Well-posedness in the critical space for the compressible Navier-Stokes equations and related problems

博　士　論　文

Well-posedness in the critical space for the compressible Navier-Stokes equations and related problems

(圧縮性 Navier-Stokes 方程式と関連する問題の臨界空間における適切性)

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Well-posedness in the critical space for the compressible Navier-Stokes equations and related problems

A thesis presented

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Summary

1.1 Compressible Navier-Stokes equations

The mathematical theory of the compressible fluid dynamics has been developed greatly in the past few years, for which the harmonic analysis has played a significant role. For the incompressible Navier-Stokes system in the whole space, regarding the system as a perturbation of the heat equation through Fourier analysis has been especially effective. The studies in recent years show that this is also true for the compressible viscous fluid. In this thesis, we consider the Cauchy problems of the compressible Navier-Stokes equations coupled with various physical perturbations and investigate their well-posedness issues in the critical spaces or *near-critical* Besov spaces.

We note that the compressible fluid models, in most cases, have the property of *quasi*-scale invariance, that is the system is left invariant under a certain dilation if we ignore the pressure. The idea of *critical regularity framework* is to solve a system of partial differential equations in a functional space where the space and time dilations remains invariant. Like the critical theory for the incompressible fluids originated from [19], the critical framework has proven extremely useful to obtain solutions with low regularity for the compressible fluids. The critical theory for the compressible viscous fluid in the Besov framework was initiated by Danchin [12]. In the barotropic case, Danchin [12] considered the critical Besov regularity in the L^2-type framework to obtain a global solution for small perturbations of a stable constant state. Since then, there have been a number of refinements as regards admissible exponents for the global existence (see [3, 6] and the references therein). The local-in-time existence issue in the critical regularity framework with large initial data (and initial density bounded away from 0) has been addressed in the barotropic case. Here, we shall see the various influences caused by a density-induced potential perturbations as well as thermic perturbations in regards the well-posedness issues of the compressible viscous model.

1.2 Blow-up criterion for the barotropic compressible Navier-Stokes equations coupled with a Yukawa potential

We first consider the Cauchy problem of the compressible Navier-Stokes system *with and without* the Yukawa-type potential term in the whole space $\mathbb{R}^n$ $(n \geq 2)$:

$$
\begin{cases}
\partial_t \rho + \mathrm{div}\,(\rho u) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\
\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla P(\rho) + \gamma \rho \nabla \psi & \\
\qquad = \mu \Delta u + (\mu + \lambda) \nabla \mathrm{div}\, u, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\
-\Delta \psi + \psi = \rho - 1, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\
(\rho, u)|_{t=0} = (\rho_0, u_0), & x \in \mathbb{R}^n.
\end{cases}
\tag{1.2.1}
$$

where $\rho = \rho(t,x)$, $u = u(t,x)$ and $\psi = \psi(t,x)$ are the unknown functions representing the density, the velocity vector field of the fluid and the potential force exerted in the fluid, respectively. The pressure $P = P(\rho)$ is given by a smooth function only depending on ρ. The deformation tensor $D(u)$ is given by $D(u) = \frac{1}{2}(Du + \nabla u)$ with $(Du)_{ij} = \partial_j u_i$ and $(\nabla u)_{ij} = ({}^t Du)_{ij} = \partial_i u_j$. The symbol $\otimes$ denotes the tensor product of two vectors. The constants μ, λ are called Lamé coefficients satisfying $\mu > 0$ and $\lambda + 2\mu > 0$, which ensure that the second order operator $2\mu \mathrm{div}\,(D(\cdot)) + \lambda\nabla(\mathrm{div}\,\cdot) = \mu\Delta + (\mu+\lambda)\nabla\mathrm{div}$ is of elliptic type. The constant $\gamma \in \mathbb{R}$ may be arbitrary. Namely, the barotropic compressible Navier-Stokes system is also considered here ($\gamma = 0$). The role of the constant γ is essential on its sign, therefore we especially choose either $\gamma = 0$ or ± 1 without losing generality. When $\gamma = 0$, as discussed in the previous Sections, system (1.2.1) describes the motion of a barotropic viscous compressible flow.

We assume that the density ρ is bounded away from 0 and tends to 1 at infinity (thus, we do not allow the presence of vacuum, or *cavity*). We may also treat the case of density-dependent coefficients, but for simplicity, we restrict ourselves to the constant case in this thesis.

Let us introduce the general Besov spaces as follows: Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition. Namely, $\{\phi_j\}_{j\in\mathbb{Z}}$ is a decomposition of unity produced by $\widehat{\phi} \in \mathcal{S}$, a non-negative radially symmetric function that satisfies $\mathrm{supp}\,\widehat{\phi} \subset \{\xi \in \mathbb{R}^n; 2^{-1} < |\xi| < 2\}$,

$$\widehat{\phi_j}(\xi) := \widehat{\phi}(2^{-j}\xi) \text{ (for all } j \in \mathbb{Z}) \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \widehat{\phi_j}(\xi) = 1 \text{ (for all } \xi \neq 0).$$

We further set $\widehat{\Phi}(\xi) := 1 - \sum_{j\geq 1} \widehat{\phi_j}(\xi)$.

