Topological approach to the stability properties of traveling waves for one-dimensional reaction diffusion systems

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博　士　論　文

Topological approach to the stability properties of traveling waves for one-dimensional reaction diffusion systems

(反応拡散系における１次元進行波解の
安定性に対する位相的手法の応用)

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平 成 26 年

Topological approach to the stability properties of traveling waves for one-dimensional reaction diffusion systems

A thesis presented
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Contents

Summary

1.1 Topological approach to an accumulation of eigenvalues associated with traveling waves for reaction diffusion systems

In this thesis, we consider the following reaction-diffusion system given by

$$U_t = BU_{xx} + F(U, \epsilon),\ U \in \mathbb{R}^N,\ t > 0,\ x \in \mathbb{R}, \tag{1}$$

where $\epsilon \in \mathbb{R}^d$ is a parameter, and

$$B = \mathrm{diag}\{d_1, \cdots, d_k, d_{k+1}, \cdots, d_N\}$$

is a diagonal diffusion matrix with nonnegative elements. Each component d_j satisfies $d_j > 0$ for $j = 1, \cdots, k$ and $d_j = 0$ for $j = k+1, \cdots, N$. Moreover $F \in C^r(\mathbb{R}^N, \mathbb{R}^N),\ r \geq 2$ for a fixed ϵ, and F is sufficiently smooth with respect to the parameter ϵ. This system appears in many fields such as the models for the super conductivity, phase transition phenomena, nerve pulse propagation and the Belousov–Zhabotinsky reaction. In particular, a special class of solutions called the traveling waves is important. Traveling waves are solutions of the form $u(t, x) = u(x + ct)$ for some nonzero constant c. Thus, if we introduce the moving coordinate $\xi = x + ct$, then traveling waves are given by the steady state solutions which do not change shape. Traveling waves of (1) satisfy the following equation

$$BU_{\xi\xi} - cU_\xi + F(U, \epsilon) = 0. \tag{2}$$

If traveling waves exist, these are affected by noise externally or internal fluctuations. Therefore the stability problem is fundamental for the observation of phenomena in nature, in fact various stationary patterns and pattern dynamics in reaction-diffusion systems are related to stability properties of solutions ([56]). One of the methods for the stability of traveling waves is to study the linearized operator associated with traveling waves $U(\xi)$ given by

$$\begin{aligned} &LV = BV_{\xi\xi} - cV_\xi + D_U F(U(\xi), \epsilon)V, \\ &L : BU^2(\mathbb{R}, \mathbb{R}^N) \to BU(\mathbb{R}, \mathbb{R}^N), \end{aligned}$$

where

$$BU(\mathbb{R}, \mathbb{R}^N) := \{v : \mathbb{R} \to \mathbb{R}^n \mid \text{ bounded uniformly continuous } \},$$

and

$$BU^2(\mathbb{R}, \mathbb{R}^N) := \{v \in BU(\mathbb{R}, \mathbb{R}^n) \mid \frac{dv}{d\xi}, \frac{d^2v}{d\xi^2} \in BU(\mathbb{R}, \mathbb{R}^n)\}.$$

Alexander–Gardner–Jones [5] showed the relationship between the number of eigenvalues of the traveling wave inside a simple closed curve and the first Chern number of vector bundle $\mathcal{E}(K)$ on the two-dimensional sphere S^2. This is called Alexander–Gardner–Jones bundle or the augmented bundle, and this characteristic number is called the stability index because it is defined for the stability problem of traveling waves. Moreover, Gardner–Jones ([28]) defined the stability index for reaction-diffusion system (1) on a bounded interval $I = [-\ell, \ell]$ with boundary conditions.

It is important to consider the difference between the properties of reaction-diffusion systems on the bounded interval and those on the unbounded interval. Sandstede–Scheel [69] defined the absolute spectrum and the asymptotical essential spectrum, and studied the accumulation of eigenvalues for several eigenvalue problems with boundary conditions. Moreover, they compared the spectral structures of relevant operators on unbounded and bounded domains. They have shown that the absolute spectrum gives a difference in the spectrum between the unbounded case and the bounded case. The reason for this is that a lot of eigenvalues accumulate on the absolute spectrum, if the bounded domain is sufficiently large but they are not on the essential spectrum for operators associated with the unbounded domain.

In the first part of this thesis, we show the relationship between the topological structure of several boundary value problems and the absolute spectrum via the Alexander–Gardner–Jones theory. Moreover, we show the accumulation of eigenvalues of the linearized operator associated with glued waves on the absolute spectrum. In particular, there is a relation between a necessary condition of the stability for glued pulses and topological structures of gluing bifurcations.

The gluing bifurcation forms one of the generating mechanisms of pulses, and it is given by the combination of the homoclinic and the heteroclinic bifurcations in dynamical systems theory. We rewrite (2) as the first order system of ordinary differential equations:

$$u' = f(u, c, \epsilon), \ u \in \mathbb{R}^n, \tag{3}$$

where $' = \frac{d}{d\xi}$ and $u = (U^1, \cdots, U^k, U^{1'}, \cdots, U^{k'}, U^{k+1}, \cdots, U^N)$ so that $n = N + k$, while $f_j(u, c, \epsilon) = u_{k+j}$ and $f_{k+j}(u, c, \epsilon) = (cu_{k+j} - F_j(U, \epsilon))/d_j$ for $j = 1, \cdots, k$ and $f_{k+j}(u, c, \epsilon) = F_j(U, \epsilon)/c$ for $j = k+1, \cdots, N$. If $u(\xi, c, \epsilon) = (U_1(\xi, c, \epsilon), \cdots, U_n(\xi, c, \epsilon))$ is a solution of (3), then $U(\xi) = (U_1(\xi, c, \epsilon), \cdots, U_k(\xi, c, \epsilon), U_{k+1}(\xi, c, \epsilon), \cdots, U_n(\xi, c, \epsilon))$ is a traveling wave of (1) with a speed c.

