2.2.3), then*

$$\mathsf{RCA} \vdash \mathrm{Seq(T)}.$$

2. *For any* $\mathrm{T} :\equiv \forall f\, (\varphi(f) \to \exists g \psi(f, g))$ *such that* $\varphi(f)$ *is in* $\mathrm{N_L}$ *and* $\psi(f, g)$ *is in* Γ_L*, if*

$$\mathsf{EL} + \mathrm{WKL} + \mathrm{GC_L} + \mathrm{CN_L} \vdash \mathrm{T},$$

where $\mathrm{GC_L}$ *is van Oosten's Lifschitz generalized continuity principle (See [13]), then*

$$\mathsf{RCA} + \mathrm{WKL} \vdash \mathrm{Seq(T)}.$$

[2]Here $\xi \le \tau$ stands for $\forall i (\xi(i) \le \tau(i))$.

3.2 Exact Formulation of Uniform Provability in RCA

Uniform Provability in RCA Consider a Π^1_2 sentence $\forall\xi(A(\xi) \to \exists\zeta B(\xi,\zeta))$. In terms of computability theory, its provability in RCA corresponds to that $\zeta : \mathbb{N} \to \mathbb{N}$ is Muchnik reducible to $\xi : \mathbb{N} \to \mathbb{N}$, namely, for all ξ satisfying $A(\xi)$, there is a program Φ (depending on ξ) which computes ζ satisfying $B(\xi,\zeta)$ with the use of ξ as oracle. On the other hand, what one intends to represent by its sequential provability in RCA is that ζ is Medvedev reducible to ξ, i.e., there is a (uniform) program Φ such that for all ξ satisfying $A(\xi)$, Φ compute ζ satisfying $B(\xi,\zeta)$ with the use of ξ as oracle. We consider the strict formulation which represents this notion in terms of reverse mathematics, and call that "**uniform provability in** RCA". Based on this observation, we propose the following two candidates of the formulation to represent uniform provability in RCA:

1. There exists a (primitive recursive) closed term t^1 of RCA such that

$$\mathsf{RCA} \vdash \forall\xi\,(A(\xi) \to t\,|\,\xi\downarrow \wedge B(\xi, t\,|\,\xi)),$$

   where $(\cdot)\,|\,(\cdot)$ is the partial continuous operation in Definition 2.2.20.

2. There exists a (Gödel primitive recursive) closed term $t^{1\to1}$ of RCA^ω such that

$$\mathsf{RCA}^\omega \vdash \forall\xi\,(A(\xi) \to B(\xi, t\xi)).$$

In fact, as we show below, these two formulations are equivalent if A is purely universal and B is not too complicated.

Remark 3.2.1. *As indicated in [13], the sequential provability in* RCA *seems not to fully represent uniform provability in* RCA*. In addition, the provability of uniform versions* $\exists\Phi\forall\xi(A(\xi) \to B(\xi,\Phi(\xi)))$ *in* RCA^ω *also seems not to be an "exact" formulation in the sense that it may just ensure the provability of* $\neg\neg\exists\Phi\forall\xi(A(\xi) \to B(\xi,\Phi(\xi)))$ *in* RCA^ω.

Remark 3.2.2. *Technically, with the aid of the term existence, the syntactical form of the sentence in question is such that the negative translation works (See the proof of Proposition 3.3.6). In fact, the proof of Proposition 3.3.6 does not work if we interpret uniform provability in* RCA *by sequential or uniform versions as in Proposition 3.1.5(1) or Proposition 4.2.3.*

3.3 Characterization of Uniform Provability in RCA

We first present the refinement of Dorais's result [13, Proposition 3.7] with a witness term, which is based on Kleene's realizability with functions (cf. Subsection 2.3.2).

Proposition 3.3.1. *Let* CN *(Definition 3.1.4) be the set of all sentences* φ *from* $\mathrm{N_K}$ *such that* $\mathsf{RCA} \vdash \varphi$*,* $A(\xi) \in \mathrm{N_K}$ *and* $B(\xi, \zeta) \in \Gamma_{\mathrm{K}}$*. If*

$$\mathsf{EL} + \mathrm{GC} + \mathrm{CN} \vdash \forall \xi \, (A(\xi) \to \exists \zeta B(\xi, \zeta)),$$

then there exists a function term t *of* RCA *such that*

$$\mathsf{RCA} \vdash \forall \xi \, (A(\xi) \to t \,|\, \xi \downarrow \wedge \, B(\xi, t \,|\, \xi)).$$

Proof. Note that for each sentence $\varphi \in \mathrm{CN}$, there exists a function term t_φ such that $\mathsf{EL} \vdash \varphi \leftrightarrow t_\varphi \text{ rf } \varphi$ (See Lemma 2.3.7), and hence $\mathsf{EL} + \mathrm{CN} \vdash t_\varphi \text{ rf } \varphi$. Then, by Theorem 2.3.8, there exists a p-functor ψ such that

$$\mathsf{EL} + \mathrm{CN} \vdash \psi \downarrow \wedge \, \psi \text{ rf } \forall \xi \, (A(\xi) \to \exists \zeta B(\xi, \zeta)).$$

By Proposition 2.2.22, there exists a function term t_ψ of EL such that $\mathsf{EL} \vdash t_\psi \,|\, \xi \simeq \psi \,|\, \xi$, and hence,

$$\mathsf{EL} + \mathrm{CN} \vdash \forall \xi \left(t_\psi \,|\, \xi \downarrow \wedge t_\psi \,|\, \xi \text{ rf } \, (A(\xi) \to \exists \zeta B(\xi, \zeta)) \right).$$

Since $A(\xi)$ is in $\mathrm{N_K}$ and $B(\xi, \zeta) \in \Gamma_{\mathrm{K}}$, as in the proof of [13, Proposition 3.7], one obtains a function term t such that

$$\mathsf{EL} + \mathrm{CN} \vdash \forall \xi \, (A(\xi) \to t \,|\, \xi \downarrow \wedge \, B(\xi, t \,|\, \xi)).$$

□

Remark 3.3.2. MP *is in* CN.

Remark 3.3.3. *Dorais' first uniformization theorem* [13, Corollary 3.9] *(Proposition 3.1.5(1)) follows from Proposition 3.3.1 as a corollary.*

Next we show the converse direction, namely that uniform provability in RCA implies intuitionistic provability by means of Kuroda's negative translation (cf. Subsection 2.3.1).

Definition 3.3.4. $\mathrm{N_M}$ is the class of formulas defined inductively as:

- A_{qf} is in $\mathrm{N_M}$.
- If A_1, A_2 are in $\mathrm{N_M}$, then $A_1 \wedge A_2$, $A_1 \vee A_2$, $\forall x^\rho A_1$, $\exists x^\rho A_1$ are in $\mathrm{N_M}$, where $\rho \in \{0, 1\}$.
- If A is in $\mathrm{N_M}$, then $\forall u^\rho \exists v^0 A_{qf} \to A$ is in $\mathrm{N_M}$, where $\rho \in \{0, 1\}$.

Lemma 3.3.5. *For any formula* $A \in \mathrm{N_M}$*,* $\mathsf{EL} + \mathrm{MP} \vdash A \to A^*$ *where* A^* *is as in Definition 2.3.1.*

Proof. The proof is by induction on the structure of $\mathrm{N_M}$. For quantifier-free A_{qf}, this is trivial. Suppose that $A_1, A_2 \in \mathrm{N_M}$ are provable in $\mathsf{EL} + \mathrm{MP}$. Then it is straightforward to see that $(A_1 \wedge A_2)^*$, $(A_1 \vee A_2)^*$, $(\forall x^\rho A_1)^*$ and $(\exists x^\rho A_1)^*$ is provable in $\mathsf{EL} + \mathrm{MP}$, where $\rho \in \{0,1\}$. For the case of $\forall u^\rho \exists v^0 A_{qf} \to A_1$, using MP and the induction hypothesis, one can see that $\forall u^\rho \exists v^0 A_{qf} \to A_1$ implies $\forall u^\rho \neg\neg\exists v^0 A_{qf} \to {A_1}^*$, which is identical to $\left(\forall u^\rho \exists v^0 A_{qf} \to A_1\right)^*$ since ${A_{qf}}^* \equiv A_{qf}$. □

Proposition 3.3.6. *Assume that $A(\xi) \in \mathrm{N_M}$ and that $B(\xi, \zeta)$ is equivalent to $\forall w^\rho \exists s^0 B_{qf}(\xi, \zeta, w, s)$ ($\rho \in \{0,1\}$) over $\mathsf{EL} + \mathrm{MP}$. If there exists a function term t of RCA such that*

$$\mathsf{RCA} \vdash \forall \xi \, (A(\xi) \to t \,|\, \xi \downarrow \wedge B(\xi, t \,|\, \xi)),$$

then

$$\mathsf{EL} + \mathrm{MP} \vdash \forall \xi \, (A(\xi) \to \exists \zeta B(\xi, \zeta)).$$

Proof. Suppose $\mathsf{RCA} \vdash \forall \xi \left(A(\xi) \to t \,|\, \xi \downarrow \wedge \forall w^\rho \exists s^0 B_{qf}(\xi, t \,|\, \xi, w, s)\right)$. Expressing more precisely, it asserts that RCA proves

$$\forall \xi \left(\begin{array}{l} A(\xi) \to \forall n \exists m (t(\langle n \rangle ^\frown \overline{\xi} m) > 0) \wedge \\ \forall \gamma^1 \left(\begin{array}{l} \forall n \exists m \left(t(\langle n \rangle ^\frown \overline{\xi} m) = \gamma(n) + 1 \wedge \forall m' < m (t(\langle n \rangle ^\frown \overline{\xi} m') = 0)\right) \\ \to \forall w \exists s B_{qf}(\xi, \gamma, w, s) \end{array} \right) \end{array} \right). \tag{3.1}$$

By Lemma 2.3.2 along with Lemma 2.2.12 and standard intuitionistic equivalences, it follows that $\mathsf{EL} + \mathrm{MP}$ proves

$$\forall \xi \left(\begin{array}{l} A^*(\xi) \to \forall n \neg\neg\exists m (t(\langle n \rangle ^\frown \overline{\xi} m) > 0) \wedge \\ \forall \gamma \left(\begin{array}{l} \forall n \neg\neg\exists m \left(t(\langle n \rangle ^\frown \overline{\xi} m) = \gamma(n) + 1 \wedge \forall m' < m (t(\langle n \rangle ^\frown \overline{\xi} m') = 0)\right) \\ \to \forall w \neg\neg\exists s B_{qf}(\xi, \gamma, w, s) \end{array} \right) \end{array} \right).$$

Therefore, using MP and Lemma 3.3.5, we have

$$\mathsf{EL} + \mathrm{MP} \vdash (3.1). \tag{3.2}$$

In the following, we reason in $\mathsf{EL} + \mathrm{MP}$. For ξ satisfying $A(\xi)$, by (3.2) with the use of $\mathrm{QF\text{-}AC}^{0,0}$, we have g^1 such that $t(\langle n \rangle ^\frown \overline{\xi}(g(n))) > 0$ and $\forall m' < m (t(\langle n \rangle ^\frown \overline{\xi} m') = 0)$ for all n. Then $\zeta := \lambda n. t(\langle n \rangle ^\frown \overline{\xi}(g(n))) \dot{-} 1$ satisfies the condition in (3.1). Thus $\mathsf{EL} + \mathrm{MP}$ proves $\forall \xi \left(A(\xi) \to \exists \zeta \forall w \exists s B_{qf}(\xi, \zeta, w, s)\right)$, equivalently, $\forall \xi \, (A(\xi) \to \exists \zeta B(\xi, \zeta))$. □

Definition 3.3.7. $\mathrm{N_{KM}}$ is the class of formulas defined inductively as:

- A_{qf} and $\exists x^\rho A_{qf}$ are in $\mathrm{N_{KM}}$, where $\rho \in \{0,1\}$.

- If A_1, A_2 are in $\mathrm{N_{KM}}$, then $A_1 \wedge A_2$, $\forall x^\rho A_1$ are in $\mathrm{N_{KM}}$, where $\rho \in \{0, 1\}$.
- If A is in $\mathrm{N_{KM}}$, then $\forall u^\rho \exists v^0 A_{qf} \to A$ is in $\mathrm{N_{KM}}$, where $\rho \in \{0, 1\}$.

Lemma 3.3.8. $\mathrm{N_{KM}} \subset \mathrm{N_K} \cap \mathrm{N_M}$.

Proof. Straightforward by induction on the construction of $\mathrm{N_{KM}}$. □

We are now prepared to state our first characterization theorem.

Theorem 3.3.9. *Let $\forall \xi\, (A(\xi) \to \exists \zeta B(\xi, \zeta))$ be a $\mathcal{L}(\mathsf{EL})$-formula where $A(\xi) \in \mathrm{N_{KM}}$ and $B(\xi, \zeta)$ is equivalent to $\forall w^\rho \exists s^0 B_{qf}(\xi, \zeta, w, s)$ ($\rho \in \{0, 1\}$) over* EL + MP. *Then there exists a function term t of* RCA *such that*

$$\mathsf{RCA} \vdash \forall \xi\, (A(\xi) \to t \,|\, \xi \downarrow \wedge\, B(\xi, t \,|\, \xi))$$

if and only if

$$\mathsf{EL} + \mathrm{MP} \vdash \forall \xi\, (A(\xi) \to \exists \zeta B(\xi, \zeta))\,.$$

Proof. By Lemma 3.3.8, this follows immediately from Proposition 3.3.1 (with Remark 3.3.2) and Proposition 3.3.6. □

Remark 3.3.10. *The Markov's principle* MP *is not provable in* EL*. However,* MP *is allowed in Markov-style constructive mathematics (see [75] for details).*[3]

Remark 3.3.11. *A lot of mathematical statements have been investigated in computable analysis ([76]). For a theorem* S *represented as a Π^1_2 sentence $\forall \xi(A(\xi) \to \exists \zeta B(\xi, \zeta))$, the fact that* S *is provable in computable analysis (in the sense of [76, 7, 8]) roughly means that there is a uniform algorithm which transforms ξ into ζ. This is conceptually the same as the intended notion expressed by sequential provability in* RCA*. However, there is a crucial difference between provability in computable analysis and uniform provability in* RCA*. In the former case, the verification that the algorithm works is carried out in a usual mathematical manner. On the other hand, in the latter case, the verification has to be carried out in the restricted mathematical universal having only the Δ^0_1 ($\approx$ computable) set existence axiom. In this sense, uniform provability in* RCA *is more restrictive than provability in computable analysis. On the other hand, the choice of* EL *as a theory for formalizing constructive mathematics is based on considering the meaning of "constructive" as the existence of algorithm.*[4] *Under this interpretation, the fact that* S *is provable in* EL *suggests that there is an algorithm which transforms ξ into ζ, and*

[3] Troelstra considers HA + $\mathrm{ECT_0}$(extended Church's thesis)+$\mathrm{MP_{PR}}$ (See Section 6.3 for definitions) to be a formalization of Markov-style constructive mathematics ([75, 4.4.12]).

[4] Troelstra [73] suggests an analogy between Weihrauch's computable analysis and constructive mathematics.

in addition, the verification is carried out in the uniformly computable manner. In this sense, the provability in EL *is seemingly further restrictive than uniform provability in* RCA*. However, Theorem 3.3.9 states that the constructive provability is equivalent to uniform provability of in* RCA *at least for 'practical'* Π^1_2 *sentences because the syntactical class which our results covers is rich enough to involve a large amount of practical statements (under the standard representation in e.g. [75, 55]) as suggested from Remark 3.4.4 and the observation at the end of Subsection 5.3.1. In addition, as we show in Theorem 3.3.18 below, even the Markov's principle can be removed for simpler statements.*

Remark 3.3.12. *From a philosophical point of view, it is remarkable that all of our proofs are constructive, namely, they are just explicit syntactic translations. Thus we constructively (from a meta-perspective) establish the equivalence between constructive provability and classical uniform provability.*

In the following, we show that in Theorem 3.3.9, if $A(\xi)$ is in particular a purely universal formula (which is of course in $\mathrm{N_{KM}}$), one can remove even MP from the intuitionistic system. We first restate Hirst and Mummert's result [37, Theorem 5.6] (Proposition 3.1.2(2)) in a for our purpose most useful form.

Proposition 3.3.13. *For a sentence* $\forall x^\rho(A(x) \to \exists y^\tau B(x,y))$ *of* $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$ *where* $A(x)$ *is purely universal (i.e., of the form* $\forall u A_{qf}(x,u)$*) and* $B(x,y)$ *is in* Γ_2*, if*

$$\mathsf{WE\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega \vdash \forall x^\rho(A(x) \to \exists y^\tau B(x,y)),$$

then there exists a term $t^{\rho\to\tau}$ *of* $\mathsf{WE\text{-}HA}^\omega$ *such that*

$$\mathsf{WE\text{-}HA}^\omega \vdash \forall x^\rho(A(x) \to B(x,tx)).$$

Proof. Without loss of generality, let $A(x) \equiv \forall u A_p(x,u)$ with prime $A_p(x,u)$ (cf. [55, Proposition 3.17]). By using $\mathrm{IP}^\omega_\forall$, we have that $\forall x \exists y \left(\forall u A_p(x,u) \to B(x,y)\right)$ is provable in $\mathsf{WE\text{-}HA}^\omega$ + AC + $\mathrm{IP}^\omega_\forall$ + M^ω. Note that $\forall x \exists y \left(\forall u A_p(x,u) \to B(x,y)\right)$ is in Γ_2 since $B(x,y)$ is in Γ_2. The discussion below is same as in the proof of [37, Theorem 5.6]. Let $(B(x,y))^D \equiv \exists \underline{v} \forall \underline{w} B_D(x,y,\underline{v},\underline{w})$ (note that $(\forall u A_p(x,u))^D \equiv \forall u A_{qf}(x,u)$). By Theorem 2.3.17, there exist closed terms $t_Y, t_{\underline{V}}, t_U$ such that $\mathsf{WE\text{-}HA}^\omega \vdash \forall x, \underline{w} \left(A_p(x,,t_U x \underline{w}) \to B_D(x, t_Y x, t_{\underline{V}} x)\right)$. Then, without difficulty, one can see

$$\mathsf{WE\text{-}HA}^\omega \vdash \forall x \exists \underline{v} \forall \underline{w} \exists u \left(A_p(x,u) \to B_D(x, t_Y x, \underline{v}, \underline{w})\right).$$

Since this is equivalent to $\mathsf{WE\text{-}HA}^\omega \vdash \forall x \left(\forall u A_p(x,u) \to (B(x,y))^D\right)$ and $B(x,y)$ is in Γ_2, applying Lemma 2.3.19, we have $\mathsf{WE\text{-}HA}^\omega \vdash \forall x(\forall u A_p(x,u) \to B(x,tx))$. □

The following conservation result is an immediate consequence from the previous proposition.

Proposition 3.3.14. *For a sentence $\forall x^{\rho}(A(x) \to \exists y^{\tau} B(x,y))$ of $\mathcal{L}(\mathsf{WE\text{-}HA}^{\omega})$ where $A(x)$ is purely universal (i.e., of the form $\forall u A_{qf}(x,u)$) and $B(x,y)$ is in Γ_2, if*

$$\mathsf{WE\text{-}HA}^{\omega} + \mathrm{AC} + \mathrm{IP}^{\omega}_{\forall} + \mathrm{M}^{\omega} \vdash \forall x^{\rho}(A(x) \to \exists y^{\tau} B(x,y)),$$

then

$$\mathsf{WE\text{-}HA}^{\omega} \vdash \exists Y^{\rho\to\tau} \forall x^{\rho}(A(x) \to B(x, Yx)),$$

and hence, $\mathsf{WE\text{-}HA}^{\omega} \vdash \forall x^{\rho}(A(x) \to \exists y^{\tau} B(x,y))$.

Remark 3.3.15. *In the same manner, one can show the analogous results obtained by replacing* $\mathsf{WE\text{-}HA}^{\omega}$ *with* $\widehat{\mathsf{WE\text{-}HA}}^{\omega}\!\upharpoonright$ *in Theorem 3.3.13 and Proposition 3.3.14 respectively (cf. [55, Section 8.3]).*

Proposition 3.3.16. *Assume that $A(\xi)$ has Π^0_1 form $\forall u A_b(\xi,u)$ and $B(\xi,\zeta)$ is equivalent to $\forall w^{\rho}\exists s^0 B_{qf}(\xi,\zeta,w,s)$ ($\rho \in \{0,1\}$) over* $\mathsf{EL} + \mathrm{MP}$*. If there exists a function term t of* RCA *such that*

$$\mathsf{RCA} \vdash \forall \xi \,(A(\xi) \to t\,|\,\xi\downarrow \wedge\, B(\xi, t\,|\,\xi))\,,$$

then

$$\mathsf{EL} \vdash \forall \xi\,(A(\xi) \to \exists \zeta B(\xi,\zeta))\,.$$

Proof. Without loss of generality, one can assume that $A(\xi)$ is a purely universal formula $\forall u A_p(\xi,u)$ with prime $A_p(\xi,u)$ by Lemma 2.2.12.

Since each purely universal formula is in $\mathrm{N_M}$, by Proposition 3.3.6, $\mathsf{EL} + \mathrm{MP}$ proves

$$\forall \xi(A(\xi) \to \exists \zeta \forall w^{\rho}\exists s^0 B_{qf}(\xi,\zeta,w,s)).$$

By identifying $\mathsf{EL} + \mathrm{MP}$ with its canonical embedding into $\mathsf{WE\text{-}HA}^{\omega} + \mathrm{M}^0$, we have

$$\mathsf{WE\text{-}HA}^{\omega} + \mathrm{M}^0 \vdash \forall \xi\left(A(\xi) \to \exists \zeta \forall w^{\rho}\exists s^0 B_{qf}(\xi,\zeta,w,s)\right).$$

Since $\forall w^{\rho}\exists s^0 B_{qf}(\xi,\zeta,w,s)$ is in Γ_2 (cf. Definition 2.3.18), by Proposition 3.3.14, we have

$$\mathsf{WE\text{-}HA}^{\omega} \vdash \forall \xi\left(A(\xi) \to \exists \zeta \forall w^{\rho}\exists s^0 B_{qf}(\xi,\zeta,w,s)\right).$$

Since $\mathsf{WE\text{-}HA}^{\omega}$ is conservative over EL for $\mathcal{L}(\mathsf{EL})$ formulas (Proposition 2.2.24), it follows that EL proves $\forall \xi\left(A(\xi) \to \exists \zeta \forall w^{\rho}\exists s^0 B_{qf}(\xi,\zeta,w,s)\right)$, equivalently, $\forall \xi\,(A(\xi) \to \exists \zeta B(\xi,\zeta))$. □

Proposition 3.3.17. *Let* $\forall\xi\,(A(\xi) \to \exists\zeta B(\xi,\zeta))$ *be a* $\mathcal{L}(\mathsf{EL})$*-formula where* $A(\xi)$ *has* Π^0_1 *form* $\forall u A_b(\xi,u)$ *and* $B(\xi,\zeta)$ *is equivalent to* $\forall w^\rho \exists s^0 B_{qf}(\xi,\zeta,w,s)$ $(\rho \in \{0,1\})$ *over* $\mathsf{EL} + \mathrm{MP}$. *Then*

$$\mathsf{EL} \vdash \forall\xi\,(A(\xi) \to \exists\zeta B(\xi,\zeta))$$

if and only if there exists a term $t^{1\to1}$ *of* RCA^ω *such that*

$$\mathsf{RCA}^\omega \vdash \forall\xi\,(A(\xi) \to B(\xi,t\xi))$$

under the canonical embedding.

Proof. Without loss of generality, one can assume that $A(\xi)$ is a purely universal formula $\forall u A_p(\xi,u)$ with prime $A_p(\xi,u)$ by Lemma 2.2.12.

Soppose $\mathsf{EL} \vdash \forall\xi\,(A(\xi) \to \exists\zeta B(\xi,\zeta))$. Then $\mathsf{E\text{-}HA}^\omega \vdash \forall\xi\,(A(\xi) \to \exists\zeta B(\xi,\zeta))$ under the canonical embedding. Since $A(\xi)$ is $\exists$-free and $\forall w^\rho \exists s^0 B_{qf}(\xi,\zeta,w,s)$ is in Γ_1, as in the proof of Proposition 4.2.3 (based on modified realizability interpretation) below, one can show that there exists a term $t^{1\to1}$ of RCA^ω such that $\mathsf{RCA}^\omega \vdash \forall\xi\,(A(\xi) \to B(\xi,t\xi))$.

The converse direction is the same as for Proposition 3.3.16 except the use of elimination of extensionality technique. Suppose that RCA^ω proves $\forall\xi\,(A(\xi) \to B(\xi,t\xi))$. Since our sentence contains quantifiers only of type 0 and 1, it follows by [55, Proposition 10.45] that WRCA^ω (i.e. $\mathsf{WE\text{-}PA}^\omega + \mathrm{QF\text{-}AC}^{1,0}$) proves $\forall\xi\,(A(\xi) \to B(\xi,t\xi))$. Then using Kuroda's negative translation [55, Proposition 10.6], the Dialectica interpretation (Proposition 3.3.14) and the conservativity (Proposition 2.2.24) just as in the proof of Proposition 3.3.16, we have that EL proves $\forall\xi\,(A(\xi) \to \exists\zeta B(\xi,\zeta))$. □

The following is our second characterization theorem.

Theorem 3.3.18. *Let* $\forall\xi\,(A(\xi) \to \exists\zeta B(\xi,\zeta))$ *be a* $\mathcal{L}(\mathsf{EL})$*-formula where* $A(\xi)$ *has* Π^0_1 *form* $\forall u A_b(\xi,u)$ *and* $B(\xi,\zeta)$ *is equivalent to* $\forall w^\rho \exists s^0 B_{qf}(\xi,\zeta,w,s)$ $(\rho \in \{0,1\})$ *over* $\mathsf{EL} + \mathrm{MP}$. *Then the following are pairwise equivalent.*

(1) $\mathsf{EL} \vdash \forall\xi\,(A(\xi) \to \exists\zeta B(\xi,\zeta))$.

(2) There exists a function term t of RCA *such that* $\mathsf{RCA} \vdash \forall\xi\,(A(\xi) \to t\,|\,\xi\downarrow \wedge B(\xi,t\,|\,\xi))$.

(3) There exists a term $t^{1\to1}$ *of* RCA^ω *such that* $\mathsf{RCA}^\omega \vdash \forall\xi\,(A(\xi) \to B(\xi,t\xi))$ *under the canonical embedding.*

Proof. The equivalence between (1) and (2) follows from Proposition 3.3.1 and Proposition 3.3.16. On the other hand, (1) is equivalent to (3) by Proposition 3.3.17. □

Remark 3.3.19. *Theorem 3.3.18 reveals that the representation of uniform provability by the partially defined application operation* $(\cdot)\,|\,(\cdot)$ *and the representation by primitive recursive functional in the sense of Gödel is equivalent for a large number of practical statements in reverse mathematics.*

3.4 Characterization of Uniform Provability in RCA_0

By a careful inspection, one observes that all proofs in the previous section also work for the fragments EL_0, RCA_0, RCA_0^ω instead of EL, RCA, RCA^ω (cf. [13, Rremark 3.10], [55, Section 8.3] and [55, Section 10.5]). Since the proofs are completely parallel to before, in this section, we state our theorems without proofs.

Firstly, the following is a counterpart of Theorem 3.3.9.

Theorem 3.4.1. *Let* $\forall\xi\,(A(\xi) \to \exists\zeta B(\xi,\zeta))$ *be a* $\mathcal{L}(\mathsf{EL}_0)$*-formula where* $A(\xi) \in \mathrm{N_{KM}}$ *and* $B(\xi,\zeta)$ *is equivalent to* $\forall w^\rho \exists s^0 B_{qf}(\xi,\zeta,w,s)$ $(\rho \in \{0,1\})$ *over* $\mathsf{EL}_0 + \mathrm{MP}$*. Then there exists a function term* t *of* RCA_0 *such that*

$$\mathsf{RCA}_0 \vdash \forall\xi\,(A(\xi) \to t\,|\,\xi\downarrow \wedge\, B(\xi, t\,|\,\xi))$$

if and only if

$$\mathsf{EL}_0 + \mathrm{MP} \vdash \forall\xi\,(A(\xi) \to \exists\zeta B(\xi,\zeta))\,.$$

The following theorem is a counterpart of Theorem 3.3.18. Notice that the class of type $1 \to 1$ functional terms of RCA_0^ω (containing only type 0 recursor R_0) is proper subclass of type $1 \to 1$ functional terms of RCA^ω (possibly containing higher type recursors) while the class of function terms of RCA_0 is the same as RCA.

Theorem 3.4.2. *Let* $\forall\xi\,(A(\xi) \to \exists\zeta B(\xi,\zeta))$ *be a* $\mathcal{L}(\mathsf{EL}_0)$*-formula where* $A(\xi)$ *has* Π^0_1 *form* $\forall u A_b(\xi,u)$ *and* $B(\xi,\zeta)$ *is equivalent to* $\forall w^\rho \exists s^0 B_{qf}(\xi,\zeta,w,s)$ $(\rho \in \{0,1\})$ *over* $\mathsf{EL}_0 + \mathrm{MP}$*. Then the following are pairwise equivalent.*

1. $\mathsf{EL}_0 \vdash \forall\xi\,(A(\xi) \to \exists\zeta B(\xi,\zeta))$.
2. *There exists a function term* t *of* RCA_0 *such that* $\mathsf{RCA}_0 \vdash \forall\xi\,(A(\xi) \to t\,|\,\xi\downarrow \wedge\, B(\xi,t\,|\,\xi))$.
3. *There exists a term* $t^{1\to1}$ *of* RCA_0^ω *such that* $\mathsf{RCA}_0^\omega \vdash \forall\xi\,(A(\xi) \to B(\xi,t\xi))$ *under the canonical embedding.*

Remark 3.4.3. *The corresponding result to the equivalence between 2 and 3 in Theorem 3.4.2 can be found in [53, Section 4], where the relation between the continuous notion in finite-type* *arithmetic and the continuous notion by means of the operation* $(\cdot)\,|\,(\cdot)$ *has been investigated with respect to higher order reverse mathematics.*

Remark 3.4.4 (on Application)**.** *For example, by a careful inspection, one can see that Kierstead's effective variant of marriage theorem [44] (see also [23]) is formalized as a* Π^1_2 *sentence such that both of the premise and the conclusion have* Π^0_1 *form. In addition, as indicated in [23], it is uniformly provable in* RCA_0 *(in particular, the verification of the solution-constructing algorithm is carried out in* RCA_0*). Therefore, by Theorem 3.3.18, it follows that Kierstead's effective marriage theorem is provable in* EL_0.