Definition 1 (Besov spaces)**.** Let $\mathcal{S}'$ be the space of tempered distributions. For $s \in \mathbb{R}$ and $1 \leq p, \sigma \leq \infty$, we define the *inhomogeneous Besov spaces* $B^s_{p,\sigma} = B^s_{p,\sigma}(\mathbb{R}^n)$ as follows:

$$B^s_{p,\sigma}(\mathbb{R}^n) := \{u \in \mathcal{S}' \; ; \; \|u\|_{B^s_{p,\sigma}} < \infty\},$$

where

$$\|u\|_{B^s_{p,\sigma}} := \begin{cases} \|\Phi * u\|_{L^p} + (\sum_{j\geq 1} 2^{js\sigma}\|\phi_j * u\|^\sigma_{L^p})^{\frac{1}{\sigma}}, & 1 \leq \sigma < \infty, \\ \|\Phi * u\|_{L^p} + \sup_{j\geq 1} 2^{js}\|\phi_j * u\|_{L^p}, & \sigma = \infty. \end{cases}$$

Let $\mathcal{P}$ be the space of all polynomials, then for $s \in \mathbb{R}$ and $1 \leq p, \sigma \leq \infty$ we define the *homogeneous Besov spaces* $\dot{B}^s_{p,\sigma} = \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ as follows:

$$\dot{B}^s_{p,\sigma}(\mathbb{R}^n) := \{u \in \mathcal{S}'/\mathcal{P} \; ; \; \|u\|_{\dot{B}^s_{p,\sigma}} < \infty\},$$

where

$$\|u\|_{\dot{B}^s_{p,\sigma}} := \begin{cases} (\sum_{j\in\mathbb{Z}} 2^{js\sigma}\|\phi_j * u\|^\sigma_{L^p})^{\frac{1}{\sigma}}, & 1 \leq \sigma < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\phi_j * u\|_{L^p}, & \sigma = \infty. \end{cases}$$

We introduce the *specific volume* $a = a(t,x)$ defined by $a := \rho^{-1} - 1$ and redefine the initial data as $a_0 := \rho_0^{-1} - 1$ accordingly. This transformation is justified by the assumption on the density. Then the equations for (a,u) are given by the following:

$$\begin{cases} \partial_t a + u \cdot \nabla a = (1+a) \mathrm{div}\, u, \\ \partial_t u - (1+a)(\mu\Delta + (\lambda+\mu)\nabla \mathrm{div}\,)u = -u\cdot\nabla u - \nabla(Q(a)) - \gamma\nabla\psi + f, \\ -\Delta\psi + \psi = \dfrac{a}{1+a}, \\ (a,u)|_{t=0} = (a_0, u_0). \end{cases} \tag{1.2.2}$$

Here, Q is a smooth function determined by P, namely, $Q(a) := -\displaystyle\int_0^t \frac{P'((1+z)^{-1})}{(1+z)^2} dz$.

We define the *critical* space for the system (1.2.2) as follows:

$$a \in L^\infty(0,T; B^{\frac{n}{p}}_{p,1}),\ u \in L^\infty(0,T; B^{\frac{n}{p}-1}_{p,1}),\ f \in L^1_{loc}\big(\mathbb{R}_+; B^{\frac{n}{p}-1}_{p,1}\big).$$

By modifying the approach used in [16] and [17], we may prove the following result concerning the existence and uniqueness of the solution for (1.2.2).

Theorem 1.2.1. *Let $\gamma \in \mathbb{R}$, $n \geq 2$ and $1 < p < 2n$. Assume that $f \in L^1_{loc}\big(\mathbb{R}_+; B^{\frac{n}{p}-1}_{p,1}\big)$, $a_0 \in B^{\frac{n}{p}}_{p,1}$ with $1 + a_0 \geq \underline{a} > 0$ and $u_0 \in B^{\frac{n}{p}-1}_{p,1}$. Then, there exists a positive time $T > 0$ such that system* (1.2.2) *has a unique solution (a,u,ψ) satisfying*

$$(a,u,\psi) \in \mathcal{C}([0,T); B^{\frac{n}{p}}_{p,1}) \times \Big(\mathcal{C}([0,T); B^{\frac{n}{p}-1}_{p,1}) \cap L^1(0,T; B^{\frac{n}{p}+1}_{p,1})\Big) \times \mathcal{C}([0,T); B^{\frac{n}{p}+2}_{p,1}).$$

Furthermore there exists some constant $C(\underline{a}) > 0$ depending only on $\underline{a}$ such that for all $(t,x) \in [0,T) \times \mathbb{R}^n$, $1 + a(t,x) > C(\underline{a})$.

Theorem 1.2.1 is known for the case $\gamma = 0$ (see [16] and [21]). Needless to say, the result above is not our main novelty in this Section. The proof of Theorem 1.2.1 is carried out similarly to the papers such as [13, 15, 21].