Then glued pulses generated by gluing bifurcations are defined as follows.

Definition 1.1. *$U_p(\xi, c, \epsilon)$ is a glued pulse from $U_f(\xi, c_0, \epsilon_0)$ and $U_b(\xi, c_0, \epsilon_0)$ if*

$$\begin{aligned}
U_f(\xi, c_0, \epsilon_0) &= (U_1^f(\xi, c_0, \epsilon_0), \cdots, U_k^f(\xi, c_0, \epsilon_0), U_{k+1}^f(\xi, c_0, \epsilon_0), \cdots, U_N^f(\xi, c_0, \epsilon_0)), \\
U_b(\xi, c_0, \epsilon_0) &= (U_1^b(\xi, c_0, \epsilon_0), \cdots, U_k^b(\xi, c_0, \epsilon_0), U_{k+1}^b(\xi, c_0, \epsilon_0), \cdots, U_N^b(\xi, c_0, \epsilon_0)), \\
U_p(\xi, c, \epsilon) &= (U_1^p(\xi, c, \epsilon), \cdots, U_k^p(\xi, c, \epsilon), U_{k+1}^p(\xi, c, \epsilon), \cdots, U_N^p(\xi, c, \epsilon)),
\end{aligned}$$

satisfy the following conditions.

At $(c_0, \epsilon_0) \in \mathbb{R}^{d+1}$*, there exist hyperbolic equilibria* p_1 *and* p_2*, and solutions*

$$\begin{aligned} u_f(\xi) &:= (U_1^f(\xi, c_0, \epsilon_0), \cdots, U_k^f(\xi, c_0, \epsilon_0), U_1^{f'}(\xi, c_0, \epsilon_0), \cdots, U_k^{f'}(\xi, c_0, \epsilon_0), \\ &\quad U_{k+1}^f(\xi, c_0, \epsilon_0), \cdots, U_N^f(\xi, c_0, \epsilon_0)), \\ u_b(\xi) &:= (U_1^b(\xi, c_0, \epsilon_0), \cdots, U_k^b(\xi, c_0, \epsilon_0), U_1^{b'}(\xi, c_0, \epsilon_0), \cdots, U_k^{b'}(\xi, c_0, \epsilon_0), \\ &\quad U_{k+1}^b(\xi, c_0, \epsilon_0), \cdots, U_N^b(\xi, c_0, \epsilon_0)), \end{aligned}$$

of (3) *satisfying*

$$\lim_{\xi \to -\infty} u_f(\xi) = p_1, \ \lim_{\xi \to \infty} u_f(\xi) = p_2, \tag{4}$$

$$\lim_{\xi \to -\infty} u_b(\xi) = p_2, \ \lim_{\xi \to \infty} u_b(\xi) = p_1, \tag{5}$$

respectively. In addition, there exists a homoclinic bifurcation set $H \subset \mathbb{R}^{d+1}$ *such that* $(c_0, \epsilon_0) \notin H$ *and* $(c_0, \epsilon_0) \in \mathrm{cl}(H)$ *and for all* $(c, \epsilon) \in H$*,* $p_1(c, \epsilon)$ *and* $p_2(c, \epsilon)$ *are families of hyperbolic equilibria with* $p_1(c_0, \epsilon_0) = p_1, p_2(c_0, \epsilon_0) = p_2$*, and*

$$\begin{aligned} u_p(\xi, c, \epsilon) &:= (U_1^p(\xi, c, \epsilon), \cdots, U_k^p(\xi, c, \epsilon), U_1^{p'}(\xi, c, \epsilon), \cdots, U_k^{p'}(\xi, c, \epsilon), \\ &\quad U_{k+1}^p(\xi, c, \epsilon), \cdots, U_N^p(\xi, c, \epsilon)), \end{aligned}$$

are solutions of (3) *satisfying*

$$\lim_{\xi \to -\infty} u_p(\xi, c, \epsilon) = p_1(c, \epsilon), \ \lim_{\xi \to \infty} u_f(\xi) = p_1(c, \epsilon). \tag{6}$$

Moreover, each orbit

$$\begin{aligned} &\mathcal{O}_f := \{u_f(\xi) \mid \xi \in \mathbb{R}\}, \\ &\mathcal{O}_b := \{u_b(\xi) \mid \xi \in \mathbb{R}\}, \\ &\mathcal{O}_{p,(c,\epsilon)} := \{u_p(\xi, c, \epsilon) \mid \xi \in \mathbb{R}\}, \end{aligned}$$

satisfies

$$d_H(\mathrm{cl}(\mathcal{O}_{p,(c,\epsilon)}), \mathrm{cl}(\mathcal{O}_f \cup \mathcal{O}_b)) \to 0 \ \text{as} \ (c, \epsilon) \to (c_0, \epsilon_0), \tag{7}$$

where $d_H(\cdot, \cdot)$ *is the Hausdorff metric.*

We consider the eigenvalue problems associated with a glued pulse $U_p(\xi, c, \epsilon)$:

$$L_p(c, \epsilon)V = BV_{\xi\xi} - cV_\xi + D_U F(U_p(\xi, c, \epsilon), \epsilon)V = \lambda V, \ V \in \mathbb{C}^N, \tag{8}$$

where

$$L_p(c, \epsilon) : BU^2(\mathbb{R}, \mathbb{R}^N) \to BU(\mathbb{R}, \mathbb{R}^N). \tag{9}$$

