Variational problems of Lagrangian submanifolds in Kahler manifolds

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URL	http://hdl.handle.net/10097/58835

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A thesis presented
by
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to
The Mathematical Institute
for the degree of
Doctor of Science

Tohoku University
Sendai, Japan

March, 2014

Acknowledgement

I would like to express my gratitude to my supervisor Professor Reiko Miyaoka for the valuable advice and constant encouragement. I am grateful to Professor Takashi Shioya, Professor Motoko Kotani, Professor Jürgen Berndt, Professor Osamu Ikawa, Professor Sadahiro Maeda, Professor Yoshihiro Ohnita, Professor Hajime Urakawa, Professor Jun-ichi Inoguchi, Professor Hiroshi Iriyeh, Professor Hajime Ono, and Professor Takashi Sakai for their valuable comments and suggestions. As for English, I am thankful to Professor Wayne Rossman for their kindhearted help.

A part of this work has been done while the author was staying at the King's College London by the "Strategic Young Researcher Overseas Visits Program for Accelerating Brain Circulation by JSPS". He expresses his sincere thanks to the hospitality of the college. The author was partially supported by Grant-in-Aid for JSPS Fellows.

I would like to thank Dr. Satoshi Ueki, Dr. Kotaro Kawai and Dr. Kaname Hashimoto for the useful discussion.

Finally, I would like to extend my indebtedness to my family for their understanding, support, and encouragement.

Contents

Chapter 1

Introduction

A submanifold L in a symplectic manifold (M^{2n}, ω) is called *Lagrangian* if the restriction of the symplectic form ω to L vanishes and $\dim L = (1/2)\dim M$. These submanifolds have been recognized as one of important objects in symplectic geometry, and investigated in various fields of mathematics. Especially, when M is a Kähler manifold, extrinsic properties of Lagrangian submanifolds have been studied by many authors. For instance, Harvey and Lawson [31] established the calibrated geometry, and introduced the notion of special Lagrangian submanifolds in Calabi-Yau manifolds. These submanifolds are calibrated, namely, volume-minimizing in each homology class.

In a Kähler manifold (M, ω, J), the well-known Wirtinger's inequality (cf. [9]) says that complex submanifolds are calibrated. On the other hand, Lagrangian submanifolds are antithetical to submanifolds, because the Lagrangian condition is $JT_pL \cap T_pL = \{0\}$ for any $p \in L$ (namely, L is *totally real*). Actually, we know few examples of stable minimal Lagrangian submanifolds in Kähler manifolds.

Let L be a Lagrangian submanifold in a Kähler manifold (M, ω, J). The Lagrangian property is preserved under Hamiltonian flows, generated by time dependent Hamiltonian vector fields on M. Therefore, it is natural to consider the variational problem under the Hamiltonian constraint. This restriction is natural from the viewpoint of Lagrangian mean curvature flows in a Kähler-Einstein manifold. In fact, the mean curvature flow preserves the Lagrangian property (see Theorem 2.1.1 in Chapter 2), and is a Hamiltonian flow if L satisfies some topological conditions.

A deformation of a Lagrangian immersion is called Hamiltonian when it is generated by a Hamiltonian vector field. A Lagrangian submanifold L which attains an extremal of the volume functional under all Hamiltonian deformations is called *Hamiltonian minimal* (H-minimal, for short). Furthermore, if the second variation under Hamiltonian deformations is non-negative, L is called *Hamiltonian stable* (H-stable, for short). This was first investigated by Y.-G. Oh ([70], [71]), and such Lagrangian submanifolds are regarded as the ‘best’ representation of Lagrangians in each Hamiltonian isotopy class. When L has the least volume in its Hamiltonian isotopy class, we call L *Hamiltonian volume minimizing* (briefly, H.V.M.). The existence and uniqueness of H.V.M. Lagrangian submanifolds in each Hamiltonian isotopy class are interesting problems, because the problem is regarded as a generalization of the classical *isoperimetric problem.* As a simplest example, consider a small or grate circle γ in the 2-sphere S^2. Then, the isoperimetric inequality is given by$l^2(\gamma) \geq (4\pi - A(\gamma))A(\gamma)$, where $l(\gamma)$ is the length of γ and $A(\gamma)$ is the area enclosed by γ. Under area-preserving deformations of γ, it follows from this inequality that γ is the (unique) length-minimizing curve. We may interpret this into H.V.M. Lagrangian submanifolds in a Kähler manifold, but only few examples of H.V.M. Lagrangians are known. Thus, as a first step, it is important to construct and classify H-minimal or H-stable Lagrangians.

In the last decades, many examples of H-minimal Lagrangian submanifold have been constructed in specific Kähler manifolds. In Chapter 2, we review the basic properties of H-minimal and H-stable Lagrangians and give some known examples. In particular, we focus on H-minimal Lagrangian submanifolds in Hermitian symmetric spaces. Typical examples are obtained by orbits of Hamiltonian actions of a Lie subgroup of isometries. If the orbit is compact Lagrangian, it is automatically H-minimal. Therefore, the classification problem of these orbits is interesting and important. Moreover, we can decide the H-stability of these orbits by using a Lie theoretical argument. Up to now, this is the only successful way to decide H-stability explicitly, because the second variation of the H-minimal Lagrangians is not simple, except for minimal Lagrangians in Kähler-Einstein manifolds.

If the ambient Kähler manifold is the complex Euclidean space $\mathbb{C}^m$, Y.-G.Oh [71] pointed out that the standard tori $T^m = S^1(r_1) \times \cdots \times S^1(r_m)$ are H-minimal and H-stable. Generalizing Oh’s results in [71], Y. Dong [26] showed that the pre-image of an H-minimal Lagrangian submanifold in $\mathbb{C}P^{m-1}$ via the Hopf fibration $\pi : S^{2m-1} \to \mathbb{C}P^{m-1}$ is H-minimal Lagrangian in $\mathbb{C}^m$. We note that there are some known H-minimal Lagrangian submanifolds in $\mathbb{C}P^{m-1}$. For instance, any compact, extrinsically homogeneous Lagrangian submanifolds in $\mathbb{C}P^{m-1}$ are H-minimal, and these are classified by Bedulli and Gori [14]. On the other hand, Anciaux and Castro [4] gave examples of H-minimal Lagrangian immersions of manifolds with various topologies by taking a product of a Lagrangian surface and Legendrian immersions in odd-dimensional unit spheres. These are compact and contained in a sphere. However, we still know few examples in $\mathbb{C}^m$.

Therefore, in Chapter 3, we give a new family of H-minimal Lagrangian submanifolds in $\mathbb{C}^m$, which are non-compact, complete and have some symmetries.

Let N^n be a submanifold in $\mathbb{R}^{n+k}$. Our example is given by the normal bundle νN of

N in the tangent space $T\mathbb{R}^{n+k}$ of the Euclidean space $\mathbb{R}^{n+k}$ which is naturally regarded as $\mathbb{C}^{n+k}$. In the following, we always use this identification. Then, the normal bundle νN is a Lagrangian submanifold in $T\mathbb{R}^{n+k}$. Harvey-Lawson [31] first noted that νN is special Lagrangian with some phase if and only if N is an *austere* submanifold, namely, the set of principal curvatures is invariant under the multiplication by -1 for any unit normal vector. In our context, the condition that a Lagrangian submanifold L is special is equivalent to that L is minimal. Hence, we can produce examples of minimal (or special) Lagrangian submanifolds from austere submanifolds. For instance, minimal surfaces and complex submanifolds are austere. However, these are not still well investigated. As for more explicit H-minimal examples, we obtain:

THEOREM 1 ([48]). *Let G be a compact, connected, semi-simple Lie group, $\mathfrak{g}$ the Lie algebra of G, and $N^n = \mathrm{Ad}(G)w$ a principal orbit of the adjoint action of G on $\mathfrak{g} \simeq \mathbb{R}^{n+k}$ through $w \in \mathfrak{g}$. Then the normal bundle νN of N is an H-minimal Lagrangian submanifold in the tangent bundle $T\mathfrak{g} \simeq \mathbb{C}^{n+k}$.*

The principal orbit N is diffeomorphic to G/T, where T is a maximal torus of G, and is called a *complex flag manifold*. Since $N = \mathrm{Ad}(G)w$ is compact, N is never austere in $\mathbb{R}^{n+k}$, nor, νN is minimal. Moreover, the mean curvature vector is not parallel. We also note that the normal bundle of $N = \mathrm{Ad}(G)w$ is always trivial, namely, νN is homeomorphic to $N \times \mathbb{R}^k$.

The principal orbits of the adjoint action of a compact semi-simple Lie group G on $\mathfrak{g}$ are known as examples of the *isoparametric submanifolds*, namely, submanifolds in $\mathbb{R}^{n+k}$ with flat normal bundles and constant principal curvatures (see Section 3.2). In the class of isoparametric submanifolds, we show that the complex flag manifolds are only those having H-minimal normal bundles. Namely, we obtain:

THEOREM 2 ([48]). *Let N be a full, irreducible isoparametric submanifold in the Euclidean space $\mathbb{R}^{n+k}$. Then the normal bundle νN is H-minimal in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$ if and only if N is a principal orbit of the adjoint action of a compact simple Lie group G.*

Note that the Riemannian product of two H-minimal Lagrangian immersions into Kähler manifolds is H-minimal in the Riemannian products of the Kähelr manifolds. Therefore, the irreducibility is essential in this Theorem.

We also give examples of non-complete H-minimal Lagrangian varieties as the twisted normal cones over the isoparametric hypersurfaces in the sphere (See Section 3.4).

The strategy of the proof of Theorem 1 and 2 is as follows. The mean curvature form $\alpha_{\tilde{H}}$ of a Lagrangian submanifold L in $\mathbb{C}^{n+k}$ is given by $\alpha_{\tilde{H}} = d\theta$, where θ is an S^1-valued function on L, called the Lagrangian angle. Thus the H-minimality is equivalent to the differential equation $\Delta\theta = 0$. We calculate θ on the normal bundle of a submanifold in $\mathbb{R}^{n+k}$, and show that the angle is given by a modification of the mean curvature of N. More precisely, it is expressed by the sum of arctangent of eigenvalues of the shape operator up to constant factor (See Section 3.1 for more details).

When N is an isoparametric submanifold, the differential equation $\Delta\theta = 0$ is expressed in terms of the principal curvatures of N, not including their differentials (Lemma 3.2.1). If the codimension of the isoparametric submanifold is equal to 1, the specification of solutions is easy (Proposition 3.2.5). When the codimension is 2, they are isoparametric hypersurfaces in the sphere, and the known examples consist of orbits of s-representations and non-homogeneous ones. We show that N is homogeneous whenever νN is H-minimal (Lemma 3.2.6). Therefore, together with the fact that isoparametric submanifolds of codimension grater than 3 are homogeneous (G. Thorbergsson [103], Olmos[77]), it is sufficient to consider the normal bundle of homogeneous submanifolds. On the other hand, the principal curvatures of these orbits are given by the restricted root systems of associated symmetric spaces. By using this, we characterize the H-minimality of normal bundles over the principal orbits of s-representations in terms of the multiplicities of roots (Proposition 3.2.12). Then we can specify the required orbits (see Section 3.3).

As we mentioned above, an austere submanifold in $\mathbb{R}^m$ is an important object related to special Lagrangians. Harvey-Lawson's result was generalized to some cotangent bundles equipped with the standard symplectic structure and a Riemannian metric. For instance, the cotangent bunlde T^*S^m of the units sphere S^m admits a Ricci-flat Kähler structure obtained by Stenzel [92]. Then, Karigiannis-Min-Oo [49] proved that the austere condition of a submanifold N in $S^m(1)$ is equivalent to that the conormal bundle ν^*N is special Lagrangian with some phase in T^*S^m. Y. Dong [27] generalized Harvey-Lawson's work to the psuedo-Riemannian complex Euclidean space and give a similar characterization of austere submanifolds in the pseudo-Euclidean space. In general Riemannian manifolds, we know some examples of austere submanifolds, but a geometric interpretation and properties of austere submanifolds are still unknown except for these cases.

In Chapter 4, we generalize Harvey-Lawson's result, and investigate the relation between the minimality of normal bundles and austere condition of the base manifolds. Let M be a smooth manifold and N a submanifold in M. It is classical that the cotangent bundle T^*M admits a standard symplectic strucure ω_0, and the conormal bundle ν^*N of N becomes a Lagrangian submanifold in (T^*M, ω_0). We introduce a Riemannian metric $\langle , \rangle$ on M, and identify T^*M with the tangent bundle TM. The Riemannian metric on M induces the Riemannian metric $\tilde{g}$ on TM, the so called *Sasaski metric*. Moreover, there exist a natural almost complex structure J on TM so that $(\tilde{g}, J)$ defines an almost Kähler structure on TM compatible with the pull-back symplectic structure ω_0 on T^*M via the metric $\tilde{g}$ (see Section 4.1). We note that the structure is Kähler if and only if

$(M, \langle , \rangle)$ is flat. When M is the standard Euclidean space $\mathbb{R}^m$, we recover the situation of Harvey-Lawson.

Let N be a submanifold in a Riemannian manifold $(M, \langle , \rangle)$, and νN be the normal bundle of N in $(TM, \tilde{g})$ which is Lagrangian. Motivated by Harvey-Lawson's result and Theorem 1, 2, we investigate the extrinsic properties of νN. As a consequence, we obtain the following:

THEOREM 3 (cf. [47]). *(1) Let N be a connected submanifold in a simply connected Riemannian symmetric space $M = U/K$. Then the normal bundle νN in $(TM, \tilde{g})$ is totally geodesic if and only if N is a reflective submanifold in M*

(2) Let N be a connected submanifold in the real space form $M(c)$. Then the normal bundle νN is minimal Lagrangian in $(TM, \tilde{g})$ if and only if N is austere in $M(c)$

(3) Let N be a Hopf hypersurface with constant principal curvatures in the non-flat complex space form $M(4c)$ with holomorphic sectional curvature $4c$. Then νN in $(TM, \tilde{g})$ is minimal if and only if N is austere in $M(4c)$.

(4) Let N be a submanifold in the non-flat complex space form $M(4c)$. If N is totally geodesic or complex, then νN is a minimal Lagrangian submanifold in $(TM, \tilde{g})$.

(5) Let N be a surface in the non-flat complex space form $M(4c)$. Then νN is minimal in $(TM, \tilde{g})$ if and only if N is totally geodesic or a complex curve. In particular, there exist an austere surface with non-minimal normal bundle.

As opposed to the case of the real space forms ((2) in this Theorem), the relation between the minimality of νN and the austere condition of N in other Riemannian manifolds is different. In fact, the minimality of normal bundles essentially depends on the curvature of the ambient space M (see Lemma 4.1.6).

As a corollary of Theorem 3 (2), we see that any tubular hypersurface of an austere submanifold in the unit sphere $S^m(1)$ has minimal Gauss maps (Section 4.3). These provide new examples of minimal Lagrangian immersions into the complex hyperquadric $Q_{m-1}(\mathbb{C})$.

Notice that, when $M = S^m(1)$, the minimality of a normal bundle νN in $(TS^m, \tilde{g})$ is equivalent to that νN is special Lagrangian with some phase in T^*S^m equipped with the Stenzel metric. Since the Stenzel metric, and thus, a Ricci-flat Kähler structure is defined on the cotangent bundle of any compact rank one symmetric space, we expect our observation to be useful to investigate special Lagrangians in these cotangent bundles.

Chapter 2

A survey of Hamiltonian minimal Lagrangian submanifolds in Kähler manifolds and Hamiltonian stability

2.1 H-minimal Lagrangian submanifolds

2.1.1 Hamiltonian deformations

Let $\iota : N \to M$ be an isometric immersion of a manifold N into a Riemannian manifold (M, g), where g denotes a Riemannian metric on M. If the immersion is an embedding, we call the image of $\iota(N)$ a *submanifold* in M, and sometimes, we identify $\iota(N)$ and N. For an immersion $\iota : N \to M$, we always consider the pull-back bundle ι^*TM and its subbundle ι_*TN over N. In the following, we often use an identification $T_pN \simeq \iota_*T_pN$ for $p \in N$.

Consider an infinitesimal deformation $\iota_t : N \times (-\epsilon, \epsilon) \to M$ of ι in M, namely, $\{\iota_t\}_t$ is a smooth family of immersions with $\iota_0 = \iota$, and ϵ is a positive number. Moreover, we assume one of the following:

1. A manifold N is *closed*, i.e., N is compact without boundary, or
2. A manifold N is compact with boundary, and any deformation of an immersion $\iota : N \to M$ fixes the boundary, or
3. A manifold N is non-compact without boundary, and any deformation $\{\iota_t\}_t$ of an immersion $\iota : N \to M$ is compactly supported, namely, the closure of $\{p \in N;\ \iota_t(p) \neq p,\ \text{for some } t \in (-\epsilon, \epsilon)\}$ is a compact subset of N.

Denote the set of all smooth functions on N by $C^\infty(N)$. Set $C^\infty_c(N) := C^\infty(N)$ when N is closed, $C^\infty_c(N) := \{f \in C^\infty(N);\ f|_{\partial N} = 0\}$ when N is compact with boundary, and

$C_c^\infty(N) := \{f \in C^\infty(N);\ f \text{ has a compact support}\}$ when N is non-compact, respectively. Denote the variational vector fields of $\{\iota_t\}_t$ by $V_t := \partial\iota_t/\partial t \in \iota_t^*TM$, and throughout this chapter, we assume each variational vector field is a normal vector field along ι_t.

We denote the Levi-Civita connection on TM by $\overline{\nabla}$. The *second fundamental form* B is defined by the *Gauss equation*:

$$\overline{\nabla}_X Y = (\overline{\nabla}_X Y)^\top + (\overline{\nabla}_X Y)^\perp =: \nabla_X Y + B(X,Y)$$

for tangent vectors $X, Y \in \Gamma(\iota_*TN)$, where $\top$ and $\perp$ means the tangential and normal components, respectively, and ∇ defines the Levi-Civita connection on ι_*TN with respect to the induced metric. The normal bundle of ι is defined by the quotient bundle $\nu N := \iota^*TM/\iota_*TN$. For an unit normal vector $u \in \nu_p N$, the *shape operator* with respect to u is a linear map $A^u : \iota_*T_pN \to \iota_*T_pN$ defined by

$$g(A^u(X), Y) = g(B(X,Y), u), \text{ for } X, Y \in \iota_*T_pN,$$

or equivalently, $A^u(X) = -(\overline{\nabla}_X u)^\top$. Since A^u is a symmetric operator, the eigenvalues of A^u take real values. We call the eigenvalues the *principal curvatures* of ι with respect to u. The *mean curvature vector* of ι is defined by $H := \mathrm{tr}B$. An isometric immersion ι is called *minimal* if $H \equiv 0$. It is a classical fact that a minimal immersion ι is a critical point of the volume functional under all infinitesimal deformations of ι. We call a minimal immersion ι *stable* if the second variation of the volume functional is non-negative under all deformations of ι.

Let (M^{2n}, ω, J) be a Kähler manifold with $\dim_{\mathbb{R}} M = 2n$, where ω and J denotes the Kähler form and the complex structure on M, respectively. We denote the associated Riemannian metric by g. Since the 2-form ω is closed, it defines a symplectic structure on M. An immersion $\iota : L \to M$ of a manifold L is called *isotropic* if $\iota^*\omega = 0$, and in addition, if $\dim_{\mathbb{R}} L = n$, then ι is called *Lagrangian*. One can show that the isotropic condition is equivalent to $JT_pL \subset \nu_pL$ for each $p \in L$. Moreover, L is Lagrangian if and only if $JT_pL = \nu_pL$. Therefore, when the immersion ι is Lagrangian, we obtain the bundle isomorphism between the normal bundle νL of L and the cotangent bundle T^*L of L since we have the standard isomorphism $TL \simeq T^*L$ via the Riemannian metric g. By this isomorphism, we identify normal vector fields and 1-forms on L via

$$\begin{aligned}
&\alpha : \Gamma(\nu L) \xrightarrow{\sim} \Omega^1(L) \\
&\alpha_V(p) := \omega(V, \cdot)|_{T_pL} = g(JV, \cdot)|_{T_pL},
\end{aligned}$$

where $\Gamma(\nu L)$ and $\Omega^1(L)$ denotes the smooth section on νL and T^*L, respectively.

In the following, we consider a Lagrangian immersion $\iota : L^n \to M^{2n}$ into a Kähler manifold. We define a tensor field S on L by

$$S(X, Y, Z) := g(JB(X, Y), Z) \tag{2.1}$$

for $X, Y, Z \in \Gamma(\iota_* TL)$. Then the tensor S is symmetric with respect to all variables. In fact, B is symmetric and

$$g(JB(X, Y), Z) = g(J(\overline{\nabla}_X Y)^\perp, Z) = g(J\overline{\nabla}_X Y, Z) = g(\overline{\nabla}_X JY, Z) = g(Y, JB(X, Z)),$$

where we use the Lagrangian and Kähler condition of L and M, respectively. For a Lagrangian immersion, the tensor S and the second fundamental form B have the same information. On the other hand, the mean curvature vector satisfies the following remarkable property:

THEOREM 2.1.1 (Dazord [25]). *Let (M, ω, J) be a Kähler manifold, and $\iota : L^n \to M^{2n}$ a Lagrangian immersion. Denote ρ the Ricci form of M. Then the mean curvature form α_H of ι satisfies the identity*

$$d\alpha_H = \iota^* \rho.$$

A proof of this theorem is given in [89]. In particular, if the Kähler manifold is *Einstein*, namely, $\rho = c\omega$ for a constant $c \in \mathbb{R}$, then the mean curvature form of a Lagrangian immersion of L is a *closed* 1-form, and hence, it determines a real cohomology class on L. One corollary of this fact is that the *mean curvature flow* for a Lagrangian submanifold in a Kähler-Einstein manifold preserves the Lagrangian condition (See Definition 2.1.2 below).

By the above motivation, it is natural to consider the variational problem for a Lagrangian immersion into a Kähler manifold under deformations with a Lagrangian constraint. The following definition is due to Y.-G. Oh [70].

DEFINITION 2.1.2 ([70]). (1) Let $\iota : L^n \to M^{2n}$ be a Lagrangian immersion. A vector field V along ι is called *Lagrangian (reap. Hamiltonian)* if the 1-form $\alpha_V \in \Omega^1(L)$ is closed (resp. exact).

(2) A smooth deformation $\{\iota_t\}_t$ of a Lagrangian immersion ι is called the *Lagrangian deformation* (resp. *Hamiltonian deformation*) if the variational vector fields V_t are Lagrangian (resp. Hamiltonian).

By Cartan's formula, we have $\iota_t^* \mathcal{L}_{V_t} \omega = d\alpha_{V_t}$ since ω is closed. Thus the Lagrangian deformations are characterized by the deformations which leave Lagrangian submanifolds Lagrangian. If the vector field V is Hamiltonian, then by definition, we have $\alpha_V = df$ for some functions $f \in C_c^\infty(L)$. Note that this is equivalent to $V = J\nabla f$ as V is a normal vector of L, where ∇f denotes the gradient of f on L. We have a characterization of Hamiltonian deformations as follows:

LEMMA 2.1.3 ([59]). *Let (M, ω) be a symplectic manifold. Suppose that there exist a nonzero constant γ such that $[\omega/\gamma] \in H^2(M, \mathbb{R})$ is an integral cohomology class, and thus there is a complex line bundle $\mathcal{L}$ over M with a $U(1)$-connection whose curvature is given by $(2\pi\sqrt{-1}/\gamma)\omega$. Then, a Lagrangian deformation $\{\iota_t\}_t$ of a Lagrangian immersion ι is Hamiltonian if and only if the holonomy homomorphism $\rho_t : \pi_1(L) \to \mathbb{R}$ of the flat bundle $\iota_t^*\mathcal{L}$ is the same for any t.*

A proof of this lemma is given in [75].

We give a geometric property of Hamiltonian deformations of a Lagrangian immersion into a Kähler-Einstein manifold M. Denote the canonical line bundle of M by K. Since M is Kähler, the Ricci form ρ represent the curvature on K. Moreover, if M is Einstein, then we have $\omega = c\rho$, and ι^*K is a flat bundle over L. Then, there exist the unique complex extension of the volume form on L denoted by $\tilde{\Omega}$.

PROPOSITION 2.1.4 ([72]). *Let $\iota : L^n \to M^{2n}$ be an oriented Lagrangian immersion into a Kähler-Einstein manifold. Then $\tilde{\Omega}$ satisfies the identity*

$$\nabla\tilde{\Omega} = \sqrt{-1}\alpha_H \otimes \tilde{\Omega},$$

where ∇ is the covariant derivative with respect to the induced connection on ι^*K.