3.5 Future Works

The author thinks that there are several possible extensions of this work. Here we list three of them.

1. Characterize the hierarchy of relative uniform provability with respect to WKL and ACA by the hierarchy of the law-of-excluded-middle over EL (cf. Section 1.5).

2. Characterize uniform provability in stronger systems (like ACA) by (semi-)intuitionistic systems, which aims the characterization of computable analysis by constructive mathematics (cf. Remark 3.3.11 as well as [73]).

3. Compare formalized Markov-style constructive mathematics with uniform provability in RCA (cf. Remark 3.3.10).

4 Metatheorems for Uniform Versions

The uniformization theorems (See Section 3.1) allow us to use sequential reverse mathematics to demonstrate the unprovability of several mathematical principles in (semi-)intuitionistic systems. The following table 4.1 gives the picture of existing uniformization theorems concerned with sequential versions.

	Higher-order setting	Second-order setting
$\mathsf{RCA} + \mathrm{WKL}$	[$\star$]	Dorais [13, Cor. 4.9]
RCA	Hirst-Mummert [37, Thm. 3.6 & Thm. 5.6]	Dorais [13, Cor. 3.9]

Table 4.1: Uniformization theorems

As shown in [13], not only the uniformization theorem for RCA but also the uniformization theorem for $\mathsf{RCA}+\mathrm{WKL}$ holds in a second-order setting. Now, what can we show in a higher-order (finite type) setting? In this chapter, we focus on the relationship between classical uniform provability and intuitionistic provability with respect to WKL in a higher-order setting ([$\star$] in table 4.1).

Let us consider a stronger form to represent uniform provability than just sequentialization. For a sentence $\mathrm{S} := \forall X\,(A(X) \to \exists Y B(X, Y))$, one can consider a sentence

$$\exists F \forall X\,(A(X) \to B(X, F(X)))\,,$$

which expresses the existence of a uniform procedure F to construct a solution for each problem X. We call this sentence the *uniform version* of S and denote it as Uni(S). Unfortunately, for a Π^1_2 sentence, this uniform version is not naturally represented in the language of second-order arithmetic since F is a third-order object. To treat uniform versions, we need systems of arithmetic in all finite types (See Subsection 2.2.2). In fact, uniform versions of ordinary mathematical theorems have been investigated in the context of higher-order reverse mathematics [54, 66].

In Section 4.2, we show the uniformization theorem concerned with uniform versions for RCA. In addition, we discuss about that for RCA + WKL (positioned at [$\star$] in table 4.1). The proof is respectively based on modified realizability and the monotone Dialectica interpretation with the use of the technique of elimination of extensionality. The uniformization results concerned with sequential versions in [37] are the immediate corollaries of our results since the sequential version follows from the uniform version.

In Section 4.3, we show a related metatheorem which states that (in particular) for every Π_2 statement S of some syntactical form, if its uniform version Uni(S) derives the uniform variant $(\exists^2)$ of ACA over a classical higher-order system with weak extensionality, then S is not provable in extremely strong semi-intuitionistic systems $\mathcal{T}$ which include bar induction BI in all types but also weak Kőnig's lemma WKL and even uniform weak Kőnig's lemma UWKL and Kőnig's lemma KL. In particular, $\mathcal{T}$ is strong enough (even without UWKL and KL) to interpret classical analysis with full dependent choice via negative translation. That is, by applying this metatheorem, one can obtain stronger unprovability results than what follows from uniformization theorems. In this sense, one can think of our metatheorem as an extended variant of the higher-order uniformization theorem positioned at [$\star$] in table 4.1. In the proof, we use (a variant of) the Dialectica interpretation, negative translation, a nonstandard axiom F^- and the model of all strongly majorizable functionals. Roughly speaking the metatheorem often allows one to detect using classical reasoning on Uni(S) that S intuitionistically implies at least the Π^0_1-law-of-excluded-middle principle Π^0_1-LEM (and so - in the presence of Markov's principle - Σ^0_1-LEM) rather than only the strictly weaker principle Σ^0_1-DML (as WKL already does; see Section 1.5).

In Section 4.4, we demonstrate that our metatheorem in Section 4.3 is applicable to concrete mathematical principles to show, using classical reasoning on the uniform versions of principles S only, the unprovability of S in the semi-intuitionistic systems mentioned above. The investigation of the strength of uniform versions in higher-order reverse mathematics plays an important role in the application of our metatheorem. In addition, as demonstrated in Section 5.4 below, our metatheorem is applicable to statements whose sequential versions imply ACA.

The content of this chapter is due to a joint work with Ulrich Kohlenbach and contained in [24]. Most of the technical tools used in this chapter are taken from Kohlenbach's monograph [55]. See Section 2.2 for the basic definitions and properties on finite type arithmetic.

4.1 Definitions

We first recall the definitions of key principles in this chapter (cf. [55, 52, 54]).

- $(\exists^2) : \exists E^2 \forall f^1 \left(Ef = 0 \leftrightarrow \exists x^0 (fx = 0) \right)$.
- $(\mu^2) : \exists \mu^2 \forall f^1 \left(\exists x^0 (fx = 0) \rightarrow f(\mu f) = 0 \right)$.

- WKL (weak Kőnig's lemma): $\forall f^1 \left(T^\infty(f) \to \exists b \leq_1 \lambda k.1 \forall x^0 \left(f\left(\overline{b}x\right) = 0\right)\right)$, where $T^\infty(f)$ expresses that f represents an infinite binary tree.

- UWKL (uniform weak Kőnig's lemma): $\exists \Phi \leq_{1(1)} 1 \forall f^1 \left(T^\infty(f) \to \forall x^0 \left(f\left(\left(\overline{\Phi f}\right)x\right) = 0\right)\right)$.

Note that $(\exists^2)$ is the uniform variant of ACA and UWKL is the uniform version of WKL.

The model $\mathcal{M}^\omega$ of all strongly majorizable functionals, which was first introduced in Bezem [4], is a crucial tool for our result in Section 4.3.

Definition 4.1.1 ([55, 48, 50])**.** The type structure $\mathcal{M}^\omega$ of all hereditarily strongly majorizable set-theoretic functionals of finite type is defined as follows:

- $$\begin{cases} n \ s\text{-}maj_0 \ m :\equiv n \geq m \wedge n, m \in \mathbb{N}; \\ M_0 := \mathbb{N}; \\ x^* \ s\text{-}maj_{\tau(\rho)} \ x \ :\equiv x^*, x \in {M_\tau}^{M_\rho} \wedge \forall y, y^*(y^* \ s\text{-}maj_\rho \ y \to x^* y^* \ s\text{-}maj_\tau \ x^* y, xy); \\ M_{\tau(\rho)} := \left\{x \in {M_\tau}^{M_\rho} \ : \ \exists x^* \in {M_\tau}^{M_\rho} \left(x^* \ s\text{-}maj_{\tau(\rho)} \ x\right)\right\} \ (\rho, \tau \in \mathbf{T}); \end{cases}$$

- $\mathcal{M}^\omega := \langle M_\rho \rangle_{\rho \in \mathbf{T}}$.

(Here ${M_\tau}^{M_\rho}$ denotes the set of all total set-theoretic functions from M_ρ to M_τ.)

In addition, we recall some principles which are used mainly in Section 4.3. See [55, Chapter 12] for the detailed discussion on $\mathrm{F}^{(-)}$ and $\Sigma^0_1\text{-UB}^{(-)}$ and see e.g. [2, 40, 41, 55] for general information on (BI) and (BR).

- For $z^{\rho(0)}$, $\left(\overline{z, n}\right)(k^0) :=_\rho \begin{cases} zk & \text{if } k <_0 n, \\ 0^\rho & \text{otherwise,} \end{cases}$

  where 0^ρ is constant-0 functional of type ρ.

- $\mathrm{F} : \forall \Phi^{2(0)}, y^{1(0)}, \exists y_0 \leq_{1(0)} y \forall k^0, z \leq_1 yk (\Phi kz \leq_0 \Phi k\,(y_0 k)\,)$.
  $\mathrm{F}^- : \forall \Phi^{2(0)}, y^{1(0)}, \exists y_0 \leq_{1(0)} y \forall k^0, z^1, n^0 (\bigwedge_{i <_0 n} zi \leq_0 yki \to \Phi k\,(\overline{z, n}) \leq_0 \Phi k\,(y_0 k)\,)$.

- $\Sigma^0_1\text{-UB} : \forall y^{1(0)} \Big(\forall k^0 \forall x \leq_1 yk \exists z^0 A(x, y, k, z) \to \exists \chi^1 \forall k^0 \forall x \leq_1 yk \exists z \leq_0 \chi k A\,(x, y, k, z)\,\Big)$,
  $\Sigma^0_1\text{-UB}^- : \forall y^{1(0)} \Big(\forall k^0 \forall x \leq_1 yk \exists z^0 A(x, y, k, z) \to \exists \chi^1 \forall k^0, x^1, n^0 (\bigwedge_{i<n} (xi \leq_0 yki) \to \exists z \leq_0 \chi k A\,((\overline{x, n})\,, y, k, z)\,)\Big)$, where $A \equiv \exists \underline{l} A_{qf}(\underline{l})$ and $\underline{l}$ is a tuple of variables of type 0 and A_{qf} is a quantifier-free formula which may contain parameters of arbitrary type.

- KL (Kőnig's lemma): $\forall f^1 \left(\tilde{T}^\infty(f) \to \exists b^1 \forall x^0 \left(f\left(\overline{b}x\right) = 0\right)\right)$, where $\tilde{T}^\infty(f)$ expresses that f represents a finitely branching infinite tree.

- $\mathrm{DC}^\rho : \forall x^0, y^\rho \exists z^\rho A(x, y, z) \to \exists f^{\rho(0)} \forall x^0 A(x, f(x), f(S\,(x)))$.
  DC (dependent choice) $:= \bigcup_{\rho \in \mathbf{T}} \{\mathrm{DC}^\rho\}$.

- $(\mathrm{BI}_\rho): \forall x^{\rho 0}, n^0 \begin{pmatrix} \exists k^0 P\left(\overline{x,k};k\right) \wedge \\ P\left(\overline{x,n};n\right) \to P\left(\overline{x,n+1};n+1\right) \wedge \\ P\left(\overline{x,n};n\right) = 0 \to Q\left(\overline{x,n};n\right) \wedge \\ \forall u^\rho Q\left(\overline{x,n} * u; n+1\right) \to Q\left(\overline{x,n};n\right) \end{pmatrix} \to Q(0^{\rho 0}; 0^0),$

  where P, Q are arbitrary formulas and

$$(\overline{x,n} * u)\, k =_\rho \begin{cases} xk & \text{if } k < n, \\ u & \text{if } k = n. \\ 0^\rho & \text{otherwise.} \end{cases}$$

  BI (bar induction) := $\bigcup_{\rho \in \mathbf{T}} \{\mathrm{BI}_\rho\}$.

  $\mathrm{BI}^{\leq 1} := \bigcup_{\rho \in \mathbf{T}, deg(\rho) \leq 1} \{\mathrm{BI}_\rho\}$.

- $\mathrm{BR}_{\rho,\tau}: \begin{cases} y\left(\overline{x,n}\right) <_0 n \to B_{\rho,\tau} yzunx =_\tau zn\left(\overline{x,n}\right) \\ y\left(\overline{x,n}\right) \geq_0 n \to B_{\rho,\tau} yzunx =_\tau u\left(\lambda D^\rho . B_{\rho,\tau} yzunx(n+1)\left(\overline{x,n} * D\right)\right) n \left(\overline{x,n}\right) \end{cases}$

  for all $x^{\rho(0)}$ and n^0.

  BR (bar recursion) := $\bigcup_{\rho,\tau \in \mathbf{T}} \{\mathrm{BR}_{\rho,\tau}\}$.

  If the system in question has BR, we implicitly assume that new constants $B_{\rho,\tau}$ for bar recursion are added. The important thing for our analysis is that BR is a purely universal axiom scheme.

Remark 4.1.2. *Bar induction in all finite types is a generalization of Brouwer's 'bar theorem' considered first by Spector* [70]. *Spector also defined the new concept of bar recursion. The precise formulations of* (BI) *and* (BR) *used above are taken from* [63] *(See also* [40] *and Section 11.1 from* [55]*).*

4.2 Uniformization Theorems Concerned with Uniform Versions

Proposition 4.2.1. 1. *Let A be an arbitrary formula of $\mathcal{L}(\textsf{E-HA}^\omega)$. Then one can construct an $\exists$-free formula B_{ef} such that*

$$\textsf{E-HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}} \vdash \neg A \leftrightarrow B_{\mathrm{ef}}.$$

2. *Let A_{ef} be an $\exists$-free formula of $\mathcal{L}(\textsf{E-HA}^\omega)$. Then*

$$\textsf{E-HA}^\omega \vdash A_{\mathrm{ef}} \leftrightarrow \neg\neg A_{\mathrm{ef}}.$$

These also hold for $\widehat{\mathsf{E}\text{-}\mathsf{HA}}^{\omega}\!\upharpoonright$ *instead of* $\mathsf{E}\text{-}\mathsf{HA}^{\omega}$.

Proof. 1. See [55, Proposition 5.14]. 2. One can show $\mathsf{E}\text{-}\mathsf{HA}^{\omega} \vdash \neg\neg A_{\mathrm{ef}} \to A_{\mathrm{ef}}$ by easy induction on the structure of A_{ef} with the use of the fact that prime formulas are decidable. The opposite direction is obvious. The same proof also works for the analogous result for $\widehat{\mathsf{E}\text{-}\mathsf{HA}}^{\omega}\!\upharpoonright$. □

As corollary to Proposition 4.2.1, we get that $\mathrm{IP}^{\omega}_{\mathrm{ef}}$ and $\mathrm{IP}^{\omega}_{\neg}$ are equivalent in the presence of AC.

Corollary 4.2.2. *1.* $\mathsf{E}\text{-}\mathsf{HA}^{\omega} + \mathrm{AC} \vdash \mathrm{IP}^{\omega}_{\mathrm{ef}} \to \mathrm{IP}^{\omega}_{\neg}$.

2. $\mathsf{E}\text{-}\mathsf{HA}^{\omega} \vdash \mathrm{IP}^{\omega}_{\neg} \to \mathrm{IP}^{\omega}_{\mathrm{ef}}$.

We first show a uniformization theorem for uniform versions via modified realizability. The proof is essentially the same as for the main theorem of Hirst and Mummert [37, Theorem 3.6].

Proposition 4.2.3. *Let* $\Delta^{\mathsf{RCA}^{\omega}}_{\mathrm{ef}}$ *be the class of* $\exists$*-free sentences provable in* RCA^{ω}. *For a sentence* $\mathrm{S} :\equiv \forall x^{\rho}(A(x) \to \exists y^{\tau} B(x,y))$ *of* $\mathcal{L}(\mathsf{E}\text{-}\mathsf{HA}^{\omega})$ *where* $A(x)$ *is* $\exists$*-free and* $B(x,y)$ *is in* Γ_1, *if*

$$\mathsf{E}\text{-}\mathsf{HA}^{\omega} + \mathrm{AC} + \mathrm{IP}^{\omega}_{\mathrm{ef}} + \Delta^{\mathsf{RCA}^{\omega}}_{\mathrm{ef}} \vdash \mathrm{S},$$

then

$$\mathsf{RCA}^{\omega} \vdash \mathrm{Uni(S)}.$$

This also holds for $\widehat{\mathsf{E}\text{-}\mathsf{HA}}^{\omega}\!\upharpoonright$ *and* $\mathsf{RCA}^{\omega}_{0}$ *instead of* $\mathsf{E}\text{-}\mathsf{HA}^{\omega}$ *and* RCA^{ω}.

Proof. Since $A(x)$ is $\exists$-free, $\forall x \exists y\,(A(x) \to B(x,y))$ is provable in $\mathsf{E}\text{-}\mathsf{HA}^{\omega} + \mathrm{AC} + \mathrm{IP}^{\omega}_{\mathrm{ef}} + \Delta^{\mathsf{RCA}^{\omega}}_{\mathrm{ef}}$. Note that $\forall x \exists y\,(A(x) \to B(x,y))$ is in Γ_1 since $B(x,y)$ is in Γ_1. Then one can construct a closed term t such that $\mathsf{E}\text{-}\mathsf{HA}^{\omega} + \Delta^{\mathsf{RCA}^{\omega}}_{\mathrm{ef}} \vdash \forall x\,(A(x) \to B(x,tx))$ by using modified realizability interpretation as in the proof of [37, Lemma 3.5] (actually using Theorem 2.3.12 and Lemma 2.3.14). Since RCA^{ω} is an extension of $\mathsf{E}\text{-}\mathsf{HA}^{\omega} + \Delta^{\mathsf{RCA}^{\omega}}_{\mathrm{ef}}$, we have $\mathsf{RCA}^{\omega} \vdash \mathrm{Uni(S)}$. The same proof works for the analogous result for $\widehat{\mathsf{E}\text{-}\mathsf{HA}}^{\omega}\!\upharpoonright$ and $\mathsf{RCA}^{\omega}_{0}$. □

Warning. Proposition 4.2.3 does not hold for every sentence $\forall x^{\rho}(A(x) \to \exists y^{\tau} B(x,y))$ in Γ_1. In fact, $\forall f^1(\exists y^1(fy = 0) \to \exists x^1(fx = 0)) \in \Gamma_1$ is logically valid, but its uniform version (μ^2) is not provable in RCA^{ω}. This means that it is essential to restrict the syntactical form of $A(x)$ to $\exists$-free in Proposition 4.2.3.

Remark 4.2.4. *All sentences* $\forall x^{\rho} \exists y^{\tau} A_{qf}(x,y)$ *provable in* RCA^{ω}, *where the degree of the type* ρ *is not greater than* 1, *the type* τ *is arbitrary and* A_{qf} *is quantifier-free, are included in* $\mathsf{E}\text{-}\mathsf{HA}^{\omega} + \mathrm{AC} + \mathrm{IP}^{\omega}_{\mathrm{ef}} + \Delta^{\mathsf{RCA}^{\omega}}_{\mathrm{ef}}$, *since one can show* $\mathsf{WE}\text{-}\mathsf{HA}^{\omega} \vdash \forall x A_{qf}(x,tx)$ $(\in \Delta^{\mathsf{RCA}^{\omega}}_{\mathrm{ef}})$ *via elimination of extensionality (Lemma 2.3.25), negative translation (Lemma 2.3.3) and the Dialectica interpretation (Theorem 2.3.17).*

Remark 4.2.5. *While the proof of Proposition 4.2.3 is based on modified realizability interpretation, by using modified realizability interpretation with truth instead (actually using [55, Theorem* 5.23 *and Lemma* 5.6*] instead of Theorem 2.3.12 and Lemma 2.3.14 in the proof of Proposition 4.2.3), one can show that for a sentence* $\mathrm{S} := \forall x\,(\neg A(x) \to \exists y B(x,y))$ *of* $\mathcal{L}(\mathsf{E\text{-}HA}^{\omega})$ *where* $A(x)$ *is arbitrary, if*

$$\mathsf{E\text{-}HA}^{\omega} + \mathrm{QF\text{-}AC}^{1,0} + \mathrm{IP}^{\omega}_{\neg} + \Delta^{\mathsf{RCA}^{\omega}}_{\neg} \vdash \mathrm{S},$$

then

$$\mathsf{RCA}^{\omega}(\textit{actually } \mathsf{E\text{-}HA}^{\omega} + \mathrm{QF\text{-}AC}^{1,0} + \mathrm{IP}^{\omega}_{\neg} + \Delta^{\mathsf{RCA}^{\omega}}_{\neg}) \vdash \mathrm{Uni(S)},$$

where $\Delta^{\mathsf{RCA}^{\omega}}_{\neg}$ *be the class of negated sentences provable in* RCA^{ω}*. Compared to Proposition 4.2.3, the syntactical restriction of* $B(x,y)$ *is dropped. In addition, since we don't have* AC *now,* $\mathrm{IP}^{\omega}_{\neg}$ *and* $\Delta^{\mathsf{RCA}^{\omega}}_{\neg}$ *seem to be proper extensions of* $\mathrm{IP}^{\omega}_{\mathrm{ef}}$ *and* $\Delta^{\mathsf{RCA}^{\omega}}_{\mathrm{ef}}$ *respectively.*

As for Proposition 4.2.3, one can also show another uniformization theorem on uniform versions via the Dialectica interpretation, which is the counterpart of [37, Theorem 5.6]. In this case, the syntactic class involved is restricted little more than in Proposition 4.2.3 and the base system is weakened to the weakly extensional one, but the non-intuitionistic scheme M^{ω} can be added to the system in the assumption. (Note that $\mathsf{WE\text{-}HA}^{\omega} \subseteq \mathsf{E\text{-}HA}^{\omega}$ and $\widehat{\mathsf{WE\text{-}HA}}^{\omega}\!\upharpoonright \subseteq \widehat{\mathsf{E\text{-}HA}}^{\omega}\!\upharpoonright$.) Furthermore, in the sense of applications to sequential reverse mathematics, one can extend this result so that the system in the assumption includes WKL and even Kőnig's lemma KL, whereas the classical system in the conclusion contains UWKL. Such an extension (in the absence of M^{ω}) is also possible for Proposition 4.2.3, but the result would be less meaningful as in the presence of extensionality UWKL already proves uniform arithmetical comprehension $(\exists^2)$ (See [52, Proposition 3.4]) while UWKL is still weak relative to WRCA^{ω} (See again [52]).

The following syntactical form is important in our results.

Definition 4.2.6 ([49])**.** Δ denotes a set of sentences of the form

$$\forall a^{\delta} \exists b \leq_{\sigma} ra \forall c^{\gamma} B_{qf}(a,b,c),$$

where $B_{qf}(a,b,c)$ is quantifier-free and does not contain any further free variables than those shown, r is a closed term (of suitable type) of $\mathsf{E\text{-}HA}^{\omega}$ (or $\widehat{\mathsf{E\text{-}HA}}^{\omega}\!\upharpoonright$ in context), the types δ, σ, γ are arbitrary, and '$b \leq_{\sigma} r\, a$' is defined pointwise, i.e $x \leq_{\sigma} y := \forall \underline{v}(x\underline{v} \leq_0 y\underline{v})$.

Moreover, $\widetilde{\Delta}$ denotes a corresponding set of the Skolem normal forms of the sentences in Δ

$$\left\{\widetilde{\mathrm{T}} :\equiv \exists B \leq r \forall a \forall c B_{qf}(a, Ba, c) : \mathrm{T} :\equiv \forall a^{\delta} \exists b \leq_{\sigma} r\, a \forall c^{\gamma} B_{qf}(a,b,c) \in \Delta\right\}.$$

Throughout this paper, for a sentence T of the form of Δ, we denote the corresponding sentence of the form of $\widetilde{\Delta}$ as $\widetilde{\mathrm{T}}$.

Remark 4.2.7. *A purely universal sentence $\forall u\, A_{qf}(u)$ has the form of Δ and $\forall u\widetilde{A_{qf}}(u) = \forall u\, A_{qf}(u)$.*

Definition 4.2.8 ([49, 55])**.** Let $\widehat{(\cdot)}^{1(1)}$ be a functional such that

$$\widehat{fn} := \begin{cases} fn & \text{if } fn \neq 0 \vee (\forall k, l\,(k * l = n \to fk = 0) \wedge \forall i \leq lth\, n\,((n)_i \leq 1)), \\ 1^0 & \text{otherwise,} \end{cases}$$

and $(\cdot)_{(\cdot)}{}^{1(1)(1)}$ be a functional such that

$$f_g n := \begin{cases} fn & \text{if } f\,(g(lth\ n)) = 0 \wedge lth\,(g(lth\ n)) = lth\ n, \\ 1^0 & \text{otherwise,} \end{cases}$$

Then

$$\mathrm{WKL}' :\equiv \forall f^1, g^1 \exists b \leq_1 \lambda k.1 \forall x^0 \left(\widehat{(\widehat{f})_g}(\bar{b}x) =_0 0\right).$$

Note that WKL′ has the form of Δ. We recall that WKL is equivalent to WKL′ over $\widehat{\text{WE-HA}}^{\omega}\restriction$ ([55, Proposition 9.18.2]). See [52] and [55, Chapter 9, 10] for the detailed discussion on WKL′.

The next proposition is the extended variant of Theorem 5.6 in Hirst-Mummert [37].

Proposition 4.2.9. *For a sentence $\mathrm{S} := \forall x^\rho\,(A(x) \to \exists y^\tau B(x, y))$ of $\mathcal{L}(\text{WE-HA}^\omega)$ where $A(x)$ is purely universal and $B(x, y)$ is in Γ_2, if*

$$\text{WE-HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \Delta \vdash \mathrm{S},$$

then

$$\text{WE-PA}^\omega + \widetilde{\Delta} \vdash \mathrm{Uni(S)}.$$

In particular, if

$$\text{WE-HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \mathrm{UWKL} + \mathrm{KL} \vdash \mathrm{S},$$

then

$$\text{WE-PA}^\omega + \mathrm{UWKL} \vdash \mathrm{Uni(S)}.$$

This also holds for $\widehat{\text{WE-HA}}^{\omega}\restriction$ and $\widehat{\text{WE-PA}}^{\omega}\restriction$ instead of WE-HA^ω and WE-PA^ω.

Proof. Since $A(x)$ is purely universal, $\forall x \exists y\,(A(x) \to B(x, y))$ is provable in $\text{WE-HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \Delta$. Let denote $A(x) \to B(x, y)$ as $C(x, y)$ for convenience. Applying Theorem 9.1

from [55] ('soundness theorem for monotone functional interpretation') to the assumption, we have

$$\mathsf{WE\text{-}HA}^{\omega} + \widetilde{\Delta} \vdash \exists Y, \underline{U} \left(\forall x, \underline{v}\, C_D(x, Yx, \underline{U}x, \underline{v}) \right)$$

where $C^D = \exists \underline{u} \forall \underline{v}\ C_D(x, y, \underline{u}, \underline{v})$. Then, a fortiori, $\exists Y \forall x \exists \underline{u} \forall \underline{v}\ C_D(x, Yx, \underline{u}, \underline{v})$ follows. Since $C(x, Yx)$ is in Γ_2, applying Lemma 2.3.19, we have

$$\mathsf{WE\text{-}HA}^{\omega} + \widetilde{\Delta} \vdash \exists Y \forall x\, C(x, Yx).$$

Then, a fortiori, $\mathsf{WE\text{-}PA}^{\omega} + \widetilde{\Delta} \vdash$ Uni(S). Taking Δ as $\widetilde{\mathrm{WKL}}'$ (equivalent to UWKL, [55, Lemma 10.32]), in particular, $\widetilde{\Delta} = \Delta$. UWKL gives WKL and so, together with AC (See [55, Lemma 9.20]), KL.

The same proof works for the analogous result for $\widehat{\mathsf{WE\text{-}HA}}^{\omega}\!\restriction$ and $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\!\restriction$.

□

Next, we refine the previous proposition to replace $\mathsf{WE\text{-}HA}^{\omega}$ by the full extensional system $\mathsf{E\text{-}HA}^{\omega}$. For that, we use the elimination of extensionality techniques developed in Subsection 2.3.5. Taking applications into account, we state the refined one in the form of being able to obtain the best possible unprovability results from the results in reverse mathematics.

Proposition 4.2.10. *Let Δ_1^* be the class of sentences $\forall a^{\rho} \exists b \leq_1 ra \forall c^{\tau} B_{qf}(a, b, c)$ (where r is a closed term and ρ, τ are arbitrary types) such that $\widetilde{\Delta_1^*}$ is provable in* $\mathsf{WE\text{-}PA}^{\omega}$+QF-AC+UWKL.

For every statement $\mathrm{S} := \forall x^{\rho} (A(x) \to \exists y^{\tau} B(x, y))$ *of* $\mathcal{L}(\mathsf{E\text{-}HA}^{\omega})$ *where $A(x)$ is purely universal, $B(x, y)$ is in Γ_2 and the types of all variables quantified in S by positively occurring $\forall$ or negatively occurring $\exists$ are not greater than* 1 *(in particular, $\rho \leq 1$), if*

$$\mathsf{E\text{-}HA}^{\omega} + \mathrm{AC!}^1 + \mathrm{AC}^0 + \mathrm{IP}_{\forall}^{\leq 1, \leq 1} + \mathrm{M}^{\leq 1} + \mathrm{KL} + \Delta_1^* \vdash \mathrm{S},$$

then

$$\mathsf{WE\text{-}PA}^{\omega} + \text{QF-AC} + \text{UWKL} \vdash \text{Uni(S)}.$$

This also holds for $\widehat{\mathsf{E\text{-}HA}}^{\omega}\!\restriction$ *and* $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\!\restriction$ *instead of* $\mathsf{E\text{-}HA}^{\omega}$ *and* $\mathsf{WE\text{-}PA}^{\omega}$.