(8) can be rewritten as the following ODE:

$$Y' = A_p(\xi, c, \epsilon; \lambda)Y \tag{10}$$

where $Y = (V_1, \cdots, V_k, V_1', \cdots, V_k', V_{k+1}, \cdots, V_N)$,

$$A_p(\xi, c, \epsilon; \lambda) \;=\; D_u f(\xi, c, \epsilon) + \lambda \mathcal{B},$$

and the matrix $\mathcal{B}$ is given in block structure with three blocks of size k, k and $N-k$, respectively, by

$$\mathcal{B} = \begin{pmatrix} O & O & O \\ B_k^{-1} Id_k & O & O \\ O & -c^{-1} Id_{N-k} & O \end{pmatrix}, \tag{11}$$

where $B_k = \mathrm{diag}\{d_1, \cdots, d_k\}$ and Id_j is a $j \times j$ the identity matrix.

Let $A_1(\lambda) = D_u f(p_1, c_0, \epsilon_0) + \lambda \mathcal{B}$ and $A_2(\lambda) = D_u f(p_2, c_0, \epsilon_0) + \lambda \mathcal{B}$ be the asymptotic matrices. We label the eigenvalues $\nu_1^j(\lambda)$ and $\nu_2^j(\lambda)$ of $A_1(\lambda)$ and $A_2(\lambda)$ according to their real parts, and repeated with multiplicity, respectively, i.e.,

$$\begin{aligned} \mathrm{Re}\,\nu_1^1(\lambda) \geq \mathrm{Re}\,\nu_1^2(\lambda) \geq \cdots \geq \mathrm{Re}\,\nu_1^n(\lambda), \\ \mathrm{Re}\,\nu_2^1(\lambda) \geq \mathrm{Re}\,\nu_2^2(\lambda) \geq \cdots \geq \mathrm{Re}\,\nu_2^n(\lambda). \end{aligned}$$

Definition 1.2. *(Absolute spectrum) Let $\Omega \subset \mathbb{C}$ be an open bounded and connected domain. Then the absolute spectrum for p_2 is defined by*

$$\Sigma_{abs}^{2,\Omega} := \{\lambda \in \Omega \mid \mathrm{Re}\,\nu_2^{i_1}(\lambda) = \mathrm{Re}\,\nu_2^{i_1+1}(\lambda)\}. \tag{12}$$

1.1.1 The topological structure of the absolute spectrum

The goal of the results of the first part is a topological characterization of the absolute spectrum. We reformulate the eigenvalue problems (10) as the following separated boundary problems. Let $\Phi(\zeta, \xi; \lambda)$ be the fundamental solution matrix of (10). Define the stable and unstable subspaces as

$$E^s(\xi; \lambda) := \{Y \in \mathbb{C}^n \mid \lim_{\zeta \to \infty} \Phi(\zeta, \xi; \lambda) Y = 0\}, \tag{13}$$

$$E^u(\xi; \lambda) := \{Y \in \mathbb{C}^n \mid \lim_{\zeta \to -\infty} \Phi(\zeta, \xi; \lambda) Y = 0\}. \tag{14}$$

For solutions $u_f(\xi)$ and $u_b(\xi)$, we take the cross sections Σ_f and Σ_b close to the equilibrium p_2. Let $\xi_f(c, \epsilon)$ and $\xi_b(c, \epsilon)$ be intersection points of $u_p(\xi, c, \epsilon)$ and Σ_*, $* = f, b$. Then we have the following boundary value problems:

$$Y' = A_p(\xi, c, \epsilon) Y, \; Y \in \mathbb{C}^n, \xi \in [\xi_f(c, \epsilon), \xi_b(c, \epsilon)] = I, \tag{15}$$

$$Y(\xi_f(c, \epsilon); \lambda) \in U_-(\lambda), \; Y(\xi_b(c, \epsilon); \lambda) \in U_+(\lambda), \tag{16}$$

where

$$U_-(\lambda) := E^u(\xi_f(c, \epsilon); \lambda), \tag{17}$$

$$U_+(\lambda) := E^s(\xi_b(c, \epsilon); \lambda). \tag{18}$$

We consider a system on $\wedge^m\mathbb{C}^n$ which is induced by (15):

$$Y^{(m)\prime} = A_p^{(m)}(\xi, c, \epsilon; \lambda)Y^{(m)},\ Y^{(m)} \in \wedge^m\mathbb{C}^n, \tag{19}$$

where $m = \dim U_-(\lambda)$.

Then, the system (19) induces a flow on the $\tau = \begin{pmatrix} n \\ m \end{pmatrix} - 1$-dimensional complex projective space $\mathbb{CP}^\tau$. For a disk $D \subset \mathbb{C}$, this flow induces a map

$$\mathcal{G}_{(c,\epsilon)} : S^2 \cong (\partial D \times I) \cup (D \times \partial I) \to \mathbb{CP}^\tau. \tag{20}$$

Alexander–Gardner–Jones theory gives the following theorem.

Theorem 1.3. *(Alexander–Gardner–Jones [5], Gardner–Jones [28], Nii [51]) Assume that m is a constant for any $\lambda \in D$. Then,*

$$[\mathcal{G}_{(c,\epsilon)}] \in \pi_2(\mathbb{CP}^\tau) \cong \mathbb{Z}$$

counts the number of eigenvalues of $L_p(c, \epsilon)$ in the interior of D.