PROOF. Since L is oriented, we may choose a positively oriented orthonormal local frame $\{E_1, \ldots, E_n\}$ of L such that $\nabla_{E_i} E_j = 0$ for any i, j at a point $p \in L$. Set $F_i := JE_i$ for $i = 1, \ldots, n$, then $\{E_1, \ldots, E_n, F_1, \ldots, F_n\}$ is a local orthonormal frame of M. Let $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$ be its dual frame. Then $\tilde{\Omega}$ is expressed by

$$\tilde{\Omega} = (\alpha_1 + \sqrt{-1}\beta_1) \wedge \ldots \wedge (\alpha_n + \sqrt{-1}\beta_n).$$

Thus, for $X \in T_pL$, we obtain

$$\nabla_X \tilde{\Omega} = \sum_{i=1}^{n} (\alpha_1 + \sqrt{-1}\beta_1) \wedge \ldots \wedge \nabla_X(\alpha_i + \sqrt{-1}\beta_i) \wedge \ldots \wedge (\alpha_n + \sqrt{-1}\beta_n). \tag{2.2}$$

Here, we have

$$\begin{aligned}
(\nabla_X \alpha_i)(E_j)(p) &= \overline{\nabla}_X(\alpha_i(E_j))(p) - \alpha_i(\overline{\nabla}_X E_j)(p) = 0, \\
(\nabla_X \alpha_i)(F_j)(p) &= \overline{\nabla}_X(\alpha_i(F_j))(p) - \alpha_i(\overline{\nabla}_X F_j)(p) \\
&= -\alpha_i(\overline{\nabla}_X JE_j)(p) = -\alpha_i(J\overline{\nabla}_X E_j)(p) \\
&= -g(JB(X, E_j), E_i)(p) \\
&= -g(JB(E_i, E_j), X)(p)
\end{aligned}$$

since the tensor $S(X, Y, Z) = g(JB(X, Y), Z)$ is symmetric. Therefore, we obtain

$$\nabla_X \alpha_i(p) = -\sum_{j=1}^{n} g(JB(E_i, E_j), X)\beta_j(p).$$

A similar calculation shows that

$$\nabla_X \beta_i(p) = \sum_{j=1}^{n} g(JB(E_i, E_j), X)\alpha_j(p).$$

Hence, we have

$$\nabla_X(\alpha_i + \sqrt{-1}\beta_i) = \sqrt{-1}\sum_{j=1}^{n} g(JB(E_i, E_j), X)(\alpha_j + \sqrt{-1}\beta_j).$$

Combining this with (2.2), we obtain

$$\nabla_X \tilde{\Omega} = \sqrt{-1}\sum_{j=1}^{n} g(JB(E_j, E_j), X)\tilde{\Omega} = \sqrt{-1}\alpha_H(X)\tilde{\Omega}.$$

Since X is arbitrary, we obtain the required equation. $\square$

COROLLARY 2.1.5 ([72]). *Under the same assumption of Proposition 2.1.4, the holonomy homomorphism $\rho : \pi_1(L) \to U(1)$ of the flat bundle $\iota^* K$ is given by*

$$\rho([\gamma]) = \exp\Big(-\sqrt{-1}\int_\gamma \alpha_H \Big).$$

PROOF. Choose a loop $\gamma : [0, 1] \to L$ with the base point $p = \gamma(0) = \gamma(1)$. Denote the parallel transport of $\tilde{\Omega}_p$ along γ by $\tilde{\Omega}'(t) = e^{\sqrt{-1}\psi(t)}\tilde{\Omega}(t)$. Then by Proposition 2.1.4, we have

$$\begin{aligned} 0 = \nabla_{\dot{\gamma}(t)}\tilde{\Omega}'(t) &= \sqrt{-1}\frac{d\psi}{dt}(t)e^{\sqrt{-1}\psi(t)}\tilde{\Omega}(t) + e^{\sqrt{-1}\psi(t)}\nabla_{\dot{\gamma}(t)}\tilde{\Omega}(t) \\ &= \sqrt{-1}e^{\sqrt{-1}\psi(t)}\Big\{\frac{d\psi}{dt}(t) + \alpha_H(\dot{\gamma}(t))\Big\}\tilde{\Omega}(t), \end{aligned}$$

and hence,

$$\frac{d\psi}{dt}(t) = -\alpha_H(\dot{\gamma}(t))$$

for any $t \in [0, 1]$. Integrating this, we obtain

$$\psi(1) = -\int_0^1 \alpha_H(\dot{\gamma}(t))dt = -\int_\gamma \alpha_H$$

since $\psi(0) = 0$. This proves the corollary. $\square$

We recall that, for a Lagrangian immersion ι into a Kähler-Einstein manifold, the mean curvature form $\alpha_H \in \Omega^1(L)$ represents a real cohomology class $[\alpha_H] \in H^1(L, \mathbb{R})$. The following theorem shows that $[\alpha_H]$ is preserved under any global Hamiltonian isotopy of ι.

THEOREM 2.1.6 ([72]). *Let $\iota : L \to M$ be a Lagrangian immersion into a Kähler-Einstein manifold. Then under the global Hamiltonian isotopy $\{\iota_t\}_{0 \leq t \leq 1}$ of $\iota = \iota_0$, the 1-forms α_{H_t} on L represent the same cohomology class, where α_{H_t} is the mean curvature form of ι_t.*

PROOF. By Lemma 2.1.3, the holonomy homomorphisms ρ_t are the same for all $t \in [0, 1]$. Hence, by Corollary 2.1.5, it follows that $\int_\gamma \alpha_{H_t}$ is constant with respect to t for any loop γ in L. This implies that α_{H_t} define the same cohomology class in $H^1(L, \mathbb{R})$. □

2.1.2 H-minimal Lagrangian submanifolds

DEFINITION 2.1.7 ([71]). A Lagrangian immersion $\iota : L^n \to M^{2n}$ is called *Lagrangian minimal* (resp. *Hamiltonian minimal, or shortly, H-minimal)* if it satisfies

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}(\iota_t(L)) = 0$$

for all Lagrangian (resp. Hamiltonian) deformations $\{\iota_t\}_t$.

Our main concern in this chapter is H-minimal Lagrangian submanifolds in a Kähler manifold. For an H-minimal Lagrangian immersion, the Euler-Lagrange equation is derived as follows:

THEOREM 2.1.8 ([71]). *A Lagrangian immersion $\iota : L^n \to M^{2n}$ is H-minimal if and only if*

$$\delta\alpha_H = 0, \ \textit{or equivalently}, \ \mathrm{div} JH = 0$$

where H is the mean curvature vector of ι.

PROOF. Recall that the first variation formula for volume functional for the general variations $\{\iota_t\}_t$ is given by (see [42])

$$\frac{d}{dt}\mathrm{Vol}(\iota_t(L)) = -\int_L g(H_t, V_t) dv_t,$$

where H_t is the mean curvature vector of ι_t. Now, we assume that the deformation is a Hamiltonian deformation. Since each ι_t are Lagrangian, we note that $g(H_t, V_t) =$

$g^*(\alpha_{H_t}, \alpha_{V_t^\perp})$ where g^* is the induced inner product on T^*L. By the definition, the variational vector field V_t is written by $\alpha_{V_t^\perp} = df_t$ for some functions $f_t \in C_c^\infty(L)$. Conversely, for any function $f \in C_c^\infty(L)$, we can find a Hamiltonian deformation so that $V_0^\perp = df$. Thus, the immersion ι is H-minimal if and only if

$$0 = \frac{d}{dt}\Big|_{t=0} \mathrm{Vol}(\iota_t(L)) = -\int_L g^*(\alpha_H, \alpha_{V^\perp})dv = -\int_L g^*(\alpha_H, df)dv = -\int_L f\delta\alpha_H dv,$$

for any function $f \in C_c^\infty(L)$ (where we denote $H := H_0, V := V_0$, and $f := f_0$), which is equivalent to $\delta\alpha_H = 0$. □

Corollary 2.1.9 (cf. [71]). *Let $\iota : L \to M$ be a Lagrangian immersion of a compact manifold L into a Kähler-Einstein manifold. If ι is non-minimal, H-minimal immersion, then $H^1(L, \mathbb{R}) \neq 0$.*

Proof. Since M is Kähler-Einstien, α_H defines a real cohomology class in $H^1(L, \mathbb{R})$ by Theorem 2.1.1. Suppose $H^1(L, \mathbb{R}) = 0$. Then, there exist a function $f \in C^\infty(L)$ such that $\alpha_H = df$. If ι is H-minimal, then we have $0 = \delta\alpha_H = \Delta f$. Since L is compact, the maximal principle implies that f is constant on L and hence, we obtain $\alpha_H = 0$. Thus, ι is indeed a minimal immersion. □

For instance, if $\iota : L \to \mathbb{C}^n$ is a compact H-minimal Lagrangian immersion, then $H^1(L, \mathbb{R}) \neq 0$ since there are no compact minimal immersions into $\mathbb{C}^n$.

Example 2.1.10. Here, we give the most basic examples of H-minimal Lagrangian submanifolds.

(1) Any *minimal* Lagrangian immersion is H-minimal. Thus, the notion of H-minimality is an extension of minimal submanifold.

(2) Any Lagrangian immersion with the *parallel mean curvature vector* (i.e., $\nabla^\perp H = 0$) is H-minimal.

(3) A curve with constant geodesic curvature in a Riemann surface.

(4) The standard tori $T^n = S^1(r_1) \times \cdots S^1(r_n)$ in $\mathbb{C}^n$ [71].

We note that there are no compact *minimal* submanifolds in the Euclidean space (cf. [53]). However, there are many examples of compact H-minimal Lagrangian submanifolds in the complex Euclidean space $\mathbb{C}^n$. The H-minimality of the example of the standard torus in $\mathbb{C}^n$ follows from the following proposition.

Proposition 2.1.11 (cf. [61]). *Let (M, ω) be the Riemannian product of two Kähler manifolds $(M_i, \omega_i)(i = 1, 2)$, where $\omega := \pi_1^*\omega_1 + \pi_2^*\omega_2$, and $\pi_i : M \to M_i$ are the projection. Assume $\iota_i : L_i \to M_i$ are two H-minimal Lagrangian immersions. Then the Riemannian product $\iota : L_1 \times L_2 \to M_1 \times M_2$ is an H-minimal Lagrangian immersion into (M, ω).*

Proof. Since the statement is local argument, we may assume all immersions are embedding. First, we show that L is Lagrangian. For any vector field V on $L = L_1 \times L_2$, there exist the unique vector field V_i on L_i so that $V = \pi_1^* V_1 + \pi_2^* V_2$. Since each L_i is Lagrangian, we have $\omega(V, W) = \omega_1(V_1, W_1) + \omega_2(V_2, W_2) = 0$ for any $V, W \in \Gamma(TL)$. Thus, L is Lagrangian in (M, ω). Next, we suppose that each L_i is H-minimal. One can check that the codifferential δ and π_i^* commute, i.e., $\delta\pi_i^*\alpha = \pi_i^*\delta\alpha$ for $\alpha \in \Omega^1(L_i)$. Since the mean curvature of L is given by $H = \pi_1^* H_1 + \pi_2^* H_2$, we have $\delta\alpha_H = \delta\pi_1^*\alpha_{H_1} + \delta\pi_2^*\alpha_{H_2} = \pi_1^*\delta\alpha_{H_1} + \pi_2^*\delta\alpha_{H_2} = 0$. Thus, L is H-minimal in M. □

The following theorem is inspired by Oh's theorem in [71].

Theorem 2.1.12 (cf. [71]). *Let L^n be a Lagrangian submanifold in $\mathbb{C}^n$. Assume that L^n is contained in the sphere S^{2n-1} (or of any radius), and has the parallel mean curvature vector in S^{2n-1} as a submanifold in the sphere. Then L^n is H-minimal in $\mathbb{C}^n$.*

Proof. Denote the mean curvature vectors of the immersions $L^n \to S^{2n-1}$ and $L^n \to \mathbb{C}^n$ by H' and H, respectively. Since the sphere is totally umbilic in $\mathbb{C}^n$, and by using the Gauss equation, we have the relation of these vectors by $H = H' - n\vec{p}$ at the point $p \in L^n$, where $\vec{p}$ is the position vector of p in $\mathbb{C}^n$. This is equivalent to $JH = JH' + n\xi$ where $\xi := -j\vec{p}$. As a vector field on $\mathbb{C}^n$, the vector field ξ generates the Hopf action on $\mathbb{C}^n$. Since L^n is Lagrangian in $\mathbb{C}^n$, ξ is tangent to L^n. Moreover, we note that L^n is invariant under the Hopf action. Then, JH' is also a tangent vector on L^n, and $\mathrm{div}JH = \mathrm{div}JH' + n\mathrm{div}\xi$. Since the Hopf action generates an isometry on S^{2n-1}, it also acts isometrically on L^n, and hence, the vector field ξ is a Killing vector field on L^n. Thus we have $\mathrm{div}\xi = 0$. On the other hand, if the vector field H' is parallel in S^{2n-1}, then we have $\mathrm{div}JH' = 0$. Thus, $\mathrm{div}JH = 0$. □

One can produce an example of Lagrangian submanifold in $\mathbb{C}^n$ which satisfies the assumption of this Theorem by taking the pre-image of minimal Lagrangian submanifolds in the complex projective space $\mathbb{C}P^{n-1}$ via the Hopf fibration $\pi : S^{2n-1} \to \mathbb{C}P^{n-1}$. Another generalization of Theorem 2.1.12 is shown in [26].

More examples and constructions of H-minimal Lagrangian submanifolds in a Kähler manifold are described in the following sections.

Finally, we give another characterization of H-minimal oriented Lagrangian submanifolds in a *Calabi-Yau* manifold (see Appendix A.1 for our definition) as an application of Proposition 2.1.4. Let M be a Calabi-Yau manifold and Ω the holomorphic $(n,0)$-form on M.

DEFINITION 2.1.13. Let $\iota : L^n \to M^{2n}$ be an oriented Lagrangian immersion into a Calabi-Yau manifold M. We define a map $\theta : L \to 2\pi\mathbb{R}/\mathbb{Z} = S^1$ by

$$\Omega|_L = e^{\sqrt{-1}\theta}\mathrm{vol}_L$$

where vol_L is the volume form on L. We call the function θ the *Lagrangian angle* of L.

We have the following:

THEOREM 2.1.14. *The Lagrangian angle θ satisfies $\alpha_H = -d\theta$.*

PROOF. By definition, we have

$$\Omega = e^{\sqrt{-1}\theta}\tilde{\Omega},$$

where $\tilde{\Omega}$ is the unique complex extension of vol_L. By Proposition 2.1.4, for any $X \in T_pL$, we have

$$\nabla_X(e^{-\sqrt{-1}\theta}\Omega) = \sqrt{-1}\alpha_H(X)e^{-\sqrt{-1}\theta}\Omega. \tag{2.3}$$

The left hand side becomes

$$\begin{aligned} \nabla_X(e^{-\sqrt{-1}\theta}\Omega) &= -\sqrt{-1}X(\theta)e^{-\sqrt{-1}\theta}\Omega + e^{-\sqrt{-1}\theta}\nabla_X\Omega \\ &= -\sqrt{-1}X(\theta)e^{-\sqrt{-1}\theta}\Omega \end{aligned}$$

since $\nabla_X\Omega = 0$ as M is Calabi-Yau. Combining this with (2.3), we obtain $\alpha_H(X) = -X(\theta) = -d\theta(X)$. Since X is arbitrary, this implies $\alpha_H = -d\theta$. $\square$

Recall that a Lagrangian submanifold L in a Calabi-Yau manifold M is *special Lagrangian* with phase $e^{\sqrt{-1}\theta}$ if L is calibrated by the calibration $\mathrm{Re}(e^{-\sqrt{-1}\theta}\Omega)$.

COROLLARY 2.1.15. *For an oriented, connected Lagrangian immersion ι into a Calabi-Yau manifold, we have (i) ι is special Lagrangian if and only if θ is constant, and (ii) ι is H-minimal if and only if θ is harmonic (as a S^1-valued function), namely, $\Delta\theta = 0$.*

REMARK 2.1.16. We emphasize that the Lagrangian angle θ is a S^1-valued function, and hence, $\alpha_H = -d\theta$ does not mean the exactness of α_H in general. If θ lifts continuously to an $\mathbb{R}$-valued function $\theta : L \to \mathbb{R}$, then L is called the *graded* Lagrangian (cf. [45]). In this case, $\alpha_H = -d\theta$ is an exact 1-form on L, and hence $[\alpha_H] = 0$ in $H^1(L,\mathbb{R})$. Conversely, if $[\alpha_H] = 0$, then L is graded. By the maximal principle (see also Corollary 2.1.9), any compact and graded Lagrangian submanifold in a Calabi-Yau manifold M is special Lagrangian.

2.2 Hamiltonian stability

2.2.1 Stability criterion

Definition 2.2.1 (cf. [2], [70], [71]). (1) An H-minimal Lagrangian immersion $\iota : L^n \to M^{2n}$ is *Hamiltonian stable (or H-stable)* if it satisfies

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Vol}(\iota_t(L)) \geq 0$$

for all Hamiltonian deformations $\{\iota_t\}_t$.

(2) ι is called *strictly Hamiltonian stable* if ι is H-stable and the nullspace $null(\iota)$ of the second variation is exactly given by

$$null(\iota) = \{V^\perp;\ V \text{ is a holomorphic Killing vector field on } (M, J, \omega)\}.$$

Remark 2.2.2. If V is a holomorphic Killing vector field on (M, J, ω), then $d\alpha_V = \mathcal{L}_V\omega = 0$, and hence, α_V is always a closed form. In particular, V generates a Lagrangian flow of ι. If in addition, α_V is exact (for instance, M is simply connected or $H^1(M, \mathbb{R}) = 0$), then V generates a Hamiltonian flow of ι, and this flow obviously preserves the volume of ι. A Hamiltonian deformation generated by a holomorphic Killing vector field is called the *trivial* deformation.

In [71], Oh derived the second variational formula for an H-minimal Lagrangian immersion into a Kähler manifold as follows.

Theorem 2.2.3 ([71]). *Let (M^{2n}, ω, J) be a Kähler manifold, and $\iota : L^n \to M^{2n}$ an H-minimal Lagrangian immersion of a manifold L^n. If $\{\iota_t\}_t$ is a Hamiltonian deformation of ι with the normal variational vector field $V = J\nabla f$ along ι for $f \in C_c^\infty(L)$, then we have*

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Vol}(\iota_t(L)) = \int_L \Big\{|\Delta f|^2 - \overline{\mathrm{Ric}}(\nabla f) - 2g(B(\nabla f, \nabla f), H) + g(J\nabla f, H)^2\Big\} dv_L,$$

where Δ is the Laplace-Bertrami operator (i.e., $\Delta = d\delta + \delta d$) acting on $C^\infty(L)$, $\overline{\mathrm{Ric}}$ is the Ricci tensor of M, B is the second fundamental form of ι, and H is the mean curvature vector of ι.

The original proof of the second variational formula is in [71]. The simplest proof is given in [89] by Schoen and Wolfson. However, we omit the proof here. We also refer to [46] for a similar argument of the second variational formula of Legendrian submanifolds in a Sasaki manifold.

When ι is *minimal* (i.e., $H \equiv 0$), the second variational formula becomes very simple. Moreover, we have the following useful theorem when M is Kähler-Einstein.

THEOREM 2.2.4 ([70]). *Let* (M^{2n}, J, ω) *be a Kähler-Einstein manifold with Einstein constant* c*, and* $\iota : L^n \to M^{2n}$ *a minimal Lagrangian immersion of a compact manifold* L^n*. Then the immersion* ι *is H-stable if and only if*

$$\lambda_1 \geq c.$$

where λ_1 *is the first eigenvalue of the Laplacian* Δ *acting on* $C^\infty(L)$.

PROOF. By the second variational formula and the assumption, we have

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Vol}(\iota_t(L)) = \int_L \Big\{|\Delta f|^2 - c|\nabla f|^2\Big\} dv_L = \int_L \Delta f(\Delta f - cf) dv_L.$$

Let $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_k < \ldots \to \infty$ be the eigenvalues of the Laplace-Beltrami operator Δ acting on $C^\infty(L)$, and $f = f_0 + \sum_{i=1}^{\infty} f_i$ the spectral decomposition of $f \in C^\infty(L)$. Then we have

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Vol}(\iota_t(L)) = \sum_{i=1}^{\infty} \lambda_i(\lambda_i - c) \int_L {f_i}^2 dv_L.$$

Thus, ι is H-stable if and only if $\lambda_1 - c \geq 0$. □

COROLLARY 2.2.5 ([70]). *Let* (M^{2n}, ω, J) *be a Kähler-Einstein manifold with non-positive Ricci curvature. Then any minimal Lagrangian immersion* $\iota : L^n \to M^{2n}$ *is H-stable.*

EXAMPLE 2.2.6 ([70]). Consider the complex projective space $\mathbb{C}P^n$ with the standard Fubini-Study metric g_{FS} of constant holomorphic sectional curvature 4. Then, the standard Kähler structure is Einstein with Einstein constant $2(n+1)$. We shall show that the totally geodesic real projective space $\mathbb{R}P^n$ is H-stable. It is well-known that the l-th eigenvalue of the Laplace-Beltrami operator Δ on the unit sphere $S^n(1)$ is given by $l(l+n-1)$ (cf. [87]). Only $2l$-th eigenvalues are realized as the eigenvalues of $S^n/\mathbb{Z}_2 = \mathbb{R}P^n$. Thus, the first eigenvalue of $S^n/\mathbb{Z}_2$ is given by $\lambda_1 = 2(n+1)$. Since the metric on the totally geodesic $\mathbb{R}P^n$ is the same as the one on $S^n/\mathbb{Z}_2$, $\mathbb{R}P^n$ is H-stable by Theorem 2.2.4.

EXAMPLE 2.2.7 ([70]). The *Clifford torus* T^n in $\mathbb{C}P^n$ is defined as follows: Consider the isometric embedding of the torus

$$T^{n+1} := S^1\Big(\frac{1}{\sqrt{n+1}}\Big) \times \cdots \times S^1\Big(\frac{1}{\sqrt{n+1}}\Big) \subset S^{2n+1}(1) \subset \mathbb{C}^{n+1}.$$

This embedding is minimal in $S^{2n+1}(1)$, and Lagrangian in $\mathbb{C}^{n+1}$ (In particular, we note that $T^{n+1} \subset \mathbb{C}^{n+1}$ is H-minimal by Theorem 2.1.12). The sphere $S^{2n+1}(1)$ and the torus

T^{n+1} are invariant under the standard Hopf S^1-action on $\mathbb{C}^{n+1}$, thus we can take the quotients of these. Then the torus $T^n := T^{n+1}/S^1$ is a minimal Lagrangian submanifold in $\mathbb{C}P^n$ which is called the *Clifford torus*. The first eigenvalue of the Laplace-Bertrami operator Δ on $C^\infty(T^n)$ is given by $\lambda_1 = 2(n+1)$ (a proof is given in [70]). Thus, by Theorem 2.2.4, the Clifford torus is H-stable. Moreover, Urbano [105] proved that the Clifford torus T^2 is the only H-stable minimal Lagrangian torus in $\mathbb{C}P^2$.

Note that any *stable* minimal submanifold in $\mathbb{C}P^n$ is a complex submanifold due to the Theorem of Lawson-Simons [54], and hence, $\mathbb{R}P^n$ and T^n in the above examples are *not* stable in the standard sense. More examples of H-stable (in particular, minimal) Lagrangian submanifold are discussed in the following sections. We also note that both the first eigenvalues of the above two examples of H-stable minimal Lagrangian submanifold are equal to the Einstein constant $c = 2(n+1)$ of $(\mathbb{C}P^n, g_{FS})$. This situation holds in more general setting. We discuss it in the next subsection.

On the other hand, we know only a few example of H-stable *H-minimal* (non-minimal) Lagrangian submanifold.

Example 2.2.8. We have already shown that the standard tori $T^n = S^1(r_1) \times \cdots \times S^1(r_n) \subset \mathbb{C}^n$ are H-minimal Lagrangian submanifolds. By analyzing the operator in the second variational formula given in Theorem 2.2.3, Oh proved that these tori are strictly H-stable in $\mathbb{C}^n$ [71].

Example 2.2.9. A natural generalization of Example 2.2.7 is a torus orbit in a toric Kähler manifold. A compact Kähler manifold (M, J, ω) with $\dim_{\mathbb{C}} M = n$ is called *toric* if there is an effective holomorphic Hamiltonian action of a real torus T^n. H. Ono investigated the H-stability of these orbits in a toric Kähler manifold [81].

Let (M, J, ω) be a compact toric Kähler manifold. Then any regular torus orbit is an H-minimal Lagrangian submanifold. If in addition, M is Einstein, then there exist a unique regular minimal orbit. H. Ono calculated the second variational formula in Theorem 2.2.2 for these orbits, and proved that *any regular T^n-orbit in $(\mathbb{C}P^n, \omega_{FS})$ is strictly H-stable*. As a special case (namely, as the unique minimal orbit), this result includes the Clifford torus (Example 2.2.6).