The main problem in this section is whether the local solution in Theorem 1.2.1 loses its regularity beyond the maximal existence time. In other words, we seek for a non-trivial sufficient condition to extend the time-local solution for (1.2.2) beyond its existence time T. For the incompressible Navier-Stokes system, the local existence of strong solution has been known since [19] and the sufficient conditions for the blow-up of such local solution were characterized by many authors. Beale-Kato-Majda [1] first showed the blow-up criterion for the incompressible Euler equations in terms of vorticity (which is applicable to the incompressible Navier-Stokes system as well). Since then, a number of significant refinements has been made, thanks to the progress in the tools of harmonic analysis. Kozono-Ogawa-Taniuchi [22] gave the sharpest criterion, in the sense that the L^∞ norm of the vorticity in [1] is replaced by the Besov norm $\dot{B}^0_{\infty,\infty}$. On the other hand, there are works that characterize the blow-up condition by certain scale-invariant quantities ([2, 11, 29]). Our result concerns the blow-up criterion of the solution of (1.2.2) in Theorem 1.2.1, which refines the spatial topology of the blow-up condition stated in paper such as [15].

Theorem 1.2.2. *Let $1 < p < n$. If the solution of* (1.2.2)

$$(a, u, \psi) \in \mathcal{C}([0, T); B^{\frac{n}{p}}_{p,1} \times B^{\frac{n}{p}-1}_{p,1} \times B^{\frac{n}{p}+2}_{p,1})$$

satisfies

(i) $a \in L^\infty(0, T; B^{\frac{n}{p}}_{p,1})$, $1 + a \geq \exists \underline{a} > 0$ *and*

(ii) either

(1) $u \in L^{\frac{2}{1-\alpha}}(0, T; \dot{B}^{-\alpha}_{\infty,\infty})$ $(-1 < \alpha < 1)$,
or

(2) $\|\nabla u\|_{\dot{B}^0_{\infty,\infty}} \log(e + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}}) \in L^1(0, T)$,

then (a, u, ψ) may be continued beyond T.

The space in $(ii) - (1)$ is a scale-invariant space which also appears in [23] and [29]. The condition $(ii) - (2)$ corresponds to the end-point case of $(ii) - (1)$ with $\alpha = -1$, but there appears a logarithmic growth with respect to time. The condition $(ii) - (2)$ should be regarded as a compressible counterpart to the Beale-Kato-Majda [1] blow-up criterion; only we are unable to replace ∇u by rot u due to the absence of Biot-Savart law.

Theorem 1.2.2 $(ii) - (1)$ is merely a consequence of a product rule applied to the convective term $u \cdot \nabla u$. To prove $(ii) - (2)$, we must use a logarithmic Sobolev inequality [22]. Our proof is partially inspired by Ogawa-Taniuchi [24], in which they give a sufficient condition for the the uniqueness of the solution of the incompressible Navier-Stokes system. We note that the logarithmic growth of $(ii) - (2)$ of Theorem 1.2.2 also appears in the incompressible case as well. See [22] and [24].

1.3 Barotropic compressible Navier-Stokes-Poisson equations

We then consider the compressible Navier-Stokes-Poisson system, i.e. the case $\kappa = 0$ and $\gamma = \pm 1$.

$$(1.3.1) \qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) & \\ \qquad = 2\mu \operatorname{div}(D(u)) + \lambda \nabla(\operatorname{div} u) + \rho \nabla \psi, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ -\Delta \psi = \rho - 1, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), & x \in \mathbb{R}^n. \end{cases}$$

Hereafter, we choose $\gamma = 1$ for the sake of notational simplicity (however, our local theory applies to $\gamma = -1$ case as well). We assume that the density ρ is bounded away from 0 and tends to some positive constant at infinity and the pressure satisfies $P'(1) > 0$.

Our main goal in this section is to solve the system (1.3.1) in the *critical* and *near-critical* regularity framework. Inspired by the recent paper [16] and [17] dedicated to the compressible barotropic flow and the incompressible inhomogeneous fluids, respectively, we aim at solving system (1.3.1) in the *Lagrangian* coordinates. The merit of introducing the Lagrangian coordinates is that by doing so we may effectively eliminate the hyperbolic component of the system, at least locally-in-time. This in turn enables us to perform the contraction estimate for the fixed point argument to obtain existence and uniqueness in the same class of *critical* functional spaces.

Let us recall the idea of scaling invariance that was briefly explained in Section 1. If we ignore the pressure and the potential term, system (1.3.1) is left invariant under the dilation $(\rho, u) \to (\rho_\ell, u_\ell)$ with

$$\rho_\nu(t,x) = \rho(\nu^2 t, \nu x) \quad \text{and} \quad u_\nu(t,x) = \nu u(\nu^2 t, \nu x) \quad \text{for} \quad \nu > 0. \tag{1.3.2}$$

This prompts us to adapt our working spaces which are norm-invariant under the above transformation, which in fact gives a candidate for the largest possible space to find solutions. Let us emphasize that the above family of transforms does not leave (1.3.1) invariant because of the pressure and the potential terms. Nevertheless, those *non-critical* terms are of lower order, at least locally-in-time, and it is thus suitable to address the solvability issue of the system in the *"near"-critical* spaces, as it is likely that this framework will provide the possibly largest space of the well-posedness.