The topological characterization of the absolute spectrum is given by the following theorem.

Proposition 1.4. *([70]) Assume that m is a constant for any $\lambda \in D$, $\Sigma_{abs}^{2,\Omega} \subset D$ satisfies several generic conditions.*

$$\hat{\mathcal{G}}_{(c,\epsilon)} = Pr \circ \mathcal{G}_{(c,\epsilon)}\big|_{\Sigma_{abs}^{2,\Omega} \times \{\xi_b(c,\epsilon)\}} : \Sigma_{abs}^{2,\Omega} \to S^1 \subset \mathbb{CP}^1 \setminus \{N \cup S\}, \tag{21}$$

is a continuous map if $|\xi_f(c, \epsilon) - \xi_b(c, \epsilon)|$ is sufficiently large, where

$$Pr : \mathbb{CP}^\tau \setminus \{[Z_1 : \cdots : Z_\tau] \mid (Z_2, \cdots, Z_\tau) \neq 0\} \to \mathbb{CP}^1 = \{[Z_1 : Z_2] \mid (Z_1, Z_2) \neq 0\},$$

is a projection. That is, the vector field (19) *induces the attractor–repeller pair decomposition for the flow on $\mathbb{CP}^\tau$.*

Using the above topological characterization, we show the accumulation of eigenvalues of L_p on the absolute spectrum $\Sigma_{abs}^{2,\Omega}$.

Theorem 1.5. *([70]) For any $K \in \mathbb{N}$, $\hat{\mathcal{G}}_{(c,\epsilon)}(\Sigma_{abs}^{2,\Omega})$ covers S^1 more than K-times and*

$$[\mathcal{G}_{(c,\epsilon)}] \geq K, \tag{22}$$

if $|\xi_f(c, \epsilon) - \xi_b(c, \epsilon)|$ is sufficiently large. That is, there is $\delta > 0$ such that for any $\alpha \in \mathbb{N}$ and $\lambda_* \in \Sigma_{abs}^{2,\Omega}$, L_p has at least K eigenvalues in $B(\lambda_*, \delta)$ if $|\xi_f(c, \epsilon) - \xi_b(c, \epsilon)|$ is sufficiently large, where $B(\lambda_*, \delta)$ is an open disk of center λ_* and radius δ.

1.1.2 Stability properties of glued pulses

Theorem 1.5 implies the following necessary condition for the stability of glued pulses.

Theorem 1.6. *([70]) Under the several generic conditions, if glued pulses $U_p(\xi, c, \epsilon)$ are stable for any $(c, \epsilon) \in H$, then for any open bounded and connected domain Ω which $A_1(\lambda)$ has no eigenvalues with zero real parts for any $\lambda \in \Omega$,*

$$\Sigma_{abs}^{2,\Omega} \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\}. \tag{23}$$

In particular, eigenvalues $\mu_2^1, \cdots, \mu_2^n$ of $D_u f(p_2, c_0, \epsilon_0) = A_2(0)$ satisfy

$$\operatorname{Re}\mu_2^1 \geq \cdots \geq \operatorname{Re}\mu_2^n$$

and μ_2^{m+1} is a complex conjugate of μ_2^m, i.e., $\mu_2^{m+1} = \overline{\mu}_2^m$, $\mu_2^m \neq \mu_2^{m+1}$ and $m = \dim W^u(p_1(c, \epsilon))$, then 0 is contained in the absolute spectrum and hence, glued pulses $U_p(\xi, c, \epsilon)$ are unstable if (c, ϵ) sufficiently close to (c_0, ϵ_0).

1.2 Topological and computational approach to eigenvalue problems for a class of one-dimensional Schrödinger operators

In the second part of this thesis, we present a powerful tool to investigate the behaviors of eigenvalues of the Schrödinger operator when a perturbation is added to the periodic potential of it. We use the topological approach in the first part combined with rigorously computational methods. The topological approach is soft and flexible, however it sometimes lacks the information on the precise location of eigenvalues. The verified numerical computations are rigorous in mathematical sense. We combine this soft machine with the verified numerical computations and show the distribution of eigenvalues of the Schrödinger operator.

The Schrödinger operator is given by:

$$Lu := -u'' + q(x)u + s(x)u, \tag{24}$$

where the periodic potential $q(x)$ with period $T(i.e., q(x+T) = q(x))$ and a localized perturbation $s(x)$ are C^r functions with r being sufficiently large. In particular, we assume that $s(x)$ satisfies $s(x) \to 0$ exponentially as $|x| \to \infty$.

It is well known that the spectrum of unperturbed operator:

$$L_0 u := -u'' + q(x)u, \tag{25}$$

consists of only the essential spectrum $\sigma_{ess}(L_0)$. In particular, it has spectral bands [25]. The essential spectrum is invariant under relatively compact perturbations and hence, the essential spectrum of L satisfies $\sigma_{ess}(L) = \sigma_{ess}(L_0) \subset \mathbb{R}$ ([42], [50]). Moreover, we can restrict λ to real values because L is a self-adjoint operator.

Let us consider the spectral problem:

$$L_0 u = \lambda u. \tag{26}$$

It can be rewritten as the first order system:

$$\begin{pmatrix} u \\ v \end{pmatrix}' = A_0(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix}, \tag{27}$$

where

$$A_0(x;\lambda) = \begin{pmatrix} 0 & 1 \\ (q(x)-\lambda) & 0 \end{pmatrix} \tag{28}$$

is a T-periodic matrix-valued function depending on a real parameter λ.