Example 2.2.10. Consider a closed curve in $S^3(1)$ given by

$$\gamma_{p,q}(s) = \frac{1}{\sqrt{p+q}}\left(\sqrt{q}e^{\sqrt{-1}(\sqrt{p/q})s}, \sqrt{-1}\sqrt{p}e^{-\sqrt{-1}(\sqrt{q/p})s}\right), \quad s \in [0, 2\pi\sqrt{pq}],$$

where p, q are relatively positive integers. Then the curve has constant geodesic curvature and torsion. We note that the curve is minimal if and only if $(p,q) = (1,1)$. The curve is Legendrian in $S^3(1)$, and the *cone* over $\gamma_{p,q}$ defined by $C(\gamma_{p,q}) := \{t\vec{p}\,; \vec{p} \in \gamma_{p,q} \subset \mathbb{C}^2,\ t \in$

$\mathbb{R}_{\geq 0}\}$ is an H-minimal Lagrangian variety in $\mathbb{C}^2$ with singularity at the origin. It is a special Lagrangian cone of some phase only if $(p,q) = (1,1)$. Schoen and Wolfson [89] investigated the H-stability of the cone $C(\gamma_{p,q})$ with $|p-q| > 0$, and they proved that *the cone $C(\gamma_{p,q})$ is H-stable if and only if $|p-q| = 1$.*

Other examples of H-stable, (non-minimal) H-minimal Lagrangian submanifolds are given in Subsection 2.2.4.

2.2.2 H-stability of minimal Lagrangian submanifolds in certain generalized flag manifolds

In this section, we estimate the first eigenvalue of the Laplace-Bertrami operator of a minimal Lagrangian submanifold in a generalized flag manifold. First, we review some facts of generalized flag manifolds (we refer to [6] and [12]).

Let G be a compact semi-simple Lie group and $\mathfrak{g}$ the Lie algebra of G. We denote an Ad_G-invariant inner product on $\mathfrak{g}$ by $\langle , \rangle$. An *adjoint orbit* of G in $\mathfrak{g}$ through $w \in \mathfrak{g}$ is defined by $M_w := \mathrm{Ad}(G)w = \{\mathrm{Ad}(g)w \ ; \ g \in G\} \subset \mathfrak{g}$. We denote the isotropy subgroup of the action G at w by K_w. Then, $M_w \simeq G/K_w$, and K_w coincides with the centralizer $C(S_w) := \{g \in G \ ; \ ghg^{-1} = h, \ \forall h \in S_w\}$ of the torus subgroup $S_w := \overline{\exp\mathbb{R}w}$. We call $M_w = G/C(S_w)$ the *generalized flag manifold.* When the torus subgroup S_w is maximal, then $T := C(S_w) = S_w$, and $M = G/T$ is called the *complex flag manifold* or *C-space.* We denote the Lie algebra of S_w by $\mathfrak{s}_w$.

The Lie algebra $\mathfrak{k}_w$ of K_w is given by $\mathfrak{k}_w = \mathrm{Ker}\ \mathrm{ad}(w) := \{X \in \mathfrak{g} \ ; \ \mathrm{ad}(w)X = 0\}$. We denote the orthogonal complement of $\mathfrak{k}_w$ in $\mathfrak{g}$ with respect to the inner product $\langle , \rangle$ by $\mathfrak{m}_w$, namely, we have the orthogonal decomposition $\mathfrak{g} = \mathfrak{k}_w \oplus \mathfrak{m}_w$. Then $\mathfrak{m}_w$ is regarded as the tangent space $T_w M_w$ via the differential of the projection $\pi : G \to M_w = G/K_w$. On the other hand, we can express $T_w M_w = \{X_w^* := d/dt|_{t=0}\mathrm{Ad}(\exp tX)w \ ; \ X \in \mathfrak{g}\} = \{-\mathrm{ad}(w)X \ ; \ X \in \mathfrak{g}\} = \mathrm{Im}\ \mathrm{ad}(w) \subset \mathfrak{g}$. Thus, we have the following identification:

$$\mathfrak{g} = \mathfrak{k}_w \oplus \mathfrak{m}_w = \mathrm{Ker}\ \mathrm{ad}(w) \oplus \mathrm{Im}\ \mathrm{ad}(w) \simeq \nu_w M_w \oplus T_w M_w = T_w \mathfrak{g},$$

where $\nu_w M$ is the normal space of M_w at w.

Since S_w is a connected center of $K_w = C(S_w)$, we have $\mathrm{Ad}(k)Z = Z$ for any $Z \in \mathfrak{s}_w$ and $k \in K_w$. This implies that $\mathrm{Ad}(gK_w)Z = \mathrm{Ad}(g)Z$ for any $g \in G$. Therefore, a map $\mathfrak{s}_w \to \mathfrak{s}_{\mathrm{Ad}(g)w}$, $Z \mapsto \mathrm{Ad}(g)Z$ for $g \in G$ defines a well-defined isomorphism, namely, it is independent of the choice of an element of gK_w. This means the following: *The disjoint union $\bigcup_{v \in M_w} \mathfrak{s}_v$ is a trivial sub-bundle of the normal bundle νM_w.*

An element w in $\mathfrak{g}$ is called *regular* if the Lie subalgebra $\mathfrak{k}_w = \mathrm{Ker}\ \mathrm{ad}(w)$ is abelian, or equivalently, $\mathfrak{s}_w$ coincides with $\mathfrak{k}_w$. Otherwise, it is said to be *singular.* We call M_w is regular (resp. singular) if it is an orbit through a regular (resp. singular) element. By the above, we see that *the normal bundle of a regular orbit M_w is always trivial.* We note that M_w is regular if and only if it is a principal orbit (see [10], p.49).

The restriction of the adjoint representation on the torus subgroup S_w is completely reducible. We denote the irreducible decomposition of the adjoint representation of the torus subgroup S_w with the representation space $\mathfrak{m}_w$ by

$$\mathrm{Ad}|_{S_w} = \sum_{j=1}^{n} \Gamma_{a_j}, \ \mathfrak{m}_w = \sum_{j=1}^{n} E_{w,j}$$

where $\Gamma_{a_j} : S_w \to GL(E_{w,j})$ $(j = 1, \cdots, n)$ are irreducible subrepresentations defined by

$$\Gamma_{a_j}(\exp s) := \begin{bmatrix} \cos a_j(s) & -\sin a_j(s) \\ \sin a_j(s) & \cos a_j(s) \end{bmatrix} \tag{2.4}$$

for $s \in \mathfrak{s}_w$, and $a_j \in \mathfrak{s}_w^*$ is the weight function. We note that $\{a_j\}_j$ are non-zero functions and $w \in \mathfrak{s}_w$. We choose an orientation of each subspace $E_{w,j}$ so that each $a_j(w)$ is a positive function, and Γ_{a_j} is a subrepresentation on $E_{w,j}$. We define the canonical almost complex structure J_w on $T_w M_w$ by $J_w e_{j1} = e_{j2}$ and $J_w e_{j2} = -e_{j1}$, where $\{e_{j1}, e_{j2}\}$ is an oriented basis on $E_{w,j}$ for $j = 1, \ldots, n$. Then we have the following relation:

$$\mathrm{ad}(s)X_j = a_j(s) J_w X_j, \ \text{ for } s \in \mathfrak{s}_w \text{ and } X_j \in E_{w,j}.$$

In fact, by the representation (2.4), we have

$$\begin{aligned} \mathrm{ad}(s)|_{E_{w,j}} &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp ts)|_{E_{w,j}} \\ &= \left.\frac{d}{dt}\right|_{t=0} \begin{bmatrix} \cos ta_j(s) & -\sin ta_j(s) \\ \sin ta_j(s) & \cos ta_j(s) \end{bmatrix} \\ &= a_j(s) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = a_j(s) J_w. \end{aligned}$$

In particular, we obtain

$$J_w X_j = \frac{1}{a_j(w)}[w, X_j], \ \text{ for } X_j \in E_{w,j}. \tag{2.5}$$

Since $w \in \mathfrak{s}_w$, an abelian subalgebra in $\mathfrak{g}$, this almost complex structure defines the G-invariant almost complex structure J on M_w. Moreover, J is integrable (see [12]), and hence, it defines the complex structure on M_w. We call J the *canonical complex structure* on M_w.

Let g be a G-invariant Riemannian metric on M_w induced from the $\mathrm{Ad}(G)$-invariant inner product $\langle , \rangle$ on $\mathfrak{g}$. It is known that, in general, the Ricci form of any Kähler metric depends only on the complex structure and the volume form. Since the G-invariant metric on M_w is unique up to homothety, we have the unique (up to homothety) G-invariant volume form on M_w which is induced from $\langle , \rangle$. Moreover, since $T_wM_w = \{X_w^* := d/dt|_{t=0}\mathrm{Ad}(\exp tX)w \ ; \ X \in \mathfrak{g}\}$, namely, any tangent vector of M_w at w is given by the *fundamental vector field* X^* for some $X \in \mathfrak{g}$, any tangent vector on M_w preserves the canonical complex structure J, i.e., $\mathcal{L}_{X^*}J = 0$. Therefore, by the Koszul formula ([12], p.89), the Ricci form on *any* G-invariant Kähler metric on M_w relative to the canonical complex structure J is given by the multiplication of

$$\rho(X^*, Y^*)_w := -\frac{1}{2}\mathrm{div}(J[X^*, Y^*])(w) \text{ for } X, Y \in \mathfrak{m}_w,$$

by a positive constant $C > 0$. Let $\{X_j, JX_j\}$ be an oriented orthonormal basis of the vector subspace $E_{w,j}$ for $j = 1, \ldots, m$. Then the 2-form ρ is written by

$$\rho(X^*, Y^*)_w = \Big\langle \sum_{j=1}^{n} [X_j, JX_j], [X, Y] \Big\rangle. \tag{2.6}$$

In particular, the vector

$$\gamma(w) := \sum_{j=1}^{n} [X_j, JX_j] \in \mathfrak{s}_w \tag{2.7}$$

does not depend on the choice of an oriented orthonormal basis $\{X_j, JX_j\}_{j=1}^n$. Moreover, the 2-form ρ is positive definite and closed (see [12]). Thus, the 2-form ρ defines itself a Kähler form of a Kähler metric g_ρ on M_w compatible with the canonical complex structure J. By definition, it is clearly Kähler-Einstein. Since the Kähler form of any Kähler-Einstein metric compatible with J is given by $C\rho$, the Kähler metric is homothetic to g_ρ. This implies an important consequence; *there exist the unique (up to homothety) G-invariant Kähler-Einstein structure on M_w compatible with the canonical complex structure J.* Moreover, it is well known that every compact, simply-connected, homogeneous Kähelr manifold is isomorphic to an orbit of the adjoint representation of its connected group of isometries endowed with the canonical complex structure (see [12]).

On the other hand, we define another symplectic structure on M_w as follows:

$$F_w(X^*, Y^*) := \langle w, [X, Y] \rangle, \text{ for } X, Y \in \mathfrak{m}_w.$$

This 2-form also defines a Kähler form of a G-invariant Kähler structure on M_w compatible with the canonical complex structure J (see [12]). As a symplectic form on M_w, we call the 2-form F the *Kirillov-Kostant-Souriau symplectic structure.* Note that, in general, F is *not* a Kähler form associated with a Kähler-*Einstein* structure on M_w.

Let g be a G-invariant Riemannian metric on $M_w \subset \mathfrak{g}$ induced from the $\mathrm{Ad}(G)$-invariant inner product $\langle , \rangle$ on $\mathfrak{g}$. We define a 2-form ω on M_w by $\omega(X^*, Y^*) := g(JX^*, Y^*) = \langle JX, Y\rangle$ for $X, Y \in \mathfrak{g}$. We call ω the *canonical 2-form* on M_w compatible with J. In general, ω does *not* define a symplectic structure on M_w.

LEMMA 2.2.11. *Let $M = M_w$ be a generalized flag manifold. Then the following three are equivalent:*

(1) The canonical 2-form ω is a Kähler form on M compatible with the canonical complex structure J.

(2) The canonical 2-form ω coincides with a constant multiple of the Kirillov-Kostant-Souriau symplectic structure F, i.e., $\omega = \alpha F$ for a positive constant α.

(3) $a_j(w) = \alpha$ for any $j = 1, \ldots, n$, where a_j is the weight function defined by (2.4).

PROOF. We use the identification $T_w M_w \simeq \mathrm{Im}\,\mathrm{ad}(w) = \mathfrak{m}_w = \sum_{j=1}^n E_{w,j}$. For $X_j \in E_{w,j}$, we have $X_j = ad(w)(-1/a_j(w)JX_j)$ by (2.5) and the $\mathrm{ad}(w)$-invariance of J. Hence, for $j \neq k$, we obtain

$$\begin{aligned} F_w(X_j, X_k) : &= (w, [\mathrm{ad}(w)^{-1}X_j, \mathrm{ad}(w)^{-1}X_k]) \\ &= \frac{1}{a_j(w)a_k(w)}(w, [JX_j, JX_k]) \\ &= \frac{1}{a_j(w)a_k(w)}(\mathrm{ad}(w)(JX_j), JX_k) = 0, \end{aligned}$$

since $\mathrm{ad}(w)(JX_j) \in E_{w,j} \perp E_{w,k}$. On the other hand,

$$\begin{aligned} F_w(X_j, JX_j) &= \frac{-1}{a_j(w)^2}(w, [JX_j, X_j]) \\ &= \frac{1}{a_j(w)}(\mathrm{ad}(w)X_j, JX_j) \\ &= \frac{1}{a_j(w)}(JX_j, JX_j) = \frac{1}{a_j(w)}\omega(X_j, JX_j) \end{aligned}$$

by (2.5). This implies

$$\omega|_{E_{w,j}} = a_j(w)F_w|_{E_{w,j}} \tag{2.8}$$

for each $j = 1, \cdots, n$. By (2.8), the equivalence of (2) and (3) obviously follows. On the other hand, since F is a Kähler form on M_w compatible with J, (2) implies (1). The converse follows from Proposition 8.76 in [12] and (2.8). □

Now, let us describe the main results in this subsection. The following theorem is due to H. Ono.

THEOREM 2.2.12 ([78]). *Let G be a compact semi-simple Lie group, and $M^{2n} := M_w = G/C(S_w)$ a generalized flag manifold. Suppose that the canonical 2-form ω defines a Kähler form on M compatible with the canonical symplectic structure J. If L^n is a closed, minimal Lagrangian submaifold in M, then the first eigenvalue λ_1 of the Laplace-Bertrami operator Δ acting on $C^\infty(L)$ satisfies*

$$\lambda_1 \leq \frac{s}{2n},$$

where s is the scalar curvature of M. Moreover, if M is Kähler-Einstein with Einstein constant c, then $\lambda_1 \leq c$.

Combining this with the result of Oh (Theorem 2.2.4), we obtain the following:

COROLLARY 2.2.13. *Let M^{2n} be a generalized flag manifold which satisfies the same assumption as in Theorem 2.2.12. Suppose M is Kähler-Einstein with Einstein constant c. Then, a closed minimal Lagrangian submanifold L^n in M^{2n} is H-stable if and only if $\lambda_1 = c$.*

We note that a typical example which satisfies the assumption in Theorem 2.2.12 is the standard embedding of a compact Hermitian symmetric space.

To prove Theorem 2.2.12, we need a lemma.

LEMMA 2.2.14. *Let $M := M_w$ be a generalized flag manifold which satisfies the same assumption as in Theorem 2.2.12. Then the second fundamental form $\overline{B}$ and the mean curvature vector $\overline{H}$ of M_w in $\mathfrak{g}$ satisfy the following equalities:*

(1) $\overline{B}_w(X^*, Y^*) = [Y, [X, w]]^\perp$ for $X, Y \in \mathfrak{g}$.

(2) $\overline{B}_w(JX^*, JY^*) = \overline{B}_w(X^*, Y^*)$ for $X, Y \in \mathfrak{g}$.

(3) $\overline{H}_w = -\dfrac{2}{\alpha}\gamma(w)$, where $\gamma(w)$ is given by (2.7).

(4) $|\overline{H}|^2 = 2s$, where s is the scalar curvature of M_w.

PROOF. First, we note that $\mathrm{ad}(w)JX = -\alpha X$ for any $X \in \mathfrak{m}_w$ from (2.5) and Lemma 2.2.11. By a direct computation, we have

$$\begin{aligned} \tilde{\nabla}_{X^*}Y^* &= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\mathrm{Ad}(\exp tX)(\mathrm{Ad}(\exp sY)w) \\ &= \mathrm{ad}(X)\mathrm{ad}(Y)w = [X, [Y, w]], \end{aligned}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $(\mathfrak{g}, \langle , \rangle)$. Hence, $\overline{B}_w(X^*, Y^*) = \overline{B}_w(Y^*, X^*) = (\overline{\nabla}_{X^*}Y^*)^\perp = [Y, [X, w]]^\perp$. Moreover,

$$\begin{aligned} \overline{B}_w(JX^*, JY^*) &= [JY, [JX, w]]^\perp = [JY, -\mathrm{ad}(w)JX]^\perp \\ &= [JY, \alpha X]^\perp = [X, J(-\alpha Y)]^\perp = [X, [Y, w]]^\perp = \overline{B}_w(X^*, Y^*). \end{aligned}$$

We choose an orthonormal basis $\{X_j, JX_j\}$ of $E_{w,j}$ for $j = 1, \ldots, n$. Then,

$$\begin{aligned}
\overline{H}_w &:= \sum_{j=1}^{n}\{\overline{B}_w(X_j, X_j) + \overline{B}_w(JX_j, JX_j)\} \\
&= 2\sum_{j=1}^{n}\overline{B}_w(X_j, X_j) \\
&= 2\sum_{j=1}^{n}[\mathrm{ad}(w)^{-1}X_j, [\mathrm{ad}(w)^{-1}X_j, w]]^{\perp} \\
&= 2\sum_{j=1}^{n}[\frac{-1}{\alpha}JX_j, -X_j]^{\perp} = -\frac{2}{\alpha}\gamma(w).
\end{aligned}$$

Since the Ricci form with respect to the Kähler metric g is given by (2.6), we have

$$\begin{aligned}
s &= 2\sum_{j=1}^{n}\rho(X_j, JX_j) \\
&= 2\sum_{j=1}^{n}(\gamma(w), [\mathrm{ad}(w)^{-1}X_j, \mathrm{ad}(w)^{-1}JX_j]) \\
&= 2\sum_{j=1}^{n}(\gamma(w), [\frac{-1}{\alpha}JX_j, \frac{1}{\alpha}X_j]) \\
&= \frac{2}{\alpha^2}|\gamma(w)|^2 \\
&= \frac{1}{2}|\overline{H}|^2.
\end{aligned}$$

□

To estimate the upper bound of the first eigenvalue, we use the next result due to B. Y. Chen.

Theorem 2.2.15 ([22]). *Let $\iota : (L^n, g) \to (\mathbb{R}^k, \langle , \rangle)$ be an isometric immersion of a closed n-dimensional Riemannian manifold into an Euclidean space, and $\tilde{H}$ the mean curvature vector of ι. Then we have the following inequality.*

$$\lambda_1 \leq \frac{1}{n}\Big(\frac{\int_L |\tilde{H}|^n dv_L}{\mathrm{Vol}(L)}\Big)^{2/n},$$

where λ_1 is the first eigenvalue of the Laplace-Bertrami operator Δ acting on $C^\infty(L)$.

We note that the equality holds if and only if there exist a vector $c \in \mathbb{R}^k$ such that $\iota - c$ is an embedding of order 1, namely, each coordinate function $\iota^j - c^j$ $(j = 1, \cdots, n)$ is an eigenfuction of the first eigenvalue.

Proof of Theorem 2.2.12. Let $\iota : L^n \to M^{2n}$ be a minimal Lagrangian immersion into the generalized flag manifold $M := M_w \subset \mathfrak{g}$. Denote the second fundamental form of $L \to M$, $M \to \mathfrak{g}(\simeq \mathbb{R}^k)$ and $L \to \mathfrak{g}$ by B, $\overline{B}$ and $\tilde{B}$, respectively. For $X, Y \in \Gamma(\iota^* TL)$, we have

$$\tilde{B}_w(X,Y) = \overline{B}_w(X,Y) + B_w(X,Y).$$

Since ι is Lagrangian, for an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_w L$, $\{e_j, Je_j\}_{j=1}^n$ gives an orthonormal basis of $T_w M$. Hence, we have

$$\begin{aligned}\tilde{H}_w &= \sum_{j=1}^n \tilde{B}_w(e_j, e_j) = \sum_{j=1}^n \overline{B}_w(e_j, e_j) \\ &= \frac{1}{2}\sum_{j=1}^n \{\overline{B}_w(e_j, e_j) + \overline{B}_w(Je_j, Je_j)\} = \frac{1}{2}\overline{H}_w,\end{aligned}$$

since $\overline{B}$ satisfies the condition in Lemma 2.2.14, and ι is a minimal immersion, where $\tilde{H}$ and $\overline{H}$ is the mean curvature vector of $L \to \mathfrak{g}$ and $M \to \mathfrak{g}$, respectively. In particular, $|\tilde{H}|^2 = (1/4)|\overline{H}|^2 = s/2$ holds by Lemma 2.2.14, and it is constant on L. Then, by Theorem 2.2.15, we have

$$\lambda_1 \leq \frac{1}{n}\Big(\frac{\int_L (s/2)^{n/2} dv_L}{\mathrm{Vol}(L)}\Big)^{2/n} = \frac{s}{2n}$$

This completes the proof. □

2.2.3 Real forms in Hermitian symmetric spaces

We have shown in Example 2.2.6, the real projective space $\mathbb{R}P^n$ in $(\mathbb{C}P^n, g_{FS})$ is an H-stable minimal Lagrangian submanifold. In this subsection, we generalize this example in a natural way. A typical method to obtain a minimal Lagrangian submanifolds in Kähler manifold is as follows:

Proposition 2.2.16 (cf. [70]). *Let (M, ω, J) be a Kähler manifold and τ an anti-holomorphic involution, namely, an anti-holomorphic isometry with $\tau^2 = Id_M$. Then the fixed point set $L := \{p \in M; \tau(p) = p\}$ of τ is a totally geodesic Lagrangian submanifold in M.*

Proof. It is well known that the fixed point set of an isometry of a Riemannian manifold is totally geodesic (see [10]), and hence, L is totally geodesic. We shall show that L is Lagrangian. Since $\tau^2 = Id_M$, for each $p \in M$, the tangent space T_pM is decomposed into $T_pM = E_+ \oplus E_-$, where E_+ and E_- are the eigenspaces of $d\tau_p$ with respect to the eigenvalues $+1$ and -1, respectively. Moreover, since τ is anti-holomorphic, namely, $d\tau_p \circ J_p = -J_p \circ d\tau_p$, we have $J_pE_+ = E_-$. On the other hand, by the definition of L, it is obvious that $E_+ = T_pL$. Thus, L is Lagrangian. □

Definition 2.2.17. If the fixed point set L of an anti-holomorphic involution τ on a Kähler manifold is not empty, we call L the *real form* of τ.

A typical example of a real form is known as the *symmetric R-space* in a Hermitian symmetric space of compact type. We recall some facts on the symmetric R-spaces (We refer to [10] and [101]).

Let $M = U/K$ be a simply connected, semi-simple symmetric space of compact type, namely, $U = I^0(M)$ is the identity component of the isometry group of M such that U is compact and semi-simple, and K is a closed subgroup of U. An *s-representation* of M is the isotropy representation $\Phi : K \to SO(T_pM)$ defined by $\Phi(k) := k_*(p)$. Let (U, K) be an associated effective Riemannian symmetric pair of M, and $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Then the isotropy representation is equivalent to the adjoint representation $\mathrm{Ad}|_{\mathfrak{p}} : K \to GL(\mathfrak{p})$. We call an orbit of s-representation the *R-space* (or *real flag manifold*). We regard orbits of s-representations as submanifolds in $\mathfrak{p} \simeq \mathbb{R}^k$, and call it *the standard embedding of R-spaces.* A (simply connected) compact semi-simple Lie group G is regarded as a symmetric space of the pair $(G\times G, \Delta G)$, where $\Delta G := \{(g, g); g \in G\}$. Under this identification, the isotropy representation of $(G \times G, \Delta G)$ is equivalent to the adjoint representation of G on $\mathfrak{g}$, and the orbit is a generalized flag manifold.

In general, an R-space is not a symmetric space. An R-space that is a symmetric space is called the *symmetric R-space.* If, in addition, $\mathfrak{g}$ is simple, it is called an *irreducible symmetric R-space.* By the results of Kobayashi-Nagano [51] and Takeuchi [96], it follows that the symmetric R-spaces consist of the Hermitian symmetric spaces of compact type and their real forms. In fact, any Hermitian symmetric space of compact type is represented as an adjoint orbit of a compact semi-simple Lie group (and hence, it is obtained by a generalized flag manifold). Moreover, any real form of a Hermitian symmetric space is a symmetric R-space, and the converse is true (See [96] and [101] for more details). These real forms were classified by Takeuchi [96], and independently by Leung [56]. The classification list is given in Table 2.1.