For the low-regularity Besov framework, Hao-Li [20] proved the unique global existence of the solution for (1.3.1) in the L^2-based Besov spaces with low regularity assumptions, using the method of [12]. Their result requires that the (small) initial data satisfies $\rho_0 - 1 \in \dot{B}_{2,1}^{\frac{n}{2}-\frac{5}{2}} \cap \dot{B}_{2,1}^{\frac{n}{2}}$ and $u_0 \in \dot{B}_{2,1}^{\frac{n}{2}-\frac{3}{2}} \cap \dot{B}_{2,1}^{\frac{n}{2}-1}$, with the restriction on the dimension $n \geq 3$. In [20], the potential term induces an extra assumption on the regularity for both ρ and u, and one is forced to impose stronger *sub-critical* (in terms of the scaling (1.3.2)) regulartiy on the low-frequency of the data, compared to the compressible Navier-Stokes equations. Recall that in $\gamma = 0$ case, the barotropic compressible flow requires only $\rho_0 - 1 \in \dot{B}_{p,1}^{\frac{n}{p}-1} \cap \dot{B}_{p,1}^{\frac{n}{p}}$ and $u_0 \in \dot{B}_{p,1}^{\frac{n}{p}-1}$ with smallness [12]. Zheng [30] proved a global result, based on the work of [5], with small inital data satisfying $\rho_0 - 1 \in \dot{B}_{2,p}^{\frac{n}{2}-2,\frac{n}{p}}$ and $u_0 \in \dot{B}_{2,p}^{\frac{n}{2}-1,\frac{n}{p}-1}$. In both [20] and [30], the two-dimensional case is excluded.

In application, system (1.3.1) in the lower dimensions is of a particular interest when used for the semiconductor devise models. Therefore, one of our motivations is to investigate the difficulty behind the two-dimensional case, in regards the more general overview of the local critical theory for system (1.3.1). Our techniques cannot treat the one-dimensional case due to failure of product estimate in Besov spaces with low regularity (more specifically the bilinear mapping $\dot{B}^{\frac{n}{p}-1}_{p,1} \times \dot{B}^{\frac{n}{p}}_{p,1} \mapsto \dot{B}^{\frac{n}{p}-1}_{p,1}$ fails when $n = 1$). However, this is the same case for the usual compressible viscous flow.

1.3.1 Notation

Before introducing the Lagrangian system, let us list some notational conventions. Throughout the proof, we denote by C a generic harmless constant the value of which may vary from line to line. For a $\mathcal{C}^1$ function $F : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$, we define $\operatorname{div} F : \mathbb{R}^n \to \mathbb{R}^m$ by $(\operatorname{div} F)^j := \sum_i \partial_i F_{ij}$, $1 \le j \le m$.

For $n \times n$ matrices $A = (A_{ij})_{1\le i,j\le n}$ and $B = (B_{ij})_{1\le i,j\le n}$, we define the trace product $A : B$ by $A : B = \operatorname{tr}(AB) = \sum_{ij} A_{ij}B_{ji}$. By $\operatorname{adj}(A)$, we denote the adjugate matrix of A, i.e. the transpose of the cofactor matrix of A. If A is invertible then $\operatorname{adj}(A) = (\det A)A^{-1}$. Given some matrix A, we define the "twisted" deformation tensor and divergence operator (acting on a vector field z) by the formulae

$$D_A(z) := \frac{1}{2}(DzA + {}^tA\nabla z)$$

$$\operatorname{div}_A z := {}^tA : \nabla z = Dz : A.$$

The flow $X = X_u$ of u is defined by

$$X_u(t, y) = y + \int_0^t u(\tau, X_u(\tau, y))d\tau. \tag{1.3.3}$$

1.3.2 Lagrangian coordinates

We set $\bar{\rho}(t, y) := \rho(t, X_u(t, y))$ and $\bar{u}(t, y) := u(t, X_u(t, y))$. With the notations $J = J_u := \det(DX_u)$ and $A = A_u := (D_yX_u)^{-1}$, (1.3.1) in Lagrangian coordinates writes as follows

$$\begin{cases} \partial_t(J\bar{\rho}) = 0, \\ \rho_0\partial_t\overline{u} - \operatorname{div}(\operatorname{adj}(DX)(2\mu D_A\overline{u} + \lambda \operatorname{div}_A\overline{u} - P(\bar{\rho})) +^t \operatorname{adj}(DX)\nabla\overline{\psi} = 0, \\ -\operatorname{div}(\operatorname{adj}(DX)A^t : \nabla\overline{\psi}) = \rho_0 - J, \\ (\bar{\rho}, \bar{u})|_{t=0} = (\rho_0, u_0). \end{cases} \tag{1.3.4}$$

From hereon, we may forget any reference to the initial Eulerian vector-field u in the equations and redefine the *flow* of $\overline{u}$ as

$$X_{\overline{u}}(t, y) = y + \int_0^t \overline{u}(\tau, y)d\tau. \tag{1.3.5}$$

We are going to solve the above system in the homogeneous Besov spaces that are close to the critical space for the barotropic case (without potential term). The Besov spaces are introduced in Definition 1 of Section 1.2.

