Concentration phenomena in singularly perturbed solutions of a spatially heterogeneous reaction-diffusion equation

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博　士　論　文

Concentration phenomena in singularly perturbed solutions of a spatially heterogeneous reaction-diffusion equation

（空間的に不均一な反応拡散方程式の特異摂動解に現れる凝集現象）

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平 成 25 年

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Summary

In 1972, A. Gierer and H. Meinhardt proposed the following activator-inhibitor system as a model of the head regeneration in hydra:

$$
\text{(GM)} \qquad \begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2 \Delta A - \mu_a(x) A + \rho_a(x)\Big(\dfrac{c_a A^p}{H^q} + \rho_0(x)\Big) & \text{in } \Omega, \\ \dfrac{\partial H}{\partial t} = D\Delta H - \mu_h(x) H + \rho_h(x)\dfrac{c_h A^r}{H^s} & \text{in } \Omega, \\ \dfrac{\partial A}{\partial \boldsymbol{\nu}} = \dfrac{\partial H}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega. \end{cases}
$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\boldsymbol{\nu}$ denotes the unit outer normal to $\partial\Omega$, $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$ is the Laplace operator, c_a, c_h, ε, D are positive constants, $\mu_a(x)$, $\rho_a(x)$, $\rho_0(x)$, $\mu_h(x)$, $\rho_h(x)$ are positive functions. They hypothesized that the head of hydra is formed at the place the activator concentrates. Moreover, since the activator grows auto-catalytically, they assumed that the inhibitor has the role of reducing the growth of activator to prevent the explosion of the activator concentration. In numerical situations, the system (GM) exhibits various type of patterns. Most typical one is the formation of spike-like patterns in which the activator concentrates in a very narrow region around finitely many points. Sometimes the activator concentrates around curves or surfaces. Some Patterns are stationary, and others are nonstationary, depending on the parameters and initial data. From a mathematical point of view, it is very difficult to understand rigorously the process of the formation of pattern in (GM). For example, we do not know how to find all stationary solutions, and hence it is hopeless to understand the global behavior of a solution with an arbitrary initial data. Therefore, it is natural to consider a simplified system. Keener [3] proposed to take the limit of $D \to \infty$. Formally speaking, in this limit, $\Delta H \to 0$ and hence $H(x,t) \to \xi(t)$ because of the no-flux boundary condition. Here $\xi(t)$ is an unknown. To derive an equation for $\xi(t)$, we integrate the second equation of (GM) over Ω to obtain

$$
\frac{\partial}{\partial t}\int_\Omega H(x,t)\,dx = -\int_\Omega \mu_h(x) H(x,t)\,dx + \int_\Omega \rho_h(x) c_h \frac{A(x,t)^s}{H(x,t)^s}\,dx.
$$

Hence, as formal limit, we are led to

$$
\text{(SS)} \qquad \begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2 \Delta A - \mu_a(x) A + \rho_a(x)\Big(\dfrac{c_a A^p}{\xi^q} + \rho_0(x)\Big) & \text{for } x \in \Omega, t > 0 \\ |\Omega|\dfrac{d\xi}{dt} = -\xi \displaystyle\int_\Omega \mu_h(x)\,dx + \dfrac{1}{\xi^s}\int_\Omega \rho_h(x) c_h A^r\,dx & \text{for } t > 0. \\ \dfrac{\partial A}{\partial \boldsymbol{\nu}} = 0 & \text{for } x \in \partial\Omega, t > 0. \end{cases}
$$

which is called the *shadow system* for (GM). This shadow system is regarded to preserve some of the essential properties of the original system, and therefore the initial-boundary value problem for (SS) is an important one that should be investigated first in theoretical studies.