Example 2.2.18. Let $Gr_p(\mathbb{C}^{p+q})$ be the complex Grassmannian manifold consistsing of all p-dimensional complex subspaces in $\mathbb{C}^{p+q}$. Define the anti-holomorphic involution by

$$\tau : Gr_p(\mathbb{C}^{p+q}) \to Gr_p(\mathbb{C}^{p+q}),\ W \ \mapsto \ \overline{W} := \{\overline{w};\ w \in W\}.$$

Then the fixed point set of τ is the real Grassmannian manifold $Gr_p(\mathbb{R}^{p+q})$. Thus, $Gr_p(\mathbb{R}^{p+q})$ is a real form, and hence, totally geodesic Lagrangian submanifolds in $Gr_p(\mathbb{C}^{p+q})$. In particular, when $p = 1$, we obtain the real projective space $\mathbb{R}P^q$ as a real form in $\mathbb{C}P^q$.

It is known that any Hermitian symmetric space of compact type with the canonical metric which comes from the Killing form is an Einstein manifold, and the Einstein constant is equal to 1/2 (cf. Proposition 9.7 in [52]). On the other hand, Takeuchi calculated the first eigenvalue for the real forms in [96]. By using his results and Corollary 2.2.13, we can determine the H-stability for all real forms in Hermitian symmetric spaces (See Table 2.1). The complete list of the H-stability for the real forms first appeared in [2] and the revised version is given in [59]. We note that Takeuchi [96] also proved the following fact: *A real form L is stable in the standard sense if and only if L is simply connected.*

Hermitian s.s. M	Real form L	λ_1	Einst. const. of L	H-st.	Stable
$Gr_p(\mathbb{C}^{p+q})$	$Gr_p(\mathbb{R}^{p+q})$	1/2	$\frac{p+q-2}{4(p+q)}$	Yes	No
$Gr_{2p}(\mathbb{C}^{2p+2q})$ $(p \leq q)$	$Gr_p(\mathbb{H}^{p+q})$	1/2	$\frac{p+q+1}{4(p+q)}$	Yes	Yes
$Gr_m(\mathbb{C}^{2m})$	$U(m)$	1/2	No	Yes	No
$SO(2m)/U(m)$	$SO(m)$ $(m \geq 5)$	1/2	$\frac{m-2}{4(m-1)}$	Yes	No
$SO(4m)/U(2m)$ $(m \geq 3)$	$U(2m)/Sp(m)$	$\frac{m}{4m-2}$	No	No	No
$Sp(2m)/U(2m)$	$Sp(m)$ $(m \geq 2)$	1/2	$\frac{m+1}{2(2m+1)}$	Yes	Yes
$Sp(m)/U(m)$	$U(m)/O(m)$	1/2	No	Yes	No
$Q_{p+q-2}(\mathbb{C})$ $(q-p \geq 3)$	$Q_{p,q}(\mathbb{R})$ $(p \geq 2)$	$\frac{p}{p+q-2}$	No	No	No
$Q_{p+q-2}(\mathbb{C})$ $(0 \leq q-p < 3)$	$Q_{p,q}(\mathbb{R})$ $(p \geq 2)$	1/2	No	Yes	No
$Q_{q-1}(\mathbb{C})$ $(q \geq 3)$	$Q_{1,q}(\mathbb{R})$	1/2	$\frac{q-2}{2(q-1)}$	Yes	Yes
$E_6/T \cdot Spin(10)$	$P_2(\mathbb{K})$	1/2	3/8	Yes	Yes
$E_6/T \cdot Spin(10)$	$Gr_2(\mathbb{H}^2)/\mathbb{Z}_2$	1/2	5/24	Yes	No
$E_7/T \cdot E_6$	$SU(8)/Sp(4)\mathbb{Z}_2$	1/2	3/8	Yes	No
$E_7/T \cdot E_6$	$T \cdot E_6/F_4$	1/6	No	No	No

Table 2.1: Real forms (totally geodesic Lagrangian submanifolds) in Hermitian symmetric spaces of compact type (cf. [59] and [96]). $Gr_p(\mathbb{F}^{p+q})$: Grassmannian manifold of all p-dimensional subspaces of $\mathbb{F}^{p+q}$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. $P_2(\mathbb{K})$: Cayley projective plane. $Q_n(\mathbb{C})$: Complex hyperquadric.

By carefully looking at Table 2.1, we see the following theorem:

THEOREM 2.2.19 ([70]). *Let L be an irreducible symmetric R-sapce canonically embedded in a Hermitian symmetric space M. If L is Einstein, then L is H-stable in M.*

Here, the Einstein condition is essential. In fact, we see:

EXAMPLE 2.2.20. Consider the complex hyperquadric

$$Q_n(\mathbb{C}) := \left\{ [z] \in \mathbb{C}P^{n+1} \ ; \ \sum_{i=0}^{n+1} z_i^2 = 0 \right\} \subset \mathbb{C}P^{n+1}.$$

Since $Q_n(\mathbb{C})$ is a complex submanifold in $\mathbb{C}P^{n+1}$, it inherits the Kähler structure induced from $(\mathbb{C}P^{n+1}, \omega_{FS}, J)$ (see [52]). In fact, $Q_n(\mathbb{C})$ is a Hermitian symmetric space which is isomorphic to the oriented 2-plane Grassmannian manifold $\tilde{Gr}_2(\mathbb{R}^{n+2}) \simeq SO(n+2)/SO(n) \times SO(2)$ via the map

$$\tilde{Gr}_2(\mathbb{R}^{n+2}) \ni V = \mathbf{a} \wedge \mathbf{b} \ \mapsto \ [\mathbf{a} + \sqrt{-1}\mathbf{b}] \in Q_n(\mathbb{C}), \tag{2.9}$$

where $\{\mathbf{a}, \mathbf{b}\}$ is an oriented orthonormal basis of the 2-plane V. Define the anti-holomorphic involution τ_k $(0 \leq k \leq n)$ on $Q_n(\mathbb{C})$ by

$$\tau_k([z]) := [\overline{z}_0, \ldots, \overline{z}_k, -\overline{z}_{k+1}, \ldots, -\overline{z}_{n+1}].$$

The involution τ_k is expressed by the following via the map (2.9):

$$\tau_k(\mathbf{a} \wedge \mathbf{b}) = (a_0, \ldots, a_k, -a_{k+1}, \ldots, -a_{n+1})^t \wedge (-b_0, \ldots, -b_k, b_{k+1}, \ldots, b_{n+1})^t,$$

where $\mathbf{a} := (a_0, \ldots, a_{n+1})^t$ and $\mathbf{b} := (b_0, \ldots, b_{n+1})^t$. Then the fixed point set of τ_k is given by

$$\begin{aligned}
\mathrm{Fix}(\tau_k) &= \{\mathbf{a} \wedge \mathbf{b} \in \tilde{Gr}_2(\mathbb{R}^{n+2}); \mathbf{a} = (a_0, \ldots, a_k, 0, \ldots, 0)^t, \mathbf{b} = (0, \ldots, 0, b_{k+1}, \ldots, b_{n+1})^t\} \\
&= \{[x] \in \mathbb{R}P^{n+1} \subset \mathbb{C}P^{n+1};\ x_0^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_{n+1}^2 = 0\} \subset Q_n(\mathbb{C}) \\
&=: Q_{k,n-k}(\mathbb{R}).
\end{aligned}$$

We note that $Q_{k,n-k}(\mathbb{R})$ is diffeomorphic to $(S^k \times S^{n-k})/\mathbb{Z}_2$. Conversely, all real forms in $Q_n(\mathbb{C})$ are given in this way. Put $n = p + q$ with $0 \leq p \leq q$. When $3 \leq p < q$, $Q_{p,q}(\mathbb{R})$ are not Einstein. Moreover, Takeuchi [96] calculated the first eigenvalue for these manifolds and proved that $\lambda_1 = p/(p+q-2) < 1/2$ when $q - p \leq 3$.

On the other hand, we see that there exist H-stable, *non-Eisntein* real forms, and hence, the converse of Theorem 2.2.19 does not hold. However, it is known that some of Einstein real forms have much stronger property than the H-stability, i.e., *Hamiltonian volume minimizing property*. This fact leads us to one of the well-known conjecture (Conjecture 2.4.3). We discuss details of this problem in Section 2.4.

2.2.4 H-Stability of parallel Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{C}P^n$

Let $\iota : L \to M$ be a Lagrangian immersion into a Kähler manifold. We call ι a *Lagrangian immersion with parallel second fundamental form* or shortly, *parallel Lagrangian immersion* if $\nabla S \equiv 0$, where S is the tensor field defined by (2.1). In fact, this definition is equivalent to the usual definition i.e.,

$$\nabla^{\perp}_X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z) = 0$$

for any $X, Y, Z \in \Gamma(TL)$, where ∇ (resp. $\nabla^{\perp}$) is the Levi-Civita connection on TL (resp. νL). Since the parallelism of the second fundamental form implies the parallelism of the mean curvature vector, a Lagrangian submanifold with parallel second fundamental form provides an example of H-minimal Lagrangian submanifold.

The parallel Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{C}P^n$ are classified by Naitoh and Takeuchi ([65], [66], [67], [68]). Their classification results assert that *the parallel Lagrangian submanifolds in $\mathbb{C}^n$ are given by the standard embedding of the irreducible symmetric R-spaces of type $U(r)$ or a Riemannian product of these embeddings.* Here, the irreducible symmetric R-spaces of type $U(r)$ are exactly the following five cases:

Riem. sym. sp. G/K	Sym. R-sp. $L = K \cdot X$	dim L
$SU(2n)/S(U(n) \times U(n))$	$U(n)$	n^2
$SO(n+2)/SO(2) \times SO(n)$	$(S^1 \times S^{n-1})/\mathbb{Z}_2$	n
$SO(4n)/U(2n)$	$U(2n)/Sp(n)$	$n(2n-1)$
$Sp(n)/U(n)$	$U(n)/SO(n)$	$n(n+1)/2$
$E_7/T \cdot E_6$	$T \cdot E_6/F_4$	27

Table 2.2: Irreducible symmetric R-spaces of type $U(r)$.

More precisely, these embeddings are described as follows (we refer to [2], [3] or [74]): Let (U, K) be a Hermitian symmetric pair of compact type with the canonical decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{u}$ and $\mathfrak{k}$ denote the Lie algebras corresponding to U and K, respectively. We take the standard inner product on $\mathfrak{u}$ by (-1)-times Killing-Cartan form of $\mathfrak{u}$. We decompose (U, K) into irreducible Hermitian symmetric pairs of compact type:

$$(U, K) = (U_1, K_1) \oplus \cdots \oplus (U_s, K_s).$$

Let $\mathfrak{u}_i = \mathfrak{k}_i + \mathfrak{p}_i$ be the canonical decomposition of $\mathfrak{u}_i$ for $i = 1, \dots, s$. Set $n_i + 1 := \dim_{\mathbb{C}} \mathfrak{p}_i = \dim_{\mathbb{C}} U_i/K_i$. Assume that there is an element $\mu_i \in \mathfrak{p}_i$ so that $(\mathrm{ad}\mu_i)^3 = -4(\mathrm{ad}\mu_i)$. Choose positive numbers $c_1, \dots, c_s$ with $\sum_{i=1}^{s} 1/c_i = 1/c$. Put $a_i := 1/\sqrt{2c_i(n_i+1)}$ for each i. Then $L_i := \mathrm{Ad}(K_i)(a_i\mu_i) \subset \mathfrak{p}_i \simeq \mathbb{C}^{n_i+1}$ is an irreducible symmetric R-space standardly embedded in the complex Euclidean space $\mathfrak{p}_i$. Moreover, the orbit $L := \mathrm{Ad}(K)\mu \subset \mathfrak{p}$ through the element $\mu := a_1\mu_1 + \cdots + a_s\mu_s \in \mathfrak{p}$ is also a symmetric R-space, and it has the form

$$L = L_1 \times \cdots \times L_s \subset S^{2n_1+1}(c_1/4) \times \cdots \times S^{2n_s+1}(c_s/4) \subset S^{2n+1}(c/4) \subset \mathbb{C}^{n+1}.$$

Then L is a compact parallel Lagrangian submanifold in $\mathbb{C}^{n+1}$. We note that L is never minimal, but H-minimal in $\mathbb{C}^{n+1}$.

If L is a parallel Lagrangian submanifold in $\mathbb{C}^{n+1}$, then the projection $\overline{L}$ is parallel Lagrangian in $\mathbb{C}P^n$. More precisely, by the result of Naitoh and Takeuchi, $\overline{L}$ is locally congruent to a symmetric space $M_0 \times M_1 \times \cdots \times M_r$, where M_0 is the Euclidean type and $M_i^{m_i} (i \geq 1)$ is one of (a) S^{m_i}, (b) $SU(p)$, $m_i = p^2$, (c) $SU(p)/SO(p)$, $m_i = (p-1)(p+2)/2$, (d) $SU(2p)/Sp(p)$, $m_i = (p-1)(2p+1)$, (e) E_6/F_4, $m_i = 26$ for some $p \geq 3$ such that $\sum_{i=0}^r m_i = n$. Here, $\overline{L}$ has no Euclidean factor if and only if L is irreducible, and in this case, $\overline{L}$ is minimal. For more details, refer to [2] and [65].

For the H-stability of the parallel Lagrangian submanifolds, Amarzaya-Ohnita proved the following by using a technique of the spherical function theory:

THEOREM 2.2.21 ([2],[3]). *Let L be an irreducible compact parallel Lagrangian submanifold in $\mathbb{C}^n$ or $\mathbb{C}P^n$. Then L is strictly H-stable.*

On the other hand, an example of H-stable, *non-parallel* Lagrangian submanifold in $\mathbb{C}P^n$ were found by Bedulli and Gori [15] and independently by Ohnita [74]:

EXAMPLE 2.2.22. Consider an irreducible unitary representation of $SU(2)$ given as follows:

$$
\begin{aligned}
&V_3 := \{f(z_1, z_2) : \mathbb{C}^2 \to \mathbb{C};\ \text{homogeneous polynomial of degree } 3\},\\
&\rho_3 : SU(2) \to GL_{\mathbb{C}}(V_3)\\
&\rho_3(x)f(z_1, z_2) := f(az_1 - \bar{b}z_2, bz_1 + \bar{a}z_2) \text{ with } x = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2).
\end{aligned}
$$

We equip V_3 with the standard Hermitian inner product such that

$$
\left\{ f_k := \frac{1}{\sqrt{k!(3-k)!}} z_1^{3-k} z_2^k;\ k = 0, 1, 2, 3 \right\}
$$

is a unitary basis of $V_3 \simeq \mathbb{C}^4 \simeq \mathbb{R}^8$. Then the orbit L of $SU(2)$ through $w := 1/\sqrt{2}(f_0 + f_3)$ is a minimal Legendrian submanifold embedded in $S^7(1)$. Consider the projection $\overline{L} := \pi(L)$ of L via the Hopf fibration $\pi : S^7(1) \to \mathbb{C}P^3$. Then $\overline{L}$ is a minimal Lagrangian submanifold with non-parallel second fundamental form in $\mathbb{C}P^3$. In [74], Ohnita calculated the first eigenvalue of the Laplacian on $\overline{L}$ by using the spherical function theory, and proved that $\overline{L}$ is strictly H-stable.

From the above results, Ohnita posed the following problem:

PROBLEM 2.2.23. Is any compact minimal Lagrangian submanifold embedded in $\mathbb{C}P^n$ H-stable? (or equivalently, $\lambda_1 = 2(n+1)$?)

Recently, Bedulli and Gori classified all the compact homogeneous Lagrangian submanifolds in $\mathbb{C}P^n$ [16]. Their classification includes some examples of minimal Lagrangian submanifold and Lagrangian submanifold with non-parallel second fundamental form.

2.3 H-minimal Lagrangian submanifolds in Kähler manifolds with symmetries

2.3.1 Moment maps and G-invariant isotropic submanifolds

Let (M, ω) be a symplectic manifold. We suppose that a Lie group G acts on M. We denote the Lie algebra of G by $\mathfrak{g}$. The action of G on (M, ω) is called *symplectic* if it acts diffeomorphically on M preserving the symplectic form ω. Moreover, a symplectic action is called *Hamiltonian* if there exists a function

$$\mu : M \to \mathfrak{g}^*,$$

where $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$, such that the following two properties holds.

(i) For $X \in \mathfrak{g}$, we define the function $\mu^X : M \to \mathbb{R}$ by $\mu^X(p) := \langle \mu(p), X \rangle$. Then, for any $X \in \mathfrak{g}$, the function μ^X satisfies $d\mu^X = \omega(X^*, \cdot)$ where X^* is the canonical vector field of X defined by

$$X^*_p := \frac{d}{dt}\Big|_{t=0} \exp(tX)p$$

for $p \in M$.

(ii) The function μ is *equivariant* to the action of G on M and the coadjoint action Ad^* on $\mathfrak{g}^*$, i.e., $\mu \circ \psi_g = Ad^*_g \circ \mu$ for any $g \in G$, where we denote the action of g on M by ψ_g.

The map μ is called the *moment map* of the Hamiltonian action G on (M, ω), and we call the quadruplet (M, ω, G, μ) the *Hamiltonian G-space.*

Let (M, ω, G, μ) be a Hamiltonian G-space. We define the center of $\mathfrak{g}^*$ by $Z(\mathfrak{g}^*) := \{X \in \mathfrak{g}^*;\ Ad^*_g(X) = X\}$. It is easy to see that the inverse image $\mu^{-1}(c)$ is G-invariant if and only if $c \in Z(\mathfrak{g}^*)$. The following two propositions are due to Joyce [43] (we also refer to [32]).

PROPOSITION 2.3.1. *Let L be a connected isotropic submanifold (i.e., $\omega|_L = 0$) in (M, ω). If L is G-invariant, then $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.*

PROOF. We denote the natural embedding of L into M by $\iota : L \to M$. Since ι is isotropic, $d(\iota^*\mu^X) = \iota^*(d\mu^X) = \omega(X^*, \cdot)|_{TL} = 0$ for any $X \in \mathfrak{g}$. By this and the assumption of connectedness of L, the function μ^X is constant on L for any $X \in \mathfrak{g}$, and hence, μ is constant on L. Thus, $L \subset \mu^{-1}(c)$ for some $c \in \mathfrak{g}^*$. Moreover, since L is G-invariant, $c \in Z(\mathfrak{g}^*)$. $\square$

Proposition 2.3.2. *Let L be a connected G-invariant submanifold of M. Suppose that the action G on L has cohomogeneity one (possibly transitive). Then L is isotropic if and only if $L \subset \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.*

Proof. By the previous proposition, if L is isotropic, then $L \subset \mu^{-1}(c)$ for $c \in Z(\mathfrak{g}^*)$. Thus, it suffices to show the converse. Suppose $L \subset \mu^{-1}(c)$ for $c \in Z(\mathfrak{g}^*)$. Then we have $\omega(X^*, Y) = d\mu^X(Y) = 0$ for any $X \in \mathfrak{g}$ and $Y \in T_pL$. Suppose L has cohomogeneity one. If a point $p \in L$ is a regular point of the action $G \curvearrowright L$, we take a vector $Y_1 \in T_pL$ so that Y_1 is transverse to the orbit of G through p since the action is cohomogeneity one. Then the tangent space T_pL is spanned by X^* for $X \in \mathfrak{g}$ and Y_1. In particular, we have $\omega|_{T_pL} = 0$. Since the set of regular points is open dense in L (cf. [10]), we have $\omega|_L = 0$, and hence, L is isotropic. When L is homogeneous, the tangent space T_pL is spanned by X^* for $X \in \mathfrak{g}$. Thus, the statement also follows by a similar argument. □

By virtue of this proposition, if p is a point of $\mu^{-1}(c)$ for $c \in Z(\mathfrak{g}^*)$, then the orbit $G \cdot p$ is an isotropic submanifold contained in $\mu^{-1}(c)$. Moreover, a Lagrangian orbit is characterized as follows:

Proposition 2.3.3 (cf. [59]). *Let p be a point of $\mu^{-1}(c)$ for $c \in Z(\mathfrak{g}^*)$, and $G \cdot p$ an isotropic orbit of G through p. Then $G \cdot p$ is Lagrangian if and only if*

$$T_q(G \cdot p) = \mathrm{Ker}(d\mu)_q \ for\ each\ q \in G \cdot p$$

that is, $G \cdot p$ is an open subset of $\mu^{-1}(c)$. Moreover, if the action of G on M is proper, then the orbit $G \cdot p$ is Lagrangian if and only if $G \cdot p$ is a connected component of $\mu^{-1}(c)$.

For the connectedness of the level set $\mu^{-1}(c)$, we have the following when G is compact:

Proposition 2.3.4 (cf. [59]). *Suppose G is compact, and it acts on a compact symplectic manifold (M, ω) in a Hamiltonian way. Then, for each $c \in Z(\mathfrak{g}^*)$, the level set $\mu^{-1}(c)$ of the moment map μ is a connected subset of M.*

In the following, we assume that (M, ω, J, g) is a Kähler manifold, and G is a connected Lie subgroup of the group of automorphism $\mathrm{Aut}(M, \omega, J, g)$. A Lagrangian submanifold L in M is called *homogeneous* if L is an orbit of the action $G \subset \mathrm{Aut}(M, \omega, J, g)$.

Proposition 2.3.5 (cf. [76]). *Every compact homogeneous Lagrangian submanifold in a Kähler manifold is H-minimal.*

Proof. Let L be a compact homogeneous Lagrangian submanifold. Since L is homogeneous, i.e., it is an orbit of G, the mean curvature vector H is G-invariant. This implies that the 1-form α_H is G-invariant, and hence, $\delta\alpha_H$ is a G-invariant function. Moreover, since the action of G on L is transitive, $\delta\alpha_H$ is a constant function on L. Because L is compact, Gauss's theorem implies $\delta\alpha_H = 0$. □

By this proposition, we get many examples of H-minimal Lagrangian submanifolds in Kähler manifold as orbits of an action of $G \subset \mathrm{Hom}(M, \omega, J, g)$. The following gives a geometric characterization of G-invariant H-minimal Lagrangians.

PROPOSITION 2.3.6 ([26]). *Let $\iota : L \to M$ be a G-invariant Lagrangian submanifold of a Kähler manifold M. Then L is H-minimal if and only if the volume functional of L is stationary under all compactly supported, G-euivariant Hamiltonian deformations.*

PROOF. If L is H-minimal, the assertion is obvious. We shall show the converse.

Let $\iota_t : L \times (-\epsilon, \epsilon) \to M$ be a G-equivariant Hamiltonian deformation with $\iota_0 = \iota$, i.e., ι_t is a Hamiltonian deformation of ι such that $\iota_t \circ g = g \circ \iota_t$ for any $g \in G$ and $t \in (-\epsilon, \epsilon)$. Then the variational vector field $V := d/dt|_{t=0}\iota_t$ is Hamiltonian, i.e., $\alpha_V = df$, and the function $f \in C_c^\infty(L)$ is G-invariant. Conversely, for any G-invariant function $f \in C_c^\infty(L)$, we define a deformation of ι by $\iota_t(p) := \exp_p(tV)$, where $JV := \nabla f$ (or equivalently, $\alpha_V = df$) and we choose ϵ small enough so that each $\iota_t, t \in (-\epsilon, \epsilon)$ is an immersion. Then the deformation ι_t is a G-equivariant Hamiltonian deformation.

Let L be a G-invariant Lagrangian submanifold. We note that the codifferential of the mean curvature $\delta\alpha_H$ of G-invariant submanifold L is a G-invariant function by the similar argument of the previous proposition. Now, we assume that L is *not* H-minimal, i.e., $\delta\alpha_H \neq 0$. Then, for any G-invariant function $\phi \in C_c^\infty(L)$, we can define the G-equivariant Hamiltonian deformation ι_t so that the variational vector field V is given by $V := df$, where $f := \phi\delta\alpha_H$. Since $\phi \in C_c^\infty(L)$ is arbitrary, we can choose ϕ so that

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}(\iota_t(L)) = -\int_L g^*(\alpha_H, \alpha_V) dv = -\int_L |\delta\alpha_H|^2 \phi dv \neq 0.$$

□

For a compact Kähler manifold, Bedulli and Gori proved the next existence result of homogeneous Lagrangian submanifolds:

THEOREM 2.3.7 ([16]). *Let (M, ω, J) be a compact Kähler manifold with $h^{1,1}(M) = 1$. Suppose G is a compact connected group of isometries acting on M in a Hamiltonian way. Then M admits a homogeneous Lagrangian submanifold with respect to G if and only if $G^{\mathbb{C}}$ has an open Stein orbit in M.*

If $h^{1,1}(M) = 1$, then there exist a nonzero constant γ so that $\omega' := \gamma\omega \in H^2(M, \mathbb{Z})$ is an integral class, and hence, M is a Hodge manifold. Therefore, by Kodaira's Embedding theorem, M is necessarily projective. Note that the assumptions of Theorem 2.3.7 are satisfied when M is a compact Hermitian symmetric space.