We consider the time-T map $\Phi(T, 0; \lambda)$.

Definition 1.7. *Let $\Phi(x, y; \lambda)$ be the fundamental solution matrix of* (27). *We define stable and unstable subspaces for the point at infinity as follows:*

$$E^s_\infty(\lambda) := \{Y \in \mathbb{R}^2 \mid \lim_{n\to\infty} \Phi(nT, 0; \lambda)Y = 0\}, \tag{29}$$

$$E^u_\infty(\lambda) := \{Y \in \mathbb{R}^2 \mid \lim_{n\to-\infty} \Phi(nT, 0; \lambda)Y = 0\}. \tag{30}$$

We consider the eigenvalue problem

$$Lu = -u'' + q(x)u + s(x)u = \lambda u. \tag{31}$$

It can be also rewritten as the first order system:

$$\begin{pmatrix} u \\ v \end{pmatrix}' = A(x;\lambda)\begin{pmatrix} u \\ v \end{pmatrix}, \tag{32}$$

or simply,

$$Y' = A(x;\lambda)Y,\ Y \in \mathbb{R}^2, \tag{33}$$

where

$$A(x;\lambda) = \begin{pmatrix} 0 & 1 \\ (q(x)+s(x)-\lambda) & 0 \end{pmatrix}. \tag{34}$$

Let $\theta = \tan^{-1}(\frac{v}{u})$, and we rewrite (32) in θ coordinates:

$$\theta' = (q(x) + s(x) - \lambda + 1)\cos^2\theta - 1. \tag{35}$$

Let $\mathcal{E}^s(\lambda)$ and $\mathcal{E}^u(\lambda)$ be points in θ coordinates corresponding to the stable and unstable subspaces $\mathbb{E}^s_\infty(\lambda)$ and $\mathbb{E}^u_\infty(\lambda)$, respectively, and $\theta(x;\lambda)$ be the unique solution of (35) satisfying $\lim_{n\to-\infty}\theta(nT;\lambda) = \mathcal{E}^u(\lambda)$, and define $\hat{\theta}(n;\lambda) := |\theta(nT;\lambda) - \theta(-nT;\lambda)|$. Let $\Lambda \subset \mathbb{R}\setminus\sigma_{ess}(L)$ be the spectral gap.

Then, the following theorem holds.

Theorem 1.8. *([71]) Let $[\lambda_1, \lambda_2] \subset \Lambda$ be an interval. For any $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that, if $|\hat{\theta}(n;\lambda_2) - \hat{\theta}(n;\lambda_1)| > 2m\pi$, then there are at least m eigenvalues of L in $[\lambda_1, \lambda_2]$.*

Let $\mathcal{N}_\lambda$ and $\mathcal{M}_\lambda$ be the compact neighborhood of $\mathcal{E}^u(\lambda)$ and $\mathcal{E}^s(\lambda)$ with $\mathcal{N}_\lambda \cap \mathcal{M}_\lambda = \emptyset$.

Theorem 1.9. *(Counting intersection number [71]). Let* $I = [\lambda_-, \lambda_+]$ *be an interval in the gap* Λ *and* $\mathcal{M} = \cup_{\lambda \in I} \mathcal{M}_\lambda$*. If* n *is sufficiently large,* $\mathcal{M} \subset \mathbb{RP}^1$*, and* I *has the following properties, then there exists at least one eigenvalue of* L *in* I*.*

$$\begin{aligned} \theta(nT; \lambda_-) &< \min \mathcal{M}, \\ \theta(nT; \lambda_+) &> \max \mathcal{M}. \end{aligned} \tag{36}$$

1.2.1 Computer assisted results

Our presented method is summarized as the following 3 steps.
Step 1. Construction of an interval $[\lambda_-, \lambda_+]$ in the spectral gap.
Step 2. Determination of an integral interval $[-nT, nT]$ and enclosure of $\mathcal{E}^u(\lambda), \mathcal{E}^s(\lambda)$.
Step 3. Enclosure of a heteroclinic orbit $\theta(nT; \lambda)$ from $\mathcal{E}^u(\lambda)$ to $\mathcal{E}^s(\lambda)$.

We verify eigenvalues of L_C by the above method for the case

$$L_C u = -u'' + 5\cos(2\pi x)u + Ce^{-x^2}u \tag{37}$$

where $C \in \mathbb{R}$ is a parameter. We obtain the following results using the software package CAPD (ver 2.0) [74] in step 2 and 3 to obtain the rigorous results.

Computer Assisted Result 1.10. *([71]) When* $C = 7.0$*, there is at least one eigenvalue of* L_C *in the interval* $[7.6151, 7.6160]$.

We are interested in the asymptotic behavior of eigenvalues in a given subinterval of the gap as the constant C tends to infinity. A variety of studies is concerned with such behavior (e.g. Deift–Hempel [35], Hempel [34]).