We shall also define the hybrid Besov space $\widetilde{B}^{s,\sigma}_{p,1}$.

Definition 2. For $s \in \mathbb{R}$ and $1 \le p, \sigma \le \infty$, we define the hybrid Besov space $\widetilde{B}^{s,\sigma}_{p,1}(\mathbb{R}^n)$ as follows:

$$\widetilde{B}^{s,\sigma}_{p,1}(\mathbb{R}^n) := \{u \in \mathcal{S}'/\mathcal{P} \ ; \ \|u\|_{\dot{B}^s_{p,\sigma}} < \infty\},$$

where

$$\|u\|_{\widetilde{B}^{s,\sigma}_{p,1}} := \sum_{j<0} 2^{js}\|\phi_j * u\|_{L^p} + \sum_{j\ge 0} 2^{j\sigma}\|\phi_j * u\|_{L^p}.$$

We denote the low frequency of u by $u_L := \dot{S}_m u = \sum_{j\le m-1} \phi_j * u$ for some fixed m and the high frequency of u by u_H. Then we may also express $\widetilde{B}^{s,\sigma}_{p,1}$ as the space in which u_L belongs to $\dot{B}^s_{p,1}$ and u_H belongs to $\dot{B}^\sigma_{p,1}$.

1.3.3 Main result

We define $E_p(T)$ as the space in which the tempered distribution $v \in \widetilde{B}^{\frac{n}{p}-1+\nu,\frac{n}{p}-1}_{p,1}$ satisfies

$$\begin{aligned} &v \in \mathcal{C}([0,T]; \widetilde{B}^{\frac{n}{p}-1+\nu,\frac{n}{p}-1}_{p,1}) \cap L^2(0,T : \widetilde{B}^{\frac{n}{p}+\nu,\frac{n}{p}}_{p,1}) \\ \text{and} \quad &\partial_t v_H, \nabla^2 v_H \in L^1(0,T; \dot{B}^{\frac{n}{p}-1}_{p,1}). \end{aligned} \tag{1.3.6}$$

The norm of $E_p(T)$ is defined by

$$\|v\|_{E_p(T)} := \|v\|_{L^\infty_T(\widetilde{B}^{\frac{n}{p}-1+\nu,\frac{n}{p}-1}_{p,1})} + \|Dv\|_{L^2_T(\widetilde{B}^{\frac{n}{p}-1+\nu,\frac{n}{p}-1}_{p,1})} + \|\partial_t v_H, \nabla^2 v_H\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}.$$

We shall obtain the existence and uniquenesss of the local-in-time solution $(\overline{\rho}, \overline{u}, \overline{\psi})$ for (1.3.4), with $\overline{a} := \overline{\rho} - 1$ in $\mathcal{C}([0,T]; \widetilde{B}^{\frac{n}{p}-2+\nu,\frac{n}{p}}_{p,1})$, $\overline{u}$ in the space $E_p(T)$, and $\overline{\psi}$ with $\nabla^2\overline{\psi} \in \mathcal{C}([0,T]; \dot{B}^{\frac{n}{p}-2+\nu}_{p,1})$.

We now state our main result:

Theorem 1.3.1. *Let $0 \le \nu \le 1$, $1 < p < \frac{2n}{2-\nu}$ and*

$$n \ge \begin{cases} 2 & \text{if } 0 < \nu \le 1, \\ 3 & \text{if } \nu = 0. \end{cases}$$

Let u_0 be a vector field in $\widetilde{B}^{\frac{n}{p}-1+\nu,\frac{n}{p}-1}_{p,1}$. Assume that the initial density ρ_0 satisfies $a_0 := (\rho_0 - 1) \in \widetilde{B}^{\frac{n}{p}-2+\nu,\frac{n}{p}}_{p,1}$ and

$$\inf_x \rho_0(x) > 0. \tag{1.3.7}$$

Then system (1.3.4) admits a unique local solution $(\overline{\rho}, \overline{u}, \overline{\psi})$ with $\overline{a} = \overline{\rho} - 1$ in $\mathcal{C}([0,T]; \widetilde{B}^{\frac{n}{p}-2+\nu,\frac{n}{p}}_{p,1})$, $\overline{u}$ in $E_p(T)$ and $\overline{\psi}$ in $\mathcal{C}([0,T]; \dot{B}^{\frac{n}{p}+\nu}_{p,1})$. Moreover, the flow map $(a_0, u_0) \mapsto (\overline{a}, \overline{u})$ is Lipschitz continuous from $\widetilde{B}^{\frac{n}{p}-2+\nu,\frac{n}{p}}_{p,1} \times \widetilde{B}^{\frac{n}{p}-1+\nu,\frac{n}{p}-1}_{p,1}$ to $\mathcal{C}([0,T]; \widetilde{B}^{\frac{n}{p}-2+\nu,\frac{n}{p}}_{p,1}) \times E_p(T)$.