If G is compact and semi-simple, the center of the Lie algebra $\mathfrak{g}$ is trivial, and thus, the Lagrangian orbit is exactly the level set $\mu^{-1}(0)$ by Proposition 2.3.3 and 2.3.4. Moreover, we have the following:

Theorem 2.3.8 ([16]). *Under the same assumptions as in Theorem 2.3.7, a Lagrangian orbit $G \cdot p$ is isolated (actually, unique) if and only if the smallest subgroup G' of G such that $G' \cdot p = G \cdot p$ is semi-simple.*

Furthermore, if M is Kähler-Einstein, we see the following:

Proposition 2.3.9 ([16]). *If G is compact, semi-simple and M is Kähler-Einstein, then the unique Lagrangian orbit is minimal.*

Proof. Let $L := G \cdot p$ be the Lagrangian orbit of G, and α_H is the mean curvature form of L. Since M is Kähler-Einstein, the 1-form α_H is closed by Theorem 2.1.1. Moreover, since α_H is G-invariant, we have

$$0 = d\alpha_H(X^*, Y^*) = X^*\alpha_H(Y^*) - Y^*\alpha_H(X^*) - \alpha_H([X^*, Y^*]) = -\alpha_H([X^*, Y^*]), \quad (2.10)$$

where X^*, Y^* denote the fundamental vector fields of $X, Y \in \mathfrak{g}$, respectively. Because G is compact, semi-simple, we have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and hence, (2.10) implies $\alpha_H = 0$. $\square$

2.3.2 Homogeneous Lagrangian submanifolds in the complex hyperquadric $Q_n(\mathbb{C})$ and isoparametric hypersurfaces in $S^{n+1}(1)$.

Gauss maps

Let $\iota : N^n \to S^{n+1}(1)$ be an immersion of an oriented hypersurface into the unit sphere. The *Gauss map* of the immersion ι is an immersion of N into the oriented 2-plane Grassmannian manifold $\tilde{Gr}_2(\mathbb{R}^{n+2})$ defined by

$$\mathcal{G} : N^n \to \tilde{Gr}_2(\mathbb{R}^{n+2}) \subset \bigwedge^2 \mathbb{R}^{n+2}, \; p \; \mapsto \iota(p) \wedge \nu(p) = \nu_p N,$$

where ν denotes the unit normal vector field of ι, and $\nu_p N$ is the normal space of N in $\mathbb{R}^{n+2}$ at p. We recall that $\tilde{Gr}_2(\mathbb{R}^{n+2})$ is diffeomorphic to the complex hyperquadric

$$Q_n(\mathbb{C}) := \left\{ [z] \in \mathbb{C}P^{n+1} \; ; \; \sum_{i=0}^{n+1} z_i^2 = 0 \right\}$$

via the map (2.9). Then $\tilde{Gr}_2(\mathbb{R}^{n+2})$ admits a Kähler-Einstein structure with Einstein constant n which is induced from $Q_n(\mathbb{C})$. The following proposition shows fundamental properties of the Gauss maps (cf. [84]).

Proposition 2.3.10 ([84]). *Let $\mathcal{G} : N^n \to \tilde{Gr}_2(\mathbb{R}^{n+2})$ be the Gauss map of an oriented hypersurface $\iota : N^n \to S^{n+1}(1)$. Then the following properties holds.*

(i) $\mathcal{G}$ *is a Lagrangian immersion.*

(ii) *Let* $\iota_t : (-\epsilon, \epsilon) \times N^n \to S^{n+1}(1)$ *be a smooth deformation of the immersion* $\iota = \iota_0$. *Then the corresponding family of Gauss maps* $\mathcal{G}_t : (-\epsilon, \epsilon) \times N^n \to \tilde{Gr}_2(\mathbb{R}^{n+2})$ *is a Hamiltonian deformation. More precisely, if the variational vector field* v_t *of* ι_t *satisfies* $v_t^{\perp} = f_t \nu_t$, *then the variational vector field* V_t *of* $\mathcal{G}_t$ *satisfies* $V_t^{\perp} = -J\nabla^{\mathcal{G}_t} f_t$, *where* $\nabla^{\mathcal{G}_t}$ *is the gradient with respect to the induced metric on* N *via* $\mathcal{G}_t$.

(iii) *If* $\mathcal{G} : N^n \to Gr_2(\mathbb{R}^{n+2})$ *is a Lagrangian immersion, then locally* $\mathcal{G}$ *arises as the Gauss map of a generalized immersion of* S^{n+1}.

It is natural to ask when the Gauss map of an oriented hypersurface gives *minimal* Lagrangian immersion. B. Palmer derived the mean curvature formula of a Gauss map as follows. Recall that the *principal curvatures* of the immersion $\iota : N^n \to S^{n+1}$ are the eigenvalues of the shape operator of ι.

THEOREM 2.3.11 ([84]). *Let* $\mathcal{G} : N^n \to \tilde{Gr}_2(\mathbb{R}^{n+2})$ *be the Gauss map of an oriented hypersurface* $\iota : N^n \to S^{n+1}(1)$. *Then the mean curvature form of the Gauss map is given by*

$$\alpha_H = -d\Big(\sum_{i=1}^{n} \arctan \kappa_i\Big) = -d\Big\{\mathrm{Im} \log \prod_{i=1}^{n} (1 + \sqrt{-1}\kappa_i)\Big\},$$

where κ_i $(i = 1, \cdots, n)$ *are the principal curvatures of the immersion* ι *with respect to the unit normal vector* ν.

We generalize this theorem in Chapter 4. Thus we omit a proof here.

Gauss maps of isoparametric hypersurfaces

In what follows, we investigate when a Lagrangian Gauss map or its image of a hypersurface in $S^{n+1}(1)$ becomes minimal Lagrangian. A typical example of a minimal Lagrangian Gauss map is obtained from the *isoparametric hypersurface*. First, we briefly review facts of isoparametric hypersurfaces in the real space forms (We refer to [10] and [19]).

Let N be an oriented hypersurface in $M(c)$ with the unit normal vector field ν. We define the map $F_t : N \to M(c)$ for $t \in \mathbb{R}$ by

$$\begin{cases} F_t(p) = \mathbf{p} + t\nu(p), \text{ if } c = 0, \\ F_t(p) = \cos t\mathbf{p} + \sin t\nu(p) \text{ if } c > 0, \\ F_t(p) = \cosh t\mathbf{p} + \sinh t\nu(p) \text{ if } c < 0, \end{cases}$$

where $\mathbf{p}$ is the position vector of the point p regarded as a vector in $\mathbb{R}^m$ when $c \geq 0$, and $\mathbb{R}^m_1$ when $c < 0$. Whenever F_t is an immersion, we call $F_t(N)$ the *parallel* hypersurface of N in $M(c)$.

Let $f : M^m(c) \to \mathbb{R}$ be a non constant smooth function on the real space form $M(c)$ with the conditions (i) $|\nabla f|^2 = a \circ f$ and (ii) $\Delta f = b \circ f$ for some smooth functions $a, b : f(M) \to \mathbb{R}$. We call the function f *isoparametric function* on M. The condition (i) means that the level hypersurfaces $f^{-1}(t)$ for regular values $t \in f(M)$ are parallel to each other in M, and the condition (ii) implies these hypersurfaces have constant mean curvatures. Moreover, an important fact is that *the hypersurface $f^{-1}(t)$ in $M(c)$ has constant principal curvatures.* We call the level set $f^{-1}(t)$ *isoparametric hypersurface* in $M(c)$. The level set of the preimage of the global maximal (resp. minimum) value of f is denoted by N_+ (resp. N_-). We call $N_\pm$ the *focal varieties* of f. We note that the focal varieties of an isoparametric function on $M(c)$ are smooth submanifolds in $M(c)$ (cf. [106]).

The isoparametric hypersurfaces in $\mathbb{R}^{n+1}$ were classified by Levi-Civita and Segre (see also Subsection 3.2.2), and for the Hyperbolic space $\mathbb{H}^{n+1}$, it is due to E.Cartan (see [10], p.86). However, the classification in $S^{n+1}(1)$ has not been completed yet.

In $S^{n+1}(1)$, extrinsically homogeneous hypersurfaces are isoparametric hypersurfaces, and these are classified by Hsiang-Lawson [31]. Their classification result asserts the following:

Theorem 2.3.12 ([34]). *Every homogeneous hypersurface in S^{n+1} is obtained by the principal orbit of the linear isotropy representation of a compact symmetric space of rank 2.*

See Table B.3 in Appendix B for the explicit classification of the homogeneous hypersurfaces.

On the other hand, isoparametric hypersurfaces in $S^{n+1}(1)$ includes infinitely many non-homogeneous examples which were discovered by Ozeki-Takeuchi and Ferus-Karcher-Münzner. These examples are called the isoparametric hypersurfaces of *OT-FKM type* (for more details, refer to monographs [19], [104] and references therein).

We return to the Gauss maps. Since an isoparametric hypersurfaces in $S^{n+1}(1)$ has constant principal curvatures, we obtain the following by Theorem 2.3.11.

Corollary 2.3.13 ([84]). *Let $\iota : N^n \to S^{n+1}(1)$ be an isoparametric hypersurface. Then the Gauss map of ι is a minimal Lagrangian immersion.*

For the Hamiltonian stability of the *Gauss maps* of isoparametric hypersurfaces, Palmer proved the following:

THEOREM 2.3.14 ([84]). *Let $\mathcal{G} : N^n \to \tilde{Gr}_2(\mathbb{R}^{n+2})$ be the Gauss map of an isoparametric hypersurface $\iota : N^n \to S^{n+1}(1)$. Then $\mathcal{G}$ is H-stable if and only if N is a hypersphere.*

PROOF. It is known that, for each isoparametric hypersurface, there exist unique minimal isoparametric hypersurface in the family of parallel hypersurfaces [21]. Since Gauss maps of parallel hypersurfaces are the same, we may assume that the isoparametric hypersurface N is minimal in $S^{n+1}(1)$.

We denote the Riemannian metric on $\tilde{Gr}_2(\mathbb{R}^{n+2})$ by $\tilde{g}$. We choose a local field of orthonormal frames $\{e_1, \cdots, e_n\}$ on an open subset U in N such that each e_i is the principal direction on U, namely, $A^\nu(e_i) = \kappa_i e_i$ for $i = 1, \cdots, n$ on U, where A^ν is the shape operator of N in $S^{n+1}(1)$. Define a local field of tangent frame $\{E_1, \cdots, E_n\}$ of N immersed in $\tilde{Gr}_2(\mathbb{R}^{n+2})$ by $E_i := \mathcal{G}_* e_i$. Then we have

$$E_i = \mathcal{G}_* e_i = \frac{d}{dt}\Big|_{t=0} c_i(t) \wedge \nu_{c_i(t)} = e_i \wedge \nu - p \wedge A(e_i) = e_i \wedge \nu + \kappa_i e_i \wedge p,$$

where $c_i(t)$ is a curve in N such that $c_i(0) = p$ and $\dot{c}_i(0) = e_i$. Moreover, we have $\tilde{g}(E_i, E_j) = (1 + \kappa_i^2)\delta_{ij}$. In particular, we obtain

$$dv_{\tilde{g}} = \Big(\prod_{i=1}^n 1 + \kappa_i^2\Big)^{n/2} dv_N = \text{const } dv_N \tag{2.11}$$

by the isoparametric condition, where dv_N is the volume form on N in $S^{n+1}(1)$. Since we assume that N^n is a minimal hypersurface in $S^{n+1}(1)$, there exist a function $f \in C^\infty(N)$ such that $\Delta^N f = nf$ by Takahasi's Theorem [95]. Then, by the Min-Max principle, we have

$$\lambda_1(N, \tilde{g}) \le \frac{\int_N |\nabla^{\tilde{g}} f|^2 dv_{\tilde{g}}}{\int_N f^2 dv_{\tilde{g}}} = \frac{\int_N |\nabla^N f|^2 dv_N}{\int_N f^2 dv_N}$$

using (2.11). Moreover, we have

$$|\nabla^{\tilde{g}} f|^2 = \sum_{i=1}^n \frac{E_i(f)^2}{1 + \kappa_i^2} \le \sum_{i=1}^n E_i(f)^2 = \sum_{i=1}^n e_i(f)^2 = |\nabla^N f|^2 \tag{2.12}$$

since $E_i = \mathcal{G}_* e_i$. If N is not a sphere, we can choose i such that $\kappa_i \neq 0$ and choosing f so that $E_i(f)$ is not identically zero, then the inequality (2.12) becomes a strict inequality. Then we obtain

$$\lambda_1(N, \tilde{g}) < \frac{\int_N |\nabla^N f|^2 dv_N}{\int_N f^2 dv_N} = \frac{\int_N f \Delta^N f dv_N}{\int_N f^2 dv_N} = n$$

since $\Delta^N f = nf$. Thus, by Theorem 2.2.4, $\mathcal{G} : N \to \tilde{Gr}_2(\mathbb{R}^{n+2})$ is H-unstable. On the other hand, if N is a sphere, one can see that $\lambda_1(N, \tilde{g}) = \lambda_1(N) = n$, and hence, the Gauss map is H-stable. $\square$

In general, the Gauss map of a hypersurface is *not* an embedding into $\tilde{Gr}_2(\mathbb{R}^{n+2})$. For instance, for an isoparametric hypersurface N, it is known that the Gauss map is a finite cover of N. More precisely, $\mathcal{G}(N)$ is diffeomorphic to $N/\mathbb{Z}_g$ for an action of a finite group $\mathbb{Z}_g$ on N (see [59]).

Classification of homogeneous Lagrangian submanifolds in $Q_n(\mathbb{C})$

Recall that every homogeneous hypersurface in $S^{n+1}(1)$ is obtained by an orbit of an s-representation of rank 2 compact symmetric space by Theorem 2.3.12. Let (U, K) be a compact Riemannian symmetric pair of rank 2. Denote the Cartan decomposition of (U, K) by $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$. The linear isotropy representation of the isotropy subgroup K is the adjoint representation $\mathrm{Ad}|_{\mathfrak{p}} : K \to GL(\mathfrak{p})$. Since $\mathfrak{u}$ admits the $\mathrm{Ad}(G)$-invariant inner product, $\mathrm{Ad}(K)$ is an isometric action on $\mathfrak{p}$, i.e., $\mathrm{Ad}(K) \subset SO(n+2)$, where $\dim\mathfrak{p} = n+2$. Thus, $\mathrm{Ad}(K)$ also acts on the unit sphere $S^{n+1} \subset \mathfrak{p}$. Since we assume that the Riemannian symmetric pair (U, K) is rank 2, a principal orbit of the action is a hypersurface in S^{n+1} and becomes an isoparametric hypersurface.

The linear isotropy action of K induces the natural action on $\tilde{Gr}_2(\mathfrak{p})$ given by

$$k \cdot (\mathbf{a} \wedge \mathbf{b}) := \mathrm{Ad}(k)\mathbf{a} \wedge \mathrm{Ad}(k)\mathbf{b}$$

for $k \in K$ and $V := \mathbf{a} \wedge \mathbf{b} \in \tilde{Gr}_2(\mathfrak{p})$, where $\{\mathbf{a}, \mathbf{b}\}$ is an oriented orthonormal basis of the 2-plane V. Since $\mathrm{Ad}(K) \subset SO(n+2)$, this action is also isometric on $\tilde{Gr}_2(\mathfrak{p}) \simeq SO(n+2)/SO(2)\times SO(n)$. If N^n is a homogeneous hypersurface in S^{n+1} which is obtained by the action of $\mathrm{Ad}(K)$, i.e., $N^n = \mathrm{Ad}(K)x$ for some $x \in S^{n+1}$, then one see that the induced action of K acts transitively on the Gauss image $L = \mathcal{G}(N)$. Combining this with Corollary 2.3.13, we conclude that *the Gauss image L of a homogeneous isoparametric hypersurface N in $S^{n+1}(1)$ is a homogeneous minimal Lagrangian submanifold in $Q_n(\mathbb{C}) \simeq \tilde{Gr}_2(\mathbb{R}^{n+2})$.*

Motivated by this, Ma-Ohnita classified all the compact homogeneous Lagrangian submanifolds in $Q_n(\mathbb{C})$ [59]. They proved the following:

Theorem 2.3.15 ([59]). *Any compact homogeneous Lagrangian submanifold in $Q_n(\mathbb{C})$ is obtained either as the Gauss image of a homogeneous isoparametric hypersurface in $S^{n+1}(1)$, or as its Lagrangian deformation consisting of compact homogeneous Lagrangian submanifolds.*

In particular, every compact homogeneous minimal Lagrangian submanifold in the complex hyperquadric $Q_n(\mathbb{C})$ is exactly the Gauss image of a homogeneous isoparametric hypersurface in S^{n+1}.

We note that there are inhomogeneous isoparametric hypersurfaces in the sphere. In that case, the Gauss image is not homogeneous, but minimal.

We briefly explain the details and the strategy of a proof of Theorem 2.3.15 according to [59]. A basic tool is the moment map.

First, we consider the Gauss image of a homogeneous isoparametric hypersurface. Let (U, K) be a compact Riemannian symmetric pair of rank 2. Consider the isometric action of K on $Q_n(\mathbb{C}) \simeq \tilde{Gr}_2(\mathfrak{p})$ as before. This action is a Hamiltonian action. In fact, the moment map $\mu : Q_n(\mathbb{C}) \to \mathfrak{k}^*$ is given by

$$\mu([\mathbf{a} + \sqrt{-1}\mathbf{b}]) = -[\mathbf{a}, \mathbf{b}] \in \mathfrak{k} \simeq \mathfrak{k}^*, \tag{2.13}$$

where $\{\mathbf{a}, \mathbf{b}\}$ is an orthonormal basis in $\mathfrak{p}$. Let N be a regular orbit of the isotropy action K on $\mathfrak{p}$, that is a homogeneous isoparametric hypersurface in the hypersphere $S^{n+1}(1) \subset \mathfrak{p}$. Then the Gauss image $L := \mathcal{G}(N)$ is a K-invariant Lagrangian submanifold. Since N is regular, a normal space $\mathfrak{a} := \nu_p N = \mathbf{a} \wedge \mathbf{b}$ is a maximal abelian subspace in $\mathfrak{p}$ (see also Subsection 3.2.4 in the next Chapter). Therefore, we have $\mu(\mathfrak{a}) = -[\mathbf{a}, \mathbf{b}] = 0$, and hence, $L \subset \mu^{-1}(0)$ by Proposition 2.3.1. Since K is compact, the level set $\mu^{-1}(0)$ is a connected set by Proposition 2.3.4. Combining this with Proposition 2.3.3, we obtain the following:

Proposition 2.3.16. *Let N be a homogeneous isoparametric hypersurface in $S^{n+1}(1)$. Then the Gauss image $L = \mathcal{G}(N)$ coincides with $\mu^{-1}(0)$, where μ is the canonical moment map defined by (2.13).*

Recall that any homogeneous Lagrangian submanifold is contained in $\mu^{-1}(c)$ for $c \in Z(\mathfrak{k}^*)$. If the center $Z(\mathfrak{k}^*)$ is trivial, Proposition 2.3.16 implies the Gauss image $L = \mathcal{G}(N)$ is the unique homogeneous Lagrangian submanifold in $Q_n(\mathbb{C})$ which is invariant the K-action. A compact Riemannian symmetric pair (U, K) of rank 2 which has the non-trivial center $\mathfrak{z} \subset \mathfrak{k}$ is one of the following:

(a) $(S^1 \times SO(3), SO(2))$.
(b) $(SO(3) \times SO(3), SO(2) \times SO(2))$.
(c) $(SO(3) \times SO(n+1), SO(2) \times SO(n))$ for $n \geq 3$.
(d) $(SO(m+2), SO(2) \times SO(m))$ for $m \geq 3$.

Now, we consider the general case. Let L' be a compact homogeneous Lagrangian submanifold in $Q_n(\mathbb{C})$ which is invariant under the Hamiltonian action of a Lie subgroup $K' \subset SO(n+2)$. A crucial fact is that *there exist a compact Riemannian symmetric pair (U, K) of rank 2 such that $L' := K' \cdot [V] = K \cdot [V]$*, where $[V]$ is a point in $Q_n(\mathbb{C}) \simeq \tilde{Gr}_2(\mathbb{R}^{n+1})$. This is a consequence of the result of Hsiang-Lawson [34] and the classification result of cohomogeneity 1 action of compact Lie groups on $S^{n+1}(1)$ due to Asoh (see for more details in [59]). Therefore, combining this with the result of Hsiang-Lawson, it is sufficient to classify the Lagrangian orbits of the above four cases.

The case (a): In this case, $n = 1$ and $Q_1(\mathbb{C}) \simeq \tilde{Gr}_2(\mathbb{R}^3) \simeq S^2(1)$. Then the Lagrangian orbits are nothing but the small circles and the equator.

The case (b): In this case, $n = 2$ and $Q_2(\mathbb{C}) \simeq S^2 \times S^2$. Then the Lagrangian orbits are exactly products of small circles.

The case (c) *and* (d): There exist a non-trivial family of Lagrangian orbits. These consist of S^1-family of Lagrangian and isotropic orbits. Each family contains Lagrangian orbits which can be never obtained as the Gauss images of homogenous isoparametric hypersurfaces. Refer to [59] for the details.

Finally, we mention the H-stability of the Gauss images of homogeneous isoparametric hypersurfaces. Let N be an isoparametric hypersurface in $S^{n+1}(1)$ (where N is not necessary homogeneous). We denote the distinct principal curvatures of N and these multiplicities by $\kappa_1, \ldots, \kappa_g$ and $m_1, \ldots, m_g$, respectively. Then Münzner proved that $g = 1, 2, 3, 4, 6$ and $m_i = m_{i+2}$ [63], [64]. In particular, the multiplicities are same when g is odd. Moreover, Münzner also showed that $m_1 = m_2$ when $g = 6$.

If N is homogeneous, Ma-Ohnita proved the following:

Theorem 2.3.17 ([59], [60]). *Let N be a homogeneous isoparametric hypersurface in $S^{n+1}(1)$. Then the Gauss image $L = \mathcal{G}(N)$ in $Q_n(\mathbb{C})$ is H-stable if and only if N satisfies one of the following:*

(i) $|m_2 - m_1| < 3$, *or*

(ii) N is a principal orbit of the Riemannian symmetric pair $(U, K) = (E_6, U(1) \cdot Spin(10))$ (in this case, $g = 4$ and $(m_1, m_2) = (6, 9)$).

In [60], they also classified all *strictly* H-stable homogeneous minimal Lagrangian submanifold in $Q_n(\mathbb{C})$. For instance, the Gauss images of Cartan hypersurfaces (namely, when $g = 3$) and a principal orbit of $(E_6, U(1) \cdot Spin(10))$ are strictly H-stable. However, we note that the case (i) includes some H-minimal Lagrangian submanifolds which are not strictly H-stable.

2.4 Hamiltonian volume minimizing problem

2.4.1 Hamiltonian volume minimizing property

In this section, we consider the global version of H-stability. For the simplicity, we always assume that a Lagrangian submanifold is compact throughout this section.

Let (M, ω) be a symplectic manifold with a symplectic form ω and L a Lagrangian submanifold in M. A diffeomorphism ϕ on M is called a *Hamiltonian diffeomorphism* of M if ϕ satisfies the following conditions:

(i) ϕ is *symplectic*, namely, $\phi^*\omega = \omega$.

(ii) ϕ is represented by the flow $\{\phi_t\}_{t\in[0,1]}$ of a time dependent Hamiltonian vector field $\{X_{H_t}\}$ on M, namely, $d/dt(\phi_t(x)) = X_{H_t}(\phi_t(x))$ with $\phi_0 = Id_M$ and $\phi_1 = \phi$, where $\omega(X_{H_t}, \cdot) = dH_t$ for $H_t \in C^\infty(M)$.

We denote the set of all Hamiltonian diffeomorphisms by $\mathrm{Ham}(M, \omega)$.

DEFINITION 2.4.1. Let (M, ω, J) be a Kähler manifold and L a Lagrangian submanifold in M. We call L *Hamiltonian volume minimizing* (or shortly, *H.V.M.* Lagrangian submanifold) if L is a volume minimizer of any Hamiltonian diffeomorphism, namely, L satisfies the inequality

$$\mathrm{Vol}(\phi(L)) \geq \mathrm{Vol(L)}$$

for any $\phi \in \mathrm{Ham}(M, \omega)$.

By definition, it follows that an H.V.M. Lagrangian submanifold is necessarily H-minimal and H-stable. We know only a few examples of H.V.M. Lagrangian submanifolds. The following examples are the origin of this problem.

EXAMPLE 2.4.2 (Kleiner-Oh [70]). The totally geodesic real projective space $\mathbb{R}P^n$ in $\mathbb{C}P^n$ is H.V.M.

A proof of Kleiner-Oh was based on the Lagrangian intersection theory and the integral geometry. We give an outline of the proof in the next subsection.

We have already shown that the Einstein real form in a Hermitian symmetric space which is a natural generalization of $\mathbb{R}P^n$ in $\mathbb{C}P^n$ (cf. Theorem 2.2.18) are H-stable Lagrangian submanifolds. Based on these examples, Oh posed the following conjecture:

CONJECTURE 2.4.3 (Oh [70]). Let (M, ω, J) be a Kähler-Einsntein manifold and L a real form of M, namely, the fixed point set of an anti-holomorphic involution of M. If L is Einstein, then L is H.V.M.