Proof. We may assume that $\mathrm{IP}_{\forall}^{\leq 1, \leq 1}$, $\mathrm{M}^{\leq 1}$ and Δ_1^* are finite, so can form the conjunction of their elements. We also denote them as $\mathrm{IP}_{\forall}^{\leq 1, \leq 1}$, $\mathrm{M}^{\leq 1}$ and Δ_1^* for readability. Take the universal closures of $\mathrm{IP}_{\forall}^{\leq 1, \leq 1}$ and $\mathrm{M}^{\leq 1}$, and denote them as $\overline{\mathrm{IP}_{\forall}^{\leq 1, \leq 1}}$ and $\overline{\mathrm{M}^{\leq 1}}$ respectively (note that $\mathrm{IP}_{\forall}^{\leq 1, \leq 1}$ and $\mathrm{M}^{\leq 1}$ may have parameters of arbitrary type). Then we have

$$\mathsf{E\text{-}HA}^{\omega} + \mathrm{AC!}^1 + \mathrm{AC}^0 \vdash \left(\overline{\mathrm{IP}_{\forall}^{\leq 1, \leq 1}} \wedge \overline{\mathrm{M}^{\leq 1}} \wedge \mathrm{KL} \wedge \Delta_1^* \right) \to \mathrm{S}$$

by the deduction theorem. Applying Lemma 2.3.25, we have

$$\mathsf{WE\text{-}HA}^{\omega} + \mathrm{AC} \vdash \left(\left(\overline{\mathrm{IP}_{\forall}^{\leq 1,\leq 1}}\right)_e \wedge \left(\overline{\mathrm{M}^{\leq 1}}\right)_e \wedge (\mathrm{KL})_e \wedge (\Delta_1^*)_e\right) \to (\mathrm{S})_e,$$

since S has no parameter as well as $\overline{\mathrm{IP}_{\forall}^{\leq 1,\leq 1}}$, $\overline{\mathrm{M}^{\leq 1}}$, KL and Δ_1^*. By a careful inspection with the use of [55, Lemma 10.41] and the restriction of the types of variables in $\mathrm{IP}_{\forall}^{\leq 1,\leq 1}$, $\mathrm{M}^{\leq 1}$ and S, one can intuitionistically show that $\overline{\mathrm{IP}_{\forall}^{\leq 1,\leq 1}} \to \left(\overline{\mathrm{IP}_{\forall}^{\leq 1,\leq 1}}\right)_e$, $\overline{\mathrm{M}^{\leq 1}} \to \left(\overline{\mathrm{M}^{\leq 1}}\right)_e$, $\mathrm{KL} \leftrightarrow (\mathrm{KL})_e$, $\Delta_1^* \to \left(\Delta_1^*\right)_e$ and $(\mathrm{S})_e \to \mathrm{S}$. Therefore we have

$$\mathsf{WE\text{-}HA}^{\omega} + \mathrm{AC} + \overline{\mathrm{IP}_{\forall}^{\leq 1,\leq 1}} + \overline{\mathrm{M}^{\leq 1}} + \mathrm{KL} + \Delta_1^* \vdash \mathrm{S}.$$

Then $\mathsf{WE\text{-}PA}^{\omega}$ + QF-AC + UWKL ⊢ Uni(S) follows from Proposition 4.2.9.

The same proof works for the analogous result for $\widehat{\mathsf{E\text{-}HA}}^{\omega}\!\upharpoonright$ and $\widehat{\mathsf{WE\text{-}PA}}^{\omega}\!\upharpoonright$. □

Remark 4.2.11. *1. Since* $\mathrm{QF\text{-}AC}^{1,0} \leftrightarrow \mathrm{QF\text{-}AC!}^{1,0}$ *over* $\widehat{\mathsf{WE\text{-}HA}}^{\omega}\!\upharpoonright$*, the intuitionistic version* $\mathrm{iWKL}_0^{\omega} := \widehat{\mathsf{E\text{-}HA}}^{\omega}\!\upharpoonright + \mathrm{QF\text{-}AC}^{1,0} + \mathrm{WKL}$ *of* WKL_0^{ω} *in* [54] *is a subsystem of* $\widehat{\mathsf{E\text{-}HA}}^{\omega}\!\upharpoonright + \mathrm{AC!}^1 + \mathrm{AC}^0 + \mathrm{IP}_{\forall}^{\leq 1,\leq 1} + \mathrm{M}^{\leq 1} + \mathrm{KL}$*. That is to say, the contrapositive of the previous proposition yields the unprovability of S in* iWKL_0^{ω} *or* $\mathrm{iWKL}^{\omega} := \mathsf{E\text{-}HA}^{\omega} + \mathrm{QF\text{-}AC}^{1,0} + \mathrm{WKL}$.

2. *All sentences* $\forall x^{\rho} \exists y^{\tau} A_{qf}(x, y)$ *provable in* RCA^{ω}*, where the degree of the type* ρ *is not greater than* 1*, the type* τ *is arbitrary and* A_{qf} *is quantifier-free, are included in* $\mathsf{E\text{-}HA}^{\omega} + \mathrm{AC!}^1 + \mathrm{AC}^0 + \mathrm{IP}_{\forall}^{\leq 1,\leq 1} + \mathrm{M}^{\leq 1} + \mathrm{KL} + \Delta_1^*$ *as in Remark 4.2.4.*

3. *The type restriction for* S *in the previous proposition still covers* Π^1_2 *sentences treated in reverse mathematics.*

Corollary 4.2.12. *Let* Δ_1^* *be the same as in Proposition 4.2.10. For every statement* $\mathrm{S} := \forall x^1 (A(x) \to \exists y^1 B(x, y))$ *of* $\mathcal{L}(\mathsf{E\text{-}HA}^2)$ *where* $A(x)$ *is purely universal and* $B(x, y)$ *is in* Γ_2*, if*

$$\mathsf{E\text{-}HA}^{\omega} + \mathrm{AC!}^1 + \mathrm{AC}^0 + \mathrm{IP}_{\forall}^{\leq 1,\leq 1} + \mathrm{M}^{\leq 1} + \mathrm{KL} + \Delta_1^* \vdash \mathrm{S},$$

then

$$\mathsf{WE\text{-}PA}^{\omega} + \mathrm{QF\text{-}AC} + \mathrm{UWKL} \vdash \mathrm{Seq(S)}\ (:= \forall x^{1(0)} \left(\forall n^0 A(xn) \to \exists y^{1(0)} \forall n B(xn, yn)\right)).$$

Proof. Immediate consequence of Proposition 4.2.10 since Uni(S) derives Seq(S). □

Remark 4.2.13. *The above corollary is applicable to some results in sequential reverse mathematics. Suppose the sequential version of a statement* S *where* $A(x)$ *is purely universal and* $B(x, y)$ *is in* Γ_2 *derives* ACA *over* RCA^{ω}*. Then we have* $\mathsf{WE\text{-}PA}^{\omega} + \mathrm{QF\text{-}AC}^{1,0} + \mathrm{Seq(S)} \vdash \mathrm{ACA}$

by elimination of extensionality [55, Proposition 10.45]. *It is known that* $\mathsf{WE\text{-}PA}^\omega + \mathrm{ACA}$ *proves the totality of the* $\alpha < \epsilon_{\epsilon_0}$*-recursive functions* ([17]). *On the other hand, the provably recursive functions of* $\mathsf{WE\text{-}PA}^\omega + \mathrm{QF\text{-}AC} + \mathrm{UWKL}$ *are the* $\alpha < \epsilon_0$*-recursive functions (See* [55, Corollary 33] *and note that Gödel primitive recursive functionals of type degree* 1 *coincide with provably recursive functions of* PA*). Therefore* Seq(S) *is not provable in* $\mathsf{WE\text{-}PA}^\omega + \mathrm{QF\text{-}AC} + \mathrm{UWKL}$. *By applying the previous corollary, we have the unprovability of* S *in* $\mathsf{E\text{-}HA}^\omega + \mathrm{AC!}^1 + \mathrm{AC}^0 + \mathrm{IP}_\forall^{\leq 1, \leq 1} + \mathrm{M}^{\leq 1} + \mathrm{KL} + \Delta_1^*$. *That is to say, one can think of Corollary 4.2.12 (essentially Proposition 4.2.10) as a kind of higher-order uniformization theorem for* $\mathsf{RCA} + \mathrm{WKL}$ *(*[$\star$] *in table 4.1). As we see in the subsequent sections, nevertheless, one can obtain the much stronger semi-intuitionistic unprovability for such statements by investigating the strength of uniform versions over a weakly extensional classical system like* WRCA^ω.

4.3 Further Extended Metatheorem

The following is our main result in this section.

Theorem 4.3.1. *Let* $\Delta^{\mathcal{M}^\omega}$ *be a set of sentences of the form* Δ *which are true in* $\mathcal{M}^\omega$. *For every statement* $\forall x^\rho \exists y^\tau A(x, y)$ *of* $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$ *in* Γ_2, *if*

$$(i) \qquad \mathsf{WE\text{-}PA}^\omega + \mathrm{QF\text{-}AC} + \mathrm{DC} + \Delta^{\mathcal{M}^\omega} + \exists Y^{\tau(\rho)} \forall x^\rho A(x, Yx) \vdash (\exists^2),$$

then

$$(ii) \qquad \mathsf{WE\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}_\forall^\omega + \mathrm{M}^\omega + \mathrm{BR} + \mathrm{F}^- + \Delta^{\mathcal{M}^\omega} \nvdash \forall x \exists y A(x, y).$$

In particular, if

$$\mathsf{WE\text{-}PA}^\omega + \mathrm{QF\text{-}AC} + \mathrm{DC} + \mathrm{UWKL} + \exists Y^{\tau(\rho)} \forall x^\rho A(x, Yx) \vdash (\exists^2),$$

then

$$\mathsf{WE\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}_\forall^\omega + \mathrm{M}^\omega + \mathrm{UWKL} + \mathrm{KL} + \Sigma_1^0\text{-}\mathrm{UB}^- + \mathrm{BI} \nvdash \forall x \exists y A(x, y).$$

We should note that F^- and Σ_1^0-UB^- are involved primarily for technical reasons (See also Remark 4.3.7). The proof is based on the Dialectica interpretation without extracting terms/bounds (Lemma 4.3.4) and negative translation along with the model $\mathcal{M}^\omega$ of all strongly majorizable functionals.

Remark 4.3.2. *To obtain the conclusion of Theorem 4.3.1 it is* **not enough** *to check that over* $\mathsf{WE\text{-}PA}^\omega + \mathrm{QF\text{-}AC}$ *the uniform version (or even just the sequential version) of* $\forall x^1 \exists y^1 A$ *implies*

ACA*: define*

$$A := \forall f^{1(0)}, \varphi^2, \psi^2 \exists g^1 (f(\varphi(g), \psi(g)) = 0 \to f(\varphi(g), g(\psi(g))) = 0) \in \Gamma_2$$

and add 'dummy variables' to get $\forall x^1 \exists y^1 A$*. Then this statement coincides with its sequential (as well as its full uniform) version and implies (using classical logic and* QF-AC*)* ACA *since it implies*

$$\forall f^{1(0)} \exists g^1 \forall n^0, k^0 (f(n, k) = 0 \to f(n, g(n)) = 0).$$

But A is provable in $\mathsf{WE\text{-}HA}^\omega$ + BR *since it has a functional interpretation in this theory.*

Prior to the proof of Theorem 4.3.1, we first show some lemmas.

Lemma 4.3.3. $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction + \mathrm{M}^0 \vdash \left(\exists^2\right)^q \to \neg\neg(\exists^2)$*, where* $\left(\exists^2\right)^q$ *denotes the negative translation (See Definition 2.3.1) of* $(\exists^2)$*.*

Proof. Reasoning in $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction + \mathrm{M}^0$,

$$\begin{aligned}
(\exists^2)^* &= \exists E \forall f \neg\neg (Ef = 0 \leftrightarrow \exists x (fx = 0)) \\
&\to \exists E \forall f ([Ef = 0 \to \neg\neg \exists x (fx = 0)] \wedge [\exists x (fx = 0) \to \neg\neg Ef = 0]) \\
&\to \exists E \forall f ([Ef = 0 \to \exists x (fx = 0)] \wedge [\exists x (fx = 0) \to Ef = 0]) \quad (\text{using } \mathrm{M}^0) \\
&= (\exists^2).
\end{aligned}$$

Therefore $\left(\exists^2\right)^q = \neg\neg(\exists^2)^* \to \neg\neg(\exists^2)$. □

The next lemma is just the simple variant of Theorem 11.9 in Kohlenbach [55], where we do not insist on the existence of witnessing terms for $(\forall \underline{a} A(\underline{a}))^D$ (nor uniform bounds) and so can add axioms Δ (without having to formalize the majorizability proof of BR as in the monotone functional interpretation).

Lemma 4.3.4 (Soundness of the Dialectica interpretation without extracting terms/bounds)**.** *Let $A(\underline{a})$ be a formula of $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$ containing only $\underline{a}$ free. Then if*

$$\mathsf{WE\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \Delta\, (+\mathrm{BR} + \mathrm{DC}^q) \vdash A(\underline{a}),$$

then

$$\mathsf{WE\text{-}HA}^\omega + \widetilde{\Delta}\, (+\mathrm{BR}) \vdash \left(\forall \underline{a}\, A(\underline{a})\right)^D,$$

where $\left(\forall \underline{a}\, A(\underline{a})\right)^D$ *is the Dialectica interpretation* [55, Definition 8.1] *of* $\forall \underline{a}\, A(\underline{a})$ *and* DC^q *denotes the negative translation of* DC*.*

Proof. As in the proof of the soundness theorem for the Dialectica interpretation extracting terms (Theorem 2.3.17), we proceed by induction on the length of the derivation. Note that our interpretation of $A(\underline{a})$ is not the Dialectica interpretation of $A(\underline{a})$ but that of the universal closure of $A(\underline{a})$. For the axioms of $\mathsf{WE\text{-}HA}^\omega$, AC, $\mathrm{IP}^\omega_\forall$ and M^ω , each induction step immediately follows from the corresponding step in the proof of Theorem 2.3.17. For the rules of $\mathsf{WE\text{-}HA}^\omega$, on the other hand, each induction step follows by imitating the construction to the witness term from the given terms in the corresponding step for Theorem 2.3.17 (using [55, Lemma 3.15] and [55, Remark 3.13.2]). In addition, the interpretation of Δ is $\widetilde{\Delta}$ (here we use that Δ only contains 'sentences'). DC^q is interpreted by BR as in the proof of [55, Theorem 11.9]. □

Remark 4.3.5. *By* [40] *(and* [63]*), classical analysis* $\mathrm{E\text{-}PA}^\omega$+DC *embeds into* $\mathsf{WE\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \mathrm{BI}$ *by (elimination of extensionality and) negative translation.*

The next lemma states that $\Delta^{\mathcal{M}^\omega}$ is closed under $\widetilde{\ }$ transformation. Note that for T of the form Δ, $\widetilde{\mathrm{T}}$ also has the form of Δ.

Lemma 4.3.6. *If a sentence* T *of* $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ *has the form of* Δ *and* $\mathcal{M} \models \mathrm{T}$, *then* $\mathcal{M} \models \widetilde{\mathrm{T}}$ *holds.*

Proof. Note that $\mathrm{T} \to \widetilde{\mathrm{T}}$ is derived from b-AC where b-AC $:\equiv \bigcup_{\rho,\tau}\{\text{b-AC}^{\rho,\tau}\}$ with

$$\text{b-AC}^{\rho,\tau} :\equiv \forall Z^{\tau\rho}\left(\forall x^\tau \exists y \leq_\rho Zx\, A(x,y,Z) \to \exists Y \leq_{\rho\tau} Z \forall x\, A(x, Yz, Z)\right).$$

Since $\mathcal{M}^\omega$ models $\mathsf{E\text{-}PA}^\omega$ + b-AC ([48, Application 3.12.1]), $\mathcal{M} \models \widetilde{\mathrm{T}}$ follows from $\mathcal{M} \models \mathrm{T}$. □

We are now in position to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. Suppose that (*i*) holds but (*ii*) does not hold for some $\forall x \exists y A(x,y) \in \Gamma_2$. Note that F^- has the form Δ. Applying Lemma 4.3.4 to (the negation of) (*ii*), we have

$$\mathsf{WE\text{-}HA}^\omega + \mathrm{BR} + \widetilde{\mathrm{F}^-} + \widetilde{\Delta^{\mathcal{M}^\omega}} \vdash \exists Y \forall x \exists \underline{u} \forall \underline{v} A_D(x, Yx, \underline{u}, \underline{v})$$

where $A^D = \exists \underline{u} \forall \underline{v} A_D(x, y, \underline{u}, \underline{v})$. Since $A(x, Yx)$ is in Γ_2, applying Lemma 2.3.19,

$$\mathsf{WE\text{-}HA}^\omega + \mathrm{BR} + \widetilde{\mathrm{F}^-} + \widetilde{\Delta^{\mathcal{M}^\omega}} \vdash \exists Y \forall x A(x, Yx)$$

follows. Since BR^q, $\left(\widetilde{\mathrm{F}^-}\right)^q$ and $\left(\widetilde{\Delta^{\mathcal{M}^\omega}}\right)^q$ are derived from BR, $\widetilde{\mathrm{F}^-}$ and $\widetilde{\Delta^{\mathcal{M}^\omega}}$ respectively, we have

$$(ii)' \qquad \mathsf{WE\text{-}HA}^\omega + \mathrm{BR} + \widetilde{\mathrm{F}^-} + \widetilde{\Delta^{\mathcal{M}^\omega}} \vdash (\exists Y \forall x A(x, Yx))^q$$

from Lemma 2.3.3. On the other hand, Lemma 2.3.3 applied to (*i*) yields

$$(i)' \qquad \mathsf{WE\text{-}HA}^\omega + \mathrm{QF\text{-}AC} + \mathrm{DC}^q + \Delta^{\mathcal{M}^\omega} + \mathrm{M}^\omega + (\exists Y \forall x A(x, Yx))^q \vdash \left(\exists^2\right)^q.$$

Combining the proofs (i') and (ii)$'$ (note that $\Delta^{\mathcal{M}^\omega}$ is derived from $\widetilde{\Delta^{\mathcal{M}^\omega}}$), we have a proof

$$\mathsf{WE\text{-}HA}^\omega + \text{QF-AC} + \text{DC}^q + \text{M}^\omega + \text{BR} + \widetilde{\text{F}^-} + \widetilde{\Delta^{\mathcal{M}^\omega}} \vdash \left(\exists^2\right)^q .$$

Now we show that this leads to a contradiction. Since we have (μ^2) intuitionistically from ($\exists^2$) by applying QF-AC^{1,0} to the formula $\forall f^1 \exists x^0 (Ex = 0 \to fx = 0)$, $\neg\neg(\exists^2)$ derives $\neg\neg(\mu^2)$ over $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction$ + QF-AC^{1,0}. Together with Lemma 4.3.3, we have

$$\mathsf{WE\text{-}HA}^\omega + \text{QF-AC} + \text{DC}^q + \text{M}^\omega + \text{BR} + \widetilde{\text{F}^-} + \widetilde{\Delta^{\mathcal{M}^\omega}} \vdash \neg\neg(\mu^2).$$

Applying (μ^2) to the functions $\lambda k^0.(\overline{1,k})$, $(\overline{1,k})\left(\mu(\overline{1,k})\right) = 0$ holds for all k. Since $\forall k \forall k' < k (\overline{1,k})(k') = 1$, $\forall k \left(\mu(\overline{1,k}) \geq k\right)$ follows. That is, $\neg\neg\exists\mu\forall k\left(\mu(\overline{1,k}) \geq k\right)$, which is intuitionistically equivalent to

$$\neg\forall\mu\neg\forall k\left(\mu(\overline{1,k}) \geq k\right),$$

follows from $\neg\neg(\mu^2)$. On the other hand, one can easily see that F$^-$ (and a-fortiori $\widetilde{\text{F}^-}$) derives $\forall\chi^2\exists b \forall n \exists z \leq_0 b \left(z = \chi\left(\overline{1,n}\right)\right)$, and hence $\forall\chi\exists b\left(\chi\left(\overline{1,b}\right) < b\right)$ follows.

Thus, we have

$$(iii) \qquad \mathsf{WE\text{-}HA}^\omega + \text{QF-AC} + \text{DC}^q + \text{M}^\omega + \text{BR} + \widetilde{\text{F}^-} + \widetilde{\Delta^{\mathcal{M}^\omega}} \vdash \perp .$$

Using again Lemma 4.3.4 applied to (iii) (note $\widetilde{\widetilde{\text{T}}} = \widetilde{\text{T}}$ for T of the form Δ), we have

$$\mathsf{WE\text{-}HA}^\omega + \text{BR} + \widetilde{\text{F}^-} + \widetilde{\Delta^{\mathcal{M}^\omega}} \vdash \perp .$$

However, $\mathcal{M}^\omega \models \mathsf{WE\text{-}HA}^\omega + \text{BR} + \widetilde{\text{F}^-} + \widetilde{\Delta^{\mathcal{M}^\omega}}$ follows from the facts $\mathcal{M}^\omega \models \mathsf{E\text{-}PA}^\omega + \text{BR}$ ([55, Theorem 11.17]) and $\mathcal{M}^\omega \models \text{F}^-$ (See [50, Proposition 4.6]) via Lemma 4.3.6. This completes the proof of Theorem 4.3.1 in the general case.

In particular, one can take $\Delta^{\mathcal{M}^\omega}$ as $\left\{\widehat{\text{WKL}'}\right\}$. To see this, it suffices to note that $\mathcal{M}^\omega \models \text{WKL}'$ by Lemma 4.3.6. Since the second-order part M_1 of $\mathcal{M}^\omega$ coincides with the class S_1 of all functions from natural numbers to natural numbers, $\mathcal{M}^\omega$ models WKL. Together with the fact $\text{WKL}' \leftrightarrow \text{WKL}$ ([55, Lemma 9.18.2]), we have $\mathcal{M}^\omega \models \text{WKL}'$. Therefore, the final assertion of Theorem 4.3.1 follows from the facts:

- $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction \vdash \widehat{\text{WKL}'} \leftrightarrow \text{UWKL}$ ([55, Lemma 10.32]),
- $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction + \text{AC} + \text{WKL} \vdash \text{KL}$ ([55, Lemma 9.20]),
- $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction + \text{QF-AC}^{1,0} \vdash \text{F}^- \to \Sigma^0_1\text{-UB}^-$ ([55, Proposition 12.6.2]),
- $\mathsf{WE\text{-}HA}^\omega + \text{AC} + \text{IP}^\omega_\forall + \text{M}^\omega + \text{BR} \vdash \text{BI}$ ([40, Theorem 3B]).

□

Remark 4.3.7. *What the proof of Theorem 4.3.1 actually establishes (together with the simple fact that formulas $B \in \Gamma_2$ intuitionistically imply B^q) is that $\mathcal{T} := \mathsf{WE\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \mathrm{BR} + \mathrm{F}^- + \Delta^{\mathcal{M}^\omega}$ is consistent but proves $\neg\forall x \exists y A(x, y)$, i.e. not only $\mathcal{T}$ but no consistent extension of $\mathcal{T}$ proves $\forall x \exists y A(x, y)$ (here we use that DC^q is - via BI - provable in $\mathcal{T}$). Note that F^- is a classically false principle (in the sense of being inconsistent with (μ^2)).*

The next corollary is the most useful form of Theorem 4.3.1 in applications to concrete mathematical principles (note Remark 4.3.9.(2) below).

Corollary 4.3.8. *For a sentence $\mathrm{S} := \forall x^\rho (A(x) \to \exists y^\tau B(x, y))$ of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ where types ρ, τ are arbitrary, $A(x)$ is purely universal and $B(x, y)$ is in Γ_2, if*

$$\mathsf{WE\text{-}PA}^\omega + \mathrm{QF\text{-}AC} + \mathrm{DC} + \mathrm{UWKL} + \mathrm{Uni(S)} \vdash (\exists^2),$$

then

$$\mathsf{WE\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \mathrm{UWKL} + \mathrm{KL} + \Sigma^0_1\text{-}\mathrm{UB}^- + \mathrm{BI} \nvdash \mathrm{S}.$$

Proof. Immediately from Theorem 4.3.1 since S is equivalent to the sentence $\forall x \exists y (A(x) \to B(x, y)) \in \Gamma_2$ in the presence of $\mathrm{IP}^\omega_\forall$. □

Remark 4.3.9.

1. *The previous corollary is false if either $\mathsf{WE\text{-}PA}^\omega$ is replaced by $\mathsf{E\text{-}PA}^\omega$ or $\mathsf{WE\text{-}HA}^\omega$ is replaced by $\mathsf{E\text{-}HA}^\omega$. One can take $S := (0 =_0 0)$ in the first case and take $S := (\exists^2)$ in the second case, since $(\exists^2)$ is provable in $\widehat{\mathsf{E\text{-}HA}}^\omega\!\upharpoonright + \mathrm{M}^\omega + \mathrm{UWKL}$ (*[55, Corollary 10.62]*).*
2. *The previous corollary does not hold for every sentence $\forall x^\rho (A(x) \to \exists y^\tau B(x, y))$ in Γ_2. In fact, $\forall f (\exists y (fy = 0) \to \exists x (fx = 0)) \in \Gamma_2$ is logically valid, but its uniform version (μ^2) derives $(\exists^2)$ over $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\upharpoonright$.*

Next, as in Proposition 4.2.10, we show the variant of Corollary 4.3.8 where $\mathsf{WE\text{-}HA}^\omega$ is replaced by the full extensional system $\mathsf{E\text{-}HA}^\omega$. The remarkable thing in the following corollary is that not only $\Sigma^0_1\text{-}\mathrm{UB}^-$ but even $\Sigma^0_1\text{-}\mathrm{UB}$ is included as well as $\mathrm{BI}^{\leq 1}$ in the extensional semi-intuitionistic system (compare to Proposition 4.2.10).

Corollary 4.3.10. *For every statement $\mathrm{S} := \forall x^\rho (A(x) \to \exists y^\tau B(x, y))$ of $\mathcal{L}(\mathsf{E\text{-}HA}^\omega)$ where $A(x)$ is purely universal, $B(x, y)$ is in Γ_2 and the types of all variables quantified by positively occurring $\forall$ or negatively occurring $\exists$ is not greater than 1 (in particular, $\rho \leq 1$), if*

$$\mathsf{WE\text{-}PA}^\omega + \mathrm{QF\text{-}AC} + \mathrm{DC} + \mathrm{UWKL} + \mathrm{Uni(S)} \vdash (\exists^2),$$

then

$$\mathsf{E\text{-}HA}^\omega + \mathrm{AC!}^1 + \mathrm{AC}^0 + \mathrm{IP}_\forall^{\leq 1, \leq 1} + \mathrm{M}^{\leq 1} + \mathrm{KL} + \Sigma^0_1\text{-UB} + \mathrm{BI}^{\leq 1} \nvdash \mathrm{S}.$$

Proof. Suppose

$$\mathsf{E\text{-}HA}^\omega + \mathrm{AC!}^1 + \mathrm{AC}^0 + \mathrm{IP}_\forall^{\leq 1, \leq 1} + \mathrm{M}^{\leq 1} + \mathrm{KL} + \Sigma^0_1\text{-UB} + \mathrm{BI}^{\leq 1} \vdash \mathrm{S}.$$

Σ^0_1-UB follows from F using QF-AC^{1,0} (and hence with AC!^{1}). Moreover,

$$\mathsf{E\text{-}HA}^\omega + \text{QF-AC}^{1,0} + \mathrm{M}^0 \vdash \mathrm{F}^- \to \mathrm{F}.$$

This follows as in [53, Proposition 3.6] (See also [55, Proposition 12.4]), where this is shown for $\mathsf{E\text{-}PA}^\omega$, since an inspection of the proof shows that only M^0 is needed. Hence

$$\mathsf{E\text{-}HA}^\omega + \mathrm{AC!}^1 + \mathrm{AC}^0 + \mathrm{IP}_\forall^{\leq 1, \leq 1} + \mathrm{M}^{\leq 1} + \mathrm{KL} + \mathrm{F}^- + \mathrm{BI}^{\leq 1} \vdash \mathrm{S}.$$

Since $\mathrm{KL} \leftrightarrow (\mathrm{KL})_e$, $\mathrm{F}^- \to (\mathrm{F}^-)_e$ and $\overline{\mathrm{BI}^{\leq 1}} \to \overline{\mathrm{BI}^{\leq 1}}_e$ over $\mathsf{WE\text{-}HA}^\omega$, as in the proof of Proposition 4.2.10, one can show

$$\mathsf{WE\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}_\forall^\omega + \mathrm{M}^\omega + \mathrm{KL} + \mathrm{F}^- + \mathrm{BI}^{\leq 1} \vdash \mathrm{S}.$$

Therefore the corollary follows from Theorem 4.3.1 analogously to Corollary 4.3.8. □

We conclude this section by briefly mentioning a variant of Theorem 4.3.1: The fact that $\exists Y \forall x A(x, Yx)$ classically implies $(\exists^2)$ is usually a reflection of the fact that $\forall x \exists y A(x, y)$ will intuitionistically imply Π^0_1-LEM or even Σ^0_1-LEM. The latter two principles are not really distinguished in our main theorem as the semi-intuitionistic theory contains Markov's principle by which they are equivalent. Markov's principle is also needed for the negative translation of QF-AC used in the proof to derive (μ^2) from $(\exists^2)$. However, if $\exists Y \forall x A(x, Yx)$ directly implies (μ^2) without the use of QF-AC (which usually will be a consequence of $\forall x \exists y A(x, y)$ intuitionistically implying Σ^0_1-LEM) then we can draw some additional information about strong semi-intuitionistic theories (not containing M^ω though) and can allow $\mathsf{E\text{-}PA}^\omega$ instead of $\mathsf{WE\text{-}PA}^\omega$. We don't state the most general result here but just give a sample:

Proposition 4.3.11. *Let $\forall x^\rho \exists y^\tau A(x, y)$ be a sentence in Γ_1.*
If

$$\mathsf{E\text{-}PA}^\omega + \exists Y \forall x A(x, Yx) \vdash (\mu^2),$$

then

$$\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}} + \mathrm{CA}_{\mathrm{ef}} \nvdash \forall x \exists y A(x, y),$$

where $\mathrm{CA_{ef}}$ *is the scheme of full comprehension (in all finite types) for* $\exists$*-free formulas (See* [55]*).*

Note that $\mathrm{CA_{ef}}$ not only implies UWKL but also e.g.

$$(\tilde{\exists}^2):\ \exists E^2 \forall f^1 (Ef = 0 \leftrightarrow \forall x^0 (fx = 0)).$$

Proof. The proof is similar (but simpler) than that of Theorem 4.3.1 using the monotone modified realizability instead of the Dialectica interpretation and we only sketch it here. By negative translation applied to the premise, we get

$$\mathsf{E\text{-}HA}^\omega + (\exists Y \forall y A(x, Yx))^q \vdash (\mu^2)^q$$

and so

$$\mathsf{E\text{-}HA}^\omega + (\exists Y \forall y A(x, Yx))^q \vdash \neg\neg(\mu^2).$$

The monotone modified realizability ([55, Theorem 7.1] applied to the negation of the conclusion gives (using that $A \in \Gamma_1$)

$$\mathsf{E\text{-}HA}^\omega + \mathrm{CA_{ef}} \vdash \exists Y \forall x A(x, Yx)$$

and so by negative translation

$$\mathsf{E\text{-}HA}^\omega + \mathrm{CA_{ef}} \vdash (\exists Y \forall x A(x, Yx))^q.$$

Hence

$$\mathsf{E\text{-}HA}^\omega + \mathrm{CA_{ef}} \vdash \neg\neg(\mu^2)$$

and so (as in the proof of Theorem 4.3.1)

$$\mathsf{E\text{-}HA}^\omega + \mathrm{CA_{ef}} + \mathrm{F}^- \vdash \perp$$

which contradicts

$$\mathcal{M}^\omega \models \mathsf{E\text{-}HA}^\omega + \mathrm{CA_{ef}} + \mathrm{F}^-.$$

$\square$

4.4 Application

In this section, we discuss the application of our metatheorems for uniform versions. The uniform versions of the following Π^1_2 statements have been investigated in higher-order reverse mathematics.