Stationary solutions of the shadow system are of particular interest, since we often observe a spike-like stationary solution in numerical simulations. We note that ξ is an unknown constant if we consider the stationary problem for (SS). Therefore it is convenient to scale the activator as $A(x) = \xi^{q/(p-1)}u(x)$, which yields

$$\text{(SSS)} \qquad \begin{cases} \varepsilon^2 \Delta u - \mu_a(x)u + \rho_a(x)c_a u^p + \xi^{-q/(p-1)}\rho_a(x)\rho_0(x) = 0 & \text{in } \Omega, \\ c_h \displaystyle\int_\Omega \rho_h(x)u^r\,dx - \xi^{s+1-\frac{qr}{p-1}} \int_\Omega \mu_h(x)\,dx = 0, & \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

If $\rho_0(x) \equiv 0$, then any (positive) solution of the Neumann problem for the single equation

$$\begin{cases} \varepsilon^2 \Delta u - \mu_a(x)u + c_a\rho_a(x)u^p = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

determines the value of ξ by the second equation of (SSS). A fundamental question is whether this Neumann problem has a nontrivial solution or not. There have been a huge amount of literature concerning this question in the case where $\mu_a(x)$ and $\rho_a(x)$ are constants. However, not much has been known about the case of variable coefficients.

The purpose of this thesis is to study the structure of nontrivial solutions of the boundary value problem for the following single equation with variable coefficients when the parameter $\varepsilon > 0$ is sufficiently small:

$$\text{(P)} \qquad \begin{cases} \varepsilon^2 \mathcal{A}(x)u - a(x)u + b(x)u^p + \delta\sigma(x) = 0, \ u > 0 & \text{in } \Omega, \\ \mathcal{B}(x)u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and p satisfies $1 < p < (n+2)/(n-2)$ if $n \geq 3$, while $1 < p < \infty$ if $n = 1, 2$, $\varepsilon > 0$ and $\delta \geq 0$ are sufficiently small constants, $\mathcal{A}(x) = \sum_{i,j=1}^n (\partial/\partial x_i)a_{ij}(x)(\partial/\partial x_j)$ is a strictly and uniformly elliptic operator with $a_{ij} \in C^{1,\alpha}(\overline{\Omega})$; $a_{ij} = a_{ji}$, both of a and b are of class C^2 on $\overline{\Omega}$ and bounded from below by positive constants; and σ is a nonnegative C^2 function on $\overline{\Omega}$ with $\|\sigma\|_{L^\infty(\Omega)} = 1$. Moreover, $\mathcal{B}(x) = \sum_{i,j=1}^n \nu_i a_{ij}(x)(\partial/\partial x_j)$ is the co-normal differential operator, and $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_n)$ is the unit outward normal to $\partial\Omega$.

We are interested in point condensation phenomena, or point concentration phenomena, observed in solutions of the problem (P) by which we mean that as $\varepsilon \downarrow 0$, the distribution of a solution concentrates around a finitely many points on $\overline{\Omega}$. In this thesis, however, we consider mainly the case of only one concentration point. Problem (P) is a generalization of [5], [6] where all coefficients are constants and [7] where only $b(x)$ is not a constant, and we would like to know the effect of the spatial heterogeneity on the concentration point, especially in the case where the inhomogeneous term $\delta\sigma(x)$ does not vanish identically.

First, we introduce an energy functional $J_\varepsilon(u)$ corresponding to (P):

$$\text{(1)} \qquad \begin{aligned} J_\varepsilon(u) := \frac{1}{2}\int_\Omega \Big(\varepsilon^2 \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + a(x)u^2\Big)dx \\ - \frac{1}{p+1}\int_\Omega b(x)u_+^{p+1}\,dx - \delta\int_\Omega \sigma(x)u\,dx, \end{aligned}$$

for $u \in W^{1,2}(\Omega)$, where $u_+(x) = \max\{u(x), 0\}$. Then we can prove the following