REMARK 2.4.4. In the original statement of this conjecture due to Oh in [70], he further assume that L has *positive Ricci-curvature*. However, there exist a *flat* H.V.M. Einstein-real form (see Subsection 2.4.3).

Before considering the Conjecture 2.4.3, we should consider the following problem which is still open:

Conjecture 2.4.5 (Oh [70]). Let L be a real form which satisfies the same assumption as in Conjecture 2.4.3. Then, the first eigenvalue λ_1 of the Laplace-Bertrami operator Δ acting on $C^\infty(L)$ satisfies $\lambda_1 \geq c$, where c is the Einstein constant of M.

2.4.2 The case $\mathbb{R}P^n$ in $\mathbb{C}P^n$

In this subsection, we review a proof of the H.V.M property for $\mathbb{R}P^n$ in $\mathbb{C}P^n$ due to the idea of Kleiner-Oh. To do this, we use the following two results. First one is a special version of the well-known Arnord-Givental conjecture in the Lagrangian intersection theory:

Theorem 2.4.6 (Oh [73]). *Let (M, ω, J) be an irreducible Hermitian symmetric space of compact type and L a real form of an anti-holomorphic involution τ. Then, for any $\phi \in \mathrm{Ham}(M, \omega)$ such that L and $\phi(L)$ intersect transversally, we have the inequality*

$$\#(L \cap \phi(L)) \geq SB(L, \mathbb{Z}_2),$$

where $SB(L, \mathbb{Z}_2)$ denote the sum of $\mathbb{Z}_2$-Betti number of L.

Next one is the Crofton type formula in the integral geometry:

Theorem 2.4.7 (Lê [55]). *Let L^n be a Lagrangian submanifold in $\mathbb{C}P^n$. Then*

$$\mathrm{Vol}(L) = c_n \int_{U(n+1)/O(n+1)} \#(L \cap g\mathbb{R}P^n) d\mu_g,$$

where c_n is a constant that does not depend on L, and $U(n+1)/O(n+1)$ is the set of all real projective spaces in $\mathbb{C}P^n$.

It is known that

$$\#(\mathbb{R}P^n \cap g\mathbb{R}P^n) = SB(L, \mathbb{Z}_2) = n + 1$$

whenever $g\mathbb{R}P^n$ and $\mathbb{R}P^n$ intersect transversally. Combining this with Theorem 2.4.7 , we obtain the following volume estimate:

Corollary 2.4.8. *For a Lagrangian submanifold L in $\mathbb{C}P^n$, we have*

$$\frac{\mathrm{Vol}(L)}{\mathrm{Vol}(\mathbb{R}P^n)} \geq \frac{\min_g \#(L \cap g\mathbb{R}P^n)}{n+1}.$$

Now, we prove the H.V.M. property for $\mathbb{R}P^n$ in $\mathbb{C}P^n$. For any $\phi \in \mathrm{Ham}(\mathbb{C}P^n, \omega)$, $\phi(\mathbb{R}P^n)$ is a Lagrangian submanifold, and hence, by Corollary 2.4.8 and Theorem 2.4.6, we have

$$\frac{\mathrm{Vol}(\phi(\mathbb{R}P^n))}{\mathrm{Vol}(\mathbb{R}P^n)} \geq \frac{\min_g \#(\phi(\mathbb{R}P^n) \cap g\mathbb{R}P^n)}{n+1} \geq \frac{SB(\mathbb{R}P^n, \mathbb{Z}_2)}{n+1} = 1$$

since $SB(\mathbb{R}P^n, \mathbb{Z}_2) = n + 1$. This implies $\mathrm{Vol}(\phi(\mathbb{R}P^n)) \geq \mathrm{Vol}(\mathbb{R}P^n)$.

2.4.3 Real forms in $Q_n(\mathbb{C})$

In [39], Iriyeh, Ono and Sakai gave the second example of an H.V.M. Lagrangian submanifold by using the idea of Kleiner-Oh.

Example 2.4.9. The product of equators $S^1 \times S^1$ in $S^2 \times S^2$ is H.V.M.

We note that $S^1 \times S^1$ is a flat torus. We also note that $S^2 \times S^2$ is isometric to the complex hyperquadric $Q_2(\mathbb{C})$ (see [18]), and the real forms in $Q_2(\mathbb{C}) \simeq S^2 \times S^2$ are exactly given by the sphere $S^2 = \{(x, -x);\ x \in S^2\}$ and the product of equators $S^1 \times S^1$ (see also Example 2.2.20). We also remark that Castro-Urbano [18] proved that *the totally geodesic sphere S^2 is the unique stable compact minimal Lagrangian surface in $Q_2(\mathbb{C})$*. Furthermore, they also showed that *the totally geodesic torus $S^1 \times S^1$ is the unique H-stable torus in $Q_2(\mathbb{C})$*.

We consider the higher dimensional case. For the complex hyperquadric, we have a Crofton type formula as follows:

Theorem 2.4.10 (Lê [55]). *Let N be a real n-dimensional submanifold in $Q_n(\mathbb{C})$. Then we have*

$$\int_{SO(n+2)} \#(gS^n \cap N) d\mu_{SO(n+2)}(g) \leq 2\frac{\mathrm{Vol}(SO(n+2))}{\mathrm{Vol}(S^n)}\mathrm{Vol}(N).$$

If we take $N := \phi(S^n)$ for $\phi \in \mathrm{Ham}(Q_n(\mathbb{C}), \omega)$, we have from Theorem 2.4.6

$$\mathrm{Vol}(\phi(S^n)) \geq \frac{\mathrm{Vol}(S^n)}{2} SB(S^n, \mathbb{Z}_2) = \mathrm{Vol}(S^n),$$

since $SB(S^n, \mathbb{Z}_2) = 2$. Therefore, we obtain

Theorem 2.4.11 (cf. [40]). *The totally geodesic sphere S^n in $Q_n(\mathbb{C})$ is H.V.M.*

We note that, *if n is even, the sphere S^n in $Q_n(\mathbb{C})$ is homologically volume minimizing* by the result of Gluck-Morgan-Ziller [29]. We also note that the real form $S^{k,n-k}$ for $0 \leq k \leq n$ in $Q_n(\mathbb{C})$ is H-stable if and only if $|(n-k) - k| < 3$ (see Table 2.1 in Subsection 2.2.3 or Theorem 2.3.18), and hence, not every real form in $Q_n(\mathbb{C})$ is H-stable.

Problem 2.4.12. Find and classify H.V.M. Lagrangian submanifolds in a specific Kähler manifold.

Chapter 3

Hamiltonian minimality of normal bundles over the isoparametric submanifolds

3.1 Preliminaries for normal bundles in $T\mathbb{R}^{n+k}$

Let $\iota : N^n \to \mathbb{R}^{n+k}$ be an isometric embedding of an n-dimensional smooth manifold into the $(n+k)$-dimensional Euclidean space $\mathbb{R}^{n+k}$ with the standard flat metric $\langle , \rangle$. In the following, we always identify N with its image under ι, and call it a submanifold in $\mathbb{R}^{n+k}$. Denote the tangent bundle of $\mathbb{R}^{n+k}$ by $T\mathbb{R}^{n+k}$. Since $T\mathbb{R}^{n+k}$ is trivial, it is identified with the direct sum $\mathbb{R}^{n+k} \oplus \mathbb{R}^{n+k}$ on which we can define the flat metric $g(,)$ induced from $\langle , \rangle$. Moreover, we define the complex structure J by $J(X, Y) = (-Y, X)$ for $(X, Y) \in T_p\mathbb{R}^{n+k} \oplus T_u\mathbb{R}^{n+k}$ where $(p, u) \in \mathbb{R}^{n+k} \oplus \mathbb{R}^{n+k}$. By this identification, we regard $T\mathbb{R}^{n+k}$ as the complex Euclidean space $\mathbb{C}^{n+k}$ with the standard Kähler form $\omega := g(J\cdot, \cdot)$. Define the *normal bundle* of N by $\nu N := \{(p, u) \in T\mathbb{R}^{n+k};\ p \in N, u \perp T_pN\}$. This is an $(n+k)$-dimensional submanifold in $T\mathbb{R}^{n+k}$. We denote the Levi-Civita connections on $\mathbb{R}^{n+k}$ and $T\mathbb{R}^{n+k}$ by $\overline{\nabla}$ and $\tilde{\nabla}$, respectively. For a normal vector $u \in \nu_pN$ at $p \in N$, the *shape operator* $A^u \in \mathrm{End}(T_pN)$ is defined by $A^u(X) := -(\overline{\nabla}_X u)^\top$ for $X \in T_pN$, where $\top$ denotes the tangent component of the vector. Since A^u is represented by a symmetric matrix, the eigenvalues of A^u are real, and we denote it by $\kappa_i(p, u)$ for $i = 1, \ldots, n$. If u is a unit normal vector, these eigenvalues are called the *principal curvatures* of N with respect to the normal direction u.

Let $\{e_1, \ldots, e_n\}$ be a local field of tangent frames of N and $\{\nu_1, \ldots, \nu_k\}$ a local field of normal frames of N in $\mathbb{R}^{n+k}$. For a point $(p, u) \in \nu N$, choose a curve $\tilde{c}_i(t) = (c_i(t), u(c_i(t)))$ $(i = 1, \ldots, n)$ on νN such that $c_i(0) = p, \dot{c}_i(0) = e_i$ and $u(c_i(t))$ is the parallel transport of the initial normal vector u along $c_i(t)$ with respect to the normal connection $\nabla^\perp$. Then we have a tangent vector of νN at (p,u) by

$$E_i(p,u) := \frac{d}{dt}\Big|_{t=0} \tilde{c}_i(t) = (e_i, \overline{\nabla}_{e_i} u(c_i(t))) = (e_i, -A^u(e_i)).$$

On the other hand, we choose another curve $\tilde{c}_\alpha(t)$ $(\alpha = 1, \dots, k)$ by $\tilde{c}_\alpha = (p, u + t\nu_\alpha(p))$. Then we obtain another tangent vector of νM at (x,u) by

$$E_{n+\alpha}(p,u) := \frac{d}{dt}\Big|_{t=0} \tilde{c}_\alpha(t) = (0, \nu_\alpha(p))$$

Thus $\{E_1, \dots, E_n, E_{n+1}, \dots, E_{n+k}\}$ is a local frame field of tangent vectors of νN. In particular, one can check that $\omega(E_s, E_t) = 0$ for any $s, t = 1, \dots, n+k$, and this implies that νN is a Lagrangian submanifold, namely, $\omega|_{\nu N} = 0$ and dim $\nu N = n + k$.

Let L be an oriented Lagrangian submanifold in $\mathbb{C}^{n+k}$. Recall that the *Lagrangian angle* $\theta : L \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$ of L is defined by (see also Definition 2.1.13)

$$e^{\sqrt{-1}\theta(p)} = dz_1 \wedge \dots \wedge dz_{n+k}(e_1, \dots, e_{n+k})(p),$$

where $z_i = x_i + \sqrt{-1}y_i$ and $\{e_1, \dots, e_{n+k}\}$ is an oriented orthonormal basis of L. By Theorem 2.3.14, the mean curvature form $\alpha_{\tilde{H}} := \omega(\tilde{H}, \cdot)|_{TL}$ where $\tilde{H}$ is the mean curvature vector of L in $\mathbb{C}^{n+k}$ satisfies the relation

$$\alpha_{\tilde{H}} = -d\theta. \tag{3.1}$$

LEMMA 3.1.1. *Let N^n be an oriented submanifold in $\mathbb{R}^{n+k}$. Then the Lagrangian angle of the normal bundle νN in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$ is given by*

$$\theta(p,u) = -\sum_{i=1}^{n} \mathrm{Arctan}\kappa_i(p,u) + \frac{k\pi}{2} \pmod{2\pi}, \tag{3.2}$$

where $\mathrm{Arctan}\kappa_i(p,u)$ *denotes the principal value of* $\arctan \kappa_i(p,u)$.

PROOF. For arbitrary point $(p,u) \in \nu N$, we can choose a local frame $\{e_1, \dots, e_n\}$ of N so that $A^u(e_i)(p) = \kappa_i(p,u)e_i(p)$ for $i = 1, \dots, n$. Define an oriented local frame of νN by $\{E_1, \dots, E_{n+k}\}$ as above so that these are orthogonal frames at (p,u). Denote the normalized frames of these frames by $\{E'_1, \dots, E'_{n+k}\}$. Then we can show that

$$\begin{aligned} dz_1 \wedge \dots \wedge dz_{n+k}(E'_1, \dots, E'_{n+k})(p,u) &= (\sqrt{-1})^k \prod_{i=1}^{n} \frac{1 - \sqrt{-1}\kappa_i(p,u)}{\sqrt{1 + \kappa_i^2(p,u)}} \\ &= \exp\Big\{\sqrt{-1}\Big(-\sum_{i=1}^{n} \mathrm{Arctan}\kappa_i(p,u) + \frac{k\pi}{2}\Big)\Big\}. \end{aligned}$$

This proves the Lemma. $\square$

By the relation (3.1) and (3.2), the mean curvature form of the normal bundle can be written by

$$\alpha_{\tilde{H}} = d\Big(\sum_{i=1}^{n} \arctan \kappa_i\Big), \tag{3.3}$$

up to constant factor of the Lagrangian angle. For convenience, we put $\tilde{\theta} := \sum_{i=1}^{n} \arctan \kappa_i$.

Remark 3.1.2. The multivalued function $\tilde{\theta}$ is a smooth function on νN even though the eigenvalues $\{\kappa_i(p,u)\}_{i=1}^{n}$ of N are not smooth on νN in general. In fact, $\tilde{\theta}$ is expressed by the elementary symmetric polynomials with respect to the eigenvalues of the shape operator (see (3.4) in the below) and these polynomials are smooth on νN.

The following necessary and sufficient conditions for the minimality of normal bundles in $\mathbb{C}^{n+k}$ was first given by Harvey-Lawson [31]:

Proposition 3.1.3 (Theorem 3.11 in [31]). *Let N^n be a connected submanifold in $\mathbb{R}^{n+k}$. Then the normal bundle νN is a minimal Lagrangian submanifold in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$ if and only if N is austere, namely, the set of principal curvatures with their multiplicities is invariant under the multiplication by -1.*

Proof. Sufficiency is obvious. Assume νN is minimal, i.e., $\tilde{H} = 0$ on νN, then we have $\tilde{\theta} \equiv c$ on νN for some constant $c \in \mathbb{R}$. Since $\kappa_i(p,-u) = -\kappa_i(p,u)$ if $\kappa_i(p,u) \neq 0$, we may assume $c \equiv 0 \pmod{\pi}$. Suppose $n = 2m$ (it is similar when the case n is odd). Then we have

$$\sum_{i=1}^{2m} \arctan \kappa_i(p,u) = \arctan\Big(\frac{S_1 - S_3 + \cdots + (-1)^{k-1} S_{2m-1}}{1 - S_2 + S_4 \cdots + (-1)^k S_{2m}}\Big), \tag{3.4}$$

where $S_l = S_l(p,u)$ $(l = 1, \cdots, 2m)$ denotes the l-th elementary symmetric polynomial with respect to $\{\kappa_i(p,u)\}_{i=1}^{2m}$. Set $u = t\nu$ for an unit normal vector $\nu \in \nu_p N$. Since $c \equiv 0 \pmod{\pi}$ and $\kappa_i(p,u) = t\kappa_i(p,\nu)$, the equality (3.4) implies

$$tS_1(p,\nu) - t^3 S_3(p,\nu) + \ldots + (-1)^{2k-1} t^{2k-1} S_{2k-1}(p,\nu) = 0$$

for any $t \in \mathbb{R}$. Thus, we obtain $S_l(p,\nu) = 0$ for $l = 1, 3, \ldots, 2k-1$. Since $(p,\nu) \in \nu_p N$ is arbitrary, this implies N is an austere submanifold. $\square$

We remark that the minimality of a Lagrangian submanifold L in $\mathbb{C}^{n+k}$ is equivalent to that L is special Lagrangian (see Proposition 2.17 in [31]). In the latter case, L is a calibrated submanifold with respect to a calibration. In fact, by using this result, one can produce many examples of special Lagrangian submanifolds in $\mathbb{C}^{n+k}$ from austere submanifolds in $\mathbb{R}^{n+k}$.

By the explicit formulation of the Lagrangian angle of νN given in Lemma 3.1.1, we improve Harvey-Lawson's result a bit as follows:

Proposition 3.1.4. *Let N^n be a submanifold in $\mathbb{R}^{n+k}$. If the mean curvature vector of the normal bundle νN is parallel in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$, then νN is minimal.*

Proof. By the equality (3.3), $\tilde{\nabla}^{\perp}\tilde{H} = 0$ implies Hess $\tilde{\theta} = 0$. Choose a curve $\tilde{\gamma} : \mathbb{R} \to \nu N$ by $\tilde{\gamma}(t) := (p, tu)$ for $(p, u) \in \nu N$. Then the curve $\tilde{\gamma}$ is a geodesic, and hence we have $0 = \text{Hess } \tilde{\theta}(\tilde{\gamma}(t)) = \ddot{\tilde{\theta}}(\tilde{\gamma}(t))$. However, from $\tilde{\theta}(\tilde{\gamma}(t)) = \sum_{i=1}^{n} \arctan t\kappa_i(p, u)$, it follows $\tilde{\theta}(\tilde{\gamma}(t)) \equiv 0$ along $\tilde{\gamma}(t)$. Then, we see that N is austere by a similar argument as in Proposition 3.1.3. □

By Proposition 3.1.3 and 3.1.4, we obtain the following.

Corollary 3.1.5. *Let N^n be a submanifold in $\mathbb{R}^{n+k}$. Then the following three are equivalent: (i) N is austere, (ii) the normal bundle νN is minimal in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$, (iii) νN has parallel mean curvature vector.*

In the following, we investigate the H-minimality of a Lagrangian submanifold in the complex Euclidean space $\mathbb{C}^{n+k}$ obtained as the normal bundle of a submanifold N^n in $\mathbb{R}^{n+k}$. By (3.3), the H-minimality of the normal bundle νN in $\mathbb{C}^{n+k}$ is equivalent to

$$\Delta\tilde{\theta} = 0, \text{ where } \tilde{\theta} := \sum_{i=1}^{n} \arctan\kappa_i. \tag{3.5}$$

We recall that there are no non-minimal, H-minimal Lagrangian normal bundles in $\mathbb{C}^{n+k}$ with parallel mean curvature vector by Corollary 3.1.5.

Besides, the normal bundle of the Riemannian product $N_1 \times N_2 \to \mathbb{R}^{n_1+k_1} \times \mathbb{R}^{n_2+k_2}$ of two embeddings $N_i \to \mathbb{R}^{n_i+k_i}$ $(i = 1, 2)$ is H-minimal if and only if each of νN_i is H-minimal (see Proposition 2.1.11). Thus, in the following, our concern is always irreducible one.

3.2 H-minimality of the normal bundle over an isoparametric submanifold

In this section, we give a characterization of the H-minimality of the normal bundle over a homogeneous isoparametric submanifold in the Euclidean space. This is an essential part of the main results in this Chapter. First, we briefly review the isoparametric submanifolds in $\mathbb{R}^{n+k}$ (For more details, refer to [10], [104] and references therein).

Let N^n be a submanifold in $\mathbb{R}^{n+k}$ of an arbitrary codimension k. There are several ways to define the notion of isoparametric submanifolds (see [104]). In this article, we consider the following two conditions.

(i) For any parallel normal vector field $u(t)$ along a piece-wise smooth curve $c(t)$ on N, the shape operator $A^{u(t)}$ has constant eigenvalues.

(ii) The normal bundle of N is flat, namely, $R^{\perp} = 0$, where $R^{\perp}$ denotes the curvature tensor with respect to the normal connection of N.

If N satisfies the condition (i), we say N *has constant principal curvatures.* If N satisfies both conditions, we call N an *isoparametric submanifold.* It is known that any non-compact complete isoparametric submanifold is a product of compact isoparametric submanifolds and the Euclidean space (see [102]). Since the Euclidean factor is obviously austere, we may assume that an isoparametric submanifold N is compact for our purpose.

In the following, we consider an isoparametric submanifold N^n in $\mathbb{R}^{n+k}$. The classification in the case $k = 1$ is classical. See Subsection 3.2.2 below. When $k = 2$, the isoparametric submanifolds are known as *isoparametric hypersurfaces* in the unit sphere $S^{n+k-1}(1)$. See more details in Subsection 3.2.3 On the other hand, it was first proved by Thorbergsson [103] that any full, irreducible, isoparametric submanifold in $\mathbb{R}^{n+k}$ with $k \geq 3$ is extrinsically homogeneous. Note that Olmos [77] gives a simple and geometrical proof of this result. Moreover, combining it with the results of Dadok [24] and Palais-Terng [83], they are principal orbits of an s-representation, namely, an isotropy orbit of semi-simple symmetric space U/K. We discuss it in Subsection 3.2.4.

3.2.1 Lemmas on isoparametric submanifolds

In this subsection, using the same notation as before, we calculate $\delta\alpha_H$ for the normal bundle νN over an isoparametric submanifold N in order to investigate the H-minimality. Since N has flat normal bundle, the shape operators are simultaneously diagonalized. Hence, we can choose a local field of orthonormal tangent frames $\{e_1, \cdots, e_n\}$ around a point $p \in N$ such that $A^u(e_i)(p) = \kappa_i(p,u)e_i(p)$ for any $u \in \nu_p N$. In particular, we can define functions $\{\kappa_i(p,u)\}_{i=1}^n$ with respect to the valuable $u \in \nu_p N$ for each $p \in N$ in this way. We note that $\{\kappa_i(p,u)\}_{i=1}^n$ are linear functions on $\nu_p N$ by the isoparametric condition.

LEMMA 3.2.1. *Let N^n be an isoparametric submanifold in $\mathbb{R}^{n+k}$. For the normal bundle νN in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$, we have*

$$\Delta\tilde{\theta}(p,u) = \sum_{\alpha=n+1}^{n+k} \Big\{ \sum_{i=1}^{n} \frac{-2\kappa_i(p,u)\kappa_i^2(p,\nu_\alpha)}{(1+\kappa_i^2(p,u))^2} + \Big(\sum_{i=1}^{n} \frac{\kappa_i(p,\nu_\alpha)}{1+\kappa_i^2(p,u)} \Big) \Big(\sum_{j=1}^{n} \frac{\kappa_j(p,u)\kappa_j(p,\nu_\alpha)}{1+\kappa_j^2(p,u)} \Big) \Big\},$$

where $\{\nu_\alpha\}_{\alpha=n+1}^{n+k}$ is an orthonormal frame of $\nu_p N$.

PROOF. Take a local orthonormal tangent frames of N around p as above and a local orthogonal frames $\{E_i, E_\alpha\}$ of νN as before. We denote $\{E'_i, E'_\alpha\}$ the normalized frames of $\{E_i, E_\alpha\}$. Then we have

$$\begin{aligned}\Delta\tilde{\theta} &= \sum_{\lambda=1}^{n+k} g(\tilde{\nabla}_{E'_\lambda}\tilde{\nabla}\tilde{\theta}, E'_\lambda) = \sum_{\lambda=1}^{n+k}\sum_{\mu=1}^{n+k} g(\tilde{\nabla}_{E'_\lambda}(E'_\mu(\tilde{\theta})E'_\mu), E'_\lambda) \\ &= \sum_{\lambda=1}^{n+k} E'_\lambda E'_\lambda(\tilde{\theta}) + \sum_{\lambda=1}^{n+k}\sum_{\mu=1}^{n+k} E'_\mu(\tilde{\theta}) g(\tilde{\nabla}_{E'_\lambda} E'_\mu, E'_\lambda),\end{aligned} \tag{3.6}$$

where we denote the standard metric on $\mathbb{C}^{n+k}$ by $g(,)$. By Remark 3.1.2, we can express $\tilde{\theta}$ by

$$\tilde{\theta}(p,u) = \sum_{i=1}^{n} \arctan\langle A^u(e_i), e_i\rangle(p).$$

Then we see that

$$E'_i(\tilde{\theta})(z) = \sum_{i=1}^{n+k} \frac{1}{(1+\kappa_i^2(z))^{3/2}} E_i(\langle A^u(e_i), e_i\rangle)(z) = 0 \tag{3.7}$$

for $i = 1, \cdots n$, since N has constant principal curvatures. Moreover, we have

$$E'_\alpha(\tilde{\theta}) = \sum_{i=1}^{n} \frac{\kappa_i(p,\nu_\alpha)}{1+\kappa_i^2(p,u)}, \quad E'_\alpha E'_\alpha(\tilde{\theta}) = \sum_{i=1}^{n} \frac{-2\kappa_i(p,u)\kappa_i^2(p,\nu_\alpha)}{(1+\kappa_i^2(p,u))^2} \tag{3.8}$$

for $\alpha = n+1, \cdots, n+k$, since N has flat normal bundle. On the other hand, it is shown that

$$g(\tilde{\nabla}_{E'_j} E'_\alpha, E'_j) = \frac{\kappa_j(p,u)\kappa_j(p,\nu_\alpha)}{1+\kappa_j^2(p,u)}, \quad g(\tilde{\nabla}_{E'_\beta} E'_\alpha, E'_\beta) = 0, \tag{3.9}$$

for $j = 1, \cdots, n$ and $\beta = n+1, \cdots, n+k$. Substituting (3.7) thorough (3.9) into (3.6), we obtain the required equality. □

By Lemma 3.2.1, we have the following crucial condition for the H-minimality of normal bundles over isoparametric submanifolds which are contained in the sphere:

COROLLARY 3.2.2. *Let N^n be an isoparametric submanifold in $\mathbb{R}^{n+k}$ which is contained in a sphere S^{n+k-1}. If the normal bundle νN is H-minimal in $\mathbb{C}^{n+k} \simeq T\mathbb{R}^{n+k}$, then N satisfies the following equality:*

$$|B|^2(p) = s_N(p) \tag{3.10}$$

for any $p \in N$, where B is the second fundamental form of N in $\mathbb{R}^{n+k}$, and s_N is the scalar curvature of N with respect to the induced metric.