1. WKL. [52, 54, 66]
2. Intermediate value theorem. [54]
3. The attainment of the maximum principle. [54]
4. Brouwer's fixed point theorem. [54]
5. Weak weak Kőnig's lemma WWKL. [66]
6. Bolzano-Weierstraß theorem. [66]
7. Infinite pigeonhole principle RT(1). [66]

Here continuous functions $\Phi : [0,1] \to \mathbb{R}$ are represented as elements in the Banach space $C[0,1]$ of (equivalence classes of) fast converging (in the uniform norm) sequences of polynomials with rational coefficients (See [68]) which is equivalent to the representation as pairs $(\Phi_r^{1(0)}, \omega^1)$ of objects of type degree 1, where Φ_r represents the restriction of Φ to the dyadic rational numbers in $[0,1]$ and ω is a modulus of uniform continuity, i.e.

$$\forall k^0, l^0, n^0 \left(|r_k -_{\mathbb{Q}} r_l| \leq_{\mathbb{Q}} 2^{-\omega(n)} \to |\Phi_r r_k -_{\mathbb{R}} \Phi_r r_l| \leq_{\mathbb{R}} 2^{-n}\right)$$

for some standard enumeration (r_k) of the dyadic rationals in $[0,1]$. Then the premises of $2, 3, 4$ are formalized as purely universal formulas since $|\Phi_r r_k -_{\mathbb{R}} \Phi_r r_l| \leq_{\mathbb{R}} 2^{-n}$ is purely universal. Note that every functional of type 1 represents a real number in Kohlenbach's representation (See [55, Section 4.1] and also [54]). In fact, it is shown in [54] that even the uniform intermediate value theorem for *uniformly continuous functions with its modulus* derives $(\exists^2)$ over RCA_0^ω. Hence this a fortiori is the case for the uniform intermediate value theorem formulated for codes of pointwise continuous functions as in [68]. In the same manner, each of the uniform versions of $3, 4$ also derives $(\exists^2)$ over RCA_0^ω [54].

For $5, 6, 7$, one has to pay attention to the formalization of uniform versions. A sentence $\forall x(\exists u \forall v A_{qf}(x,u,v) \to \exists y B(x,y))$ (like $5, 6, 7$) is intuitionistically equivalent to $\forall x, u(\forall v A_{qf}(x,u,v) \to \exists y B(x,y))$. But their uniform versions may have different strength as suggested in Section 5.1. Here we call the uniform version of the latter one '*strict*' uniform version. By inspecting the proofs in [66], one can easily see that each of the *strict* uniform versions of $5, 6, 7$ derives $(\exists^2)$ over RCA_0^ω.

Based on these observations along with the fact that $(\exists^2)$ is not provable in RCA^ω, it follows from Proposition 4.2.3 that all of 1-7 are not provable in $\mathsf{E\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}} + \Delta^{\mathsf{RCA}^\omega}_{\mathrm{ef}}$.

Next we turn to discuss some applications of Corollary 4.3.8 and Corollary 4.3.10. As described in Subsection 2.2.2, RCA^ω_0 has the full extensionality scheme (E). To show the unprovability of a Π^1_2 statement in the strong semi-intuitionistic system via Corollary 4.3.8 or Corollary 4.3.10, we have to show that the (strict) uniform version derives $(\exists^2)$ over the **weakly** extensional system. However, some of the proofs in [66] are carried out still over the weakly extensional version WRCA^ω_0 (See Subsection 2.2.2) of RCA^ω_0. In fact, the strict uniform versions of Bolzano-Weierstraß theorem or RT(1) derives $(\exists^2)$ over WRCA^ω_0 respectively, and hence, it follows from Corollary 4.3.8 and Corollary 4.3.10 that they are provable neither in $\mathsf{WE\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \mathrm{UWKL} + \mathrm{KL} + \Sigma^0_1\text{-}\mathrm{UB}^- + \mathrm{BI}$ nor in $\mathsf{E\text{-}HA}^\omega + \mathrm{AC}!^1 + \mathrm{AC}^0 + \mathrm{IP}^{\leq 1, \leq 1}_\forall + \mathrm{M}^{\leq 1} + \mathrm{KL} + \Sigma^0_1\text{-}\mathrm{UB} + \mathrm{BI}^{\leq 1}$. Such other examples will be presented in Section 5.4 below. On the other hand, it immediately follows from Theorem 4.3.1 that the uniform version of WWKL, as well as WKL, does not derive $(\exists^2)$ over WRCA^ω_0 regardless of their formalization (i.e. strict or not). One can actually see that the proofs for $1, 2, 3, 4, 5$ in [54, 66] (i.e. the proofs that their uniform versions imply $\exists^2$) use the extensionality axioms of type 1(1) or 2.

Conversely, Corollary 4.3.10 can be used to show the underivability of $(\exists^2)$ from certain uniform principles over WRCA^ω_0(+UWKL + DC). In fact, one can show that each of $2, 3, 4$ is provable in $\widehat{\mathsf{E\text{-}HA}}^\omega\!\upharpoonright + \mathrm{QF\text{-}AC}^{0,0} + \mathrm{WKL}$ by imitating its uniform proof in WKL_0 (See [68]) respectively. Hence it follows via Corollary 4.3.10 that neither of the uniform versions of $2, 3, 4$ under representing its continuity as uniform continuity with the modulus, derives $(\exists^2)$ over WRCA^ω_0. However, it is still open whether each of the uniform versions of $2, 3, 4$ in the usual sense of continuity derives $(\exists^2)$ over WRCA^ω_0(+UWKL + DC).

5 Reverse Mathematics from the Perspective of Uniformity

In this chapter, we investigate some concrete examples from the viewpoint of uniform provability in the context of classical reverse mathematics. As the framework for our investigation in Section 5.1 and Section 5.3, we employ the second-order system with set-based language [68], but one can imitate the discussion also in the second-order system with function-based language (cf. Section 2.2). We use the standard notation and terminology, and sometimes suppress the tedious formal treatments and state our assertions informally as usual in the literature of classical reverse mathematics. See Simpson's monograph [68] for basic knowledge of (classical) reverse mathematics including techniques for encoding mathematical statements in second-order arithmetic, and see [29] for the basic discussions on cardinality in weak first-order arithmetic and the first-order hierarchy. We recall that $\mathsf{WKL}_0 = \mathsf{RCA}_0 + \mathrm{WKL}$ (weak Kőnig's Lemma) and $\mathsf{ACA}_0 = \mathsf{RCA}_0 + \mathrm{ACA}$ (arithmetical comprehension).

The contents in Section 5.3 and Section 5.4 suggest that our metatheorems in Chapter 3 and Chapter 4 are applicable to a large number of mathematical statements.

Section 5.1 and Section 5.2 are basically from [22], which is a joint work with Keita Yokoyama. The results in Subsection 5.3.1 is basically from [23], which is a joint work with Kojiro Higuchi and Takayuki Kihara. The results in Subsection 5.3.2 are presented here for the first time. Section 5.4 is from [24], which is a joint work with Ulrich Kohlenbach.

5.1 General Remark

Many mathematical statements formalized as Π^1_2 sentences of the form (♠) in Definition 3.1.1 are provable in RCA_0. As already mentioned in Section 1.3, it is revealed that the non-uniformity of some proofs in RCA_0 cannot be avoided by showing that their sequential versions implies some non-constructive principles not provable in RCA. However, the sequential or uniform version may imply such a principle for another reason. In this section, we illustrate this phenomenon by investigating the sequential strength of some concrete examples.

Now we concentrate our attention on Π^1_2 statements having the following syntactical form:

$$(\natural') \qquad \forall X\,(\exists Z\theta(X,Z) \to \exists Y\psi(X,Y))\,,$$

where $\theta(X,Z)$ is arithmetical. Despite the fact that $(\natural')$ is, even in intuitionistic predicate logic[1], equivalent to the following statement:

$$(\natural) \qquad \forall X,Z\,(\theta(X,Z) \to \exists Y\psi(X,Y))\,,$$

the sequential version of $(\natural')$ is occasionally stronger than that of $(\natural)$ even if $\theta(X,Z)$ has a very weak complexity such as Π^0_1. This is caused by the difficulty of obtaining the sequence of Z in $(\natural')$. Using the finite marriage theorem and the bounded Kőnig's lemma, we illustrate this phenomenon. On the other hand, the sequential version of a statement of the form $(\natural')$ is not always stronger than that of $(\natural)$ as we see in the case of the weak weak Kőnig's lemma. The important point is that the sequential form of $(\natural')$ captures the difficulty of obtaining a solution Y from X alone while that of $(\natural)$ captures the difficulty of obtaining a solution Y using both X and Z.

That is to say, whenever we consider the sequential version or the uniform version of a Π^1_2 statement, we must pay attention to the formalization and what information can be used to obtain a solution.[2]

Notation 5.1.1. *As usual, we denote the sequential version of a statement* T *as* Seq(T)*. In addition, we use a prime mark, like* Seq(F′MT)*, to indicate* **which** *assumption of uniformity is dropped by sequentializing.*

The Finite Marriage Theorem The so-called marriage theorem for finite graphs (See Theorem 5.3.1) states that *a finite binary graph (B,G,R) satisfying the Hall condition:*

$$\forall \mathbf{x} \subset_{\mathrm{fin}} B \exists \mathbf{y} \subset_{\mathrm{fin}} G\,(|\mathbf{x}| \leq |\mathbf{y}| \wedge \forall g \in \mathbf{y} \exists b \in \mathbf{x}\,((b,g) \in R))\,,$$

has an injection $M \subseteq R$ from B to G. It is well-known that there is a uniform algorithm to construct an injection from a given finite bipartite graph satisfying the Hall condition, which suggests that the sequential version of the finite marriage theorem is provable in RCA_0. However, it depends on the formalization. We provide the following two formalizations of the finite

[1] This intuitionistic equivalence will be used to show the best possibility of Dorais' uniformization results below (See Section 5.2).

[2] Kohlenbach [54] indicates that the investigation of uniform versions reveals the difference between principles from intuitionistic point of view. On the other hand, our investigation suggests that even the uniform strength of two intuitionistically equivalent statements may be different each other.

marriage theorem.

$$\text{FMT}:\\ \forall B,G,R,k\left(\left(\begin{array}{c}(B,G,R)\text{ is a bipartite graph}\\ \text{which satisfies the Hall condition}\\ \text{and }k\text{ bounds }B\cup G\end{array}\right)\to\exists M\left(\begin{array}{c}M\subseteq R\\ \text{is injective}\end{array}\right)\right),$$

$$\text{F}'\text{MT}:\\ \forall B,G,R\left(\exists k\left(\begin{array}{c}(B,G,R)\text{ is a bipartite graph}\\ \text{which satisfies the Hall condition}\\ \text{and }k\text{ bounds }B\cup G\end{array}\right)\to\exists M\left(\begin{array}{c}M\subseteq R\\ \text{is injective}\end{array}\right)\right),$$

where "k bounds $B\cup G$" denotes that for all $v\in B\cup G$, $v<k$. Note that the premise of $(\ldots\to\ldots)$ in FMT can be written by a purely universal formula.

Proposition 5.1.2.

1. $\mathsf{RCA}_0\vdash$ Seq(FMT).

2. $\mathsf{RCA}_0\vdash$ Seq(F′MT) $\leftrightarrow$ ACA.

Proof. (1) A slight recasting of the proof of the finite marriage theorem in RCA_0 ([34, Theorem 2.1]).

(2) ACA $\vdash$ Seq(F′MT) follows from the fact that the infinite marriage theorem is provable in ACA ([35, Theorem 2.2]). For the reverse direction, it suffices to find the range of an injection $f:\mathbb{N}\to\mathbb{N}$ ([68, Lemma Ⅲ.1.3]). The basic idea is to construct, simultaneously in RCA_0, infinite numbers of finite bipartite graphs $\langle(B_n,G_n,R_n)\rangle_{n\in\mathbb{N}}$ such that the solution of the i-th graph indicates whether i is in the range of f or not. By Σ^0_0 comprehension, take $\langle B_n\rangle_{n\in\mathbb{N}}$ and $\langle G_n\rangle_{n\in\mathbb{N}}$ as

$$b\in B_n\Leftrightarrow b=0\vee f\left(\frac{b-2}{2}\right)=n,$$
$$g\in G_n\Leftrightarrow g=1\vee f\left(\frac{g-3}{2}\right)=n,$$

which means that in addition to the underlying sequence $\{0,1\}_{n\in\mathbb{N}}$ of vertices, the odd numbers are divided into $\{B_n\}_{n\in\mathbb{N}}$ and the even numbers are divided into $\{G_n\}_{n\in\mathbb{N}}$ according to f, and take $\langle R_n\rangle_{n\in\mathbb{N}}$ as

$$\begin{aligned}(b,g)\in R_n\Leftrightarrow{}&(b,g)=(0,1)\\ &\vee\left(b=0\wedge f\left(\frac{g-3}{2}\right)=n\right)\vee\left(g=1\wedge f\left(\frac{b-2}{2}\right)=n\right).\end{aligned}$$

Then it is easy to see that (B_n, G_n, R_n) satisfies the Hall condition for each $n \in \mathbb{N}$. Moreover if n is in the range of f via j, $B_n \cup G_n$ is bounded by $2j + 4$, and otherwise, $B_n \cup G_n$ is bounded by 2. Thus, by Seq(F′MT), there exists a sequence $\langle M_n \rangle_{n \in \mathbb{N}}$ of injections. Define $S := \{n : M_n(0) \neq 1\}$, then S is the range of f by the above construction. □

The previous proposition indicates that ACA is not needed to construct an injection from a finite bipartite graph satisfying the Hall condition, and only used to take a sequence of bounds. In fact, the next proposition follows from the previous proposition immediately. (One can even prove it directly.)

Proposition 5.1.3. *The following assertion* SeqB *is equivalent to* ACA *over* RCA_0.

(SeqB) *For any sequence of sets $\langle X_n \rangle_{n \in \mathbb{N}}$, if X_n is finite for all n, then there exists a function $g : \mathbb{N} \to \mathbb{N}$ such that $g(n)$ bounds X_n.*

Proof. ACA $\vdash$ SeqB is straightforward. For the reverse direction, it suffices to show Seq(F′MT) from SeqB over RCA_0. Let $\langle (B_n, G_n, R_n) \rangle_{n \in \mathbb{N}}$ be a sequence of finite bipartite graphs satisfying the Hall condition. Using SeqB, we have a function $g : \mathbb{N} \to \mathbb{N}$ such that $g(n)$ bounds $B_n \cup G_n$ for all $n \in \mathbb{N}$. Then the existence of a sequence of injections follows from Seq(FMT). □

The Bounded Kőnig's lemma It is known that the bounded Kőnig's lemma, which states that *an infinite tree having a bounding function has an infinite path*, is equivalent to WKL [68, Lemma IV.1.4]. As in the previous section, we provide the two formalizations of it.

$$\mathrm{BKL} : \forall T, g \left(\left(\begin{array}{l} T \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is an infinite tree} \\ \text{and } g : \mathbb{N} \to \mathbb{N} \text{ bounds } T \end{array} \right) \to \exists P \left(\begin{array}{l} P \text{ is an infinite} \\ \text{path of } T \end{array} \right) \right),$$

$$\mathrm{B'KL} : \forall T \left(\exists g \left(\begin{array}{l} T \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is an infinite tree} \\ \text{and } g : \mathbb{N} \to \mathbb{N} \text{ bounds } T \end{array} \right) \to \exists P \left(\begin{array}{l} P \text{ is an infinite} \\ \text{path of } T \end{array} \right) \right),$$

where "g bounds T" denotes that for all $\sigma \in T$ and $i < \mathrm{lh}(\sigma)$, $\sigma(i) < g(i)$. In addition, we now treat a weaker version of the bounded Kőnig's lemma in which a tree in question is bounded by a constant.

$$\mathrm{B_cKL} : \forall T, k \left(\left(\begin{array}{l} T \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is an infinite tree} \\ \text{and } k \text{ bounds } T \end{array} \right) \to \exists P \left(\begin{array}{l} P \text{ is an infinite} \\ \text{path of } T \end{array} \right) \right),$$

$$\mathrm{B'_cKL} : \forall T \left(\exists k \left(\begin{array}{l} T \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is an infinite tree} \\ \text{and } k \text{ bounds } T \end{array} \right) \to \exists P \left(\begin{array}{l} P \text{ is an infinite} \\ \text{path of } T \end{array} \right) \right),$$

where "k bounds T" denotes that for all $\sigma \in T$ and $i < \mathrm{lh}(\sigma)$, $\sigma(i) < k$. Note that the premise of $(\ldots \to \ldots)$ in $\mathrm{B_cKL}$ can be written by a purely universal formula.

Proposition 5.1.4.

1. $\mathsf{RCA}_0 \vdash \mathrm{Seq}(\mathrm{BKL}) \leftrightarrow \mathrm{Seq}(\mathrm{B_cKL}) \leftrightarrow \mathrm{WKL}$.

2. $\mathsf{RCA}_0 \vdash \mathrm{Seq}(\mathrm{B'KL}) \leftrightarrow \mathrm{Seq}(\mathrm{B'_cKL}) \leftrightarrow \mathrm{ACA}$.

Proof. We reason in RCA_0.

(1) WKL implies Seq(WKL) ([36, Lemma 5]), and Seq(WKL) implies Seq(BKL) by imitating the proof of BKL in WKL ([68, Lemma IV.1.4]). The implication from Seq(BKL) to Seq($\mathrm{B_cKL}$) is obvious. That from Seq($\mathrm{B_cKL}$) to WKL follows immediately from the fact that binary trees are bounded by 2.

(2) It is straightforward that ACA implies Seq(B′KL) by imitating the proof of Kőnig's lemma in ACA ([68, Lemma III.7.2]). Seq(B′KL) implies Seq($\mathrm{B'_cKL}$). The implication from Seq($\mathrm{B'_cKL}$) to ACA follows from Lemma 5.1.11 below. □

In the reverse mathematics of analysis, the bounded Kőnig's lemma corresponds to the Heine/Borel compactness of effectively totally bounded complete separable metric spaces. Thus, to consider the strength of a sequential version of a mathematical statement which is related to Heine/Borel compactness, it is important to check which version of bounded Kőnig's lemma is needed. Here, we will consider the maximum principle of continuous functions as an example. The following statement is equivalent to WKL over RCA_0. (See [68, Section IV].)

(MP) For any f, if f is a continuous function from $[-1, 1]$ to $\mathbb{R}$, then there exists $a \in [-1, 1]$ such that

$$\max\{f(x) : x \in [-1, 1]\} = f(a).$$

By an easy consideration, we can see that MP is equivalent to the following.

(MP^+) For any f, if f is a continuous function from $(-1, 1)$ to $\mathbb{R}$ such that $f(0) > 0$ and $\lim_{x\to\pm 1} f(x) = 0$, then there exists $a \in (-1, 1)$ such that

$$\max\{f(x) : x \in (-1, 1)\} = f(a).$$

For the sequential version of MP, the following is well-known, actually, it is an easy consequence of $\mathsf{RCA}_0 \vdash \mathrm{WKL} \leftrightarrow \mathrm{Seq}(\mathrm{WKL})$ ([36, Lemma 5]).

Proposition 5.1.5. Seq(MP) *is equivalent to* WKL *over* RCA_0.

However, the sequential version of MP^+ is strictly stronger than that of MP. (In general, ACA is required to extend a continuous function $f : (-1,1) \to \mathbb{R}$ with $\lim_{x\to\pm1} f(x) = 0$ into a continuous function from $[-1,1]$ to $\mathbb{R}$.)

Proposition 5.1.6. *The following are equivalent over* RCA_0.

1. ACA.
2. *The sequential version of the following statement: for any f, if f is a bounded support continuous function from $\mathbb{R}$ to $\mathbb{R}$, then there exists $a \in \mathbb{R}$ such that* $\max\{f(x) : x \in \mathbb{R}\} = f(a)$. (*Here, f is said to have bounded support if there exists $k \in \mathbb{N}$ such that the closure of* $\{x \in \mathbb{R} : f(x) \neq 0\}$ *is a subset of* $[-k,k]$.)
3. *The sequential version of the following statement: for any f, if f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that $f(0) > 0$ and $\lim_{x\to\pm\infty} f(x) = 0$, then there exists $a \in \mathbb{R}$ such that* $\max\{f(x) : x \in \mathbb{R}\} = f(a)$.
4. $\mathrm{Seq}(\mathrm{MP}^+)$.

Proof. By modifying the proof of MP $\leftrightarrow$ WKL, we can easily see that 2 is equivalent to the sequential version of the following statement: if $T \subseteq \mathbb{N}^{\mathbb{N}}$ is an infinite tree such that $T \subseteq 2k \times 2^{<\mathbb{N}}$ for some k, then T has an infinite path. Note that this is a weaker version of $\mathrm{Seq}(\mathrm{B}'_c\mathrm{KL})$, and still is equivalent to ACA as in the proof of Lemma 5.1.11 below. For a given continuous function f from $\mathbb{R}$ to $\mathbb{R}$ such that $f(0) > 0$ and $\lim_{|x|\to\infty} f(x) = 0$, define a continuous function g as $g(x) = \max\{0, f(x) - f(0)/2\}$. Then, g has bounded support and $\max\{g(x) : x \in \mathbb{R}\} + f(0)/2 = \max\{f(x) : x \in \mathbb{R}\}$, hence we have $2 \leftrightarrow 3$. By an easy rescaling, we have $3 \leftrightarrow 4$. Thus, they are all equivalent to ACA. □

The Weak Weak Kőnig's Lemma The weak weak Kőnig's lemma, which states that *a binary tree with positive measure has an infinite path*, has an intermediate strength between RCA_0 and WKL_0 ([68, Remark X.1.8]). In this case, both of sequential versions are stronger than the pointwise version and actually equivalent to WKL.

$$\mathrm{WWKL} : \forall T, m\left(\left(\begin{array}{c} T \subseteq 2^{<\mathbb{N}} \text{ is a tree and} \\ m \in \mathbb{Q}^+ \text{ satisfies } (\mathrm{W}_2) \end{array}\right) \to \exists P\left(\begin{array}{c} P \text{ is an infinite} \\ \text{path of } T \end{array}\right)\right),$$

$$\mathrm{W'WKL} : \forall T\left(\exists m\left(\begin{array}{c} T \subseteq 2^{<\mathbb{N}} \text{ is a tree and} \\ m \in \mathbb{Q}^+ \text{ satisfies } (\mathrm{W}_2) \end{array}\right) \to \exists P\left(\begin{array}{c} P \text{ is an infinite} \\ \text{path of } T \end{array}\right)\right),$$

where (W_2) denotes

$$\lim_{n\to\infty} \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = n\}|}{2^n} \geq m.$$

Proposition 5.1.7.

1. $\mathsf{RCA}_0 \vdash \mathrm{Seq}(\mathrm{WWKL}) \leftrightarrow \mathrm{WKL}$. ([15, Theorem 4.1.(2)])

2. $\mathsf{RCA}_0 \vdash \mathrm{Seq}(\mathrm{W'WKL}) \leftrightarrow \mathrm{WKL}$.

Proof of 2. It is easy to show within RCA_0 that for binary tree T, if there exists $m \in \mathbb{Q}^+$ such that $\lim_{n\to\infty} \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = n\}|}{2^n} \geq m$, then T is infinite. Therefore WKL $\vdash$ Seq(W′WKL) immediately follows from WKL $\vdash$ Seq(WKL) ([36, Lemma 5]). For the reverse direction, Seq(W′WKL) obviously implies Seq(WWKL), which is equivalent to WKL over RCA_0 from (1). □

Remark 5.1.8. *Note that the previous proposition does not suggest that the sequential strength of a mathematical statement equivalent to* WWKL *is* WKL *in general. Here, we will consider Riemann integrability for bounded functions as an example. The following statement is equivalent to* WWKL *over* RCA_0. (*See [65].*)

(Int) *For any f, if f is a continuous function from* $[0,1]$ *to* $[0,1]$*, then there exists $r \in \mathbb{R}$ such that*

$$\int_0^1 f(x)\,dx = r.$$

However, Seq(Int) *does not imply* WKL*. This is because* Seq(Int) *follows from the following sequential contrapositive of* W′WKL :

$$(A) \qquad \forall T \left(\forall n \begin{pmatrix} T_n \subseteq 2^{<\mathbb{N}} \text{ is a tree} \\ \text{which has no path} \end{pmatrix} \to \forall n \lim_{k\to\infty} \frac{|\{\sigma \in T_n : \mathrm{lh}(\sigma) = k\}|}{2^k} = 0 \right).$$

The contrapositive of W′WKL *does not have the form* (♠) *in Definition 3.1.1 any more and* (A) *is trivially equivalent to* WWKL*. Therefore* Seq(Int) *is actually equivalent to* WWKL*. In fact, for many sequential versions of mathematical statements which are equivalent to* WWKL*, we do not need* Seq(WWKL) *or* Seq(W′WKL) *but* (A).

Next, we will investigate the effect of uniformity for positive measure more precisely. For this, we shall consider some more variants, namely, bounded Kőnig's lemmas with respect to measure.

- WBKL : $\forall T, m, g \left(\begin{pmatrix} T \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is a tree,} \\ m \in \mathbb{Q}^+ \text{ satisfies } (\mathrm{W_g}), \\ g : \mathbb{N} \to \mathbb{N} \text{ bounds } T \end{pmatrix} \to \exists P \begin{pmatrix} P \text{ is an infinite} \\ \text{path of } T \end{pmatrix} \right)$,

  where $(\mathrm{W_g})$ denotes

$$\lim_{n\to\infty} \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = n\}|}{\prod_{i<n} g(i)} \geq m.$$

- $\mathrm{WB_cKL} : \forall T, m, k \left(\left(\begin{array}{l} T \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is a tree,} \\ m \in \mathbb{Q}^+ \text{ satisfies } (\mathrm{W_k}), \\ k \text{ bounds } T \end{array} \right) \to \exists P \left(\begin{array}{l} P \text{ is an infinite} \\ \text{path of } T \end{array} \right) \right),$

  where $(\mathrm{W_k})$ denotes

$$\lim_{n\to\infty} \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = n\}|}{k^n} \geq m.$$

Proposition 5.1.9. WBKL *and* $\mathrm{WB_cKL}$ *are equivalent to* WWKL *over* RCA_0.

Proof. We reason in RCA_0. WBKL to $\mathrm{WB_cKL}$ to WWKL is trivial. We will show WBKL from WWKL. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a tree bounded by $g : \mathbb{N} \to \mathbb{N}$ such that for some $q \in \mathbb{Q}^+$,

$$\lim_{n\to\infty} \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = n\}|}{\prod_{i<n} g(i)} \geq q.$$

For $\sigma \in \mathbb{N}^{<\mathbb{N}}$, define $l_g(\sigma)$ and $r_g(\sigma)$ as follows:

$$l_g(\sigma) = \sum_{k<\mathrm{lh}(\sigma)} \frac{\sigma(k)}{\prod_{i\leq k} g(i)}, \qquad r_g(\sigma) = l_g(\sigma) + \frac{1}{\prod_{i<\mathrm{lh}(\sigma)} g(i)}.$$

Similarly, for $\sigma \in 2^{<\mathbb{N}}$, define $l_2(\sigma)$ and $r_2(\sigma)$ as follows:

$$l_2(\sigma) = \sum_{k<\mathrm{lh}(\sigma)} \sigma(k) 2^{-k-1}, \qquad r_2(\sigma) = l_2(\sigma) + 2^{-\mathrm{lh}(\sigma)}.$$

Note that $\bigcup_{\sigma\in T, \mathrm{lh}(\sigma)=m} [l_g(\sigma), r_g(\sigma)]$ are disjoint intervals in $[0, 1]$ whose lengths sum to the measure of level m of T and these intervals can be approximated arbitrarily well from within by intervals with dyadic rational endpoints. That is, for $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that

$$\frac{\left| \left\{ \sigma \in 2^{<\mathbb{N}} : \begin{array}{l} \mathrm{lh}(\sigma) = N \wedge \\ \exists \tau \in T (\mathrm{lh}(\tau) = m \wedge l_g(\tau) \leq l_2(\sigma) \wedge r_2(\sigma) \leq r_g(\tau)) \end{array} \right\} \right|}{2^N}$$
$$> \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = m\}|}{\prod_{i<m} g(i)} - \frac{q}{2^{m+2}}.$$

We define $h(m)$ as the least such N.