Proposition 1 (Minimal Solution) *There exists a positive number δ_* such that for each $\delta \in [0, \delta_*)$ the functional $J_\varepsilon(u)$ has a unique local minimizer $u_{m,\varepsilon}$ in $W^{1,2}(\Omega)$, regardless the size of $\varepsilon > 0$. Moreover, if $\delta = 0$, then $u_{m,\varepsilon}(x) \equiv 0$, while if $\delta > 0$, then*

$$0 < u_{m,\varepsilon}(x) \leq \frac{\delta}{\min_{x\in\overline{\Omega}} a(x)} \quad \textit{for all } x \in \overline{\Omega}.$$

Definition 1 We call the solution obtained in Proposition 1 the *minimal solution* for the problem (P).

Next, we put

$$I_\varepsilon(v) := J_\varepsilon(u_{m,\varepsilon} + v) - J_\varepsilon(u_{m,\varepsilon}) \quad \text{for } v \in W^{1,2}(\Omega). \tag{2}$$

We can apply the mountain pass lemma ([1], [**?**, Theorem 2.2]) to this functional I_ε and conclude as follows:

Lemma 2 (Mountain Pass Solution) *Let δ_* be the positive constant given by Proposition 1 and $0 \leq \delta < \delta_*$. Then zero is a local minimum of I_ε in $W^{1,2}(\Omega)$ for each $\varepsilon > 0$. In addition, there exists an $e \in W^{1,2}(\Omega)$ such that $I_\varepsilon(e) < 0$. Let $\Gamma = \{\gamma \in C^0([0,1]; W^{1,2}(\Omega)) \mid \gamma(0) = 0, \gamma(1) = e\}$. Then*

$$c_\varepsilon = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\varepsilon(\gamma(t))$$

is a positive critical value of I_ε. Moreover, c_ε is the smallest positive critical value of I_ε.

We remark here that a critical point $u_c \in W^{1,2}(\Omega)$ of J_ε is a weak solution of Problem (P). Then by the elliptic regularity theory we conclude that u_c is a classical solution of (P). In particular, $u_c \in C^{2,\alpha}(\overline{\Omega})$ (see [**?**, Theorem 6.31 and pp.130]). Clearly, a classical solution of (P) gives rise to a critical point of J_ε. Hence, finding a solution of (P) is equivalent to finding a critical point of J_ε. On the other hand, $v_c \in W^{1,2}(\Omega)$ is a critical point of I_ε if and only if $u_{m,\varepsilon} + v_c$ is a critical point of J_ε. Consequently our problem is reduced to finding a critical point of I_ε.

Now let v_ε be a critical point of I_ε corresponding to c_ε: $I_\varepsilon(v_\varepsilon) = c_\varepsilon$ and $I'_\varepsilon(v_\varepsilon) = 0$. Then

$$u_\varepsilon = u_{m,\varepsilon} + v_\varepsilon$$

is a solution of (P). We call u_ε a *ground-state solution* of (P).

To treat a point-concentration phenomenon in a family of ground-state solutions of (P), we define the following:

Definition 2 A family $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ of solutions of (P) is said to exhibit *a point concentration phenomenon* if there exists positive constants c_0 and C_0 with $c_0 < C_0$ such that

$$c_0\varepsilon^n \leq J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_{m,\varepsilon}) \leq C_0\varepsilon^n. \tag{3}$$

Moreover, a point $P_0 \in \overline{\Omega}$ is said to be a *concentration point* of $\{u_\varepsilon\}$ if $\{u_\varepsilon\}$ satisfies (3) and there is a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $\varepsilon_k \downarrow 0$ and $P_{\varepsilon_k} \to P_0$ as $k \to \infty$ where P_ε is a local maximum point of u_ε on $\overline{\Omega}$.

The purpose of this thesis is (i) to show that the ground-state solutions $\{u_\varepsilon\}$ exhibit a point-condensation phenomenon, and they concentrate at exactly one point $P_0 \in \overline{\Omega}$; and (ii) to give a method to locate P_0 by introducing a *locator function*.