PROOF. Assume $\Delta\tilde{\theta} = 0$ on νN. Since N is contained in S^{n+k-1}, the position vector $\mathbf{p}$ of $p \in N$ is regarded as an unit normal vector of N in $\mathbb{R}^{n+k}$. Then, for a point $(p, \mathbf{p}) \in \nu N$, we have from Lemma 3.2.1,

$$\begin{aligned} 0 = \Delta\tilde{\theta}(p, t\mathbf{p}) &= \frac{t\kappa}{(1+t^2\kappa^2)^2} \sum_{\alpha=n+1}^{n+k} \Big\{ \sum_{i=1}^{n} 2\kappa_i^2(p, \nu_\alpha) - \Big(\sum_{i=1}^{n} \kappa_i(p, \nu_\alpha) \Big)^2 \Big\}, \\ &= \frac{t\kappa}{(1+t^2\kappa^2)^2} \Big\{ 2|B|^2(p) - |H|^2(p) \Big\} \end{aligned}$$

for $t \in \mathbb{R}$, where H is the mean curvature vector of N in $\mathbb{R}^{n+k}$, and $\kappa = \kappa_i(p, \mathbf{p})$ for $i = 1, \ldots, n$, since S^{n+k-1} is totally umbilic. Therefore, we have $2|B|^2(p) - |H|^2(p) = 0$. By the Gauss equation, this is equivalent to (3.10). $\square$

REMARK 3.2.3. For a complete isoparametric submanifold N in $\mathbb{R}^{n+k}$, the condition that N is contained in the unit sphere is equivalent to the compactness of N (see Corollary 5.2.10 in [10]).

REMARK 3.2.4. By carefully looking at the proofs of Lemma 3.2.1 and Corollary 3.2.2, one can check that the statement of Corollary 3.2.2 holds when N satisfies only (i), namely, N is a submanifold with constant principal curvatures. We note that not only any principal orbit of adjoint representation of a compact semi-simple Lie group G in $\mathfrak{g}$, but also some singular orbits satisfy (3.10) (see Lemma 2.2.13).

3.2.2 Isoparametric hypersurfaces in $\mathbb{R}^{n+1}$

The isoparametric hypersurfaces in $\mathbb{R}^{n+1}$ are classified by Levi-Civita [57] for $n = 3$, and Segre [90] for the general dimension. We denote the number of distinct principal curvatures by g. Then g is at most two, and an isoparametric hypersurface in $\mathbb{R}^{n+1}$ is one of the following:

$g = 1$: An affine hyperplane $\mathbb{R}^n$ or a hypersphere $S^n(r)$, where $r > 0$.

$g = 2$: A spherical cylinder $\mathbb{R}^k \times S^{n-k}(r)$, i.e., a tube around an affine plane $\mathbb{R}^k$, where $r > 0$.

By the classification, it is obvious that an isoparametric hypersurface N^n in $\mathbb{R}^{n+1}$ is austere if and only if it is an open part of an affine hyperplane. On the other hand, the irreducible (or compact) one is the hypersphere. Then we have the following:

PROPOSITION 3.2.5. *The normal bundle of the n-dimensional hypersphere $N^n = S^n(r)$ with radius $r > 0$ in $\mathbb{R}^{n+1}$ is H-minimal if and only if $n = 2$.*

One can prove this by a direct calculation by using the formula in Lemma 3.2.1. However, we shall prove more general statement Theorem 3.3.1 in Section 3.3, and we omit the direct proof. We note that the normal bundles of 2-spheres $S^2(r)$ with $r > 0$ in $\mathbb{R}^3$ generate an 1-parameter family of ruled H-minimal Lagrangian submanifolds in $\mathbb{C}^3$.

3.2.3 Isoparametric hypersurfaces in $S^{n+1}(1)$

The codimension 2 isoparametric submanifolds in $\mathbb{R}^{n+2}$ are known as isoparametric hypersurfaces in the unit sphere $S^{n+1}(1)$. One of large subclass of these hypersurfaces are extrinsically homogeneous hypersurfaces in $S^{n+1}(1)$ and these are classified by Hsiang-Lawson [31]. This result asserts that all homogeneous hypersurfaces in $S^{n+1}(1)$ are obtained by principal orbits of s-representations of compact symmetric spaces of rank 2 (see also the next subsection). Other classes includes infinitely many non-homogeneous examples due to Ozeki-Takeuchi and Ferus-Karcher-Münzner. These are the so called isoparametric hypersurfaces of *OT-FKM type* (for more details, refer to monographs [19], [104] and references therein). The classification of isoparametric hypersurfaces in $S^{n+1}(1)$ have not been completed yet.

Let N be an isoparametric hypersurface in the unit sphere $S^{n+1}(1)$, and ν the unit normal vector field on N. We denote the distinct principal curvatures of N with respect to ν by $\kappa_i = \cot\theta_i$ with $0 < \theta_1 < \cdots < \theta_g < \pi$, and these multiplicities by m_i for $i = 1, \ldots, g$, respectively. Then, Münzner showed the following [63]:

$$\theta_i = \theta_1 + \frac{i-1}{g}\pi, \text{ for } i = 1, \ldots, g, \tag{3.11}$$

$$m_i = m_{i+2}, \text{ modulo } g \text{ indexing.} \tag{3.12}$$

In particular, $0 < \theta_1 < \pi/g$, and the multiplicities are same if g is odd. Münzner also proved that the number of distinct principal curvatures g is equal to $1, 2, 3, 4$ or 6 [64].

The unit normal vector ν and the position vector $\mathbf{p}$ of $p \in N$ are regarded as orthonormal frames of the normal space of N in $\mathbb{R}^{n+2}$. Let $\{e_1, \cdots, e_n\}$ be a local field of orthonormal frames of N in S^{n+1} such that $\overline{A}^{\nu}(e_i) = \kappa_i e_i$, where $\overline{A}$ is the shape operator of N in S^{n+1}. Since the unit sphere is totally umbilic, we have $A^{\nu}(e_i) = \overline{A}^{\nu}(e_i) = \kappa_i e_i$ and $A^{\mathbf{p}}(e_i) = -e_i$. Thus, the principal curvatures of N in $\mathbb{R}^{n+2}$ are given by $\kappa_1, \ldots, \kappa_n$, and -1. The multiplicities of the principal curvatures $\kappa_1, \ldots, \kappa_n$ are $m_1, \ldots, m_g$ as before, and the multiplicity of the principal curvature -1 is equal to n. In particular, N is never austere in $\mathbb{R}^{n+2}$, and hence, the normal bundle is not a minimal submanifold in $T\mathbb{R}^{n+2} \simeq \mathbb{C}^{n+2}$ by Proposition 3.1.3. The following lemma is crucial in our argument.

Lemma 3.2.6. *Let N^n be an isoparametric hypersurface in the unit sphere $S^{n+1}(1) \subset \mathbb{R}^{n+2}$. Suppose that the normal bundle νN of N as a submanifold in $\mathbb{R}^{n+2}$ is H-minimal* *in $\mathbb{C}^{n+2} \simeq T\mathbb{R}^{n+2}$. Then the multiplicities of the distinct principal curvatures in $\{\kappa_i\}_{i=1}^n$ are all equal to 2. In particular, N is a homogeneous hypersurface in $S^{n+1}(1)$.*

PROOF. By Corollary 3.2.2, an isoparametric hypersurface N^n in $S^{n+1}(1)$ must satisfy the following equality.

$$2|\overline{B}|^2 - |\overline{H}|^2 = n(n-2), \tag{3.13}$$

where $\overline{B}$ and $\overline{H}$ are the second fundamental form and the mean curvature vector of N^n in S^{n+1}, respectively. By (3.11) and (3.12), the length of $\overline{B}$ and the mean curvature $\overline{h} := \sum_{i=1}^n \kappa_i$ of N in S^{n+1} are given as follows (cf. [63]):

$$|\overline{B}|^2 = \begin{cases} gm\{g\cot^2(g\theta_1) + g - 1\}, \text{ if } g \text{ is odd}, \\ \frac{g}{2}m_1\Big\{\frac{g}{2}\cot^2\Big(\frac{g}{2}\theta_1\Big) + \frac{g}{2} - 1\Big\} + \frac{g}{2}m_2\Big\{\frac{g}{2}\tan^2\Big(\frac{g}{2}\theta_1\Big) + \frac{g}{2} - 1\Big\}, \text{ if } g \text{ is even}, \end{cases}$$

$$\overline{h} = \begin{cases} gm\cot(g\theta_1), \text{ if } g \text{ is odd}, \\ \frac{g}{2}m_1\cot\Big(\frac{g}{2}\theta_1\Big) - \frac{g}{2}m_2\tan\Big(\frac{g}{2}\theta_1\Big), \text{ if } g \text{ is even}, \end{cases}$$

where $gm = n$ if g is odd, and $(g/2)(m_1 + m_2) = n$ if g is even. By these relations, we see that the equality (3.13) is equivalent to the following equality:

$$\begin{cases} m(m-2)\csc^2(g\theta_1) = 0, \text{if } g \text{ is odd}, \\ m_1(m_1-2)\csc^2\Big(\frac{g}{2}\theta_1\Big) + m_2(m_2-2)\sec^2\Big(\frac{g}{2}\theta_1\Big) = 0, \text{if } g \text{ is even}. \end{cases} \tag{3.14}$$

If g is odd, (3.14) implies $m = 2$. If g is even, (3.14) implies that (i) $m_1 = m_2 = 2$ or (ii) $m_i = 1$ and $m_j > 2$ with $i \neq j$. We shall show that (ii) is not the case. When $g = 6$, it is known that the multiplicities are same (see [63]) which takes values $m = 1$ or 2, and only the latter case is possible by (3.14). Hence, it is sufficient to consider the cases $g = 2$ and 4. First, we have from Lemma 3.2.1,

$$0 = \Delta\tilde{\theta}(p, s\nu) = \sum_{i=1}^n \frac{-2(1+\kappa_i^2)\kappa_i}{(1+s^2\kappa_i^2)^2}s + \Big(\sum_{i=1}^n \frac{\kappa_i}{1+s^2\kappa_i^2}\Big)\Big(\sum_{j=1}^n \frac{1+\kappa_j^2}{1+s^2\kappa_j^2}\Big)s \tag{3.15}$$

for the unit normal vector ν and any $s \in \mathbb{R}$.

The case $g = 2$: Substituting $s = 1$ in (3.15), we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^n \frac{-2\kappa_i}{1+\kappa_i^2} + \Big(\sum_{i=1}^n \frac{\kappa_i}{1+\kappa_i^2}\Big)n = (n-2)\Big(\sum_{i=1}^n \frac{\kappa_i}{1+\kappa_i^2}\Big) \\ &= \frac{1}{2}(n-2)(m_1\sin 2\theta_1 + m_2\sin 2\theta_2) = \frac{1}{2}(n-2)(m_1 - m_2)\sin 2\theta_1, \end{aligned} \tag{3.16}$$

where we use the relation $\theta_2 = \theta_1 + \pi/2$. Since $0 < \theta_1 < \pi/2$, the equality (3.16) implies $n = 2$ or $m_1 = m_2$. However, if $n = 2$, then $m_1 = m_2 = 1$. Consequently, the multiplicities are same. Therefore (ii) is not the case.

The case $g = 4$: For $i = 1, 2$, we have

$$\begin{aligned}
\frac{(1+\kappa_i^2)\kappa_i}{(1+s^2\kappa_i^2)^2} + \frac{(1+\kappa_{i+2}^2)\kappa_{i+2}}{(1+s^2\kappa_{i+2}^2)^2} &= \frac{\sin 4\theta_i(1-s^4)}{4P_i^2} \\
\frac{\kappa_i}{1+s^2\kappa_i^2} + \frac{\kappa_{i+2}}{1+s^2\kappa_{i+2}^2} &= \frac{\sin 4\theta_i(1-s^2)}{4P_i} \\
\frac{1+\kappa_i^2}{1+s^2\kappa_i^2} + \frac{1+\kappa_{i+2}^2}{1+s^2\kappa_{i+2}^2} &= \frac{1+s^2}{P_i},
\end{aligned}$$

where we put $P_i := (\sin^2\theta_i + s^2\cos^2\theta_i)(\cos^2\theta_i + s^2\sin^2\theta_i)$. Moreover, we have $\sin 4\theta_2 = -\sin 4\theta_1$. Thus, we obtain from (3.15)

$$\begin{aligned}
0 &= \left\{\left(-\frac{m_1}{2P_1^2} + \frac{m_2}{2P_2^2}\right) + \left(\frac{m_1}{4P_1} - \frac{m_2}{4P_2}\right)\left(\frac{m_1}{P_1} + \frac{m_2}{P_2}\right)\right\}\sin 4\theta_1(1-s^4)s \qquad (3.17)\\
&= \left\{\frac{m_1(m_1-2)}{4P_1^2} - \frac{m_2(m_2-2)}{4P_2^2}\right\}\sin 4\theta_1(1-s^4)s
\end{aligned}$$

for any $s \in \mathbb{R}$. Since $\sin 4\theta_1 \neq 0$, the equality (3.17) implies that the case (ii) $m_i = 1$ and $m_j > 2$ with $i \neq j$ does not occur.

When the multiplicities of distinct principal curvatures in $\{\kappa_i\}_{i=1}^n$ are equal to 2, the isoparametric hypersurface N in S^{n+1} is a homogeneous hypersurface. In fact, E. Cartan proved this for $g \leq 3$, and Ozeki-Takeuchi for the case $(g, m) = (4, 2)$ [82] . The remaining case $(g, m) = (6, 2)$ was settled by R. Miyaoka [62]. This completes the proof. □

More precisely, by the classification theorem of homogeneous hypersurfaces in S^{n+1} due to Hsiang-Lawson [34], an isoparametric hypersurfaces N in $S^{n+1}(1)$ with constant multiplicity 2 is a principal orbit of an isotropy representation of one of the Riemannian symmetric pair (U, K) in Table 3.1.

g	(U, K)	$N \simeq K/K_0$	dim N	N_+	N_-
1	$(S^1 \times SO(4), SO(3))$	S^2	2	$\{pt\}$	$\{pt\}$
2	$(SO(4) \times SO(4), SO(3) \times SO(3))$	$S^2 \times S^2$	4	S^2	S^2
3	$(SU(3) \times SU(3), SU(3))$	$SU(3)/T^2$	6	$\mathbb{C}P^2$	$\mathbb{C}P^2$
4	$(SO(5) \times SO(5), SO(5))$	$SO(5)/T^2$	8	$\mathbb{C}P^3$	$\mathbb{Q}^3$
6	$(G_2 \times G_2, G_2)$	G_2/T^2	12	$\mathbb{Q}^5$	$\mathbb{Q}^5$

Table 3.1: Isoparametric hypersurfaces in S^{n+1} with constant multiplicity 2, where $N_\pm$ are the singular orbits (these are so called *focal manifolds* of N). $\mathbb{Q}^n$ denotes the complex hyperquadric.

Note that the 2-spheres S^2 in $\mathbb{R}^4$ are not full, and $S^2 \times S^2$ in $\mathbb{R}^6$ are reducible. In terms of the representation, these isotropy representations are reducible. However, the irreducible component is isometric to 2-sphere in $\mathbb{R}^3$ in both cases, and hence, the H-minimality of these examples essentially follows from Proposition 3.2.5.

On the other hand, it is shown that other hypersurfaces in Table 1 are complex flag manifolds standardly embedded in the Lie algebra $\mathfrak{k}$ of K as adjoint orbits of K. In Section 3.3, we show that these isoparametric hypersurfaces give a family of non-minimal, H-minimal Lagrangian submanifolds in $\mathbb{C}^{n+k}$.

3.2.4 Principal orbits of s-representations

Let N be a full, irreducible isoparametric submanifold in the Euclidean space. By virtue of Lemma 3.2.6 and the homogeneity of the case codimension is grater than or equal to 3, we may consider principal orbits of an s-representation. First we recall some general arguments for orbits of s-representation. We refer to [58] and [94] (see also [33] and [40]).

Let (U, K) be a Riemannian symmetric pair of compact type, where U is a compact, connected real semi-simple Lie group and K a closed subgroup of U such that there exist an involutive automorphism σ of U so that $\mathrm{Fix}(\sigma, U)^0 \subset K \subset \mathrm{Fix}(\sigma, U)$, where $\mathrm{Fix}(\sigma, U) := \{g \in U;\ \sigma(g) = g\}$ and $\mathrm{Fix}(\sigma, U)^0$ is the identity component of $\mathrm{Fix}(\sigma, U)$. Denote the Lie algebra of U and K by $\mathfrak{u}$ and $\mathfrak{k}$, respectively. Let $(\mathfrak{u}, \sigma)$ be the orthogonal symmetric Lie algebra which corresponds to (U, K), namely, σ is an involution on $\mathfrak{u}$ such that the $+1$-eigenspace coincides with $\mathfrak{k}$ and $\mathfrak{k}$ is a compactly embedded Lie algebra in $\mathfrak{u}$.

We take an inner product $\langle , \rangle$ of $\mathfrak{u}$ which is invariant under σ and $\mathrm{Ad}(U)$ on $\mathfrak{u}$. Then we have the orthogonal decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$. Since the subspace $\mathfrak{p}$ is invariant under $\mathrm{Ad}(K)|_{\mathfrak{p}}$, K acts on $\mathfrak{p}$ as an orthogonal transformation. We call this action of K the *s-representation* of the symmetric pair (U, K).

Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. For an 1-form λ on $\mathfrak{a}$, set

$$\begin{aligned}
\mathfrak{k}_\lambda &:= \{X \in \mathfrak{k};\ (\mathrm{ad}H)^2 X = -\lambda(H)^2 X \text{ for all } H \in \mathfrak{a}\},\\
\mathfrak{p}_\lambda &:= \{X \in \mathfrak{p};\ (\mathrm{ad}H)^2 X = -\lambda(H)^2 X \text{ for all } H \in \mathfrak{a}\}.
\end{aligned}$$

Then $\mathfrak{p}_{-\lambda} = \mathfrak{p}_\lambda$, $\mathfrak{k}_{-\lambda} = \mathfrak{k}_\lambda$, $\mathfrak{p}_0 = \mathfrak{a}$, and $\mathfrak{k}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. A non-zero 1-form λ is called a *root* of $(\mathfrak{u}, \sigma)$ with respect to $\mathfrak{a}$ if $\mathfrak{p}_\lambda \neq \{0\}$. We denote the set of all roots of $(\mathfrak{u}, \sigma)$ by R, and call R the *restricted root system* on $\mathfrak{a}$. We take a basis of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ and define the lexicographic ordering on $\mathfrak{a}^*$ with respect to the basis. We call a root $\lambda \in R$ a *positive root* if $\lambda > 0$, and put $R_+ := \{\lambda \in R;\ \lambda > 0\}$. Then we have restricted root space decompositions

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in R_+} \mathfrak{k}_\lambda, \quad \mathfrak{p} = \mathfrak{a} + \sum_{\lambda \in R_+} \mathfrak{p}_\lambda. \tag{3.18}$$

These are orthogonal direct sums with respect to $\langle , \rangle$. We put $m_\lambda := \dim_{\mathbb{R}} \mathfrak{p}_\lambda$, and call it the *multiplicity* of $\lambda \in R_+$.

Let us consider orbits of the s-representation. Since any s-representation is polar (see [10]) and the section is given by $\mathfrak{a}$, it is sufficient to consider the orbits through a point $w \in \mathfrak{a}$. The point w is called a *regular* element if $\lambda(w) \neq 0$ for any $\lambda \in R$ (otherwise, it is called *singular*). We note that regular orbits are orbits of maximal dimension [94]. Since the isotropy action does not have any exceptional orbits, an orbit is regular if and only if it is principal.

When w is a regular element, we have the following:

LEMMA 3.2.7 (cf. [94]). *Let $w \in \mathfrak{a}$ be a regular element, and N_w the orbit through w in $\mathfrak{p}$. Then N_w has the following properties:*

(1) The tangent space $T_w N_w$ and the normal space $\nu_w N_w$ of N_w at w in $\mathfrak{p}$ are given by

$$T_w N_w = \sum_{\lambda \in R_+} \mathfrak{p}_\lambda, \quad \nu_w N_w = \mathfrak{a}.$$

In particular, $\mathrm{codim} N_w = \dim \mathfrak{a}$.

(2) The shape operator A^u of N_w in $\mathfrak{p}$ in the direction $u \in \nu_w N_w$ satisfies

$$A^u(X_\lambda) = -\frac{\lambda(u)}{\lambda(w)} X_\lambda \text{ for } X_\lambda \in \mathfrak{p}_\lambda \text{ and } \lambda \in R_+.$$

(2) in the lemma shows that the shape operators of N_w in $\mathfrak{p}$ are simultaneously diagonalized, and the eigenvalues are given by $\{-\lambda(u)/\lambda(w)\}_{\lambda \in R_+}$ for $u \in \nu_w N_w$. In particular, N_w has the flat normal bundle by the Ricci equation. Moreover, since N_w has constant principal curvatures, it is an isoparametric submanifold.

In general, there might exist an element $u \in \nu_w N$ such that not all eigenvalues in $\{-\lambda(u)/\lambda(w)\}_{\lambda \in R_+}$ are distinct. To distinguish such an element, we set up a subset on $\nu_w N$. For the root system R, we set

$$r := \{\lambda \in R;\ \lambda/2 \notin R\}, \text{ and } r_+ := r \cap R_+.$$

Then r is a reduced root system, namely, if two roots $\lambda, \mu \in r$ are proportional, then $\mu = \pm\lambda$. We also set $l_\lambda := m_\lambda + m_{2\lambda}$, where $m_{2\lambda} = 0$ unless $2\lambda \in r$. Then, the vector subspace $V_\lambda := \mathfrak{p}_\lambda + \mathfrak{p}_{2\lambda}$ is the common eigenspace of the eigenvalue $-\lambda(u)/\lambda(w)$ for $u \in \nu_w N_w$ with $\dim_{\mathbb{R}} V_\lambda = l_\lambda$. Put

$$\mathcal{R} := \{(\lambda, \mu) \in r_+ \times r_+;\ \lambda \neq \mu\}.$$

We define a subset of $\mathfrak{a} = \nu_w N_w$ by

$$\mathcal{U} := \Big\{u \in \nu_w N_w;\ -\frac{\lambda(u)}{\lambda(w)} \neq -\frac{\mu(u)}{\mu(w)} \text{ for any } (\lambda, \mu) \in \mathcal{R}\Big\},$$

namely, $\mathcal{U}$ is the set of normal vectors such that the eigenvalues are distinct for each of eigenspaces V_λ.