Now we define $T^* \subseteq 2^{\mathbb{N}}$ as

$$\sigma \in T^* \Leftrightarrow$$
$$\forall m < \mathrm{lh}(\sigma)\left(\begin{array}{l} h(m) \leq \mathrm{lh}(\sigma) \to \exists \tau \in T(\mathrm{lh}(\tau) = m \wedge \\ l_g(\tau) \leq l_2(\sigma \restriction h(m)) \wedge r_2(\sigma \restriction h(m)) \leq r_g(\tau)) \end{array}\right).$$

Then, T^* is a tree such that for all $n \in \mathbb{N}$,

$$\frac{|\{\sigma \in T^* : \mathrm{lh}(\sigma) = n\}|}{2^n} > \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = n\}|}{\prod_{i<n} g(i)} - \sum_{m<n} \frac{q}{2^{m+2}} \geq \frac{q}{2}.$$

Thus, by WWKL, there exists a path P^* through T^*. For any $m \in \mathbb{N}$, there exists a unique $\tau_m \in T$ such that $\mathrm{lh}(\tau_m) = m$ and $l_g(\tau_m) \leq l_2(P \restriction h(m)) \wedge r_2(P \restriction h(m)) \leq r_g(\tau_m)$. Then, $P = \bigcup_{m \in \mathbb{N}} \tau_m$ is a path through T. □

Next we investigate the sequential strength of the statements in question. The following proposition means that the uniformity for positive-measure does not help to weaken the sequential strength of the bounded Kőnig's lemma.

Proposition 5.1.10.

1. Seq(W′BKL), Seq(WBKL), Seq(W′$\mathrm{B_c}$KL) *and* Seq(W$\mathrm{B_c}$KL) *are equivalent to* WKL *over* RCA_0.
2. Seq(W′B′KL), Seq(WB′KL), Seq(W′$\mathrm{B'_c}$KL) *and* Seq(W$\mathrm{B'_c}$KL) *are equivalent to* ACA *over* RCA_0.

Here WB′KL, W′BKL, W′B′KL, W$\mathrm{B'_c}$KL, W′$\mathrm{B_c}$KL, *and* W′$\mathrm{B'_c}$KL *are defined in the same manner as before, that is,* W′ *(resp.* B′, $\mathrm{B'_c}$*) means that the universal quantifier* $\forall m$ *(resp.* $\forall g$, $\forall k$*) is moved into* $(\ldots \to \ldots)$ *as the existential quantifier* $\exists m$ *(resp.* $\exists g$, $\exists k$*).*

To show the previous proposition, we first show the following lemma.

Lemma 5.1.11. $\mathsf{RCA}_0 \vdash$ Seq(W$\mathrm{B'_c}$KL) $\to$ ACA, *that is, the following statement implies* ACA *over* RCA_0 :

$$\forall \langle T_n \rangle_{n\in\mathbb{N}}, \langle m_n \rangle_{n\in\mathbb{N}} \left(\forall n \exists k \left(\begin{array}{l} T_n \subseteq \mathbb{N}^{<\mathbb{N}} \text{ is a tree,} \\ m_n \in \mathbb{Q}^+ \text{ satisfies } (\mathrm{W_k}) \text{ for } T_n, \\ k \text{ bounds } T_n \end{array} \right) \right.$$
$$\left. \longrightarrow \exists \langle P_n \rangle_{n\in\mathbb{N}} \forall n \, (P_n \text{ is an infinite path of } T_n) \right).$$

Proof. As in the proof of Proposition 5.1.2.(2), we will find the range of an injection $f : \mathbb{N} \to \mathbb{N}$ ([68, Lemma III.1.3]). By Σ^0_0 comprehension, we take a sequence $\langle T_n \rangle_{n\in\mathbb{N}}$ of trees from the given injection f as

$$\sigma \in T_n \;\Leftrightarrow\; \begin{array}{l} \forall i < \mathrm{lh}(\sigma)\Big(\sigma(0) = 0 \wedge \sigma(i+1) \leq 1 \wedge f(i) \neq n\Big) \vee \\ \exists j < \sigma(0)\Big(\forall i < \mathrm{lh}(\sigma)\,(\sigma(i) \leq 2j+1) \wedge f(j) = n\Big). \end{array}$$

Then, each $T_n \subseteq \mathbb{N}^{<\mathbb{N}}$ clearly forms a tree. Define $m_n :\equiv 1/2$. We need to find a required bound k for each n. For given n, if there exists j such that $f(j) = n$, then define $k := 2j+2$, and otherwise, define $k := 2$. In either case, we can check that k bounds T_n and $m_n (= 1/2)$ satisfies ($\mathrm{W_k}$) for T_n. Thus, by Seq(WB$'_{\mathrm{c}}$KL), there exists a sequence $\langle P_n \rangle_{n\in\mathbb{N}}$ of paths. Define $S := \{n : P_n(0) \neq 0\}$. It is easy to see that $P_n(0) \neq 0 \leftrightarrow \exists j\,(f(j) = n)$, namely, S is the range of f. □

Proof of Proposition 5.1.10. We reason in $\mathsf{RCA_0}$.

(1) Each of Seq(W$'$BKL), Seq(WBKL), Seq(W$'$B$_{\mathrm{c}}$KL), Seq(WB$_{\mathrm{c}}$KL) follows from Seq(BKL), then also from WKL by Proposition 5.1.4.(1). On the other hand, each of them implies Seq(WWKL) which is equivalent to WKL.

(2) Each of Seq(W$'$B$'$KL), Seq(WB$'$KL), Seq(W$'$B$'_{\mathrm{c}}$KL), Seq(WB$'_{\mathrm{c}}$KL) follows from Seq(B$'$KL), then also from ACA by Proposition 5.1.4.(2). On the other hand, each of Seq(W$'$B$'$KL), Seq(WB$'$KL), Seq(W$'$B$'_{\mathrm{c}}$KL) implies Seq(WB$'_{\mathrm{c}}$KL) and Seq(WB$'_{\mathrm{c}}$KL) implies ACA by Lemma 5.1.11. □

5.2 Best Possibility of Dorais' Uniformization Results

In this section, we discuss about the class of formulas which is covered by uniformization theorems described in Section 3.1. Dorais's uniformization theorems (Proposition 3.1.5) are the following:

(1) For any $\mathrm{T} :\equiv \forall f\,(\varphi(f) \to \exists g \psi(f, g))$ such that $\varphi(f)$ is in $\mathrm{N_K}$ and $\psi(f, g)$ is in Γ_{K}, if

$$\mathsf{EL} + \mathrm{GC} + \mathrm{CN} \vdash \mathrm{T},$$

then

$$\mathsf{RCA} \vdash \mathrm{Seq(T)}.$$

(2) For any $\mathrm{T} :\equiv \forall f\,(\varphi(f) \to \exists g \psi(f, g))$ such that $\varphi(f)$ is in $\mathrm{N_L}$ and $\psi(f, g)$ is in Γ_{L}, if

$$\mathsf{EL} + \mathrm{WKL} + \mathrm{GC_L} + \mathrm{CN_L} \vdash \mathrm{T},$$

then

$$\mathsf{RCA} + \mathrm{WKL} \vdash \mathrm{Seq}(\mathrm{T}).$$

As mentioned in [13], an advantage of Dorais' uniformization theorems than Hirst and Mummert's one is the richness of the formula class to which the uniformization theorem is applicable. In fact, the classes Γ_K and Γ_L seems to be rich enough for practical Π^1_2 statements. Then the interest is only in the possibility of extending $\mathrm{N_K}$ and $\mathrm{N_L}$. All purely existential and purely universal formulas are included in $\mathrm{N_K}$ and all formulas of the form $\exists u \leq t \forall v A_{qf}$ are included in $\mathrm{N_L}$. In addition, it should be remarkable that the formulas of the form $\forall u \exists v A_{qf}$ are involved in $\mathrm{N_K}$ (and hence also in $\mathrm{N_L}$). In the following, we show using the investigation in Section 5.1 that $\mathrm{N_K}$ and $\mathrm{N_L}$ cannot be extended to the class involving all formulas of the form $\exists u \forall v A_{qf}$ in Proposition 3.1.5. This reveals that Dorais' uniformization theorems (Proposition 3.1.5) are the best possible for the syntactical classes involved.

Proposition 5.2.1. *In Proposition 3.1.5.(1), the formula class* $\mathrm{N_K}$ *cannot be extended to involve all the formulas of the form* $\exists u \forall v A_{qf}$.

Proof. Suppose not. Since a bounded formula is equivalent to some prime formula in $\mathsf{EL_0}$ (Lemma 2.2.12), the assumption of the finite marriage theorem $\mathrm{F'MT}$ (intuitionistically equivalent to FMT) in Section 5.1 has the form $\exists u \forall v A_{qf}$. Then Proposition 5.1.2.(2) derives that function-based $\mathrm{F'MT}$ is not provable in $\mathsf{EL} + \mathrm{GC} + \mathrm{CN}$. However, it is provable in $\mathsf{EL_0}$ by the standard proof of the finite marriage theorem in $\mathsf{RCA_0}$ ([34, Theorem 2.1]). □

Proposition 5.2.2. *In Proposition 3.1.5.(2), the formula class* $\mathrm{N_L}$ *cannot be extended to involve all the formulas of the form* $\exists u \forall v A_{qf}$.

Proof. Suppose not. As in the proof of Proposition 5.2.1, Proposition 5.1.4.(2) derives that $\mathrm{B'_cKL}$ (intuitionistically equivalent to $\mathrm{B_cKL}$) in Section 5.1 is not provable in $\mathsf{EL} + \mathrm{WKL} + \mathrm{GC_L} + \mathrm{CN_L}$. However, it is provable in $\mathsf{EL_0} + \mathrm{WKL}$ by the standard proof of the bounded Kőnig's lemma in $\mathsf{WKL_0}$ ([68, Lemma IV.1.4]).

□

5.3 Investigation of Sequential Marriage Theorems

A marriage problem[3] is a bipartite graph $\mathbb{G} = (B, G, R)$ which consists of a set of vertices partitioned into B and G and a set of edges such that $R \subset B \times G$. Intuitively, B is a set of boys, G is a set of girls, and $(b, g) \in R$ means boy b knows girl g. Throughout this section, we use

[3] Matrimonial interpretations of Theorem 5.3.1 and 5.3.3 are popularized by Halmos and Vaughan [32].

the notation like $N_{\mathbb{G}}(X)$ for the set of acquaintances of X. The subscript $\mathbb{G}$ may be dropped when it is clear from the context. The following classical theorem states that any finite marriage problem has a solution provided that it satisfies the Hall condition, i.e., every n boys knows at least n girls.

Theorem 5.3.1 (P. Hall [30])**.** *If $\mathbb{G} = (B, G, R)$ is a finite bipartite graph such that for all $X \subset B$, $|N_{\mathbb{G}}(X)| \geq |X|$, then there exists a matching of B.*

Definition 5.3.2. Let $\mathbb{G} = (B, G, R)$ is a (possibly infinite) bipartite graph.

- $\mathbb{G}$ satisfies *the Hall condition* (for B) if $\forall X \subset_{\mathrm{fin}} B(|N_{\mathbb{G}}(X)| \geq |X|)$.
- $\mathbb{G}$ is *B-locally finite* if $\forall b \in B(|N_{\mathbb{G}}(b)| < \infty)$.
- $\mathbb{G}$ has a *solution* if there exists an injection $M \subset R$ from B to G.

Intuitively, B-locally finite means that every boy knows at most finitely many girls. Theorem 5.3.1 can be extended to infinite graphs, but it requires adding B-locally finiteness in assumption.[4]

Theorem 5.3.3 (**Infinite marriage theorem**, M. Hall [31])**.** *If $\mathbb{G} = (B, G, R)$ is a bipartite graph which is B-locally finite and satisfies the Hall condition, then $\mathbb{G}$ has a solution.*

In the early age of recursive graph theory (cf. [26]), Manaster and Rosenstein [64] found that a *computable* bipartite graph satisfying the conditions need not have a *computable* solution, even if its locally finiteness is *computably* confirmed. To render the marriage theorem *computable*, Kierstead [44] introduced the notion of expanding Hall condition, which indicates that the difference between $|N_{\mathbb{G}}(X)|$ and $|X|$ tends to infinity as $|X|$ tends to infinity, where X ranges over all finite subsets of B. Then, he found that there is an effective procedure to obtain a solution from a computable bipartite graph which is computably locally finite and satisfies the computable expanding Hall condition.

Based on these facts, Hirst [35] (See also [34]) investigated marriage theorems in the context of reverse mathematics and showed that the infinite marriage theorem is equivalent to ACA over RCA_0. Moreover, he showed that the infinite marriage theorem under the assumption of computably locally finiteness is equivalent to WKL over RCA_0. Furthermore, Fujiwara, Higuchi and Kihara [23] indicated that Kierstead's effective variant of infinite marriage theorem [B$''$, G$''$, H$''$] (so symbolized in [23]) is provable in RCA_0 and investigated all of the considerable marriage theorems in the context of reverse mathematics. As we have already mentioned in Remark 3.4.4, one can apply Theorem 3.4.2 to [B$''$, G$''$, H$''$], and obtains that [B$''$, G$''$, H$''$] is intuitionistically provable. Other investigations of marriage theorems with respect to reverse mathematics can be found in [11, 38].

[4] If "B-locally finite" is dropped, the assertion does not hold.

5.3.1 Constant Bounded Marriage Theorems

While Kierstead's computable expanding Hall condition makes a computable marriage problem have a computable solution, we introduce another combinatorial condition which also achieves this.

Definition 5.3.4. A bipartite graph $\mathbb{G} = (B, G, R)$ satisfies *the constant bounded Hall condition* (expressed by $\mathrm{H_{cb}}$) if there exists k such that for all $X \subset_{\mathrm{fin}} B$, $|X| \leq |N_{\mathbb{G}}(X)| \leq |X| + k$ holds.

Note that the B-locally finiteness follows from this condition. In fact, if a computable marriage problem fulfills the constant bounded Hall condition, then this problem will have a computable solution. However, our results below reveals that there is no uniform algorithm to obtain a solution from a computable marriage problem fulfilling the constant bounded Hall condition.

Definition 5.3.5. A bipartite graph $\mathbb{G} = (B, G, R)$ is *computably B-locally finite* if there is a function $f : B \to \mathbb{N}$ such that $f(b) = |N_{\mathbb{G}}(b)|$ for all $b \in B$. The *G-locally finiteness* (cf. Definition 5.3.2) and *computably G-locally finiteness* are defined in the same manner.

Hereafter, we use the following symbols for readability:

- X : no locally finiteness for X, for $X \in \{\mathrm{B}, \mathrm{G}\}$,
- X' : X-locally finite, for $X \in \{\mathrm{B}, \mathrm{G}\}$,
- X'' : computably X-locally finite, for $X \in \{\mathrm{B}, \mathrm{G}\}$,
- $\mathrm{H_{cb}}$: the constant bounded Hall condition.

Then we investigate all marriage theorems having the following form:

Statement ($\mathrm{B}^{(\cdot)}_{\mathrm{H_{cb}}}\mathrm{G}^{(\cdot)}$-M)**.** If a bipartite graph $\mathbb{G}$ satisfies $\mathrm{B}^{(\cdot)}$, $\mathrm{G}^{(\cdot)}$ and the constant bounded Hall condition, then $\mathbb{G}$ has a solution.

In the proofs below, we often use (implicitly in many cases) the following fact:

Lemma 5.3.6. $\mathsf{RCA}_0 \vdash$ FPP(finite pigeonhole principle) : *for all $n \in \mathbb{N}$, there is no (code of) injection from* $\{0, \ldots, n+1\}$ *to* $\{0, \ldots, n\}$.

Proof. We reason within RCA_0. Suppose not. Take the least n such that there exists a (code of the graph of) injection r from $\{0, ..., n+1\}$ to $\{0, ..., n\}$ (this is possible only by Σ^0_0 induction). If $n = 0$, this contradicts the injectivity of r. Let $n > 0$. In the case that there is no $i < n+1$ such that $r(i) = n$. Then $r \setminus (n+1, r(n+1))$ is an injection from $\{0, \ldots, n\}$ to $\{0, \ldots, n-1\}$. This contradicts the leastness of n. Otherwise, define $r' : \{0, \ldots, n\} \to \{0, \ldots, n-1\}$ as

$$r'(i) := \begin{cases} r(i) & \text{if } r(i) \neq n, \\ r(n+1) & \text{if } r(i) = n. \end{cases}$$

Then it is straightforward to see that r' is an injection from $\{0, \ldots, n\}$ to $\{0, \ldots, n-1\}$. Again this contradicts the leastness of n. $\square$

In contrast to the finite pigeonhole principle, the infinite pigeonhole principle is not provable in RCA_0 as follows:

Lemma 5.3.7 (Theorem 6.4 in [34]). $\mathsf{RCA}_0 \vdash \mathrm{B}\Pi^0_1 \leftrightarrow \mathrm{RT}(1)$(infinite pigeonhole principle) : *for all $k \in \mathbb{N}$ and function $f : \mathbb{N} \to k$ (i.e. $\{0, \ldots, k-1\}$), there exists $i < k$ such that $f(j) = i$ for infinitely many j.*

We start with proving that all of the marriage theorems with the constant bounded Hall condition are provable in RCA_0.

Theorem 5.3.8. $\mathsf{RCA}_0 \vdash \mathrm{B}_{\mathrm{H_{cb}}}\text{G-M}$ *(equivalently* $\mathrm{B}'_{\mathrm{H_{cb}}}\text{G-M}$*), that is, the following is provable within* RCA_0. *If $\mathbb{G}$ is a bipartite graph which satisfies the constant bounded Hall condition, then $\mathbb{G}$ has a solution.*

Proof. We reason in RCA_0. Let

$$\Phi(c) \equiv \forall X \subset_{\mathrm{fin}} B \left(|N_{\mathbb{G}}(X)| \leq |X| + c \right).$$

Note that the statement $|N_{\mathbb{G}}(X)| \leq |X| + c$ is written as a Π^0_1 formula, and then so is $\Phi(c)$. Since $\mathbb{G}$ satisfies the constant bounded Hall condition, $\exists c \Phi(c)$ holds. By Π^0_1 least number principle, which can be carried out in RCA_0, there exists a least c_1 such that $\Phi(c_1)$ holds. By the leastness of c_1, there exists $X_1 \subset_{\mathrm{fin}} B$ such that $|N_{\mathbb{G}}(X_1)| = |X_1| + c_1$. We fix such X_1. Then the set $N_{\mathbb{G}}(X_1)$ exists by Σ^0_0 comprehension, and $|N_{\mathbb{G}}(X_1)| < \infty$. We first note the following:

$$\text{For all } b \in B \setminus X_1, \text{ there is at most one } g \in N_{\mathbb{G}}(b) \setminus N_{\mathbb{G}}(X_1) \tag{5.1}$$

since if not, $|N_{\mathbb{G}}(X_1 \cup \{b\})| \geq |X_1 \cup \{b\}| + c_1 + 1$. Moreover, we claim that

$$X_2 := \{b \in B \setminus X_1 : N_{\mathbb{G}}(b) \subseteq N_{\mathbb{G}}(X_1)\}$$

is finite, hence exists by bounded Π^0_1 comprehension in RCA_0 [68, Theorem II.3.9]. Indeed, X_2 has at most c_1 many elements. Otherwise, for such a finite set X' of size $c_1 + 1$ with $N_{\mathbb{G}}(X') \subseteq N_{\mathbb{G}}(X_1)$, we have $|X_1 \cup X'| \geq |X_1| + c_1 + 1 > |N_{\mathbb{G}}(X_1)| = |N_{\mathbb{G}}(X_1 \cup X')|$. Next, we claim that

$$Y_1 := \{g \in G \setminus N_{\mathbb{G}}(X_1) : |N_{\mathbb{G}}(g) \setminus X_1| \geq 2\}$$

is finite (actually, of size at most c_1), and exists by bounded Σ^0_1 comprehension in RCA_0 [68, Theorem II.3.9]. Suppose not. Then there exists a finite set Y' of such girls such that $|Y'| = c_1+1$.

Moreover, by (5.1) with $Y' \cap N_{\mathbb{G}}(X_1) = \emptyset$, for every different $g_1, g_2 \in Y'$, two sets $N_{\mathbb{G}}(g_1) \setminus X_1$ and $N_{\mathbb{G}}(g_2) \setminus X_2$ are disjoint. It follows that $N_{\mathbb{G}}(Y') \geq 2|Y'| = 2(c_1 + 1)$ holds. Let X' be a finite subset of $N_{\mathbb{G}}(Y')$ such that $|X'| \geq 2(c_1 + 1)$. By (5.1), each boy in X' knows just one girl in Y'. Therefore,

$$|N_{\mathbb{G}}(X_1 \cup X')| \leq |N_{\mathbb{G}}(X_1)| + |Y'| = (|X_1| + c_1) + (c_1 + 1) = |X_1| + 2c_1 + 1.$$

On the other hand,

$$|X_1 \cup X'| = |X_1| + |X'| \geq |X_1| + 2c_1 + 2.$$

These contradict the Hall condition.

Now note that the condition (5.1) implies $N_{\mathbb{G}}(N_{\mathbb{G}}(Y_1)) \subseteq N_{\mathbb{G}}(X_1) \cup Y_1$. Therefore, we have

$$\begin{aligned}|X_1| + |N_{\mathbb{G}}(Y_1)| &= |X_1 \cup N_{\mathbb{G}}(Y_1)| \leq |N_{\mathbb{G}}(X_1 \cup N_{\mathbb{G}}(Y_1))| \\ &\leq |N_{\mathbb{G}}(X_1) \cup Y_1| = |N_{\mathbb{G}}(X_1)| + |Y_1| \leq (|X_1| + c_1) + c_1 = |X_1| + 2c_1.\end{aligned}$$

Hence, $N_{\mathbb{G}}(Y_1)$ has at most $2c_1$ many elements, and $N_{\mathbb{G}}(Y_1)$ exists by Σ^0_0 comprehension. Moreover, $|X_1 \cup X_2 \cup N_{\mathbb{G}}(Y_1)|$ is finite. On the other hand, $N_{\mathbb{G}}(X_1 \cup X_2 \cup N_{\mathbb{G}}(Y_1)) \subseteq N_{\mathbb{G}}(X_1) \cup Y_1$ holds by our choice of X_1, X_2 and Y_1. Thus the following finite subgraph

$$(X_1 \cup X_2 \cup N_{\mathbb{G}}(Y_1), N_{\mathbb{G}}(X_1) \cup Y_1, R)$$

of $\mathbb{G}$ satisfies the Hall condition with the aid of the Hall condition for the original graph $\mathbb{G}$. Then it has a matching M by the finite marriage theorem in RCA_0 ([35, Theorem 2.1]). Again by (5.1), each boy $b \notin X_1 \cup X_2 \cup N_{\mathbb{G}}(Y_1)$ knows just one girl $g_b \notin N_{\mathbb{G}}(X_1) \cup Y_1$. Moreover, for any such boys b and b', if $b \neq b'$, then $g_b \neq g_{b'}$, since $b, b' \notin N_{\mathbb{G}}(Y_1)$. Therefore

$$M \cup \{(b, g_b) \in R : b \in B \setminus (X_1 \cup X_2 \cup N_{\mathbb{G}}(Y_1))\}$$

is a solution of $\mathbb{G}$. This completes the proof of our theorem. □

Consequently, all of $\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}'$-M, $\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}''$-M, $\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}$-M, $\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}'$-M and $\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}''$-M are probable in RCA_0. As a corollary, it follows that a computable bipartite graph fulfilling the constant bounded Hall condition has a computable solution. However, the algorithm in the proof of Theorem 5.3.8 to give a solution for a given instance of $\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}$-M is *not* uniform, in contrast to the uniformity of the algorithm in the proof of Kierstead's effective marriage theorem [44].

Our proof of Theorem 5.3.8 in RCA_0 contains an implicit non-uniformity in the use of least number principle. The next theorem suggests that this non-uniformity can not be avoided. As before, we use a notation Seq(A) for the sequential version of A.

Theorem 5.3.9. *The following are pairwise equivalent over* RCA_0.

1. ACA.

2. $\mathrm{Seq}(\mathrm{B}'_{\mathrm{H_{cb}}}\text{G-M})$*, that is, for all sequence* $\langle B_n, G_n, R_n, k_n\rangle_{n\in\mathbb{N}}$ *such that* (B_n, G_n, R_n) *satisfies the* B_n*-constant bounded Hall condition via* k_n*, then there exists a sequence* $\langle M_n\rangle_{n\in\mathbb{N}}$ *of their solutions.*

3. $\mathrm{Seq}(\mathrm{B}'_{\mathrm{H_{cb}}}\text{G}'\text{-M})$*, that is, for all sequence* $\langle B_n, G_n, R_n, k_n\rangle_{n\in\mathbb{N}}$ *such that* (B_n, G_n, R_n) *is* G_n*-locally finite and satisfies the* B_n*-constant bounded Hall condition via* k_n*, then there exists a sequence* $\langle M_n\rangle_{n\in\mathbb{N}}$ *of their solutions.*

Warning (See Section 5.1)**.** In our formalization of the sequential versions of $\mathrm{B}'_{\mathrm{H_{cb}}}\text{G-M}$ and $\mathrm{B}'_{\mathrm{H_{cb}}}\text{G}'\text{-M}$, the sequence of k_n is given on ahead. This is the suitable way of sequentializing because our interest is in the non-uniformity of the construction of a solution from given constant bounded Hall condition via k.

Prior to the proof of Theorem 5.3.9, we first prove the following WKL counterpart:

Theorem 5.3.10. *The following are pairwise equivalent over* RCA_0.

1. WKL.

2. $\mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\text{G-M})$*, that is, for all sequence* $\langle B_n, G_n, R_n, p_n, k_n\rangle_{n\in\mathbb{N}}$ *such that* (B_n, G_n, R_n) *is computably* B_n*-locally finite via* p_n *and satisfies the* B_n*-constant bounded Hall condition via* k_n*, then there exists a sequence* $\langle M_n\rangle_{n\in\mathbb{N}}$ *of their solutions.*

3. $\mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\text{G}'\text{-M})$*, that is, for all sequence* $\langle B_n, G_n, R_n, p_n, k_n\rangle_{n\in\mathbb{N}}$ *such that* (B_n, G_n, R_n) *is computably* B_n*-locally finite via* p_n*,* G_n*-locally finite and satisfies the* B_n*-constant bounded Hall condition via* k_n*, then there exists a sequence* $\langle M_n\rangle_{n\in\mathbb{N}}$ *of their solutions.*

Proof. $(1 \to 2)$ holds by the facts that $\mathrm{WKL} \vdash \mathrm{B}''_{\mathrm{H}}\text{G-M}$ ([35, Theorem 2.3]) and that $\mathsf{RCA}_0 \vdash \mathrm{WKL} \leftrightarrow \mathrm{Seq}(\mathrm{WKL})$ ([36, Lemma 5]). $(2 \to 3)$ is trivial. We shall show $(3 \to 1)$. It is suffices to separate the range of disjoint functions ([68, Lemma IV.4.4]). Let $f, g : \mathbb{N} \to \mathbb{N}$ be given injections with disjoint ranges.

We construct a sequence of bipartite graphs $\langle (B_n, G_n, R_n)\rangle_{n\in\mathbb{N}}$ in RCA_0. For each $n \in \mathbb{N}$, put $B_n = G_n = \mathbb{N}$. At first, $(0, 0)$ and $(0, 1)$ are enumerated into each R_n. At the j-th step in the construction of R_i, if $f(j) = i$ occurs, then put $(j + 1, 1) \in R_i$. If $g(j) = i$ occurs, then put $(j + 1, 0) \in R_i$. Otherwise, put $(j + 1, j + 2) \in R_i$.

We put $\langle p_n\rangle_{n\in\mathbb{N}} := \langle p\rangle_{n\in\mathbb{N}}$ where $p : \mathbb{N} \to \mathbb{N}$ such that $p(n) = n + 1$ and $\langle k_n\rangle_{n\in\mathbb{N}} := \langle 1\rangle_{n\in\mathbb{N}}$ in RCA_0. Then each i graph (B_n, G_n, R_n) is G_n-locally finite and computably B_n-locally finite via p_n, and it is also easy to see that for all n and $X \subset_{\text{fin}} B_n$, $|X| \le |R_n[X]| \le |X| + k_n$ holds within RCA_0. Then Seq($\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}'$-M) implies the existence of a sequence $\langle M_i \rangle_{i\in\mathbb{N}}$ of solutions for $\langle (B_n, G_n, R_n) \rangle_{n\in\mathbb{N}}$. Define $V := \{i : (0,0) \in M_i\}$ by Σ^0_0 comprehension. Then V separates the ranges of f and g because of the above construction. ◻

Proof of Theorem 5.3.9. ($1 \to 2$) is shown straightforwardly by revising the proof of $\mathsf{ACA} \vdash \mathrm{B}'_{\mathrm{H}}\mathrm{G}$-M by Hirst ([35, Theorem 2.2]) a bit. ($2 \to 3$) is trivial. We show ($3 \to 1$) by revising a proof of ($3 \to 1$) of Theorem 5.3.10 by using "liberation method" as in the proofs of Lemma 2.3 and Lemma 2.6 in [23].

Let $f : \mathbb{N} \to \mathbb{N}$ be an injection and for each $n \in \mathbb{N}$, put $B_n = G_n = \mathbb{N}$. At first, put $(0,0), (0,1), (1,0) \in R_n$. At the j-th step in the construction of R_i, if $f(j) = i$ occurs, then put $(j+2, 1), (1, j+2) \in R_i$. Otherwise, put $(j+2, j+2) \in R_i$. Then $\langle B_n, G_n, R_n, 1 \rangle_{n\in\mathbb{N}}$ satisfies our assumptions, so has a sequence $\langle M_n \rangle_{n\in\mathbb{N}}$ of solutions by Seq($\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}'$-M). It is easy to see that $V := \{i : (0,0) \in M_i\}$ is the range of f. ◻

By inspecting the proofs of Theorem 5.3.9 and Theorem 5.3.10, one soon notices that if the Hall condition is bounded just by $k = 1$, there is no uniform algorithm to obtain a solution of the marriage problem. In contrast, under the assumption of computably G-locally finiteness, even if the constant bound of the Hall condition is arbitrarily big k, the marriage problem is solvable uniformly in RCA_0.