Let $\mathbb{F}$ be the set of floating point numbers and $\mathbb{IF}$ be the set of intervals whose end-points are in $\mathbb{F}$. Similarly, $\mathbb{IF}^n$ is the set of n-dimensional cubes, that is,

$$\mathbb{IF}^n := \{I_1 \times I_2 \times \cdots \times I_n | I_i \in \mathbb{IF}\}. \tag{38}$$

We define the parameter space as $(\lambda, C) \in \Lambda \times \mathcal{C} = \cup_i \Lambda_i \times \cup_i \mathcal{C}_i \subset \mathbb{IF}^2$ where $\Lambda_i = [\lambda_i, \lambda_{i+1}]$ and $\mathcal{C}_i = [C_i, C_{i+1}]$. First, using our method, we obtain intervals $\mathcal{I}_C$ containing eigenvalues for each fixed C. Next, for any j, we check off $\mathcal{Q}_{k\ell} = \Lambda_k \times \mathcal{C}_\ell$ if $\Lambda_k \times \{C_\ell\}$ or $\Lambda_k \times \{C_{\ell+1}\}$ contains eigenvalues. By the above procedures, we obtain enclosures of eigenvalue branches in $\mathcal{C} \times \Lambda$ as follows.

Computer Assisted Result 1.11. *([71]) There exists at least one pair* (λ, C) *in each shaded rectangle* $\mathcal{Q}$ *in* $[7.3, 12.2] \times [6.0, 30.0]$ *in Figure 1, such that the connecting orbit from* $\mathcal{E}^s(\lambda)$ *to* $\mathcal{E}^u(\lambda)$ *exists. Therefore, at least one eigenvalue of* L_C *exists with the parameter* C*.*

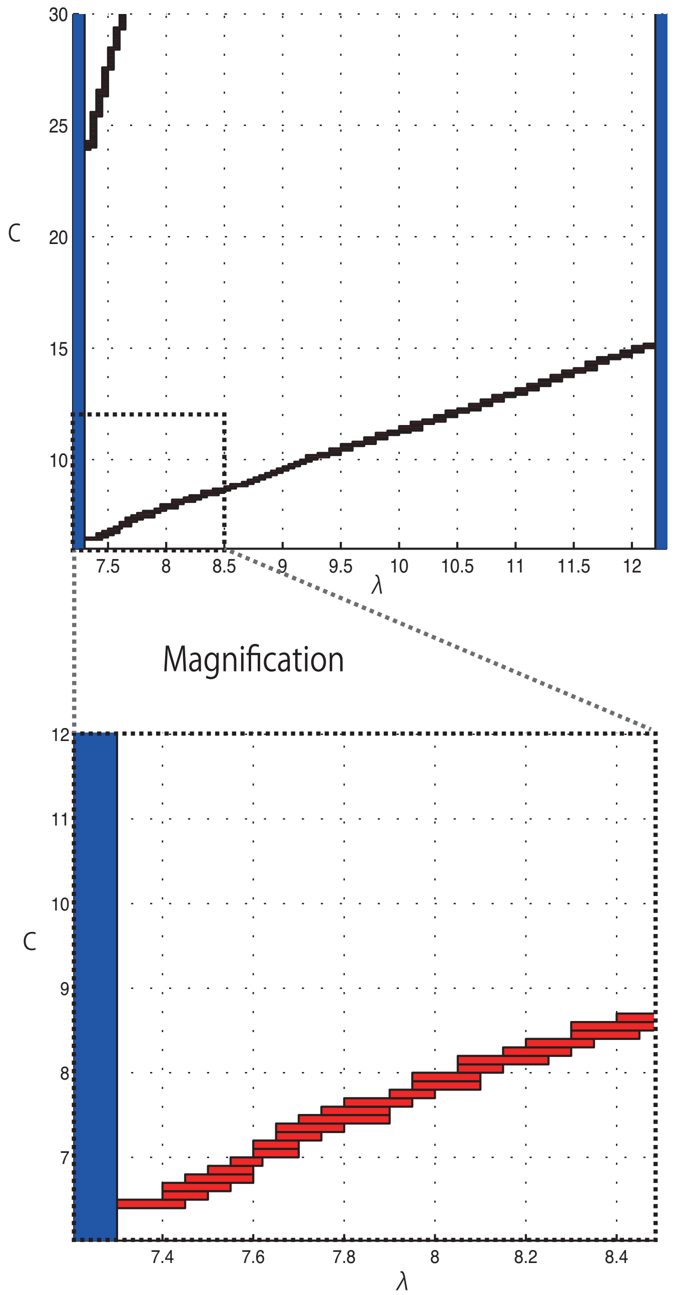

Figure 1: Distribution of the eingenvalues: The horizontal axis is λ-coordinate and the vertical axis is C-coordinate. The upper figure indicates the behavior of eigenvalues and the spectral gap. Two shaded lines in $\Lambda \times \mathcal{C} = [7.3, 12.2] \times [6.0, 30.0]$ consist of rectangles $\mathcal{Q}_{k\ell}$ which contain at least one eigenvalue of L_C, and the outer region (blue-colored region) of region $\Lambda \times \mathcal{C}$ contains essential spectra (two different spectral bands) of L_C. The lower figure is a magnification of the area enclosed by dotted line in the upper figure, and each shaded (red-colored) rectangle $\mathcal{Q}_{k\ell}$ contains at least one eigenvalue of L_C.

References

[1] V. I. Arnol'd, *Characteristic class entering in quantization conditions*, Functional Analysis and Its Applications, **1**, (1967), 1-13.

[2] V. I. Arnol'd, *Mathematical methods of classical mechanics*, second edition, Springer, (1989).

[3] N. W. Ashcroft and N. D. Mermin, *Solid state physics*, HRW, New York, (1976).

[4] A. Abbondandolo, *Morse theory for Hamiltonian systems*, CHAPMAN HALL/CRC, (2001).