As a general empirical law, in order to establish a local theory, one is only required to control the high frequency of the initial data. In our assumptions of data, the critical regularity is imposed only on the high-frequency part. We note that in two dimensions, $\nu = 1$ provides the best result as far as the admissibility of the Lebesgue exponent p is concerned. In this special case, Theorem 1.3.1 states: *Let* $1 < p < 2n$, $n \geq 2$, $u_0 \in \widetilde{B}^{\frac{n}{p},\frac{n}{p}-1}_{p,1}$ *and* $a_0 \in \widetilde{B}^{\frac{n}{p}-1,\frac{n}{p}}_{p,1}$ *satisfies Condition* (1.3.7). *Then system* (1.3.4) *admits the unique local solution.*

In Eulerian coordinates, Theorem 1.3.1 recasts in:

Theorem 1.3.2. *Under the same assumptions as in Theorem* 1.3.1*, system* (1.3.1) *has a unique local solution* (ρ, u, ψ) *with* $u \in E_p(T)$*,* ρ *bounded away from 0 and* $\rho - 1 \in \mathcal{C}([0,T]; \widetilde{B}^{\frac{n}{p}-2+\nu,\frac{n}{p}}_{p,1})$*, and* $\nabla^2\psi \in \mathcal{C}([0,T]; \widetilde{B}^{\frac{n}{p}-2+\nu,\frac{n}{p}}_{p,1})$*.*

In two dimensions, in order to compensate the gap of regularity caused by the presence of potential term, we are required to impose an additional regularity (compared to the standard compressible Navier-Stokes system) to the low frequency of the initial density $a_0 \in \widetilde{B}^{\frac{n}{p}-2+\nu,\frac{n}{p}}_{p,1} = \dot{B}^{\frac{n}{p}-2+\nu}_{p,1} \cap \dot{B}^{\frac{n}{p}}_{p,1}$. On the other hand for the initial velocity u_0, the assumption $u_0 \in \widetilde{B}^{\frac{n}{p}-1+\nu,\frac{n}{p}-1}_{p,1} = \dot{B}^{\frac{n}{p}-1+\nu}_{p,1} + \dot{B}^{\frac{n}{p}-1}_{p,1}$ is of lower order (*sub-critical* in terms of scaling) and only becomes *critical* when $\nu = 0$.

1.4 Full compressible Navier-Stokes system

In Chapter 5, we consider the Cauchy problem of the following full compressible Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 2$:

$$(1.4.1) \quad \begin{cases} \partial_t\rho + \mathrm{div}\,(\rho u) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla P = \mathrm{div}\,\tau, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ \partial_t\Big[\rho\Big(\frac{|u|^2}{2} + e\Big)\Big] + \mathrm{div}\,\Big[u\Big(\rho(\frac{|u|^2}{2} + e) + P\Big)\Big] & \\ \qquad\qquad = \mathrm{div}\,(\tau \cdot u) - \mathrm{div}\,q, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ (\rho, u, e)|_{t=0} = (\rho_0, u_0, e_0), & x \in \mathbb{R}^n, \end{cases}$$

where $\rho = \rho(t,x) \in \mathbb{R}_+$, $u = u(t,x) \in \mathbb{R}^n$ and $e = e(t,x) \in \mathbb{R}$ are the unknown functions, representing the fluid density, the velocity vector field and the internal energy per unit mass, respectively. We restrict ourselves to the case of Newtonian gases, namely we assume the viscous stress tensor τ to be given by

$$\tau := \lambda \mathrm{div}\, u \;\mathrm{Id} + 2\mu D(u),$$

where $D(u)$ designates the deformation tensor defined by

$$D(u) := \frac{1}{2}(Du + \nabla u) \quad \text{with} \quad (Du)_{ij} := \partial_j u^i \quad \text{and} \quad (\nabla u)_{ij} := ({}^t(Du))_{ij} = \partial_i u^j.$$

The viscosity coefficients λ and μ satisfy $\mu > 0$ and $\lambda + 2\mu > 0$, which ensure the ellipticity of the second order operator in the velocity equation. We assume the *Fourier law*; that is the heat conduction q is given by $q = -k\nabla\theta$ where k is a given positive constant and $\theta = \theta(t, x)$, the temperature. The given function P represents the pressure depending on ρ and θ. In this paper, we restrict ourselves to the following pressure law:

$$P(\rho, \theta) := \pi_0(\rho) + \theta\pi_1(\rho),$$

where π_0 and π_1 are given smooth functions. We suppose that that the internal energy e is given by $e = C_v\theta + h(\rho)$, where C_v is a (positive) specific heat constant and h some smooth function. *Maxwell's relation* (that is $\rho^2\partial_\rho e = P - \theta\partial_\theta P$) then gives that h should satisfy the relation $\rho^2 h'(\rho) = \pi_0(\rho)$.