Definition 3 For any $Q \in \overline{\Omega}$, let

$$\Phi(Q) := a(Q)^{1-n/2+2/(p-1)}\, b(Q)^{-2/(p-1)} (\det A_Q)^{1/2},$$
$$\text{where} \quad A_Q := (a_{ij}(Q))_{1\le i,j\le n}.$$

We call $\Phi(Q)$ the *primary locator function*.

Let $u_m(Q)$ denote the smaller of the two non-negative roots of the algebraic equation

$$-a(Q)\zeta + b(Q)\zeta^p + \delta\sigma(Q) = 0. \tag{4}$$

Put

$$\gamma(Q) := \left\{\frac{b(Q)}{a(Q)}\right\}^{1/(p-1)} u_m(Q). \tag{5}$$

Finally we define an important integral as follows:

$$\begin{aligned} &\mathcal{I}(\gamma(Q)) := \mathcal{I}(\gamma(Q); w) \\ &= \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla w|^2 + w^2\right) dy - \frac{1}{p+1}\int_{\mathbb{R}^n} \left\{(\gamma(Q)+w)^{p+1} - \gamma(Q)^{p+1} - (p+1)\gamma(Q)^p w\right\} dy \end{aligned} \tag{6}$$

where $w = w_{\gamma(Q)}$ is a unique positive solution of the following boundary value problem:

$$\begin{cases} \Delta w - w + (\gamma(Q)+w)^p - \gamma(Q)^p = 0 \quad \text{in } \mathbb{R}^n, \\ \lim\limits_{|y|\to\infty} w(y) = 0, \ w(0) = \max\limits_{y\in\mathbb{R}^n} w(y). \end{cases} \tag{GS-γ}$$

Definition 4 For each $Q \in \overline{\Omega}$, let

$$\Lambda(Q) := \Phi(Q)\mathcal{I}(\gamma(Q)).$$

We call $\Lambda(Q)$ the *locator function* for the boundary value problem (P).

A few remarks are in order here. First, $w_{\gamma(Q)}$ is known to be spherically symmetric with respect to the origin, and decays exponentially as $|y| \to \infty$ (see [**?**]). Second, we shall prove that (GS-γ) has at most one solution if δ is sufficiently small by making use of the implicit function theorem and the uniqueness of solution of $\Delta w - w + w^p = 0$ (due to, e.g., [4]). Third, note that $\gamma(Q)$ is constant on $\overline{\Omega}$ if and only if either (i) $\delta = 0$ or (ii) $\sigma(x) = Ca(x)^{p/(p-1)}b(x)^{-1/(p-1)}$ where C is constant. In the case where $\gamma(Q)$ is a constant function, the locator function $\Lambda(Q)$ reduces to a constant multiple of the primary locator function $\Phi(Q)$. Here, in the case of $\delta > 0$, we do not know what the upper bound of γ depends on since we use the implicit function theorem to prove the uniqueness of solution of (GS-γ). However, we can know that the upper bound of $\|\gamma\|_{L^\infty(\Omega)}$ depends on only p, n and γ in the cases of $1 < p < \infty$ if $n = 1$, $p \le 2$ if $n = 2$, and $1 < p \le n/(n-2)$ if $n \ge 3$ by the shooting theory of the ordinary equation, [2] and [4].

The main results of this thesis are stated as follows.

Theorem 3 *Suppose that $P_0 \in \overline{\Omega}$ is a concentration point of a family $\{u_\varepsilon\}_{\varepsilon>0}$ of the ground-state solutions. Then, the following holds:*

(i) If $\min_{Q\in\partial\Omega} \Lambda(Q) < 2\min_{Q\in\overline{\Omega}} \Lambda(Q)$, then $P_0 \in \partial\Omega$. Moreover, P_0 is a minimum point of the locator function $\Lambda(Q)$ over $\partial\Omega$,

(ii) If $\min_{Q\in\partial\Omega} \Lambda(Q) > 2\min_{Q\in\overline{\Omega}} \Lambda(Q)$, then $P_0 \in \Omega$. Moreover, P_0 is a minimum point of $\Lambda(Q)$ over $\overline{\Omega}$.