Theorem 5.3.11. $\mathsf{RCA}_0 \vdash$ Seq($\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}''$-M)*, that is, the following is provable in* RCA_0. *For all sequence $\langle B_n, G_n, R_n, p_n, k_n \rangle_{n\in\mathbb{N}}$ such that (B_n, G_n, R_n) is computably G_n-locally finite via p_n and satisfies the B_n-constant bounded Hall condition via k_n, then there exists a sequence $\langle M_n \rangle_{n\in\mathbb{N}}$ of their solutions.*

We first introduce some notions used in the proof of Theorem 5.3.11.

Definition 5.3.12 (R-chain, Properness)**.** Let (B, G, R) be an infinite bipartite graph.

1. A finite sequence $s = \langle s^B_j, s^G_j \rangle_{j<k}$ is a *R-chain with starting point b of length k* (> 0) if $\langle s^B_j \rangle_{j<k}$ and $\langle s^G_j \rangle_{j<k}$ are nondecreasing sequences of finite subsets of B and G respectively, where $s^B_0 = \{b\}$, $s^G_j \subseteq N_{\mathbb{G}}(s^B_j)$, $s^B_{j+1} = N_{\mathbb{G}}(s^G_j)$, and (s^B_j, s^G_j, R) satisfies the Hall condition for each $j < k$.

2. A R-chain $\langle s^B_j, s^G_j \rangle_{j<k}$ is called *proper* if $\langle s^B_j \rangle_{j<k}$ is strictly increasing.

Proof of Theorem 5.3.11. To remove the illegibility from sequentializing a formal proof, we just give, in RCA_0, a uniform proof of $\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}''$-M for a graph of which B is infinite. It can be straightforwardly transformed to the proof of Seq($\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}''$-M) in RCA_0. Let $\mathbb{G} = (B, G, R)$ be a bipartite graph which is computably G-locally finite and satisfies the constant bounded Hall condition via k, and $\{b_i : i \in \mathbb{N}\}$ be an enumeration of B. We shall now construct a solution of $\mathbb{G}$ by a procedure like primitive recursion.

Let $\theta(u, v)$ express that u encodes a sequence $\langle u_i\rangle_{i<v+1}$ of length $v+1$ of chains $u_i = \langle u^B_{i,j}, u^G_{i,j}\rangle_{j<lh(u_i)}$, where each u_i is a least non-proper R_i-chain of finite length in the remaining graph $(B_i, G_i, R_i) := (B, G, R) \setminus \bigcup_{i'<i} u_{i'}$ and b_i is contained in $\bigcup_{i'\leq i} u^B_{i'}$. Now $\theta(u, v)$ is written as Σ^0_0 formula with the aid of the computably G-locally finiteness of $\mathbb{G}$.

Suppose that we have shown $\forall v \exists u \theta(u, v)$. Then the witness u^v for each v is unique and u^{v_1} is an initial segment of u^{v_2} for $v_1 \leq v_2 \leq v$ because of the minimality of each u_i in the description of $\theta(u, v)$. Take a function which outputs the unique u^v for each $v \in \mathbb{N}$ by Δ^0_1 comprehension as in the proof of [68, Theorem II.3.4], and take (by Σ^0_0 comprehension) a function $g : \mathbb{N} \to \mathbb{N}$ as $g(v)$ is the least matching of $((u^v)_v)^B$ in $R \setminus \bigcup_{i<v}(u^v)_i$. Since the description of $\theta(u, v)$ ensures the Hall condition for each subgraph $(u^v)_v$ in each remaining graph, by the finite marriage theorem ([35, Theorem 2.1]), this g is well-defined. Define M as $\bigcup_{v\in\mathbb{N}} g(v)$, then one can straightforwardly verify in RCA_0 that M is an injection from B to G. Therefore, it suffices to show $\forall v \exists u \theta(u, v)$ by Σ^0_1 induction on v. To show $\exists u \theta(u, 0)$, we first show the following key claim.

Claim (RCA_0)**.** If a bipartite graph $\mathbb{G} = (B, G, R)$ satisfies the constant bounded Hall condition via $k \in \mathbb{N}$, then there is no proper R-chain $s = \langle s^B_i, s^G_i\rangle_{i<lh(s)}$ (with a starting point from B) of length more than $t(k + 1)$, where $t(k) := k(k + 3)/2$.

(Proof of Claim.) Suppose not, i.e., assume that $s = \langle s^B_i, s^G_i\rangle_{i<lh(s)}$ be a proper R-chain of length more than $t(k+1)$. Note that $t(k+1)-(t(k)+1) = k+1$. Now we shall show that for all $n \leq k+1$ there exists $X \subseteq s^B_{t(n)}$ and $Y \subseteq s^G_{t(n)}$ such that $Y \subseteq N_{\mathbb{G}}(X)$ and $|X| + n \leq |Y|$ holds by induction on n. Note that the above statement can be written as Σ^0_0 sentence by using s, then this induction can be carried out in our system RCA_0. The initial step is accomplished obviously. Let X_n and Y_n be witnesses of the case of n, i.e., $X_n \subseteq s^B_{t(n)}$, $Y_n \subseteq s^G_{t(n)}$, $Y_n \subseteq N_{\mathbb{G}}(X_n)$ and $|X_n| + n \leq |Y_n|$ hold. By properness of R-chain, we can choose $g_j \in s^G_j \setminus s^G_{j-1} \neq \emptyset$ for each $t(n) < j \leq t(n + 1)$.

In the case that $s^B_{t(n)+1} \cap N_{\mathbb{G}}(g_{j_1}) = \emptyset$ for some $t(n) + 1 < j_1 \leq t(n + 1)$, $N_{\mathbb{G}}(s^B_{t(n)+1}) \cap \{g_{j_1}\} = \emptyset$ holds. Then, $g_{j_1} \in s^G_{j_1} \subseteq N_{\mathbb{G}}(s^B_{j_1})$ implies that there is $\hat{b} \in s^B_{j_1} \setminus s^B_{t(n)+1}$ such that $g_{j_1} \in N_{\mathbb{G}}(\hat{b})$. Now $\hat{b} \in s^B_{j_1} \setminus s^B_{t(n)+1} = N_{\mathbb{G}}(s^G_{j_1-1}) \setminus N_{\mathbb{G}}(s^G_{t(n)})$ implies that there is $\hat{g} \in s^G_{j_1-1} \setminus s^G_{t(n)}$ such that $\hat{b} \in N_{\mathbb{G}}(\hat{g})$. As $g_{j_1} \notin s^G_{j_1-1}$, the girls g_{j_1} and $\hat{g}$ are different, and they are not contained in $s^G_{t(n)}$. Hence, the boy $\hat{b} \notin X_n \subseteq s^B_{t(n)}$ knows two different girls $g_{j_1}, \hat{g} \notin Y_n \subseteq s^G_{t(n)}$. Therefore, for $X_{n+1} = X_n \cup \{\hat{b}\}$ and $Y_{n+1} = Y_n \cup \{g_{j_1}, \hat{g}\}$, we have $Y_{n+1} \subseteq N_{\mathbb{G}}(X_{n+1})$ and $|X_{n+1}| + n + 1 \leq |Y_n|$.

Otherwise, i.e., $s^B_{t(n)+1} \cap N_{\mathbb{G}}(g_j) \neq \emptyset$ for every $t(n)+1 < j \leq t(n+1)$. Choose $x_j \in s^B_{t(n)+1} \cap N_{\mathbb{G}}(g_j)$ for each $t(n) + 1 < j \leq t(n + 1)$, and put $\hat{X} = \{x_j\}_{t(n)+1<j\leq t(n+1)}$. Since $(s^B_{t(n)+1}, s^G_{t(n)+1})$ satisfies the Hall condition, there exists $\hat{Y} \subseteq s^G_{t(n)+1}$ such that $|\hat{X}| \leq |\hat{Y}|$ holds. Then we can verify that $\hat{X}$ and $\hat{Y} \cup \{g_j\}_{t(n)+1<j\leq t(n+1)}$ are witnesses of the case of $n + 1$ straightforwardly. Therefore the induction step is also accomplished. Then there exists $X \subseteq s^B_{t(k)}$ and $Y \subseteq s^G_{t(k)}$ such that $Y \subseteq N_{\mathbb{G}}(X)$ and

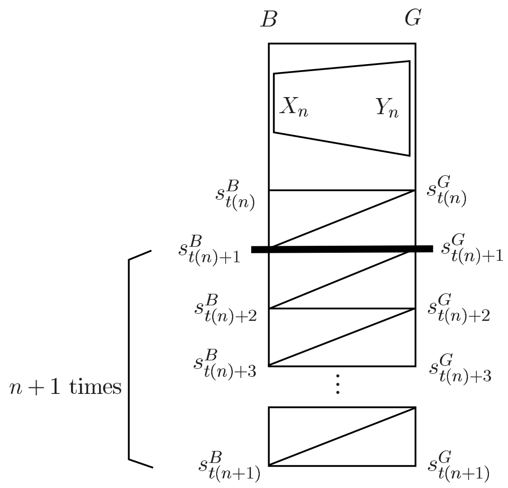

Figure 5.1: Proof of Claim

$|X| + k < |Y|$ holds. This contradicts our assumption that $\mathbb{G}$ satisfies the constant bounded Hall condition via k and complete the proof of our claim. $\square$

Because of the computably G-locally finiteness of $\mathbb{G}$, we can effectively produce a non-proper R-chain s with starting point $b_0 \in B$ by the following procedure: Let s^B_0 be the set consisting only of b_0, take the fast witnessed set of girls s^G_j such that $\langle s^B_{j'}, s^G_{j'} \rangle_{j' \leq j}$ forms an R-chain, and put $s^B_{j+1} = N_{\mathbb{G}}(s^G_j)$. Claim 5.3.1 ensures that this procedure would stop eventually until j is up to $t(k+1)$, i.e., $\langle s^B_j, s^G_j \rangle_{j \leq t(k+1)}$ is non-proper. Then, by Σ^0_0 least number principle, there exists u_0 such that $\theta(u_0, 0)$ holds. Thus the initial step is accomplished.

Next we turn to the induction step. Assume that $\exists u \theta(u, v)$ holds, and let u' be u such that $\theta(u, v)$ holds. Then $R' = R \setminus \bigcup_{j \leq v} u'_j$ satisfies the constant bounded Hall condition by the disjoint property:

$$N_{\mathbb{G}}(B \setminus B[v]) \cap N_{\mathbb{G}}(B[v]) = \emptyset$$

where $B[v]$ denotes the set of boys in $\bigcup_{j \leq v} u'_j$. As in the initial step, we can effectively produce a non-proper R'-chain s' of finite length, where we take b_{v+1} as the starting point of s' if $b_{v+1} \notin (u'_v)^B$. Let u_{v+1} be such a least s', then $\theta(u'^\frown u_{v+1}, v+1)$ holds. This completes the proof of our theorem. $\square$

Corollary 5.3.13. $\mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}''\text{-M})$ *is provable in* RCA_0.

We summarize the sequential strength of marriage theorems with the constant bounded Hall condition in Table 5.1.

ACA	$\mathrm{Seq}(\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}\text{-}\mathrm{M})$	$\mathrm{Seq}(\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}'\text{-}\mathrm{M})$	$\mathrm{Seq}(\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}''\text{-}\mathrm{M})$
WKL	$\mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}\text{-}\mathrm{M})$	$\mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}'\text{-}\mathrm{M})$	$\mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}''\text{-}\mathrm{M})$
RCA_0			

Table 5.1: The sequential strength of constant bounded marriage theorems

Applications: As we have already seen in Section 3.1, the uniformization theorems [37, 13] assert that for Π^1_2 statements of some syntactical form, its provableness in (semi-)intuitionistic systems guarantees its sequential provableness in weak subsystems of second-order arithmetic. Hirst-Mummert's uniformization theorems (Proposition 3.1.2) can be applied for Π^1_2 statements of the following syntactical form:

$$\forall X\,(\varphi(X) \to \exists Y \psi(X, Y))\,,$$

where $\varphi(X)$ is $\exists$-free and $\psi(X, Y)$ is in Γ_1. On the other hand, Dorais' uniformization theorems (Proposition 3.1.5) can be applied for more Π^1_2 statements, namely, for Π^1_2 statements with $\varphi(X)$ including purely existential formulas as subformula. (See [13, Section 4] for details.) By a careful inspection, one can see (via Lemma 2.2.12) that the assertion that "a bipartite graph (B, G, R) satisfies the constant bounded Hall condition via k" has the syntactical form of

$$\forall x \exists y A_{qf}$$

and the assertion that "M is a solution of (B, G, R)" is purely universal[5] (and hence, is in Γ_{K}). That is to say, Dorais' uniformization theorems can be applied to our marriage theorem with the constant bounded Hall condition while Hirst-Mummert's uniformization theorems can not. As a consequence of Theorem 5.3.9 and 5.3.10, we have the following.

Corollary 5.3.14.

1. $\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}\text{-}\mathrm{M}$ *and* $\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}'\text{-}\mathrm{M}$ *are not provable in* $\mathsf{EL} + \mathrm{WKL} + \mathrm{GC_L} + \mathrm{CN_L}$.
2. $\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}\text{-}\mathrm{M}$ *and* $\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}'\text{-}\mathrm{M}$ *are not provable in* $\mathsf{EL} + \mathrm{GC} + \mathrm{CN}$.

On the other hand, one can also see that the assertion that "a bipartite graph (B, G, R) is B-locally finite, computably G-locally finite via p and satisfies the constant bounded Hall condition via k" still has the syntactical form of $\forall x \exists y A_{qf}$. Therefore one Theorem 3.4.1 can be applied to (the proofs of) Theorem 5.3.11 and we obtain the following.

[5] If we formalize this in the system with set-based language, this has the syntactical form of $\forall x \exists y A_{qf}$ because one has to describe "M is a function" additionally.

Corollary 5.3.15. $\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}''$-M *and* $\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}''$-M *are provable in* EL_0

5.3.2 Constant Bounded Symmetric Marriage Theorems

Hirst [35] showed that the symmetric marriage theorem is equivalent to ACA over RCA_0 and that for computably locally finite graphs is equivalent to WKL over RCA_0, which were the threshold for the enriched development of the reverse mathematics of graph coloring. Fujiwara [21] investigated symmetric marriage theorems with Kierstead's expanding Hall condition in the context of reverse mathematics. Here we investigate symmetric marriage theorems with the constant bounded Hall condition.

Definition 5.3.16. Let $\mathbb{G} = (B, G, R)$ is a (possibly infinite) bipartite graph.

- $\mathbb{G}$ satisfies *the symmetric Hall condition* if $\forall X \subset_{\mathrm{fin}} B \cup G\,(|N_{\mathbb{G}}(X)| \geq |X|)$.
- $\mathbb{G}$ is *locally finite* if $\forall x \in B \cup G\,(|N_{\mathbb{G}}(x)| < \infty)$.
- $\mathbb{G}$ is *computably locally finite*[6] if there is a function $f : B \cup G \to \mathbb{N}$ such that $f(x) = |N_{\mathbb{G}}(x)|$ for all $x \in B \cup G$.
- $\mathbb{G}$ has a *symmetric solution* if there exists a **bijection** $M \subset R$ from B to G.

The basic statement of the symmetric marriage theorem is the following:

Theorem 5.3.17 (Symmetric Marriage Theorem)**.** *If $\mathbb{G} = (B, G, R)$ is a bipartite graph which is locally finite and satisfies the symmetric Hall condition, then $\mathbb{G}$ has a symmetric solution.*

Definition 5.3.18. Let $\mathbb{G} = (B, G, R)$ is a (possibly infinite) bipartite graph.

- $\mathbb{G}$ satisfies *the Y-Hall condition*, which is expressed by $\mathrm{Y_H}$, if $\forall X \subset_{\mathrm{fin}} Y\,(|N_{\mathbb{G}}(X)| \geq |X|)$, where $Y \in \{B, G\}$.
- $\mathbb{G}$ satisfies *the constant bounded Y-Hall condition*, which is expressed by $\mathrm{Y_{H_{cb}}}$, if there exists k such that for all $X \subset_{\mathrm{fin}} Y$, $|X| \leq |N_{\mathbb{G}}(X)| \leq |X| + k$ holds, where $Y \in \{B, G\}$.
- $\mathbb{G}$ satisfies *the constant bounded symmetric Hall condition* if there exists k such that for all $X \subset_{\mathrm{fin}} B \cup G$, $|X| \leq |N_{\mathbb{G}}(X)| \leq |X| + k$ holds.

As in Subsection 5.3.1, we investigate all symmetric marriage theorems having the following form:

[6] The corresponding notion in recursive graph theory [26] is called "highly recursive".

Statement ($\mathrm{B}^{(\cdot)}_{\mathrm{H}_{(\cdot)}}\mathrm{G}^{(\cdot)}_{\mathrm{H}_{(\cdot)}}$-$\mathrm{M}_\mathrm{s}$)**.** If a bipartite graph $\mathbb{G}$ satisfies $\mathrm{B}^{(\cdot)}$, $\mathrm{G}^{(\cdot)}$, $\mathrm{B}_{\mathrm{H}_{(\cdot)}}$ and $\mathrm{G}_{\mathrm{H}_{(\cdot)}}$, then $\mathbb{G}$ has a symmetric solution.

We first show that the constant bounded Hall condition for one side already makes the symmetric marriage theorem provable in $\mathsf{RCA}_0 + \mathrm{B}\Pi^0_1$.

Proposition 5.3.19. $\mathsf{RCA}_0 + \mathrm{B}\Pi^0_1 \vdash \mathrm{B}'_{\mathrm{H}_{\mathrm{cb}}}\mathrm{G}'_{\mathrm{H}}$-$\mathrm{M}_\mathrm{s}$ *(equivalently* $\mathrm{B}_{\mathrm{H}_{\mathrm{cb}}}\mathrm{G}'_{\mathrm{H}}$-$\mathrm{M}_\mathrm{s}$*), that is, the following is provable in* $\mathsf{RCA}_0 + \mathrm{B}\Pi^0_1$*. If* $\mathbb{G} = (B, G, R)$ *is a locally finite bipartite graph which satisfies the constant bounded B-Hall condition and G-Hall condition, then* $\mathbb{G}$ *has a symmetric solution.*

Proof. The proof proceeds by extending the proof of Theorem 5.3.8. Just as in the proof of Theorem 5.3.8, take a least c_1 such that $\Phi(c_1)$ holds and consider X_1, X_2 and Y_1 as before. Then the finiteness of $X^* := N_{\mathbb{G}}(N_{\mathbb{G}}(X_1) \cup Y_1)$ follows from the G-locally finiteness with the use of $\mathrm{B}\Pi^0_1$ as follows. Since $\mathbb{G}$ is G-locally finite,

$$\forall g \in (N_{\mathbb{G}}(X_1) \cup Y_1)\exists r \forall b((b, g) \in R \to b < r) \tag{5.2}$$

holds (note that the (code of) finite set $N_{\mathbb{G}}(X_1) \cup Y_1$ exists in RCA_0 as mentioned in the proof of Theorem 5.3.8). Applying $\mathrm{B}\Pi^0_1$ to (5.2), we have the finiteness of X^*. Take a matching M^* of X^* by the finite marriage theorem ([35, Theorem 2.1]). By Σ^0_0 comprehension, $M := M^* \cup \{(b, g) \in R : b \in B \setminus X^*\}$ exists. Reasoning in RCA_0, we show that M is a symmetric solution. First one can show that M is an injective function from B to G as in Theorem 5.3.8. Furthermore, each girl $g \in N_{\mathbb{G}}(X^*) \setminus (N_{\mathbb{G}}(X_1) \cup Y_1)$ only knows some boys (in fact, just one boy) in X^*, since otherwise g should be in Y_1. Therefore, by the G-Hall condition, $|X^*| = |N_{\mathbb{G}}(X^*)|$ holds and hence, it follows that M^* is also a matching of $N_{\mathbb{G}}(X^*)$. In addition, each girl $g' \in G \setminus N_{\mathbb{G}}(X^*)$ knows one boy $b' \in B \setminus X^*$ by the G-Hall condition. Thus M is surjective. □

Remark 5.3.20. *If the G-locally finiteness is dropped from the assumption of* $\mathrm{B}'_{\mathrm{H}_{\mathrm{cb}}}\mathrm{G}'_{\mathrm{H}}$-$\mathrm{M}_\mathrm{s}$*, the assertion is already false.*

Remark 5.3.21. *In the proof of Proposition 5.3.19,* $\mathrm{B}\Pi^0_1$ *is only used to verify the finiteness of* $X^* := N_{\mathbb{G}}(N_{\mathbb{G}}(X_1) \cup Y_1)$*. Since the finiteness is guaranteed from the computably G-locally finiteness or the constant bounded G-Hall condition without using* $\mathrm{B}\Pi^0_1$*, this proof reveals that* $\mathrm{B}_{\mathrm{H}_{\mathrm{cb}}}\mathrm{G}''_{\mathrm{H}}$-$\mathrm{M}_\mathrm{s}$ *or* $\mathrm{B}_{\mathrm{H}_{\mathrm{cb}}}\mathrm{G}_{\mathrm{H}_{\mathrm{cb}}}$-$\mathrm{M}_\mathrm{s}$ *is already provable in* RCA_0*. We show below that this weak variant* $\mathrm{B}'_{\mathrm{H}_{\mathrm{cb}}}\mathrm{G}'_{\mathrm{H}}$-$\mathrm{M}_\mathrm{s}$ *of the symmetric marriage theorem is equivalent to* $\mathrm{B}\Pi^0_1$ *over* RCA_0 *(Theorem 5.3.28).*

Notice that the proof of Theorem 5.3.19 also contains a non-uniformity in the use of least number principle. In fact, our investigation of its sequential version reveals that this non-uniformity can not be avoided.

Theorem 5.3.22. *The following are pairwise equivalent over* RCA_0.

1. ACA.
2. $\mathrm{Seq}(\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H}}\text{-}\mathrm{M_s})$*, that is, for all sequence* $\langle B_n, G_n, R_n, k_n\rangle_{n\in\mathbb{N}}$ *such that* (B_n, G_n, R_n) *is locally finite, satisfies the* B_n*-constant bounded Hall condition via* k_n *and satisfies the G-Hall condition, then there exists a sequence* $\langle M_n\rangle_{n\in\mathbb{N}}$ *of their symmetric solutions.*
3. $\mathrm{Seq}(\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H_{cb}}}\text{-}\mathrm{M_s})$*, that is, for all sequence* $\langle B_n, G_n, R_n, k_n\rangle_{n\in\mathbb{N}}$ *such that* (B_n, G_n, R_n) *is locally finite and satisfies the constant bounded symmetric Hall condition via* k_n*, then there exists a sequence* $\langle M_n\rangle_{n\in\mathbb{N}}$ *of their symmetric solutions.*

Proof. By inspecting the proof of Theorem 5.3.9. □

On the other hand, the next theorem states that if we have the G-locally finiteness, the symmetric marriage theorem is uniformly provable in RCA_0.

Theorem 5.3.23. $\mathsf{RCA}_0 \vdash \mathrm{Seq}(\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}''_{\mathrm{H}}\text{-}\mathrm{M_s})$*, that is, the following is provable in* RCA_0*. For all sequence* $\langle B_n, G_n, R_n, p_n, k_n\rangle_{n\in\mathbb{N}}$ *such that* (B_n, G_n, R_n) *is* B_n*-locally finite, computably* G_n*-locally finite via* p_n*, satisfies the* B_n*-constant bounded Hall condition via* k_n *and satisfies the G-Hall condition, then there exists a sequence* $\langle M_n\rangle_{n\in\mathbb{N}}$ *of their symmetric solutions.*

Proof. Extending the proof of Theorem 5.3.11, we shall give a uniform proof of $\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}''_{\mathrm{H}}\text{-}\mathrm{M_s}$ in RCA_0. We reason in RCA_0 and let $\mathbb{G} = (B, G, R)$ be a bipartite graph satisfying the conditions. As in the proof of Theorem 5.3.11, consider the same $\theta(u, v)$ and take a function $u^{(\cdot)} : \mathbb{N} \to \mathbb{N}$ such that $\theta(v, u^v)$ holds for each $v \in \mathbb{N}$. Then we take (by Σ^0_0 comprehension) a function $g_s : \mathbb{N} \to \mathbb{N}$ as $g_s(v)$ is the least complete matching of $\left(((u^v)_v)^B, ((u^v)_v)^G, R \setminus \bigcup_{i<v}(u^v)_i\right)$. To verify the well-definedness of this g_s, it suffices to show

$$\left|((u^v)_v)^B\right| = \left|((u^v)_v)^G\right| \tag{5.3}$$

for each $v \in \mathbb{N}$. If $\mathbb{G}_v := (B_v, G_v, R_v) := \mathbb{G}\setminus\bigcup_{i<v}(u^v)_i$ satisfies the G_v-Hall condition for all $v \in \mathbb{N}$, then (5.3) holds for each $v \in \mathbb{N}$ since $((u^v)_v)^B = N_{\mathbb{G}_v}\left(((u^v)_v)^G\right)$ and $\left(((u^v)_v)^B, ((u^v)_v)^G\right)$ satisfies the B_v-Hall condition from the definition of non-proper chain. Then we claim by induction that $\mathbb{G}_v$ satisfies G_v-Hall condition for all $v \in \mathbb{N}$. Note that the assertion "$\mathbb{G}_v$ satisfies G_v-Hall condition" can be written as a Π^0_1 formula with the aid of the computably G-locally finiteness of the original graph $\mathbb{G}$, and hence this induction is carried out in RCA_0. For the induction step, suppose that $\mathbb{G}_{v+1} = \mathbb{G}\setminus\bigcup_{i<v+1}(u^{v+1})_i$ does not satisfy the G_{v+1}-Hall condition. Then there exists $Y \subset_{\mathrm{fin}} G \setminus \bigcup_{i<v+1}\left((u^{v+1})_i\right)^G$ such that $\left|N_{\mathbb{G}_{v+1}}(Y)\right| < |Y|$. Since $\left((u^{v+1})_v\right)^B = N_{\mathbb{G}_v}\left(\left((u^{v+1})_v\right)^G\right)$, we have

$\left|\left((u^{v+1})_v\right)^B\right| = \left|\left((u^{v+1})_v\right)^G\right|$ as before, and hence

$$\left|N_{\mathbb{G}_v}\left(\left((u^{v+1})_v\right)^G \cup Y\right)\right| = \left|\left((u^{v+1})_v\right)^B \cup N_{\mathbb{G}_{v+1}}(Y)\right| < \left|\left((u^{v+1})_v\right)^G \cup Y\right|.$$

This means that $\mathbb{G}_v$ does not satisfy the G_v-Hall condition. Thus the proof of our claim is complete, and consequently g_s is well-defined.

Define M_s as $\bigcup_{v\in\mathbb{N}} g_s(v)$. The injectivity of M_s is straightforwardly verified. In the following, we show the surjectivity. Suppose not. Then there exists $g' \in G$ such that $g' \notin \bigcup_{i<v} ((u^v)_i)^G$ for all $v \in \mathbb{N}$. On the other hand, by the description of $\theta(u, v)$,

$$\forall b \in N_{\mathbb{G}}(g')\exists v_b \left(b \in \bigcup_{i<v_b} ((u^{v_b})_i)^B\right).$$

Therefore, by $\mathrm{B}\Sigma^0_0$ (provable in RCA_0), there exists v' such that

$$\forall b \in N_{\mathbb{G}}(g')\left(b \in \bigcup_{i<v'} \left((u^{v'})_i\right)^B\right).$$

Since $\left|\left((u^{v'})_i\right)^B\right| = \left|\left((u^{v'})_i\right)^G\right|$ for all $i < v'$ and they are disjoint, we have

$$\left|\bigcup_{i<v'} \left((u^{v'})_i\right)^B\right| = \left|\bigcup_{i<v'} \left((u^{v'})_i\right)^G\right|$$

by Σ^0_0 induction. Since $g' \notin \bigcup_{i<v'} \left((u^{v'})_i\right)^G$, we have consequently

$$\left|N_{\mathbb{G}}\left(\bigcup_{i<v'} \left((u^{v'})_i\right)^G \cup \{g'\}\right)\right| = \left|\bigcup_{i<v'} \left((u^{v'})_i\right)^B\right| < \left|\bigcup_{i<v'} \left((u^{v'})_i\right)^G\right| + 1 = \left|\bigcup_{i<v'} \left((u^{v'})_i\right)^G \cup \{g'\}\right|,$$

which contradict the G-Hall condition of the original graph $\mathbb{G}$. □

Corollary 5.3.24. $\mathsf{RCA}_0 \vdash \mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}''_{\mathrm{H}}\text{-}\mathrm{M_s}), \mathrm{Seq}(\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}''_{\mathrm{H_{cb}}}\text{-}\mathrm{M_s}), \mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}''_{\mathrm{H_{cb}}}\text{-}\mathrm{M_s})$.

The remaining problem is the sequential strength of $\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H}}\text{-}\mathrm{M_s}$. As we will see below, this symmetric marriage theorem is uniformly provable in RCA, but the verification for the termination of algorithm requires a kind of induction axiom $\mathrm{B}\Pi^0_1$ not provable in RCA_0.

Theorem 5.3.25. $\mathsf{RCA}_0 + \mathrm{B}\Pi^0_1 \vdash \mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H}}\text{-}\mathrm{M_s})$*, that is, the following is provable in* $\mathsf{RCA}_0 + \mathrm{B}\Pi^0_1$*. For all sequence $\langle B_n, G_n, R_n, k_n\rangle_{n\in\mathbb{N}}$ such that (B_n, G_n, R_n) is computably B_n-locally finite and G_n-locally finite, satisfies the B_n-constant bounded Hall condition via k_n and satisfies the G_n-Hall condition, then there exists a sequence $\langle M_n\rangle_{n\in\mathbb{N}}$ of their symmetric solutions.*

For the proof of Theorem 5.3.25, the exact same construction as for Theorem 5.3.23 works, but in this case, we interchange the roles between genders, namely, we start to construct chains from G (not from B as for Theorem 5.3.11 and Theorem 5.3.23). The same verification also works except for the termination of the chain construction (See **Claim** in the proof of Theorem 5.3.11). Thus it suffices for Theorem 5.3.25 to show the following lemma in $\mathsf{RCA}_0 + \mathrm{B}\Pi^0_1$.