Important examples of such pressure laws are ideal fluids (for which $\pi_0(\rho) = 0$ and $\pi_1(\rho) = R\rho$ for some positive constant R), barotropic gases ($\pi_1(\rho) = 0$) and Van der Waals gases ($\pi_0(\rho) = -\alpha\rho^2$ and $\pi_1(\rho) = \beta\rho/(\gamma - \rho)$ for some positive constant α, β and γ).

The boundary conditions at infinity are that u and θ tend to 0, and that ρ tends to some positive constant ρ^*. The exact meaning of the convergence will follow from the functional framework we shall work in. For simplicity, we assume $C_v = 1$ and $\rho^* = 1$ in all that follows. With no loss of generality, one can impose in addition that $\pi_0(1) = 0$.

Our main goal is to solve the full Navier-Stokes equations in the so-called *critical regularity framework*. As discussed in Section 1.1, this approach originates from a paper by Fujita-Kato [19] devoted to the well-posedness issue for the incompressible Navier-Stokes equations. In our context, the idea is to solve (1.4.1) in a function space having the same invariance by time and space dilations as (1.4.1), namely $(\rho, u, \theta) \to (\rho_\nu, u_\nu, \theta_\nu)$ with

$$\rho_\nu(t, x) = \rho(\nu^2 t, \nu x), \quad u_\nu(t, x) = \nu u(\nu^2 t, \nu x) \text{ and } \theta_\nu(t, x) = \nu^2\theta(\nu^2 t, \nu x). \tag{1.4.2}$$

The above family of transforms does not quite leave (1.4.1) invariant (as P has to be changed into $\nu^2 P$). Nevertheless, the pressure term is, to some extent, lower order, and it is thus suitable to address the solvability issue of the system in 'critical' spaces, that is in spaces with norm invariant for all $\nu > 0$ by the scaling transformation (1.4.2).

Prompted by the recent paper dedicated to the compressible barotropic flow [16] or by the work in [17] concerning incompressible inhomogeneous fluids, we here aim at solving the full compressible system (1.4.1) in the *Lagrangian* coordinates. Let us emphasize that this approach has already been successfully applied in the case of smooth data (see e.g. [25–28]). We here want to perform it *in the critical regularity framework.*

The motivation behind introducing Lagrangian coordinates is to effectively eliminate the hyperbolic part of the system, given that the density equation becomes explicitly solvable once the flow of the velocity field has been determined. At the same time, the system for the velocity and energy in Lagrangian coordinates remains of parabolic type (at least for small enough time), and the Banach fixed point theorem turns out to be applicable for obtaining the existence *and* uniqueness of the solution in *the same class of spaces* as in the Eulerian framework. This is the key to improving the set of data leading to well-posedness, compared to [13].

1.4.1 Lagrangian coordinates

We use the same notations given in Section 1.3.1. As before, the flow $X = X_u$ of u is defined by

$$X_u(t,y) = y + \int_0^t u(\tau, X_u(\tau,y))d\tau. \tag{1.4.3}$$

Let $\bar\rho(t,y) := \rho(t, X_u(t,y))$, $\bar u(t,y) := u(t, X_u(t,y))$ and $\bar E(t,y) := E(t, X_u(t,y))$ denote the density, velocity and energy functions in Lagrangian coordinates, respectively.

Looking at the energy equation, it is natural to introduce the *total energy along the flow* defined by

$$\overline{K} := J\overline{E} = \rho_0(\bar\theta + \frac{|\bar u|^2}{2} + h(\overline{\rho})). \tag{1.4.4}$$

Setting $J = J_u := \det(DX_u)$ and $A = A_u := (DX_u)^{-1}$, system (1.4.1) recasts in

$$\begin{cases} \partial_t(J\overline{\rho}) = 0, \\ \rho_0\partial_t\overline{u} - \operatorname{div}\Big[\operatorname{adj}(DX)(2\mu D_A\bar u + \lambda\operatorname{div}_A\bar u - \overline{P}\mathrm{Id})\Big] = 0, \\ \partial_t\overline{K} - k\operatorname{div}\Big[\operatorname{adj}(DX)\big({}^tA\nabla(\frac{\overline{K}}{\rho_0} - \frac{|\bar u|^2}{2} - h(\overline{\rho})) + \bar\tau\cdot\bar u - \bar u\overline{P}\big)\Big] = 0, \\ (\overline{\rho}, \overline{u}, \overline{K})|_{t=0} = (\rho_0, u_0, K_0), \end{cases} \tag{1.4.5}$$

where we have redefined the initial data K_0 as

$$K_0 := E_0 = \rho_0\Big(\theta_0 + \frac{|u_0|^2}{2} + h(\rho_0)\Big), \tag{1.4.6}$$

and the pressure function $\overline{P}$ as

$$\overline{P} = \overline{P}(\overline{\rho}, \overline{u}, \overline{K}) := \pi_0(\bar\rho) + \Big(\frac{\overline{K}}{\rho_0} - \frac{|\bar u|^2}{2} - h(\overline{\rho})\Big)\pi_1(\bar\rho).$$