[5] J. Alexander, R. Gardner and C. K. R. T. Jones, *A topological invariant arising in the stability analysis of traveling waves*, J. Reine Angew. Math, **410**, (1990), 167-212.

[6] J. Alexander and C. K. R. T. Jones, *Existence and stability of asymptotically oscillatory double pulses*, J. Reine Angew. Math, **446**, (1994), 49-79.

[7] M. Atiyah, *K-theory*, New York, (1967).

[8] M. Atiyah, V. Patodi and I. Singer, *Spectral asymmetry and Riemannian geometry*, Math. Proc. Cambride Phillos. Soc., **79**, (1976), 79-99.

[9] P. Bates and C. K. R. T. Jones, *Invariant manifolds for semilinear partial differential equations*, Dynamical reported., **2**, (1989), 1-38.

[10] V. Bykov, *The bifurcations of separatrix contours and chaos*, Phisica D, **62**, (1993), 290-299.

[11] E. A. Coddington, N. Levinson,, *Theory of ordinary differential equations*, New York, Toront, London: McGill-Hill Book Company, Inc, **XII**, (1955).

[12] F. Chardard, F. Dias, T. J. Bridges, *Computing the Maslov index of solitary waves. Part 1: Hamiltonian systems on a 4-dimensional phase space*, Physica D, **238**, (2010), 1841-1867.

[13] F. Chardard, F. Dias and T. J. Bridges, *Fast computation of the Maslov index for hyperbolic periodic orbits*, J. Phys. A: Math. Gen, **39**, (2006), 4545-14557.

[14] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Lecture Notes, 38, (1978).

[15] W. Coppel, *Dichotomies in stability theory*, Lecture Notes in Mathematics, Springer Berlin **629**, (1978).

[16] E. Davies and M. Plum, *Spectral pollution*, IMA J. Numer. Anal, **24**, (2004), 417-438.

[17] P. A. Deift and R. Hempel, *On the existence of eigenvalues of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$*, Commun. Math. Phys, **103**, (1986), 461-490.

[18] B. Deng, *The existence of infinite traveling front and back waves in FitzHugh-Nagumo equations from a twisted heteroclinic loop*, SIAM J.Math. Anal, **22**, (1991), 1631-1650.

[19] J. Deng and S. Nii, *Infinite-dimensional Evans function theory for elliptic eigenvalue problems in a channel*, J. Diff. Eqs., **225**, (2006), 57-89.

[20] J. Deng and S. Nii, *An infinite-dimensional Evans function theory for elliptic eigenvalue problems*, J. Diff. Eqs., **244**, (2008), 753-765.

[21] J. Evans, *Nerve axon equations, Linear approximations*, Indiana Univ. Math.J. **21**, (1972), 877-885.

[22] J. Evans, *Nerve axon equations, II: Stability at rest*, Indiana Univ. Math.J. **22**, (1972), 75-90.

[23] J. Evans, *Nerve axon equations, III: Stability of the nerve impulse*, Indiana Univ. Math.J. **22**, (1972), 577-594.

[24] J. Evans, *Nerve axon equations, IV: The stable and unstable impulses*, Indiana Univ. Math.J. **24**, (1975), 1169-1190.

[25] M. Eastham, *The spectral theory of periodic differential equations*, Scottish Achademic Press, (1973).

[26] R. Gardner, *On the structure of the spectra of periodic travelling waves*, J. reine angew. Math. **491**, (1997), 149-181.

[27] R. Gardner, *Spectral analysis of long wavelength periodic waves and applications*, J. Math. Pures Appl. **72**, (1993), 415-439.

[28] R. Gardner and C. K. R. T. Jones, *A stability index for steady state solutions of boundary value problems for parabolic systems*, J. Diff. Eqs. **91**, (1991), 181-203.

[29] R. Gardner and C. K. R. T. Jones, *A stability index for steady state solutions of boundary value problems for parabolic systems*, J. Diff. Eqs. **91**, (1991), 181-203.

[30] I. Gel'fand and V. Lidskii, *On the structure of the regions of stability of linear canonical systems of diffrential equations with periodic coefficients*, Amer. Math. Soc. Transl. Ser, **28**, (1958), 143-181.

[31] P. Glendinning and C. Sparrow , *T-points:a codimension two heteroclinic bifurcation*, J. Stat. Phys **43**, (1986), 479-488.

[32] P. Griffiths and J. Harris, *Princeples of algebraic geometry*, New York: Wiley, (1978).

[33] J. Hale, *Ordinary differential equation*, Krieger, Second edition, (1980).

[34] R. Hempel, *On the asymptotic distribution of the eigenvalue branches of a Shrödinger operator* $H - \lambda W$ *in a spectral gap of* H, J. Reine Angew. Math, **399**, 38-59.

[35] D. Henry, *Geometric theory of semi linear parabolic equations*, Springer, (1981).

[36] D. Husemoller, *Fibre bundles*, Springer Verlag, Third edition, (1993).

[37] H. Ikeda, *Singular pulse wave bifurcations from front and back waves in bistable reaction diffusion systems*, Methods Appl. Anal. **3**, (1996), 285-317.

[38] H. Ikeda, Y. Nishiura and H. Suzuki, *Stability of traveling waves and a relation between the Evans function and SLEP equaton*, J. Reine. Angew. Math. **475**, (1996), 1-37.

[39] C. K. R. T. Jones, *Stability of the travelling wave solution of the FitzHugh–Nagumo system*, Trans. A.M.S. **286**, (1984), 431-469.