Lemma 5.3.26 ($\mathsf{RCA}_0 + \mathrm{B}\Pi^0_1$)**.** *If a bipartite graph $\mathbb{G} = (B, G, R)$ satisfies the constant bounded B-Hall condition and G-Hall condition, then for all $g \in G$, the R-chain starting from g will be eventually non-proper, in other words, there is no infinite proper R-chain $\langle s^B_i, s^G_i\rangle_{i<\in\mathbb{N}}$ such that $s^G_0 = \{g\}$.*

Proof. The proof is similar to that for **Claim** in the proof of Theorem 5.3.11 except that there is no explicit bound in this case. Suppose not, i.e., there exists an infinite proper R-chain $\langle s^B_i, s^G_i\rangle_{i\in\mathbb{N}}$ such that $s^G_0 = \{g^*\}$ for some $g^* \in G$.

We claim that for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $X \subseteq s^B_m$ and $Y \subseteq s^G_m$ such that $Y \subseteq N_{\mathbb{G}}(X)$ and $|X| + n \leq |Y|$ holds by Σ^0_1 induction on n. The initial step is obvious. For the induction step, assume that $m \in \mathbb{N}$, $X \subseteq s^B_m$, $Y \subseteq s^G_m$, $Y \subseteq N_{\mathbb{G}}(X)$ and $|X| + n \leq |Y|$. If $|X| + n < |Y|$, we are done. Then assume $|X| + n = |Y|$. Since our R-chain $\langle s^B_i, s^G_i\rangle_{i\in\mathbb{N}}$ is proper, for all $i \in \mathbb{N}$, there exists $b_i \in s^B_{m+i} \setminus s^B_{m+i-1}$ and $g_i \in s^G_{m+i+1} \setminus s^G_{m+i}$ such that $(b_i, g_i) \in R$, and such functions $b_{(\cdot)} : \mathbb{N} \to \mathbb{N}$ and $g_{(\cdot)} : \mathbb{N} \to \mathbb{N}$ exist in RCA_0. If we have shown

$$\exists i', g'(g' \in s^G_{m+i'} \wedge (b_{i'}, g') \in R \wedge g' \notin Y), \tag{5.4}$$

then $|X \cup \{b_{i'}\}| + n = |Y| + 1 < |Y \cup \{g', g_{i'}\}|$, and hence the induction step is established. To show (5.4), assume that for all $i \in \mathbb{N}$ and $g \in s^G_{m+i}$ such that $(b_i, g) \in R$, g is in Y. By the definition of proper R-chain, there exists $g \in s^G_{m+i}$ such that $(b_i, g) \in R$ for all $i \in \mathbb{N}$. Thus for all $i \in \mathbb{N}$, there exists $g^Y_i \in Y$ such that $(b_i, g^Y_i) \in R$, and such function $g^Y_{(\cdot)}$ exists in RCA_0. However, it follows from the G-locally finiteness and $\mathrm{B}\Pi^0_1$ (See the proof of Proposition 5.3.19) that $N_{\mathbb{G}}(Y)$ is finite, which is a contradiction. This competes the proof of our claim.

Obviously, our claim contradicts the constant bounded B-Hall condition. □

Next we discuss the "reverse" direction. The following results are also interesting in the sense that our Π^1_2 statements (which assert the existence of sets) imply an induction scheme despite the fact that a Π^1_2 statement is usually equivalent to some set existence axiom in the practice of reverse mathematics.

Lemma 5.3.27. $\mathsf{RCA}_0 \vdash \mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H}}\text{-}\mathrm{M_s} \to \mathrm{B}\Pi^0_1$. *That is, the following assertion implies* $\mathrm{B}\Pi^0_1$ *over* RCA_0. *If $\mathbb{G} = (B, G, R)$ is a finite bipartite graph which is computably B-locally finite and G-locally finite, satisfying the B-constant bounded Hall condition and satisfying the G-Hall condition, then $\mathbb{G}$ has a symmetric solution.*

Proof. By Lemma 5.3.7, it suffices to show $\neg\mathrm{RT}(1) \to \neg\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H}}\text{-}\mathrm{M_s}$ over RCA_0. We reason in RCA_0 and suppose that there exists $f : \mathbb{N} \to k$ $(k > 1)$ such that

$$\forall i < k \exists j' \forall j (f(j) = i \to j < j'). \tag{5.5}$$

We construct a graph $\mathbb{G} := (B, G, R)$ which satisfies the conditions but has no symmetric solution. Define $B := \mathbb{N}$ and $G := \mathbb{N}$ and construct R as follows:

- put $(j, j), (j, j + 1)$ in R for $j < k - 1$;
- put $(j, j + 1)$ in R for $j \geq k - 1$;
- put $(j + k - 1, f(j))$ in R for all $j \in \mathbb{N}$.

Then it is obvious that $\mathbb{G}$ is computably B-locally finite and satisfies the constant bounded B-Hall condition by k. The G-locally finiteness follows from the property (5.5). In the following, we show that $\mathbb{G}$ satisfies the G-Hall condition. Let Y be an arbitrary finite subset of G. In the case that there exists $i^* < k$ such that $i^* \notin Y$. Take

$$X := \{i : i < i^* \wedge i \in Y\} \cup \{i - 1 : i > i^* \wedge i \in Y\},$$

then it is straightforward to see $|Y| = |X|$ and $X = N_{\mathbb{G}}(Y)$. Assume that there is no $i^* < k$ such that $i^* \notin Y$, in other words, $\{0, \ldots, k - 1\} \subset Y$. Since Y is finite, there exists $i' \notin Y$ $(i' > 0)$. Take

$$X := \{i : i < f(i' - 1) \wedge i \in Y\} \cup \{i - 1 : i > f(i' - 1) \wedge i \in Y\} \cup \{(i' - 1, f(i' - 1))\},$$

then it is also straightforward to see that that $|Y| = |X|$ and that $X = N_{\mathbb{G}}(Y)$ from our assumption.

On the other hand, if $\mathbb{G}$ has a symmetric solution $M_s : G \to B$, then $M_s(i)$ must be $i - 1$ for all $i \geq k$, and hence $M_s(i)$ must be less than $k - 1$ for all $i < k$, which contradicts the finite pigeonhole principle (See Lemma 5.3.6).

This completes the proof of our lemma. □

Theorem 5.3.28. $\mathsf{RCA}_0 \vdash \mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H}}\text{-}\mathrm{M_s} \leftrightarrow \mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H}}\text{-}\mathrm{M_s} \leftrightarrow \mathrm{B}\Pi^0_1$.

Proof. Immediate from Proposition 5.3.19 and Lemma 5.3.27. □

Theorem 5.3.29. $\mathsf{RCA}_0 \vdash \mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H}}\text{-}\mathrm{M_s}) \leftrightarrow \mathrm{B}\Pi^0_1$, *that is, the following are pairwise equivalent to* $\mathrm{B}\Pi^0_1$ *over* RCA_0. *For all sequence* $\langle B_n, G_n, R_n, k_n\rangle_{n\in\mathbb{N}}$ *such that* (B_n, G_n, R_n) *is computably* B_n*-locally finite and* G_n*-locally finite, satisfying the* B_n*-constant bounded Hall condition via* k_n *and satisfying the* G_n*-Hall condition, then there exists a sequence* $\langle M_n\rangle_{n\in\mathbb{N}}$ *of their symmetric solutions.*

Proof. Immediate from Lemma 5.3.25 and Lemma 5.3.27 since $\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H}}\text{-}\mathrm{M_s}$ of course follows from $\mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H}}\text{-}\mathrm{M_s})$. □

We summarize the sequential strength of symmetric marriage theorems with the constant bounded Hall condition in Table 5.2.

ACA	$\mathrm{Seq}(\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H_{cb}}}\text{-}\mathrm{M_s})$		$\mathrm{Seq}(\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H}}\text{-}\mathrm{M_s})$
	$\mathrm{Seq}(\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}''_{\mathrm{H_{cb}}}\text{-}\mathrm{M_s})$	$\mathrm{Seq}(\mathrm{B}'_{\mathrm{H_{cb}}}\mathrm{G}''_{\mathrm{H}}\text{-}\mathrm{M_s})$	
	$\mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}''_{\mathrm{H_{cb}}}\text{-}\mathrm{M_s})$	$\mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}''_{\mathrm{H}}\text{-}\mathrm{M_s})$	$\mathrm{Seq}(\mathrm{B}''_{\mathrm{H_{cb}}}\mathrm{G}'_{\mathrm{H}}\text{-}\mathrm{M_s})$
RCA_0			$\mathrm{B}\Sigma^0_2$ ($\leftrightarrow$ $\mathrm{B}\Pi^0_1$)

Table 5.2: The sequential strength of constant bounded symmetric marriage theorems

5.4 Investigation of Some Uniform Versions over WRCA^{ω}_0

We mentioned at the beginning of Chapter 4 that our metatheorems in Section 4.3 are extensively applicable to statements which are provable in RCA_0 but whose sequential versions derive ACA. For the purpose of confirming that, we investigate the following principles studied in preceding papers.

1. Jordan decomposition for 2×2 matrices.
2. Principle of trichotomy for reals.
3. Π^0_1 least number principle.

In the following, we see that each of them has a syntactical form to which Corollary 4.3.8 and Corollary 4.3.10 are applicable, and that each of their uniform versions derives $(\exists^2)$ over WRCA^{ω}_0 (reflecting the fact that the pointwise versions intuitionistically imply Π^0_1-LEM or even Σ^0_1-LEM). The proofs are similar to those in [66]. At first, we consider the Jordan decomposition for 2×2 matrices. As shown in [37, Section 4], it is provable in RCA_0 but its sequential version is equivalent to ACA over RCA_0. Note that using the representation of real numbers by Kohlenbach [55, Section 4.1], every functional of type 1 can be seen to represent a unique real number. Furthermore, since a complex number is naturally defined as a pair of real numbers, every functional of type 1 also represents a 2×2 complex matrix via the standard encoding.

Definition 5.4.1 (See [37] for details)**.**

$$\begin{aligned}\mathrm{JD}_2 : \quad & \forall M^1\big(M \text{ is a } 2 \times 2 \text{ complex matrix} \to \exists U^1, J^1 \\ & \big(U, J \text{ are } 2 \times 2 \text{ complex matrices such that } M = UJU^{-1} \text{ and } J \text{ consists of Jordan blocks}\big)\big).\end{aligned}$$

Note that the tuple of U^1 and J^1 can be coded as a single variable of type 1.

Theorem 5.4.2. WRCA_0^ω + Uni(JD$_2$) $\vdash (\exists^2)$.

Proof. We reason in WRCA_0^ω. By primitive recursion with a parameter of type 1, define a functional Ξ of type $1 \to 1$ such that

$$\Xi(f)(m) := \begin{cases} \frac{1}{2^{n_l+1}} \text{ where } n_l \text{ is a least number such that } f(n_l) = 0 & \text{if } \exists n \le m f(n) = 0, \\ \frac{1}{2^{m+1}} & \text{otherwise.} \end{cases}$$

Then for every f of type 1, $\Xi(f)$ represents a real number. Furthermore, there exists n such that $f(n) = 0$ if and only if $\Xi(f) \neq_{\mathbb{R}} 0$. By a standard discussion as in linear algebra, one can show that for every $x \in \mathbb{C}$, the Jordan canonical form of $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if $x \neq_{\mathbb{C}} 0$, and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if $x =_{\mathbb{C}} 0$. As mentioned in the proof of [37, Lemma 4.4], it is effectively decided whether the upper right-hand entry of the Jordan canonical form of $\begin{pmatrix} 1 & 0 \\ \Xi(f) & 1 \end{pmatrix}$ is 0 or 1. Then $\exists n\, (f(n) = 0)$ is equivalent to some quantifier-free formula with Ξ, so by [55, Proposition 3.17], one can construct a term t of type 2 such that $t(f) = 0 \leftrightarrow \exists n\, (f(n) = 0)$ for every f. Therefore we have $(\exists^2)$. □

Corollary 5.4.3. JD$_2$ *is provable neither in* $\mathsf{WE\text{-}HA}^\omega$+AC+$\mathrm{IP}_\forall^\omega$+$\mathrm{M}^\omega$+UWKL+KL+$\Sigma_1^0$-UB$^-$+BI *nor in* $\mathsf{E\text{-}HA}^\omega$ + AC!^{1} + AC^0 + $\mathrm{IP}_\forall^{\le 1, \le 1}$ + $\mathrm{M}^{\le 1}$ + KL + Σ_1^0-UB + BI$^{\le 1}$.

Proof. This follows immediately from the previous theorem applied to Corollary 4.3.8 and 4.3.10 (note that 'M is a 2×2 complex matrix' is dropped in Kohlenbach's representation and the conclusion of JD$_2$ is in Γ_2). □

Remark 5.4.4.
1. *The above corollary extends the unprovability result mentioned at the end of Hirst and Mummert* [37].
2. *The Jordan canonical form depends on the eigenvalues of the given matrix. As mentioned in* [13]*, the fundamental theorem of algebra is provable in* $\mathsf{E\text{-}HA}^\omega$ + QF-AC^{0,0} + WKL. *That is, one can show that every complex matrix has complex eigenvalues within* $\mathsf{E\text{-}HA}^\omega$ + QF-AC^{0,0} + WKL *but cannot construct its Jordan canonical form even in* $\mathsf{E\text{-}HA}^\omega$ + AC!^{1} + AC^0 + $\mathrm{IP}_\forall^{\le 1, \le 1}$ + $\mathrm{M}^{\le 1}$ + KL + Σ_1^0-UB + BI$^{\le 1}$.

Next, we consider the principle of trichotomy for reals. As shown in [14, Section 2], it is provable in RCA_0 but its sequential version is equivalent to ACA over RCA_0. Note that $\alpha^1 \ge_{\mathbb{R}} \beta^1$ and $\alpha^1 =_{\mathbb{R}} \beta^1$ are represented as purely universal formulas and $\alpha^1 >_{\mathbb{R}} \beta^1$ is represented as a purely existential formula.

Definition 5.4.5.

$$\mathrm{LPO}: \ \forall f^1 \, (\forall n(f(n) \le 1) \to (\exists n(f(n) = 1) \vee \forall n(f(n) = 0)))\,.$$

$$\mathrm{TRI}: \ \forall \alpha^1 \, (\alpha \in \mathbb{R} \to \alpha >_{\mathbb{R}} 0 \vee \alpha =_{\mathbb{R}} 0 \vee \alpha <_{\mathbb{R}} 0)\,.$$

$$\mathrm{TRI}^- : \ \forall \alpha^1(\alpha \in \mathbb{R} \wedge \alpha \ge_{\mathbb{R}} 0 \to \alpha >_{\mathbb{R}} 0 \vee \alpha =_{\mathbb{R}} 0).$$

Here we think of $A \vee B$ as an abbreviation of $\exists k\,((k = 0 \to A) \wedge (k \neq 0 \to B))$. Note that $\alpha \in \mathbb{R}$ is dropped in Kohlenbach's [55] representation (in which **every** α^1 represents a unique real number).

Theorem 5.4.6. $\mathsf{WRCA}_0^\omega + \mathrm{Uni}(\mathrm{TRI}^-) \vdash (\exists^2)$.

Proof. We reason in WRCA_0^ω. By primitive recursion with a parameter of type 1, define a functional Ξ of type $1 \to 1$ as in the proof of Theorem 5.4.2. Then for every f of type 1, $\Xi(f)$ is a real number and $\Xi(f) \ge_{\mathbb{R}} 0$. Let Ψ be a witness of $\mathrm{Uni}(\mathrm{TRI}^-)$. One can easily show $\exists n f(n) = 0 \leftrightarrow \Psi(\Xi(f)) \neq 0$. Note that the right side in this equivalence is a quantifier-free formula. Then we have $(\exists^2)$. $\square$

Since '$\alpha \ge_{\mathbb{R}} 0$' is purely universal and the proof of [14, Theorem 1] shows $\mathrm{LPO} \leftrightarrow \mathrm{TRI} \leftrightarrow \mathrm{TRI}^-$ over $\widehat{\mathsf{WE\text{-}HA}}^\omega\!\restriction + \mathrm{QF\text{-}AC}^{0,0}$, the next corollary immediately follows as before.

Corollary 5.4.7. *Each of* LPO, TRI *and* TRI^- *is provable neither in* $\mathsf{WE\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}_\forall^\omega + \mathrm{M}^\omega + \mathrm{UWKL} + \mathrm{KL} + \Sigma_1^0\text{-}\mathrm{UB}^- + \mathrm{BI}$ *nor in* $\mathsf{E\text{-}HA}^\omega + \mathrm{AC!}^1 + \mathrm{AC}^0 + \mathrm{IP}_\forall^{\le 1, \le 1} + \mathrm{M}^{\le 1} + \mathrm{KL} + \Sigma_1^0\text{-}\mathrm{UB} + \mathrm{BI}^{\le 1}$.

At the end, we consider the least number principle with a parameter of type 1. In the practice of reverse mathematics, the least number principle often appears in non-uniform proofs within RCA. It is known that Π_n^0-least number principle is equivalent to Σ_n^0-induction over RCA_0^* (See [68]). On the other hand, one can show that the sequential version of Π_1^0 least number principle with a set parameter is equivalent to ACA over RCA_0 using the idea of the proof of Theorem 5.4.9 below.

Definition 5.4.8.

$$\mathrm{L}\Pi_1^0(\varphi): \ \forall x^0, g^1 \left(\varphi(x, g) \to \exists x_l^0 \, (\varphi(x_l, g) \wedge \forall x' < x_l \neg\varphi(x', g))\right),$$

where φ is a Π_1^0-formula which may have more parameters.

Theorem 5.4.9. *Let $\varphi_1(x, \alpha)$ be the Π_1^0-formula expressing $\alpha \le_{\mathbb{R}} \langle x \rangle_{i \in \mathbb{N}}$ where $\langle x \rangle_{i \in \mathbb{N}}$ denotes the infinite constant-x sequence. Then* $\mathsf{WRCA}_0^\omega + \mathrm{Uni}\left(\mathrm{L}\Pi_1^0(\varphi_1)\right) \vdash (\exists^2)$.

Proof. By primitive recursion with a parameter of type 1, define a functional Ξ of type $1 \to 1$ as in the proof of Theorem 5.4.2. Then for every f of type 1, $\Xi(f) \leq_{\mathbb{R}} 1$ holds. Let Ψ be a witness of $\mathrm{Uni}\left(\mathrm{L}\Pi^0_1(\varphi_1)\right)$. One can easily show $\exists n f(n) = 0 \leftrightarrow \Psi(\Xi(f)) \neq 0$. Hence we can take $E(f) := \overline{sg}(\Psi(\Xi(f)))$ to derive $(\exists^2)$. □

Corollary 5.4.10. $\mathrm{L}\Pi^0_1$ *is provable neither in* $\mathsf{WE\text{-}HA}^\omega + \mathrm{AC} + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \mathrm{UWKL} + \mathrm{KL} + \Sigma^0_1\text{-}\mathrm{UB}^- + \mathrm{BI}$ *nor in* $\mathsf{E\text{-}HA}^\omega + \mathrm{AC}!^1 + \mathrm{AC}^0 + \mathrm{IP}^{\leq 1, \leq 1}_\forall + \mathrm{M}^{\leq 1} + \mathrm{KL} + \Sigma^0_1\text{-}\mathrm{UB} + \mathrm{BI}^{\leq 1}$.

Question 5.4.11. *As suggested from [55] and [66], the hierarchy of the reverse mathematics of sequential versions collapses if we investigate uniform versions over* RCA^ω_0. *However, our observation in Section 4.4 suggests that the hierarchy seems to be maintained if we work over* WRCA^ω_0.[7] *On the other hand, our observation in this subsection suggests that for an ordinary existence theorem whose sequential version implies* ACA *over* RCA_0, *its uniform version implies* $(\exists^2)$ *over* WRCA^ω_0. *Then how about the case for* WKL*? Thus is the following thesis true in general for an ordinary existence theorem* S*:*

$$\mathsf{RCA}_0 + \mathrm{Seq}(\mathrm{S}) \vdash \mathrm{WKL} \quad \Rightarrow \quad \mathsf{WRCA}^\omega_0 + \mathrm{Uni}(\mathrm{S}) \vdash \mathrm{UWKL}?$$

[7] While WRCA^ω_0 does not satisfy the deduction theorem [51], this causes no problems in terms of the basic standpoint of reverse mathematics that the strength of a theorem S is compared by the size of the axiom system RCA_0+S.

6 Logical Principles Weaker than Markov's Principle

In this chapter, we investigate some **semi-classical** principles weaker than the Markov's principle MP in the spirit of constructive reverse mathematics.

One motivation of this investigation is from **classical** reverse mathematics. We mentioned in Notation 3.0.26 that EL_0 + LEM can be identified with the most popular base system RCA_0 (presented in [68]) for reverse mathematics. To distinguish RCA_0 in [68] from EL_0 + LEM, here we call the former as "set-based RCA_0" and the latter "function-based RCA_0" for convenience. The aforementioned identification between the function-based RCA_0 and the set-based RCA_0 is due to the fact that the function-based RCA_0 implies Σ^0_1 induction scheme Σ^0_1-IND and Δ^0_1 comprehension scheme Δ^0_1-CA:

$$\forall \alpha, \beta (\forall y (\exists x(\alpha(y,x) \neq 0) \leftrightarrow \neg \exists x(\beta(y,x) \neq 0)) \rightarrow \exists \gamma \forall y (\gamma(y) = 0 \leftrightarrow \exists x(\alpha(y,x) \neq 0))).[1]$$

In fact, Σ^0_1-IND intuitionistically follows from QF-IND and QF-AC^{0,0} by inspecting the proof of [55, Proposition 3.21]. On the other hand, the following observation suggests that the situation is somewhat different for Δ^0_1-CA. Since quantifier-free formulas are decidable in EL_0 (See Lemma 2.2.13), Δ^0_1-CA intuitionistically implies a weak law-of-excluded-middle principle Δ^0_1-LEM :

$$\forall \alpha, \beta (\forall y (\exists x(\alpha(y,x) \neq 0) \leftrightarrow \neg \exists x(\beta(y,x) \neq 0)) \rightarrow \forall y (\exists x(\alpha(y,x) \neq 0) \vee \neg \exists x(\alpha(y,x) \neq 0))).$$

However, as shown in [1], the first-order variant of Δ^0_1-LEM is not provable in HA. This means that Δ^0_1-CA is not provable in EL. On the other hand, it is straightforward to see that Δ^0_1-CA is provable in EL_0 + MP. Then one can think of Δ^0_1-CA as a logical principle, and hence it is natural to ask the logical strength of Δ^0_1-CA. In addition, there is a further important reason to explore Δ^0_1-CA. As we have mentioned in Section 1.5 (and also at the end of Section 4.3), the hierarchy of the law-of-excluded-middle principles is closely related to uniform provability in classical reverse mathematics. In the recent development of reverse mathematics, many (recur-

[1] In this definition (and others as well), one can use equality "=" instead of non-equality "≠", but here we use non-equality following the convention from constructive mathematics.

sion theoretic or combinatorial) principles have been found in between RCA_0 and WKL_0. In the light of our results in Chapter 3 along with the correspondence between WKL and Σ^0_1-DML (See Section 1.5), it is expected that such intermediate principles (in between RCA_0 and WKL_0) also correspond to some weak logical axioms weaker than Σ^0_1-DML. Thus it is natural to investigate the logical strength of Δ^0_1-CA as a first target also from this standpoint.

Another motivation is from the interest of the arithmetical hierarchy of the law-of-excluded-middle itself. As shown in [1], Δ^0_1-LEM is implied by both of MP (also follows from the discussion in the previous paragraph) and Σ^0_1-DML (See Figure 1.1 in Section 1.5). On the other hand, Ishihara [42] showed that MP is intuitionistically equivalent to the combination of WMP :

$$\forall\alpha(\forall\beta(\neg\neg\exists n(\beta(n) \neq 0) \vee \neg\neg\exists n(\alpha(n) \neq 0 \wedge \beta(n) = 0)) \rightarrow \exists n(\alpha(n) \neq 0)).$$

with Π^0_1-DML(called $\mathrm{MP}^\vee$):

$$\forall\alpha,\beta(\neg(\neg\exists x(\alpha(x) \neq 0) \wedge \neg\exists x(\beta(x) \neq 0)) \rightarrow (\neg\neg\exists x(\alpha(x) \neq 0) \vee \neg\neg\exists x(\beta(x) \neq 0))),$$

and that Σ^0_1-DML(called SEP) intuitionistically implies Π^0_1-DML. Therefore, a natural question occurs on the relation between Δ^0_1-LEM and Π^0_1-DML.

In conclusion, we provide the complete classification of the principles presented above. The reader is assumed to be familiar with a modicum of intuitionistic logic. We often suppress a cumbersome formal discussion and just give an informal proof.

The content of this chapter is a joint work with Hajime Ishihara and Takako Nemoto (Some of the contents will be contained in [25]).

6.1 Basic Results

Post's famous theorem from computability theory, which is the motivation for Δ^0_1-CA, states that if a set and its complement are both recursively enumerable, then the set is recursive. Troelstra and van Dalen [75, 4.5.3 and Exercise 4.5.1] discuss the abstract version of Post's Theorem PT:

$$\forall\alpha,\beta(\forall y(\neg\exists x(\alpha(y,x) \neq 0) \leftrightarrow \exists x(\beta(y,x) \neq 0)) \rightarrow \forall y(\exists x(\alpha(y,x) \neq 0) \vee \neg\exists x(\alpha(y,x) \neq 0))).$$

This looks very similar to our formulation of Δ^0_1-LEM. However, as we show below, PT is equivalent to MP whereas Δ^0_1-LEM is strictly weaker than MP. In the following, the subscript "c" is assigned to the **closed** variant and the subscript "u" is assigned to the **universal** variant.

Proposition 6.1.1. *The following are pairwise equivalent over* EL_0.

1. MP : $\forall\alpha(\neg\neg\exists x(\alpha(x) \neq 0) \rightarrow \exists x(\alpha(x) \neq 0))$.

2. PT^{c} : $\forall\alpha,\beta((\neg\exists x(\alpha(x)\neq 0)\leftrightarrow\exists x(\beta(x)\neq 0))\rightarrow(\exists x(\alpha(x)\neq 0)\vee\neg\exists x(\alpha(x)\neq 0)))$.
3. PT.

Proof. We reason in EL_0 (note that we do not use any law-of-excluded-middle principles and induction scheme in the following discussion).

We first show that MP implies PT^{c}. Suppose that $\neg\exists x(\alpha(x)\neq 0)\leftrightarrow\exists x(\beta(x)\neq 0)$. Define $\gamma:\mathbb{N}\rightarrow\mathbb{N}$ as

$$\gamma(x):=\begin{cases}1 & \text{if } \alpha(x)\neq 0\vee\beta(x)\neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

Then it is straightforward to see that $\neg\forall x(\gamma(x)=0)$, equivalently $\neg\neg\exists x(\gamma(x)\neq 0)$ holds. Therefore, by MP, we have $\exists x(\gamma(x)\neq 0)$, equivalently $\exists x(\alpha(x)\neq 0)\vee\exists x(\beta(x)\neq 0)$. In the latter case, we have $\neg\exists x(\alpha(x)\neq 0)$ by our assumption.

Next we show the converse direction. Suppose $\neg\neg\exists x(\alpha(x)\neq 0)$. Since now $\neg\exists x(\alpha(x)\neq 0)\leftrightarrow\exists x(\lambda n.0(x)\neq\lambda n.0(x))$ holds, by PT^{c}, we have $\exists x(\alpha(x)\neq 0)\vee\neg\exists x(\alpha(x)\neq 0)$. In the latter case, we have contradiction again by our assumption. Thus $\exists x(\alpha(x)\neq 0)$ holds.

To show that PT^{c} implies PT, consider the universal variant PT^{u}:

$$\forall y\forall\alpha,\beta((\neg\exists x\alpha(y,x)\neq 0\leftrightarrow\exists x\beta(y,x)\neq 0)\rightarrow(\exists x\alpha(y,x)\neq 0\vee\neg\exists x\alpha(y,x)\neq 0)).$$

Since PT^{u} implies PT, it suffices to show that PT^{c} implies PT^{u}. Fix y, α and β such that $\neg\exists x\alpha(y,x)\neq 0\leftrightarrow\exists x\beta(y,x)\neq 0$. Take α' as $\alpha'(x):=\alpha(y,x)$ and β' as $\beta'(x):=\beta(y,x)$. Then $\alpha(y,x)\neq 0\leftrightarrow\alpha'(x)\neq 0$ and $\beta(y,x)\neq 0\leftrightarrow\beta'(x)\neq 0$. Therefore, by PT^{c}, we have $\exists x\alpha(y,x)\neq 0\vee\neg\exists x\alpha(y,x)\neq 0$.

Finally we show that PT implies PT^{c}. Fix α and β such that $\neg\exists x\alpha(x)\neq 0\leftrightarrow\exists x\beta(x)\neq 0$. Take α' as $\alpha'(y,x):=\alpha(x)$ and β' as $\beta'(y,x):=\beta(x)$. Then for all y,

$$\alpha(x)\neq 0\leftrightarrow\alpha'(y,x)\neq 0 \text{ and } \beta(x)\neq 0\leftrightarrow\beta'(y,x)\neq 0. \tag{6.1}$$

Therefore, by PT, we have $\forall y(\exists x\alpha'(y,x)\neq 0\vee\neg\exists x\alpha'(y,x)\neq 0)$. Again by (6.1), we have $\exists x\alpha(x)\neq 0\vee\neg\exists x\alpha(x)\neq 0$. □

It follows from [1] that Σ^0_1-DML implies Δ^0_1-LEM over EL_0. The next proposition states that even the weaker principle Π^0_1-DML implies Δ^0_1-LEM.

Proposition 6.1.2. $\mathsf{EL}_0+\Pi^0_1\text{-DML}\vdash\Delta^0_1\text{-LEM}$.