Let us finally emphasize that one may forget any reference to the initial Eulerian vector-field u by defining directly the "flow" X of $\overline{u}$ by the formula

$$X(t,y) = y + \int_0^t \overline{u}(\tau,y)\,d\tau. \tag{1.4.7}$$

We obtain the existence and uniquenesss of a local-in-time solution $(\overline{\rho},\overline{u},\overline{K})$ for (1.4.5), with $\overline{a}:=\overline{\rho}-1$ in $\mathcal{C}([0,T];\dot{B}^{\frac{n}{p}}_{p,1})$ and $(\overline{u},\overline{K})$ in the space

$$E_p(T):=\left\{(v,\psi)\;\middle|\;\begin{array}{l} v\in\mathcal{C}([0,T];\dot{B}^{\frac{n}{p}-1}_{p,1}),\ \partial_t v,\nabla^2 v\in L^1(0,T;\dot{B}^{\frac{n}{p}-1}_{p,1})\\ \psi\in\mathcal{C}([0,T];\dot{B}^{\frac{n}{p}-2}_{p,1}),\partial_t\psi,\nabla^2\psi\in L^1(0,T;\dot{B}^{\frac{n}{p}-2}_{p,1})\end{array}\right\} \tag{1.4.8}$$

endowed with the norm

$$\|(v,\psi)\|_{E_p(T)}:=\|v\|_{L^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\|\partial_t v,\nabla^2 v\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})}+\|\psi\|_{L^\infty_T(\dot{B}^{\frac{n}{p}-2}_{p,1})}+\|\partial_t\psi,\nabla^2\psi\|_{L^1_T(\dot{B}^{\frac{n}{p}-2}_{p,1})}.$$

It is easily checked that $E_p(T)$ is critical in the meaning of (1.4.2).

Let us now state our main result.

Theorem 1.4.1. *Let $1<p<2n$ and $n\geq 2$. Let u_0 be a vector field in $\dot{B}^{\frac{n}{p}-1}_{p,1}$ and K_0, a real valued function in $\dot{B}^{\frac{n}{p}-2}_{p,1}$. Assume that ρ_0 satisfies $a_0:=(\rho_0-1)\in\dot{B}^{\frac{n}{p}}_{p,1}$ and*

$$\inf_x \rho_0(x)>0. \tag{1.4.9}$$

Then system (1.4.5) admits a unique local solution $(\overline{\rho},\overline{u},\overline{K})$ with $\overline{\rho}$ bounded away from zero, $\overline{a}:=\overline{\rho}-1$ in $\mathcal{C}([0,T];\dot{B}^{\frac{n}{p}}_{p,1})$ and $(\overline{u},\overline{K})$ in $E_p(T)$.

Moreover, the flow map $(a_0,u_0,K_0)\mapsto(\overline{a},\overline{u},\overline{K})$ is Lipschitz continuous from $\dot{B}^{\frac{n}{p}}_{p,1}\times\dot{B}^{\frac{n}{p}-1}_{p,1}\times\dot{B}^{\frac{n}{p}-2}_{p,1}$ to $\mathcal{C}([0,T];\dot{B}^{\frac{n}{p}}_{p,1})\times E_p(T)$.

In Eulerian coordinates, the above theorem implies:

Theorem 1.4.2. *Under the same assumptions as in Theorem 1.4.1, with in addition $n\geq 3$ and $1<p<n$, system (1.4.1) has a unique local solution (ρ,u,θ) with $(u,\theta)\in E_p(T)$, ρ bounded away from 0 and $\rho-1\in\mathcal{C}([0,T];\dot{B}^{\frac{n}{p}}_{p,1})$.*

In dimension $n=2$, or if $n\leq p<2n$, only partial results are available. First, in the critical functional framework, prescribing (a_0,u_0,θ_0) or (a_0,u_0,E_0) is no longer equivalent since the product does not map $\dot{B}^{\frac{n}{p}}_{p,1}\times\dot{B}^{\frac{n}{p}-2}_{p,1}$ in $\dot{B}^{\frac{n}{p}-2}_{p,1}$ any longer, and the data are interrelated through (1.4.4). Second, even if one chooses to work with (a,u,E) rather than with (a,u,θ), having (u,E) in $E_p(T)$ *does not* quite imply that $(\bar{u},\bar{K})$ is in $E_p(T)$ (and the converse is false, too).

The restriction that $1<p<n$ and $n\geq 3$ in Theorem 1.4.2 is consistent with the recent paper by Chen-Miao-Zhang [7]. There, the authors established the ill-posedness of the full compressible Navier-Stokes system in three dimension in the sense that the continuity of data-solution map fails at the origin in the critical Besov framework that we used, if $p>n$. In other words, up to the limit case $p=n$, Theorem 1.4.2 is optimal as regards the local well-posedness issue with unknowns (ρ,u,θ).

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