We prove this theorem by finding the limit of $I_\varepsilon(v_\varepsilon)$ as $\varepsilon \downarrow 0$, assuming that a family $\{u_\varepsilon\}$ of ground-state solutions of (P) exhibits a point concentration phenomenon around a point P_0. Here, we note that $I_\varepsilon(v_\varepsilon) = J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_{m,\varepsilon})$. To do this, we derive an asymptotic form u_ε near a local minimum point P_ε. From this limit formula, we see that $I_\varepsilon(v_\varepsilon)$ has to converge to the minimum of $\Lambda(Q)$ either over the boundary or over $\overline{\Omega}$ as $\varepsilon \downarrow 0$ because $I_\varepsilon(v_\varepsilon)$ is the minimum of the positive critical values of $I_\varepsilon(v)$.

By the definition of Λ, if γ is constant, then $\mathcal{I}(\gamma(\cdot))$ is constant, hence Λ becomes the product of Φ and a constant $\mathcal{I}(\gamma)$. We note that $\gamma(x)$ is constant in both cases of (i) $\delta = 0$ and (ii) $\sigma(x) = Ca(x)^{p/(p-1)}b(x)^{-1/(p-1)}$ by (4) and (5). Thus, we have the following corollary:

Corollary 4 *Assume either* (i) *that $\delta = 0$ or* (ii) *that $\sigma(x) = Ca(x)^{p/(p-1)}b(x)^{-1/(p-1)}$ where C is constant. Suppose that $P_0 \in \overline{\Omega}$ is a concentration point of a family $\{u_\varepsilon\}_{\varepsilon>0}$ of the ground-state solutions of* (P). *Then, the following holds:*

(I) *If $\min_{Q\in\partial\Omega} \Phi(Q) < 2\min_{Q\in\overline{\Omega}} \Phi(Q)$, then $P_0 \in \partial\Omega$. Moreover, P_0 is a minimal point of the primary locator function $\Phi(Q)$ over $\partial\Omega$.*

(II) *If $\min_{Q\in\partial\Omega} \Phi(Q) > 2\min_{Q\in\overline{\Omega}} \Phi(Q)$, then $P_{0\in\Omega}$. Moreover, P_0 is a minimal point of $\Phi(Q)$ over $\overline{\Omega}$.*

Although we can locate the concentration point P_0 by finding the minimum points of Λ over $\overline{\Omega}$ and $\partial\Omega$, it is in general very difficult to calculate these minimum points. For, we must solve the boundary value problem (GS-γ) in $\mathbb{R}^n$ and know the dependence of the energy $\mathcal{I}(\gamma(Q); \mathbb{R}^n)$ on Q explicitly. However, if δ is sufficiently small, then the minimal points of the primary locator function Φ gives us a first approximation:

Theorem 5 *Suppose that $P_0 \in \overline{\Omega}$ is a concentration point of a family $\{u_\varepsilon\}_{\varepsilon>0}$ of the ground-state solutions. In addition, in the case of $p < 2$, assume that*

$$\text{(S)} \qquad \textit{if} \quad \sigma(x) = 0 \quad \textit{for some, } x_0 \in \partial\Omega \quad \textit{then} \quad \frac{\partial\sigma}{\partial\nu}(x_0) = 0.$$

Then, the following holds:

(I) If $\min_{Q\in\partial\Omega} \Phi(Q) < 2\min_{Q\in\overline{\Omega}} \Phi(Q)$, then $P_0 \in \partial\Omega$. Moreover, if all the minimum points of $\Phi|_{\partial\Omega}$ on $\partial\Omega$ are nondegenerate (as a critical point), then there exists a minimum point Q_0 of Φ over $\partial\Omega$ such that $|P_0 - Q_0| = O(\delta)$.