Proof. We reason in EL_0. It suffices to show that Π^0_1-DML implies $\Delta^0_1\text{-LEM}^{\mathrm{c}}$:

$$\forall\alpha,\beta((\exists x(\alpha(x)\neq 0)\leftrightarrow\neg\exists x(\beta(x)\neq 0))\rightarrow(\exists x(\alpha(x)\neq 0)\vee\neg\exists x(\alpha(x)\neq 0)))$$

by the discussion in the proof of Proposition 6.1.1. Suppose $\exists x(\alpha(x) \neq 0) \leftrightarrow \neg\exists x(\beta(x) \neq 0)$. Then it is straightforward to see $\neg(\neg\exists x(\alpha(x) \neq 0) \wedge \neg\exists x(\beta(x) \neq 0))$. By Π^0_1-DML, we have $\neg\neg\exists x(\alpha(x) \neq 0) \vee \neg\neg\exists x(\beta(x) \neq 0)$. In the former case, $\neg\neg\exists x(\alpha(x) \neq 0)$ is equivalent to $\neg\neg\neg\exists x(\beta(x) \neq 0)$ by our assumption, which is intuitionistically equivalent to $\neg\exists x(\beta(x) \neq 0)$, which is equivalent to $\exists x(\alpha(x) \neq 0)$ again by our assumption. In the latter case, $\neg\neg\exists x(\beta(x) \neq 0)$ is equivalent to $\neg\exists x(\alpha(x) \neq 0)$ by our assumption. □

Remark 6.1.3. *It was recently shown by Kohlenbach [56] that* Δ^0_1-LEM *is strictly weaker than* Π^0_1-DML.

6.2 Equivalence between Δ^0_1-LEM and Δ^0_1-CA

Our original aim was to decompose Δ^0_1-CA to some logical axiom and some choice scheme in the spirit of **constructive** reverse mathematics [43, 3]. As we already mentioned at the beginning of this chapter, Δ^0_1-CA intuitionistically implies Δ^0_1-LEM. We discuss the converse direction. First we sketch the proof of Δ^0_1-CA from Δ^0_1-LEM using the countable choice scheme:

Suppose $\forall y(\exists x(\alpha(y,x) \neq 0) \leftrightarrow \neg\exists x(\beta(y,x) \neq 0))$. Then $\forall y(\exists x(\alpha(y,x) \neq 0) \vee \neg\exists x(\alpha(y,x) \neq 0))$ holds. By Δ^0_1-LEM, for all y, $\exists x(\alpha(y,x) \neq 0)$ is decidable, and hence there exists $z[y]^0$ such that $z = 0 \leftrightarrow \exists x(\alpha(y,x) \neq 0)$. Therefore, using the choice scheme, we have a function γ such that $\gamma(y) \leftrightarrow \exists x(\alpha(y,x) \neq 0)$ for all y.

This proof shows that if we have the choice scheme Π^0_1-$\mathrm{AC}^{0,0}$ for purely universal formulas, Δ^0_1-CA is derivable from Δ^0_1-LEM, since one can see that $z = 0 \leftrightarrow \exists x(\alpha(y,x) \neq 0)$ is intuitionistically equivalent to some purely universal formula using our assumption $\exists x(\alpha(y,x) \neq 0) \leftrightarrow \neg\exists x(\beta(y,x) \neq 0)$. However it is not clear that Δ^0_1-CA is derivable from Δ^0_1-LEM without the use of the choice scheme. Despite the fact, we show in the following that Δ^0_1-CA is equivalent to Δ^0_1-LEM over EL_0 using modified realizability interpretation.

Theorem 6.2.1. $\mathsf{EL}_0 \vdash \Delta^0_1$-CA $\leftrightarrow \Delta^0_1$-LEM.

Proof. $\mathsf{EL}_0 \vdash \Delta^0_1$-CA $\rightarrow \Delta^0_1$-LEM is obvious. In the following, we show $\mathsf{EL}_0 \vdash \Delta^0_1$-LEM $\rightarrow \Delta^0_1$-CA. First note that as in the proof of Proposition 6.1.1, we have the equivalence (over $\widehat{\mathsf{E\text{-}HA}}^\omega\!\upharpoonright$) between Δ^0_1-CA, the closed variant Δ^0_1-CA^{c}:

$$\forall \alpha, \beta((\exists x \alpha(x) \neq 0 \leftrightarrow \neg\exists x \beta(x) \neq 0) \rightarrow \exists s(s = 0 \leftrightarrow \exists x \alpha(x) \neq 0))$$

and the universal variant Δ^0_1-CA^{u}:

$$\forall \alpha, \beta, y((\exists x \alpha(x,y) \neq 0 \leftrightarrow \neg\exists x \beta(x,y) \neq 0) \rightarrow \exists \gamma(\gamma(y) = 0 \leftrightarrow \exists x \alpha(x,y) \neq 0)).$$

On the other hand, Δ^0_1-CA$^{\mathrm{u}}$ implies Δ^0_1-CA$^{\mathrm{u}}_-$:

$$\forall \alpha, \beta, y \left(\begin{array}{l} \exists x'' \forall x' (\alpha(x', y) \neq 0 \to \neg \exists x \beta(x, y) \neq 0 \wedge \neg \exists x \beta(x, y) \neq 0 \to \alpha(x'', y) \neq 0) \\ \to \exists \gamma (\gamma(y) = 0 \leftrightarrow \exists x \alpha(x, y) \neq 0) \end{array} \right).$$

Thus $\widehat{\mathsf{E\text{-}HA}}^{\omega}{\restriction} + \Delta^0_1\text{-CA} \vdash \Delta^0_1\text{-CA}^{\mathrm{u}}_-$. Since Δ^0_1-CA is mr-interpretable in $\widehat{\mathsf{E\text{-}HA}}^{\omega}{\restriction}$ (See [56, Lemma 3]), there exists a term t of $\widehat{\mathsf{E\text{-}HA}}^{\omega}{\restriction}$ such that $\widehat{\mathsf{E\text{-}HA}}^{\omega}{\restriction} \vdash t \text{ mr } \Delta^0_1\text{-CA}^{\mathrm{u}}_-$. Since Δ^0_1-CA$^{\mathrm{u}}_-$ is in Γ_1, by [55, Lemma 5.20], we have

$$\widehat{\mathsf{E\text{-}HA}}^{\omega}{\restriction} \vdash \Delta^0_1\text{-CA}^{\mathrm{u}}_-. \tag{6.2}$$

As in the proof of Proposition 6.1.1, one can see that Δ^0_1-LEM implies

$$\Delta^0_1\text{-LEM}^{\mathrm{u}}: \ \forall \alpha, \beta, y((\exists x \alpha(y, x) \neq 0 \leftrightarrow \neg \exists x \beta(y, x) \neq 0) \to (\exists x \alpha(y, x) \neq 0 \vee \neg \exists x \alpha(y, x) \neq 0)).$$

In addition, one can see without difficulty that Δ^0_1-LEM$^{\mathrm{u}}$ implies IP$(\Delta^0_1, \infty)^{\mathrm{c}}$:

$$\forall \alpha, \beta, y \left(\begin{array}{l} ((\exists x \alpha(y, x) \neq 0 \leftrightarrow \neg \exists x \beta(y, x) \neq 0) \wedge (\neg \exists x \beta(y, x) \neq 0 \to \exists z \psi(z, y))) \\ \to \exists z (\neg \exists x \beta(y, x) \neq 0 \to \psi(z, y)) \end{array} \right).$$

Since Δ^0_1-CA$^{\mathrm{u}}_-$ implies Δ^0_1-CA$^{\mathrm{u}}$ assuming IP$(\Delta^0_1, \infty)^{\mathrm{c}}$, we have $\widehat{\mathsf{E\text{-}HA}}^{\omega}{\restriction} + \Delta^0_1\text{-LEM} \vdash \Delta^0_1\text{-CA}^{\mathrm{u}}_- \to \Delta^0_1\text{-CA}$. Combining this with (6.2), we have $\widehat{\mathsf{E\text{-}HA}}^{\omega}{\restriction} + \Delta^0_1\text{-LEM} \vdash \Delta^0_1\text{-CA}$. Therefore, by the conservativity of $\widehat{\mathsf{E\text{-}HA}}^{\omega}{\restriction}$ over EL_0 (Proposition 2.2.24), $\mathsf{EL}_0 \vdash \Delta^0_1\text{-LEM} \to \Delta^0_1\text{-CA}$ follows. □

6.3 Underivability of Δ^0_1-LEM from WMP

In this section, we show that Δ^0_1-LEM is not derivable from WMP. Our result itself does not conflict with classical mathematics. In the proof, however, we use some principles from constructive mathematics, which are false in classical mathematics. In addition, all proofs are completely syntactical, namely, we provide a constructive proof that WMP does not imply Δ^0_1-LEM from a meta-perspective.

Note that HA is the usual system of intuitionistic first-order (Heyting) arithmetic in [72, Section 1.3]. We first recall some definitions, where T denotes Kleene's (primitive recursive) T-predicate and U denotes the (primitive recursive) result-extracting function, namely, $T(x, y, z)$ expresses that the Turing machine with Gödel number x applied to the input y terminates with a computation whose Gödel number is z and $U(z)$ is its output (See e.g. [75] for more information).

- MP$_{\mathrm{PR}}$: $\neg\neg\exists x(t(x) \neq 0) \to \exists x(t(x) \neq 0)$, where t is a (primitive recursive) term of EL.

- $\mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}}$: $\neg\neg\exists x(t(x) \neq 0) \to \exists x(t(x) \neq 0)$, where t is a closed (primitive recursive) term of EL (following the notation in [72]).

- CT (Church's Thesis) : $\forall\alpha\exists e\forall y\exists z(T(e,y,z) \wedge \alpha(y) = U(z))$.

- CT_0 : $\forall x\exists y B(x,y,\underline{z}) \to \exists u\forall x\exists v(T(u,x,v) \wedge B(x,U(v),\underline{z}))$,
  where $\underline{z}$ stands for a tuples of number variables

- ECT_0 (Extended Church's Thesis) :
  $\forall x(A(x,\underline{z}) \to \exists y B(x,y,\underline{z})) \to \exists u\forall x(A(x,\underline{z}) \to \exists v(T(u,x,v) \wedge B(x,U(v),\underline{z})))$,
  where $\underline{z}$ stands for a tuples of number variables and $A(x,\underline{z})$ is almost negative.

Warning. We treat the axiom scheme ECT_0 for first-order arithmetic HA also in second-order arithmetic (EL or EL_0). That is, we mean by CT_0 and ECT_0 Church's thesis and the extended Church's thesis only for $\mathcal{L}(\mathsf{HA})$-formulas instead of those for $\mathcal{L}(\mathsf{EL})$-formulas.

Definition 6.3.1. We consider the following slightly extended variants of CT_0 and ECT_0 in second-order arithmetic.

- CT_0^+ : $\forall x\exists y B(x,y,\underline{\alpha}) \to \exists u\forall x\exists v(T(u,x,v) \wedge B(x,U(v),\underline{\alpha}))$,
  where $\underline{\alpha}$ stands for a tuples of function variables, $A(x,\underline{\alpha})$ is almost negative and all of quantifiers in B are number quantifiers.

- ECT_0^+ : $\forall x(A(x,\underline{\alpha}) \to \exists y B(x,y,\underline{\alpha})) \to \exists u\forall x(A(x,\underline{\alpha}) \to \exists v(T(u,x,v) \wedge B(x,U(v),\underline{\alpha})))$,
  where $\underline{\alpha}$ stands for a tuples of function variables, $A(x,\underline{\alpha})$ is almost negative (See Definition 2.3.6) and all of quantifiers in A and B are number quantifiers.

Remark 6.3.2. ECT_0^+ *implies* CT_0^+.

Proposition 6.3.3. $\mathsf{EL}_0 \vdash \mathrm{MP} \to \mathrm{MP}_{\mathrm{PR}} \to \mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}}$.

Proof. Straightforward. □

Proposition 6.3.4. $\mathsf{EL}_0 + \mathrm{CT}_0^+ \vdash \mathrm{WMP}$.

Proof. By inspecting the proof of Lemma 1 and Proposition 2 in [42]. □

Proposition 6.3.5 (due to Hajime Ishihara)**.** $\mathsf{EL}_0 + \mathrm{ECT}_0^+ + \Delta^0_1\text{-LEM} \vdash \Pi^0_1\text{-DML}$.

Proof. We first show that $\mathsf{EL}_0 + \mathrm{ECT}_0^+ + \Delta^0_1$-LEM proves

$$\forall\alpha,\beta((\neg\neg\exists x(\alpha(x)) \neq 0 \leftrightarrow \neg\beta(x) \neq 0) \to (\neg\neg\exists x(\alpha(x)) \neq 0 \vee \neg\exists x(\alpha(x)) \neq 0)). \qquad \text{(K)}$$

Note that Δ^0_1-LEM implies Δ^0_1-LEM$^{\mathrm{c}}$:

$$\forall \alpha, \beta((\exists x(\alpha(x) \neq 0) \leftrightarrow \neg\exists x(\beta(x) \neq 0)) \to (\exists x(\alpha(x) \neq 0) \vee \neg\exists x(\alpha(x) \neq 0))).$$

We reason in $\mathsf{EL}_0 + \mathrm{ECT}_0^+ + \Delta^0_1$-LEM. Suppose $\neg\neg\exists x(\alpha(x) \neq 0) \leftrightarrow \neg\exists x(\beta(x) \neq 0)$. If $\neg\neg\exists x(\alpha(x) \neq 0) \to \exists x(\alpha(x) \neq 0)$, then $\exists x(\alpha(x)) \neq 0 \leftrightarrow \neg\exists x(\beta(x) \neq 0)$. Hence, by Δ^0_1-LEM$^{\mathrm{c}}$, we have

$$(\neg\neg\exists x(\alpha(x) \neq 0) \to \exists x(\alpha(x) \neq 0)) \to (\exists x(\alpha(x) \neq 0) \vee \neg\exists x(\alpha(x) \neq 0)).$$

Since the premise of the previous formula is almost negative and all quantifiers are type 0, ECT_0^+ yields a number e such that

$$\forall m \left(\begin{array}{l} (\neg\neg\exists x(\alpha(x) \neq 0) \to \exists x(\alpha(x) \neq 0)) \to \\ \exists u(T(e, m, u) \wedge (U(u) = 0 \to \exists x(\alpha(x) \neq 0)) \wedge (U(u) \neq 0 \to \neg\exists x(\alpha(x) \neq 0))) \end{array} \right). \tag{6.3}$$

Consider the primitive recursive function $\gamma^e : \mathbb{N} \to \mathbb{N}$ such that

$$\gamma^e(u) := \begin{cases} 1 & \text{if } T(e, 0, u) \wedge U(u) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In the following, we show the equivalence between $\exists u(\gamma^e(u) \neq 0)$ and $\neg\exists x(\alpha(x) \neq 0)$. Suppose $\neg\exists x(\alpha(x) \neq 0)$. Then $\neg\neg\exists x(\alpha(x) \neq 0) \to \exists x(\alpha(x) \neq 0)$ holds, and hence by (6.3) there exists u_0 such that

$$T(e, 0, u_0) \wedge (U(u_0) = 0 \to \exists x(\alpha(x) \neq 0)) \wedge (U(u_0) \neq 0 \to \neg\exists x(\alpha(x) \neq 0)).$$

Therefore, we have $T(e, 0, u_0) \wedge U(u_0) \neq 0$, and so $\gamma^e(u_0) \neq 0$. For the converse direction, suppose $\gamma^e(u') \neq 0$ and $\exists x(\alpha(x) \neq 0)$. Then $T(e, 0, u') \wedge U(u') \neq 0$ holds. On the other hand, since $\neg\neg\exists x(\alpha(x) \neq 0) \to \exists x(\alpha(x) \neq 0)$ holds, again by (6.3), there exists u'' such that

$$T(e, 0, u'') \wedge (U(u'') = 0 \to \exists x(\alpha(x) \neq 0)) \wedge (U(u'') \neq 0 \to \neg\exists x(\alpha(x) \neq 0)).$$

By the uniqueness of u in $T(e, 0, u)$, $u' = u''$ holds. Therefore we have $U(u'') \neq 0$, and hence $\neg\exists x(\alpha(x) \neq 0)$. This contradicts our assumption $\exists x(\alpha(x) \neq 0)$.

Therefore, by Δ^0_1-LEM$^{\mathrm{c}}$, we have $\exists u(\gamma^e(u) \neq 0) \vee \neg\exists u(\gamma^e(u) \neq 0)$, equivalently, $\neg\exists x(\alpha(x) \neq 0) \vee \neg\neg\exists x(\alpha(x) \neq 0)$. Thus (K) is proved.

To complete the proof of our proposition, it suffices to show that (K) implies Π^0_1-DML. Suppose that $\neg(\neg\exists x(\alpha(x) \neq 0) \wedge \neg\exists x(\beta(x) \neq 0))$. In EL_0, one can define $\alpha', \beta' : \mathbb{N} \to \mathbb{N}$ such that

$$\alpha'(x) := \begin{cases} 1 & \text{if } \alpha(x) \neq 0 \wedge \forall k < x(\alpha(k) = 0 \wedge \beta(k) = 0), \\ 0 & \text{otherwise;} \end{cases}$$

$$\beta'(x) := \begin{cases} 1 & \text{if } \alpha(x) = 0 \wedge \beta(x) \neq 0 \wedge \forall k < x(\alpha(k) = 0 \wedge \beta(k) = 0), \\ 0 & \text{otherwise.} \end{cases}$$

Then obviously $\exists x(\alpha'(x) \neq 0) \to \neg\exists x(\beta'(x) \neq 0)$ holds. On the other hand, since $\neg\exists x(\alpha'(x) \neq 0) \wedge \neg\exists x(\beta'(x) \neq 0)$ implies $\neg\exists x(\alpha(x) \neq 0) \wedge \neg\exists x(\beta(x) \neq 0)$, we have $\neg\exists x(\beta'(x) \neq 0) \to \neg\neg\exists x(\alpha'(x) \neq 0)$ by our assumption. Thus $\neg\neg\exists x(\alpha'(x) \neq 0) \leftrightarrow \neg\exists x(\beta'(x) \neq 0)$ holds. By (K), we have $\neg\neg\exists x(\alpha'(x) \neq 0) \vee \neg\exists x(\alpha'(x) \neq 0)$. Since $\exists x(\alpha'(x) \neq 0) \to \exists x(\alpha(x) \neq 0)$ and $\exists x(\beta'(x) \neq 0) \to \exists x(\beta(x) \neq 0)$, it is straightforward to see $\neg\neg\exists x(\alpha(x) \neq 0) \vee \neg\neg\exists x(\beta(x) \neq 0)$. □

Proposition 6.3.6 (Proposition 1(1) in [42]). $\mathsf{EL} \vdash \mathrm{MP} \leftrightarrow \mathrm{WMP} + \Pi^0_1\text{-DML}$.

In the following, we shall show that $\mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}}$ is not provable in $\mathsf{EL} + \mathrm{ECT}^+_0$. Combined this with the above propositions, we have that Δ^0_1-LEM is not derivable from WMP (Theorem 6.3.11).

Lemma 6.3.7. $\mathsf{EL}_0 \vdash \mathrm{ECT}^+_0 \leftrightarrow \mathrm{ECT}_0 + \mathrm{CT}$.

Proof. We reason in EL_0. It is obvious that ECT^+_0 implies ECT_0. In addition, one can easily see that ECT^+_0 implies CT by taking $A(x, \alpha)$ as $0 = 0$ and $B(x, y, \alpha)$ as $\alpha(x) = y$.

We show that ECT^+_0 can be derived from ECT_0 and CT. Without loss of generality, one can assume $\forall x(A(x, \alpha) \to \exists y B(x, y, \alpha))$ with only one function parameter α. For each prime subformula $t[x, y, \underline{w}, \alpha] = 0$ in $A(x, \alpha)$ or $B(x, y, \alpha)$[2], by (the proof of) Lemma 2.2.18, there exists an equivalent formula of the form $\exists n \forall \underline{i} < \underline{m}[x, y, \underline{w}, \alpha](t'[n, \underline{i}, x, y, \underline{w}, \alpha] = 0)$ where $\underline{m}$ and t' contain neither recursors nor λ-operators. Then α occurs in t' and $\underline{m}$ only in the form of $(\alpha(y))^0$ since t' contains neither recursors nor λ-operators. On the other hand, there exists e such that

$$\forall y \exists z(T(e, y, z) \wedge \alpha(y) = U(z)) \tag{6.4}$$

by CT. By (6.4) together with the uniqueness of z in $T(e, y, z)$ and Lemma 2.2.11, it is not hard to see that

$$A[\alpha(y)] \leftrightarrow \exists z(T(e, y, z) \wedge A[U(z)/\alpha(y)]) \text{ for any formula } A[\alpha(y)]. \tag{6.5}$$

In the following, we claim that there is a (primitive recursive) function symbol f of EL such that $\forall \underline{i} < \underline{m}[x, y, \underline{w}, \alpha](t'[n, \underline{i}, x, y, \underline{w}, \alpha] = 0) \leftrightarrow \exists z f(z, n, x, y, \underline{w}, e) = 0$. We only discuss in the

[2]Note that all prime subformulas in $A(x, \alpha)$ or $B(x, y, \alpha)$ do not contain function parameters except α since all of quantifiers in A and B are number quantifiers.

case that $\underline{i}$ and $\underline{m}$ are single tuples for simplicity (one can show the case for k-tuples just by k iterations of the discussion below). By repeatedly applying (6.5) for each occurrence of α in t', we have that $t'[n,i,x,y,\underline{w},\alpha]=0$ is equivalent to some formula $\exists z A_{qf}(z,n,i,x,y,\underline{w},e)$ and contains neither recursors nor λ-operators. Therefore there is a primitive recursive function symbol f_0 of EL such that $A_{qf}(z,n,i,x,y,\underline{w},e)\leftrightarrow f_0(z,n,i,x,y,\underline{w},e)=0$ (cf. [55, Proposition 3.8]). Thus $\forall i<m(t'[n,i,x,y,\underline{w},\alpha]=0)$ is equivalent to $\forall i<m\exists z(f_0(z,n,i,x,y,\underline{w},e)=0)$, which is equivalent to $\exists z'\forall i<m\exists z<z'(f_0(z,n,i,x,y,\underline{w},e)=0)$ by Lemma 2.2.15. By the proof of Lemma 2.2.12, there is an primitive recursive function symbol f_1 of EL such that $\forall i<m\exists z<z'(f_0(z,n,i,x,y,\underline{w},e)=0)\leftrightarrow f_1(m,z',n,x,y,\underline{w},e)=0$. Note that m may contain α. Again by repeatedly applying (6.5) for each occurrence of α in m, we have that $f_1(m,z',n,x,y,\underline{w},e)=0$ is equivalent to some formula $\exists z'' B_{qf}(z'',z',n,x,y,\underline{w},e)$, and hence there is a primitive recursive function symbol f_2 such that $B_{qf}(z'',z',n,x,y,\underline{w},e)\leftrightarrow f_2(z'',z',n,x,y,\underline{w},e)=0$ as before. Consequently, it follows that $\forall i<m(t'[n,i,x,y,\underline{w},\alpha]=0)$ is equivalent to $\exists z'\exists z'' f_2(z'',z',n,x,y,\underline{w},e)=0$, which establishes our claim.

Therefore, each prime subformula $t[x,y,\underline{w},\alpha]=0$ in $A(x,\alpha)$ or $B(x,y,\alpha)$ is equivalent to some purely existential $\mathcal{L}(\mathsf{HA})$-formula containing e. Applying this procedure to all prime subformulas in $A(x,\alpha)$ and $B(x,y,\alpha)$, we obtain the equivalent $\mathcal{L}(\mathsf{HA})$-formulas $A'(x,e)$ and $B'(x,y,e)$ respectively. In addition, by the construction above, $A'(x,e)$ is still almost negative. Therefore, by ECT_0, we have ECT_0^+. □

Lemma 6.3.8 (due to Takako Nemoto)**.** $\mathsf{HA}+\mathrm{ECT}_0\nvdash \mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}}$

Proof. Suppose that $\mathsf{HA}+\mathrm{ECT}_0\vdash \mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}}$. Then by [72, Theorem 3.2.18(ii)], $\mathsf{HA}\vdash\exists u(u\ \mathrm{r}\ \mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}})$. Thus $\forall v((v\ \mathrm{r}\ \neg\neg\exists x(t(x)\neq 0))\to !\{u\}(v)\wedge\{u\}(v)\ \mathrm{r}\ \exists x(t(x)\neq 0))$ holds. On the other hand, since $\neg\neg\exists x(t(x)\neq 0)$ is almost negative, $\mathsf{HA}\vdash\neg\neg\exists x(t(x)\neq 0)\to\exists v(v\ \mathrm{r}\ \neg\neg\exists x(t(x)\neq 0))$ by [72, Lemma 3.2.11]. Therefore $\mathsf{HA}\vdash\neg\neg\exists x(t(x)\neq 0)\to\exists u,v(!\{u\}(v)\wedge\{u\}(v)\ \mathrm{r}\ \exists x(t(x)\neq 0))\to\exists x(t(x)\neq 0)$. Thus $\mathsf{HA}\vdash\mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}}$, which is a contradiction [72, 1.11.5]. □

Lemma 6.3.9. $\mathsf{EL}+\mathrm{ECT}_0^+\nvdash\mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}}$.

Proof. Suppose $\mathsf{EL}+\mathrm{ECT}_0^+\vdash\mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}}$. Then by Lemma 6.3.7, $\mathsf{EL}+\mathrm{CT}+\mathrm{ECT}_0\vdash\mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}}$. Therefore, there exists a conjunction F of closed instances of ECT_0 such that $\mathsf{EL}+\mathrm{CT}\vdash F\to\mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}}$ (using the deduction theorem). Since $\mathsf{EL}+\mathrm{CT}$ is conservative over HA for $\mathcal{L}(\mathsf{HA})$-formulas (cf. [72, Theorem 3.6.2]), we have $\mathsf{HA}\vdash F\to\mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}}$, and hence $\mathsf{HA}+\mathrm{ECT}_0\vdash\mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}}$ follows. This contradicts Lemma 6.3.8. □

Proposition 6.3.10. $\mathsf{EL}+\mathrm{ECT}_0^+\nvdash\Delta^0_1$-LEM.

Proof. Suppose $\mathsf{EL}+\mathrm{ECT}_0^+\vdash\Delta^0_1$-LEM. Then by Proposition 6.3.5, $\mathsf{EL}+\mathrm{ECT}_0^+\vdash\Pi^0_1$-DML. By Proposition 6.3.4, Proposition 6.3.6 and Remark 6.3.2, we have $\mathsf{EL}+\mathrm{ECT}_0^+\vdash\mathrm{MP}$. This contradicts the combination of Lemma 6.3.9 with Proposition 6.3.3. □

Theorem 6.3.11. EL + WMP $\nvdash$ Δ^0_1-LEM.

Proof. Suppose EL + WMP $\vdash$ Δ^0_1-LEM. Then EL + ECT^+_0 $\vdash$ Δ^0_1-LEM by Proposition 6.3.4 and Remark 6.3.2. This contradicts Proposition 6.3.10. □

Remark 6.3.12. *Using the extensional model* ECF *of the hereditarily continuous functionals, one can show the conservativity of* $\mathsf{E\text{-}HA}^\omega$ + QF-AC + CT *over* EL + CT *(cf. [72, Section 2.6]). Therefore, Lemma 6.3.9 can be extended to* $\mathsf{E\text{-}HA}^\omega$ + QF-AC + ECT^+_0 $\nvdash$ $\mathrm{MP}^{\mathrm{c}}_{\mathrm{PR}}$*, and hence,* $\mathsf{E\text{-}HA}^\omega$ + QF-AC + ECT^+_0 $\nvdash$ Δ^0_1-LEM *and* $\mathsf{E\text{-}HA}^\omega$ + QF-AC + WMP $\nvdash$ Δ^0_1-LEM *follows in the same manner.*

Remark 6.3.13. *Recently, Hendtlass and Lubarsky [33] have shown the underivability of* MP *from* WMP *by using a semantical method. In fact, there is an alternative proof for the underivability of* Δ^0_1-LEM *from* WMP *in the semantical method ([62]).*

On the other hand, the converse underivability, namely, the underivability of WMP from Δ^0_1-LEM follows from the next strong result due to Kohlenbach.[3]

Proposition 6.3.14 ([52])**.** EL + Π^0_1-LEM $\nvdash$ WMP.

6.4 Conclusion and Questions

In conclusion, we summarize the established interrelations between logical principles in Figure 6.1.

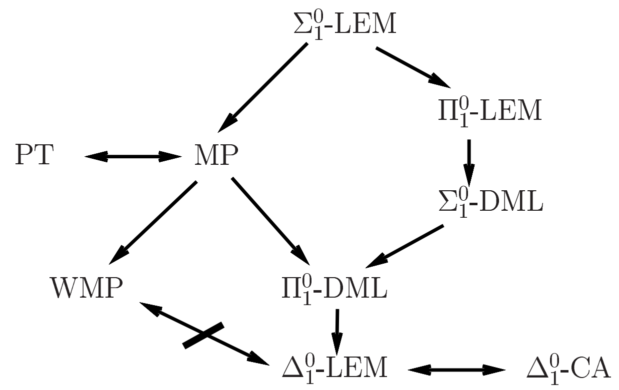

Figure 6.1: Interrelations between Logical Principles over EL

[3]Proposition 6.3.14 is optimal in the sense that Σ^0_1-LEM already imply MP, and hence WMP.

Question 6.4.1. *In the formulation of* RCA_0*, is there any role of* Δ^0_1-CA *more than just ensuring that the universe of sets (or functions) is closed under "recursive in"? In fact, for the function-based systems,* QF-AC^{0,0} *requires that the universe of functions is closed under "recursive in". Then our concrete question is the following:*

Is there some mathematical statement which is equivalent to Δ^0_1-CA *over* EL_0 *(or* EL*)?*

Question 6.4.2. *As we already mentioned at the beginning of this chapter, the existing results suggest that intermediate principles in between* RCA_0 *and* WKL_0 *correspond to some logical axioms weaker than* Σ^0_1-DML*. Then a possible question is the following*:

What is the classical (computational) counterpart of Π^0_1-DML*?*

In fact, the author thinks that it is still open whether the logical principle corresponding to RCA_0 *is* Δ^0_1-LEM *because of the issue mentioned in the previous question.*

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