[40] C. K. R. T. Jones, *Geometrical perturbation theory*, Lecture Notes in Mathematics. Springer-Verlag. **1609**, (1995), 44-118.

[41] T. Kapitula, *The Evans function and generalized Melnikov integrals*, SIAM. J. Math. Anal. **30-2**, (1998), 273-297.

[42] T. Kato, *Perturbation theory for linear operators*, Berlin/Heidelverg, Germany: Springer, (Reprint of the 1980 edition).

[43] H. Kokubu, *Heteroclinic bifurcations associated with different saddle indices*, Adb. Ser. Dyn. Syst **9**, (1991).

[44] H. Kokubu, Y. Nishiura and H. Oka, *Heteroclinic and homoclinic bifurcations in bistable reaction diffusion systems*, J. Diff. Eqs. **86**, (1990), 260-341.

[45] X. Lin, *Using Melnikov's method to solve Silnikov's problems*, Proc. Roy. Soc. Edinburgh **116A**, (1990), 295-325.

[46] H. Mainhardt and M. Kingler, *A model of pattern formation on the shells on molluscs*, J.Theoretical Biology **126**, (1987), 63-87.

[47] J. Milner and J. Stahseff, *Characteristic classes*, Annals of Math. Studies. Princeton Univ. Press. (1974).

[48] K. Mischaikow and M. Mrozak, *Conley index*, Handbook of Dynamical Systems(ed. B. Fiedler), North-Holland, (2002), 393-460.

[49] K. Nagatou, *A numerical method to verify the elliptic eigenvalue problems including a uniqueness property*, Computing 63, **157**, (1999), 109-130.

[50] K. Nagatou, M. Plum and M. Nakao, *Eigenvalue excluding for perturbed-periodic one-dimensional Schrödinger operators*, Proc. R. Soc. A, **159**, (2011).

[51] S. Nii, *An extension of the stability index for travelling wave solutions and its application for bifurcations*, SIAM J.Math. Anal. **28**, (1997), 402-433.

[52] S. Nii, *Accumlation of eigenvalues for linear stability problem if traveling pulses bifurcating from coexisting unstable front and back waves*, Phycica D **142**, (2000), 70-86.

[53] S. Nii, *Stability of travelling multiple-front (multiple back) wave solutions of the FitzHugh-Nagumo equations*, SIAM J.Math. Appl. Anal. **28**, (1997), 1094-1112.

[54] Y. Nishiura, *Singular limit approach to stability and bifurcation for bistable reaction diffusion systems*, Rocky Mountain J.Math **21**, (1991), 727-767.

[55] Y. Nishiura, *Coexistence of infinitely many stable solutions to reaction diffusion systems in the singular limit*, Dynamics reported (New series), Springer **3**, (1994).

[56] Y. Nishiura, *Far from equilibrium dynamics, Translations of Mathematical Monographs*, AMS **209**, (2002).

[57] J. Palis and W. de Melo, *Geometric theory of Dynamical systems*, Springer, New York (1982).

[58] K. J. Palmer , *Exponential dichotomies and transversal homoclinic points*, J. Diff. Eqs **55**, (1984), 224-256.

[59] K. J. Palmer , *Exponential dichotomies and Fredholm operators*, Proc. Amer. Math. Soc. **104**, (1988), 149-156.

[60] J. Rademacher, *Homoclinic bifurcation from heterclinic cycles with periodic orbits and tracefiring of pulses*, Ph.D thesis (2004).

[61] J. Rademacher, *Homoclinic orbits near heteroclinic cycles with one equilibrium and one periodic orbit*, J. Diff. Eqs **218**, (2005), 390-443.

[62] J. Rademacher, B.Sandstede and A. Scheel, *Computing absolute and essential spectra using continuation*, J. Diff. Eqs **218**, (2005), 390-443.

[63] J. Robbin, D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27**, (1995), 1-33.

[64] C. Robinson, *Dynamical systems: stability, symbolic dynamics, and chaos, second edition*, CRC press, Boca Raton (1998)

[65] B. Sandstede, *Stability of multiple-pulse solutions*, Trans. Amer. Math. Soc **350**, (1998), 429-472.

[66] B. Sandstede, *Stability of traveling waves, In Handbook of Dynamical Systems II (B Fiedler, ed.).* North-Holland, (2002), 983-1055.

[67] B. Sandstede, *Stability of N-fronts bifurcating from a twisted heteroclinic loop and an application to the FitzHugh-Nagumo equations*, SIAM J.Math. Anal **29**, (1998), 183-207.

[68] B. Sandstede and A. Scheel, *Gluing unstable fronts and back together can produce stable pulses*, Nonlinearity **13**, (2000).

[69] B. Sandstede and A. Scheel, *Absolute and convective instabilities of waves on unbounded and large bounded domains*, Phycica D **145**, (2000).

[70] A. Sekisaka, *Topological relationship between the absolute spectrum and eigenvalue problem for pulse near heteroclinic cycle*, Preprint.

[71] A. Sekisaka and S. Nii, *Computer assisted verification of the eigenvalue problem for one-dimensional Schrödinger operator*, Preprint.

[72] E. Yanagida, *Stability of fast travelling pulse solutions of the FitzHugh–Nagumo equations*, J. Math. Biology **22**, (1985), 81-104.

[73] K. Yokota, *Pulse solutions and time-space self-replaced pattern for three component reaction diffusion systems (in Japanese)*, Master thesis of Hokkaido University (2002).

[74] Computer Assisted Proofs in Dynamics group (CAPD) website, *http://capd.ii.uj.edu.pl/index.php.*