(II) If $\min_{Q\in\partial\Omega}\Phi(Q) > 2\min_{Q\in\overline{\Omega}}\Phi(Q)$*, then* $P_0 \in \Omega$*. Moreover, if all the minimum points of* Φ *in* Ω *are nondegenerate, then there exists a minimum point* Q_0 *of* Φ *over* $\overline{\Omega}$ *such that* $|P_0 - Q_0| = O(\delta)$*.*

Consequently, we know the location of P_0 by calculating the minimum of Φ over $\overline{\Omega}$ and that over $\partial\Omega$. Moreover, we find that if the inhomogeneous term $\delta\sigma$ is sufficiently small, then $\delta\sigma$ does not affect much the location of the concentration point.

So far, we have been concerned with a concentration phenomena observed in groud-state solutions whose existence is guranteed by the mountain pass lemma. However, it is quite possible that solutions with higher energy $J_\varepsilon(u) > c_\varepsilon$ exist and exhibit a point-concentration phenomenon, as in the case of spatially homogeneous equations. The following result reveals the role of the primary locator function Φ in locating the concentration point.

Theorem 6 *Let* $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ *be a family of positive solutions of the following Neumann problem:*

$$\text{(7)} \qquad \begin{cases} \varepsilon^2\Delta u - a(x)u + b(x)u^p = 0,\ u(x) > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that there exists a positive constant C_0 *such that* $0 < J_\varepsilon(u_\varepsilon) \le C_0\varepsilon^n$ *for* $0 < \varepsilon < \varepsilon_0$*. Assume that* u_ε *attains a local maximum at* $P_\varepsilon \in \Omega$ *and* $P_\varepsilon \to P_0 \in \Omega$ *as* $\varepsilon \downarrow 0$*. Then* P_0 *is a critical point of the primary locator function* Φ*, that is,* $\nabla\Phi(P_0) = 0$*. Moreover, for any* $R > 0$*,*

$$u_\varepsilon(P_\varepsilon + \varepsilon z) = v_{P_\varepsilon}(z) + O(\varepsilon) \quad \text{in } C^2(\overline{B}_R(0)) \quad \text{as } \varepsilon \downarrow 0,$$

where $v_Q(z) = (a(Q)/b(Q))^{1/(p-1)}w(a(Q)^{1/2}z)$ *and* w *is a unique positive solution of the boundary value problem*

$$\text{(GS-0)} \qquad \begin{cases} \Delta w - w + w^p = 0 & \text{in } \mathbb{R}^n, \\ \lim\limits_{|y|\to\infty} w(y) = 0,\ w(0) = \max\limits_{y\in\mathbb{R}^n} w(y). \end{cases}$$

This theorem says that any solution u_ε with $0 < J_\varepsilon(u_\varepsilon) \le C_0\varepsilon^n$ looks like $v_{\mathbb{P}_\varepsilon}((x - P_\varepsilon)/\varepsilon)$ near a local maximum point P_ε as long as P_ε stays away from the boundary. Moreover, this result is a counterpart of that by Wei [8] where the case of constant coefficients is considered.

This thesis is organized as follows: In Chapter 2, we construct the minimal solution $u_{m,\varepsilon}$ and then prove the existence of mountain-pass solution stated in Lemma 2. Moreover, we prove the uniqueness of entire solution which appears as the first approximation of ground-state solutions. In the last section of Chapter 2 we derive an upper bound of energy of a ground-state solution, which is crucial in proving Theorem 3. Chapter 3 is concerned with the asymptotic behavior of ground-state solutions as $\varepsilon \downarrow 0$. In Chapter 4 we prove Theorem 3, Corollary 4 and Theorem 5. Finally in Chapter 5 we consider the boundary value problem (7) and prove Theorem 6.

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