LEMMA 3.2.8. *For the regular orbit N_w through a regular element $w \in \mathfrak{p}$, the subset $\mathcal{U}$ is dense in $\mathfrak{a}$.*

PROOF. Since w is a regular element, it is shown that

$$-\frac{\lambda(u)}{\lambda(w)} = -\frac{\mu(u)}{\mu(w)} \text{ if and only if } \lambda \wedge \mu(w, u) = 0.$$

Thus we obtain

$$\mathcal{U} = \mathfrak{a} \setminus \bigcup_{(\lambda,\mu)\in\mathcal{R}} \mathrm{Ker}(w \rfloor (\lambda \wedge \mu)). \tag{3.19}$$

Since the root system r is reduced, two distinct positive roots $\lambda, \mu \in r_+$ are linearly independent. Then we have $w\rfloor(\lambda \wedge \mu) = \lambda(w)\mu - \mu(w)\lambda \neq 0$ as w is a regular element. Thus, $\mathrm{Ker}(w\rfloor(\lambda \wedge \mu))$ defines a subspace of $\mathfrak{a}$ which does not coincide with $\mathfrak{a}$. Therefore, by (3.19) and since $\mathcal{R}$ is a finite set, the complement of $\mathcal{U}$ has no interior point in $\mathfrak{a}$. This implies the lemma. $\square$

It is useful to divide the set $\mathcal{R}$ into some disjoint subsets. In the following, we refer to Chapter V in [91] for the general facts of the reduced root system. For $(\lambda, \mu) \in \mathcal{R}$, the angle $\angle(\lambda, \mu)$ of (λ, μ) is defined by $\langle \lambda, \mu \rangle = |\lambda||\mu| \cos \angle(\lambda, \mu)$ with $0 < \angle(\lambda, \mu) < \pi$. Define subsets of $\mathcal{R}$ as follows:

If $\angle(\lambda, \mu) = \pi/2$ and $|\lambda| = |\mu|$, then we set

$$O(\lambda; \mu) := \{(\lambda, \mu)\}.$$

If $\angle(\lambda, \mu) = 2\pi/3$ and $|\lambda| = |\mu|$, then we set

$$A(\lambda; \mu) := \{(\lambda, \mu), (\mu, \lambda + \mu), (\lambda + \mu, \lambda)\}.$$

If $\angle(\lambda, \mu) = 3\pi/4$ and $|\lambda| = (1/\sqrt{2})|\mu|$, then we set

$$\begin{aligned} B_1(\lambda; \mu) &:= \{(\lambda, \mu), (\mu, \lambda + \mu), (\lambda + \mu, 2\lambda + \mu), (2\lambda + \mu, \lambda)\}, \\ B_2(\lambda; \mu) &:= \{(\mu, \lambda), (\lambda, 2\lambda + \mu), (2\lambda + \mu, \lambda + \mu), (\lambda + \mu, \mu)\}. \end{aligned}$$

If $\angle(\lambda, \mu) = 5\pi/6$ and $|\lambda| = (1/\sqrt{3})|\mu|$, then we set

$$\begin{aligned} G_1(\lambda; \mu) &:= \{(\lambda, \mu), (\mu, \lambda + \mu), (\lambda + \mu, 3\lambda + 2\mu), (3\lambda + 2\mu, 2\lambda + \mu), \\ &\quad (2\lambda + \mu, 3\lambda + \mu), (3\lambda + \mu, \lambda)\}. \\ G_2(\lambda; \mu) &:= \{(\mu, \lambda), (\lambda, 3\lambda + \mu), (3\lambda + \mu, 2\lambda + \mu), (2\lambda + \mu, 3\lambda + 2\mu), \\ &\quad (3\lambda + 2\mu, \lambda + \mu), (\lambda + \mu, \mu)\}. \end{aligned}$$

These subsets are well-defined, namely, each of elements is the pair of positive roots. We call these subsets the *cyclic subsets* of $\mathcal{R}$. We note that the cyclic subsets $G_i(\lambda;\mu)$ $(i=1,2)$ are contained in $\mathcal{R}$ if and only if the reduced root system r is isomorphic to the root system of the exceptional simple Lie group G_2. The following lemma follows from an argument of the reduced root system (A proof is given in Appendix A.2).

LEMMA 3.2.9. *$\mathcal{R}$ is a disjoint union of cyclic subsets.*

We need the following:

LEMMA 3.2.10. *Define maps $l : \mathcal{R} \to \mathbb{R}$ and $\Psi : \mathcal{R} \setminus \bigcup O(\lambda,\mu) \to \Lambda^2(\mathfrak{a})$ by*

$$l(\lambda,\mu) := l_\lambda l_\mu \text{ and } \Psi(\lambda,\mu) := \Psi_{\lambda,\mu} := \frac{\lambda \wedge \mu}{\langle \lambda,\mu\rangle}.$$

Then l and Ψ take the same value on each cyclic subset.

PROOF. The assertion on the map Ψ follows from a direct calculation. We consider the map l. It is known that, for two restricted roots $\lambda,\mu \in R$ with the same length, there exists an element s of the Weyl group such that $\mu = s\lambda$. Moreover, we have $m_\lambda = m_\mu$ (cf. [40]). Since the Weyl group acts on the root system, $2\mu \in R$ whenever $2\lambda \in R$, and these length are also the same. Therefore, we have $l_\lambda = l_\mu$. On the other hand, if $(\alpha,\beta) \in \mathcal{R}$ belongs to $O(\lambda;\mu)$ or $A(\lambda;\mu)$, then $|\alpha| = |\beta|$. If (α,β) belongs to $B_i(\lambda;\mu)$ or $G_i(\lambda;\mu)$ $(i=1,2)$, then $(|\alpha|,|\beta|) = (|\lambda|,|\mu|)$ or $(|\mu|,|\lambda|)$. This implies $l_\alpha l_\beta = l_\lambda l_\mu$ on each of cyclic subsets. $\square$

REMARK 3.2.11. If N^n is a homogeneous isoparametric hypersurface in $S^{n+1}(1)$, it is shown that

$$\Psi_{\lambda_i,\lambda_j}(w,\nu)^{-1} = \frac{1+\kappa_i\kappa_j}{\kappa_i - \kappa_j} = \cot(\theta_j - \theta_i),$$

where we use the same notation as in the subsection 3.2.3 and set $\kappa_i = \cot\theta_i = -\lambda_i(\nu)/\lambda_i(w)$ for $i = 1,\dots,g$. It is known that the quantity $\cot(\theta_j - \theta_i) = (1+\kappa_i\kappa_j)/(\kappa_i - \kappa_j)$ coincides with a principal curvature of the focal manifolds $N_\pm$ of N. A geometrical consequence from the assertion of Lemma 3.2.10 for the map Ψ is that every focal manifold N_i of N focalized by the eigenspace V_{λ_i} has the same principal curvatures (see also Chapter 5 in [10]).

Now, we give a characterization of the H-minimality of the normal bundle over the regular orbit of an s-representation.

PROPOSITION 3.2.12. *Let $N^n = N_w$ be a regular orbit of an s-representation through an element $w \in \mathfrak{p} \simeq \mathbb{R}^{n+k}$. Then the normal bundle νN is H-minimal in $T\mathfrak{p} \simeq \mathbb{C}^{n+k}$ if and only if $l_\lambda = 2$ for all $\lambda \in r_+$.*

PROOF. For any normal vector $u \in \nu_w N$, we put $\kappa_\lambda(w,u) := -\lambda(u)/\lambda(w)$ for $\lambda \in r_+$. Then, by Lemma 3.2.1, we have

$$\begin{aligned}
\Delta\tilde{\theta}(w,u) &= \sum_{\alpha=n+1}^{n+k} \Big\{ \sum_{\lambda\in r_+} (-2l_\lambda) \frac{\kappa_\lambda(w,u)\kappa_\lambda^2(w,\nu_\alpha)}{(1+\kappa_\lambda^2(w,u))^2} \\
&\quad + \Big(\sum_{\lambda\in r_+} l_\lambda \frac{\kappa_\lambda(w,\nu_\alpha)}{1+\kappa_\lambda^2(w,u)} \Big) \Big(\sum_{\mu\in r_+} l_\mu \frac{\kappa_\mu(w,u)\kappa_\mu(w,\nu_\alpha)}{1+\kappa_\mu^2(w,u)} \Big) \Big\} \\
&= \sum_{\alpha=n+1}^{n+k} \sum_{\lambda\in r_+} l_\lambda(l_\lambda-2) \frac{\kappa_\lambda(w,u)\kappa_\lambda^2(w,\nu_\alpha)}{(1+\kappa_\lambda^2(w,u))^2} \qquad (3.20) \\
&\quad + \sum_{\alpha=n+1}^{n+k} \sum_{(\lambda,\mu)\in\mathcal{R}} l_\lambda l_\mu \frac{\kappa_\lambda(w,\nu_\alpha)\kappa_\mu(w,\nu_\alpha)\kappa_\mu(w,u)}{\{1+\kappa_\lambda^2(w,u)\}\{1+\kappa_\mu^2(w,u)\}}.
\end{aligned}$$

Note that

$$\sum_{\alpha=n+1}^{n+k} \kappa_\lambda(w,\nu_\alpha)\kappa_\mu(w,\nu_\alpha) = \frac{\langle\lambda,\mu\rangle}{\lambda(w)\mu(w)} \qquad (3.21)$$

for any $\lambda,\mu \in r_+$. We denote the first and the second term of (3.20) by $\mathrm{I}(w,u)$ and $\mathrm{II}(w,u)$, respectively. We first assert that $\mathrm{II}(w,u)$ vanishes for any $u \in \nu_w N$. Take an element $u \in \mathcal{U}$, then we have

$$\begin{aligned}
\mathrm{II}(w,u) &= \frac{1}{2} \sum_{\alpha=n+1}^{n+k} \sum_{(\lambda,\mu)\in\mathcal{R}} l_\lambda l_\mu \frac{\kappa_\lambda(w,\nu_\alpha)\kappa_\mu(w,\nu_\alpha)\{\kappa_\lambda(w,u)+\kappa_\mu(w,u)\}}{\{1+\kappa_\lambda^2(w,u)\}\{1+\kappa_\mu^2(w,u)\}}. \\
&= -\frac{1}{2} \sum_{(\lambda,\mu)\in\mathcal{R}} l_\lambda l_\mu \frac{\langle\lambda,\mu\rangle/\lambda(w)\mu(w)}{\kappa_\lambda(w,u)-\kappa_\mu(w,u)} \Big\{ \frac{1}{1+\kappa_\lambda^2(w,u)} - \frac{1}{1+\kappa_\mu^2(w,u)} \Big\} \\
&= -\frac{1}{2} \sum_{\substack{(\lambda,\mu)\in\mathcal{R} \\ \langle\lambda,\mu\rangle\neq 0}} l_\lambda l_\mu \Psi_{\lambda,\mu}(w,u)^{-1} \Big\{ \frac{1}{1+\kappa_\lambda^2(w,u)} - \frac{1}{1+\kappa_\mu^2(w,u)} \Big\}, \qquad (3.22)
\end{aligned}$$

where we assume $\kappa_\lambda(w,u) \neq \kappa_\mu(w,u)$ for any $\lambda,\mu \in r_+$ with $\lambda \neq \mu$ since $u \in \mathcal{U}$. By virtue of Lemma 3.2.9 and 3.2.10, the right hand side of (3.22) is the alternating sums for the elements of each of cyclic subsets, and hence, $\mathrm{II}(w,u)$ vanishes on $\mathcal{U}$. However, since $\mathrm{II}(w,u)$ is a continuous function on $\mathfrak{a} = \nu_w N$, and since $\mathcal{U}$ is dense in $\mathfrak{a}$ by Lemma 3.2.8, we conclude $\mathrm{II}(w,u) = 0$ on $\mathfrak{a} = \nu_w N_w$. Therefore, by (3.20) and (3.21), we obtain

$$\Delta\tilde{\theta}(w,u) = \mathrm{I}(w,u) = \sum_{\lambda\in r_+} l_\lambda(l_\lambda-2) \frac{|\lambda|^2}{\lambda(w)^2} \frac{\kappa_\lambda(w,u)}{(1+\kappa_\lambda^2(w,u))^2} \qquad (3.23)$$

on $\nu_w N$. Therefore, if $l_\lambda = 2$ for all $\lambda \in r_+$, then $\Delta\tilde{\theta} = 0$ on $\nu_w N$. On the other hand, we have an isometric action of K on νN which is naturally induced from the action of K on

N, i.e., $k \cdot (w, u) := (\mathrm{Ad}(k)w, \mathrm{Ad}(k)u)$ for $k \in K$. Since $\Delta\tilde{\theta} = \delta\alpha_{\tilde{H}}$ is invariant under the action, this implies $\Delta\tilde{\theta} = 0$ on νN. Therefore, νN is H-minimal.

We shall show the converse. Assume that $\Delta\tilde{\theta} = 0$ on νN. Let $\mathcal{W}$ be a Weyl chamber of the root system R. By Lemma 3.2.8, we see that $\mathcal{U} \cap \mathcal{W} \neq \{\phi\}$. Take a normal vector $u_0 \in \mathcal{U} \cap \mathcal{W}$, and set $u(s) := su_0 + w \in \nu_w N$ with $s \in \mathbb{R}$. Then, by (3.23) and by the assumption, we have an identity

$$0 = \sum_{\lambda \in r_+} l_\lambda(l_\lambda - 2)\frac{|\lambda|^2}{\lambda(w)^2}\kappa_\lambda(w, u(s)) \prod_{\substack{\mu \in r_+ \\ \mu \neq \lambda}} \{1 + \kappa_\mu^2(w, u(s))\}^2. \tag{3.24}$$

Since $\kappa_\lambda(w, u(s)) = -1 + \kappa_\lambda(w, u_0)s$, the right hand side is a polynomial with respect to $s \in \mathbb{R}$. We can extend this polynomial over the complex valuable, and choose a complex number $s_\lambda \in \mathbb{C}$ for $\lambda \in r_+$ so that $1 + \kappa_\lambda(w, u(s_\lambda))^2 = 2 - 2\kappa_\lambda(w, u_0)s_\lambda + \kappa_\lambda(w, u_0)^2 s_\lambda^2 = 0$ since $\kappa_\lambda(w, u_0) = -\lambda(u_0)/\lambda(w) \neq 0$ for $u_0 \in \mathcal{W}$. Moreover, since $u_0 \in \mathcal{U}$, we can show that $1 + \kappa_\mu(w, u(s_\lambda))^2 \neq 0$ for any $\mu \in r_+ \setminus \{\lambda\}$. Then, substituting s_λ into (3.24), we obtain $l_\lambda = 2$, since l_λ is a positive integer. Because $\lambda \in r_+$ is arbitrary, this proves the proposition. $\square$

3.3 Main theorem

In this section, we describe the main results of this paper. The notation is the same as in the section 3.2.4.

Let G be a compact semi-simple Lie group and $N_w = \mathrm{Ad}(G)w$ a principal orbit of the adjoint action of G in $\mathfrak{g}$ through the regular element w, namely, N_w is the standard embedding of a complex flag manifold (see also Subsection 2.2.2). Since $\mathfrak{g}$ admits a G-invariant inner product, we identify $\mathfrak{g}$ with the Euclidean space $\mathbb{R}^{n+k}$. We consider the H-minimality of the normal bundle of the orbit N_w in $\mathfrak{g}$.

Since the compact Lie group G is regarded as a symmetric space of the Riemannian symmetric pair $(G \times G, \Delta G)$, where $\Delta G = \{(g, g) \in G \times G; g \in G\} \simeq G$, and the isotropy representation is equivalent to the adjoint representation of G, we can apply the general setting given in the subsection 3.2.4. More precisely, we have

$$\begin{aligned}
&\mathfrak{u} = \mathfrak{g} + \mathfrak{g}, \quad \sigma = \sigma_0 : \mathfrak{g} + \mathfrak{g} \to \mathfrak{g} + \mathfrak{g},\ (X, Y) \mapsto (Y, X), \\
&\mathfrak{k} = \{(X, X);\ X \in \mathfrak{g}\} \simeq \mathfrak{g} \text{ (as a Lie algebla)}, \\
&\mathfrak{p} = \{(X, -X);\ X \in \mathfrak{g}\} \simeq \mathfrak{g} \text{ (as a vector space)}.
\end{aligned}$$

We define a linear isomorphism between $\mathfrak{g}$ and $\mathfrak{p}$ by

$$\phi : \mathfrak{g} \to \mathfrak{p},\ X \mapsto (X, -X).$$

Since G is semi-simple, any orbit of the adjoint representation splits into a Riemannian product of adjoint orbits of some simple Lie groups. Thus, without loss of generality, we may assume that G is a compact, simple Lie group.

In this case, the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ is a complex simple Lie algebra. Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$ and $\tilde{R}$ the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$. Then we have a direct sum decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \tilde{R}} \mathfrak{g}^{\mathbb{C}}_{\alpha},$$

where $\mathfrak{g}^{\mathbb{C}}_{\alpha} := \{X \in \mathfrak{g}^{\mathbb{C}};\ \mathrm{ad}(H)X = \sqrt{-1}\alpha(H)X\ \forall H \in \mathfrak{t}\}$ (see [58], Chapter V). We note that $\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}}_{\alpha} = 1$ for any $\alpha \in \tilde{R}$. Take an ordering on $\mathfrak{t}$ and denote the set of positive roots by $\tilde{R}_{+}$. We put

$$\mathfrak{g}_{\alpha} := \mathfrak{g} \cap (\mathfrak{g}^{\mathbb{C}}_{\alpha} + \mathfrak{g}^{\mathbb{C}}_{-\alpha})$$

for $\alpha \in \tilde{R}_{+}$. Then, we have a direct sum decomposition

$$\mathfrak{g} = \mathfrak{t} + \sum_{\alpha \in \tilde{R}_{+}} \mathfrak{g}_{\alpha}. \tag{3.25}$$

The space $\mathfrak{g}_{\alpha}$ is a real vector subspace of $\mathfrak{g}$ with real dimension 2. In fact, we can take a basis of $\mathfrak{g}_{\alpha}$ by $V_1 := X_{\alpha} + X_{-\alpha}$ and $V_2 := \sqrt{-1}(X_{\alpha} - X_{-\alpha})$, where X_{α} and $X_{-\alpha}$ are complex basis of $\mathfrak{g}^{\mathbb{C}}_{\alpha}$ and $\mathfrak{g}^{\mathbb{C}}_{-\alpha}$, respectively. We note that the basis satisfy

$$\mathrm{ad}(H)V_1 = \alpha(H)V_2, \quad \mathrm{ad}(H)V_2 = -\alpha(H)V_1$$

for any $H \in \mathfrak{t}$.

Since $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$, $\mathfrak{a} := \phi(\mathfrak{t})$ is a maximal abelian subspace of $\mathfrak{p}$. Consider the restricted root system R of $(\mathfrak{g} + \mathfrak{g}, \sigma)$ with respect to $\mathfrak{a}$. We take an ordering on $\mathfrak{a}$ which is compatible with the ordering on $\mathfrak{t}$ via the isomorphism $\phi|_{\mathfrak{t}}$. For any $X \in \mathfrak{g}_{\alpha}$ and $H \in \mathfrak{t}$, we see

$$\begin{aligned}
\mathrm{ad}(\phi(H))^2(\phi(X)) &= [(H, -H), [(H, -H), (X, -X)]] \\
&= [(H, -H), ([H, X], [H, X])] \\
&= (\mathrm{ad}(H)^2(X), -\mathrm{ad}(H)^2(X)) \\
&= -\alpha(H)^2(X, -X) \\
&= -\alpha \circ \phi^{-1}(\phi(H))(\phi(X)).
\end{aligned}$$

This implies that $\lambda := \alpha \circ \phi^{-1} \in R_{+}$ and $\phi(X) \in \mathfrak{p}_{\lambda}$. Combining this with the decomposition (3.25), we see that there exist a one-to-one correspondence between R_{+} and $\tilde{R}_{+}$, and under this correspondence, $\mathfrak{g}_\alpha$ is isomorphic to $\mathfrak{p}_\lambda$. In particular, $m_\lambda := \dim_{\mathbb{R}}\mathfrak{p}_\lambda = \dim_{\mathbb{R}}\mathfrak{g}_\alpha = 2$ for all $\lambda \in R$, and R is itself a reduced system since the root system $\tilde{R}$ of a complex simple Lie algebra is reduced. Therefore, combining this with Proposition 3.2.12, we obtain the following.

THEOREM 3.3.1. *Let G be a compact, semi-simple Lie group and $\mathfrak{g}$ the Lie algebra of G and $N^n = \mathrm{Ad}(G)w$ a principal orbit of the adjoint action of G on $\mathfrak{g} \simeq \mathbb{R}^{n+k}$ through $w \in \mathfrak{g}$. Then the normal bundle νN is an H-minimal Lagrangian submanifold in the tangent bundle $T\mathfrak{g} \simeq \mathbb{C}^{n+k}$.*

We note that any normal bundle νN over the principal orbit $N = \mathrm{Ad}(G)$ is trivial as a vector bundle (see the argument in Subsection 2.2.2). Thus, νN is homeomorphic to $N \times \mathbb{R}^k$. Moreover, it is foliated by k-dimensional totally geodesic fiber. Such a submanifold is called the *k-ruled* in $\mathbb{C}^{n+k}$ (If $k = 1$, it is nothing but the *ruled* submanifold in the standard sense).

Moreover, we see the following:

THEOREM 3.3.2. *Let N be a full, irreducible isoparametric submanifold in the Euclidean space $\mathbb{R}^{n+k}$. Then the normal bundle νN is H-minimal in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$ if and only if N is a principal orbit of the adjoint action of a compact simple Lie group G.*

This theorem essentially follows from Proposition 3.2.12 and the following proposition which follows from the general argument of symmetric spaces. We refer to [33] for the general facts of symmetric spaces. The following proof is based on the argument of Loos [58] Theorem 4.4 in Chapter VI and Takagi-Takahashi [94].

PROPOSITION 3.3.3 (cf. [58], [94]). *Let $(\mathfrak{u}, \sigma)$ be an effective, irreducible orthogonal symmetric Lie algebra of compact type, and $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the ± 1-eigenspaces decomposition of σ. Then the following are equivalent:*

(i) For the restricted root system R of $(\mathfrak{u}, \sigma)$, $m_\lambda = 2$ for all $\lambda \in R_+$.

(ii) The dual $\mathfrak{u}^* := \mathfrak{k} + \mathfrak{p}^*$ of $\mathfrak{u}$, where $\mathfrak{p}^* := \sqrt{-1}\mathfrak{p}$, has a complex structure. Namely, there exist an automorphism J on $\mathfrak{u}^*$ such that $J^2 = -Id_{\mathfrak{u}^*}$ and $J[X, Y] = [X, JY]$ for any $X, Y \in \mathfrak{u}^*$.

(iii) $\mathfrak{u}$ is not simple.

(iv) $(\mathfrak{u}, \sigma)$ is isomorphic to an irreducible orthogonal symmetric Lie algebra of Type II*, namely, $\mathfrak{k}$ is a compact simple Lie algebra, $\mathfrak{u}$ is isomorphic to $\mathfrak{k} + \mathfrak{k}$ (direct sum), and under this isomorphism, σ is equivalent to the involution $\sigma_0 : \mathfrak{k} + \mathfrak{k} \to \mathfrak{k} + \mathfrak{k}$ defined by $(X, Y) \mapsto (Y, X)$.*

PROOF. $(i) \Rightarrow (iii)$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{u}$ which contains $\mathfrak{a}$. Then $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of the semi-simple complex Lie algebra $\mathfrak{u}^{\mathbb{C}}$. We denote the root system of $\mathfrak{u}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$ by $\tilde{R}$, namely, an element α of $\tilde{R}$ is an linear form on $\mathfrak{t}$ such that

$$\mathfrak{u}^{\mathbb{C}}_{\alpha} := \{X \in \mathfrak{u}^{\mathbb{C}};\ \mathrm{ad}(H)X = \sqrt{-1}\alpha(H)X \text{ for any } H \in \mathfrak{t}\} \neq \{0\}.$$

For $\alpha \in \tilde{R}$, we denote the restriction of α to $\mathfrak{a}$ by $\overline{\alpha}$. Putting $\tilde{R}_0 := \{\alpha \in \tilde{R};\ \overline{\alpha} = 0\}$, we have

$$R = \{\overline{\alpha};\ \alpha \in \tilde{R} \setminus \tilde{R}_0\},$$

where R is the set of all roots of $(\mathfrak{u}, \sigma)$ with respect to $\mathfrak{a}$ (see Proposition 3.3 in [58], Chapter VI). Taking an order on $\mathfrak{a}$, we denote the set of all positive roots in R by R_+. For $\lambda \in R_+$, we set $\tilde{R}_\lambda := \{\alpha \in \tilde{R};\ \overline{\alpha} = \lambda\}$. Then the number of elements in $\tilde{R}_\lambda$ is equal to m_λ. Now, we assume $m_\lambda = 2$ for all $\lambda \in R_+$, i.e., $\tilde{R}_\lambda$ has exactly two elements α and β such that $\lambda = \overline{\alpha} = \overline{\beta}$.

CLAIM 1. *$-\sigma(\alpha) = \beta$ and $\langle \alpha, \beta \rangle = 0$.*

Since $-\sigma(\alpha)(H) = -\alpha(\sigma(H)) = \alpha(H)$ for any $H \in \mathfrak{a} \subset \mathfrak{p}$, we have $\overline{-\sigma(\alpha)} = \overline{\alpha}$. This implies $-\sigma(\alpha) = \alpha$ or $-\sigma(\alpha) = \beta$. If $-\sigma(\alpha) = \alpha$, then $-\sigma(\beta) = \beta$, and it follows that $\alpha|\mathfrak{t} \cap \mathfrak{k} = \beta|\mathfrak{t} \cap \mathfrak{k} = 0$. Combining this with $\overline{\alpha} = \overline{\beta}$, we obtain $\alpha = \beta$, and this is a contradiction. Therefore, $-\sigma(\alpha) = \beta$.

To prove $\langle \alpha, \beta \rangle = 0$, it is sufficient to show that $\alpha \pm \beta \notin \tilde{R}$.

Suppose $\alpha - \beta (= \alpha + \sigma(\alpha)) \in \tilde{R}$. Then, $\sigma(\alpha) \in \tilde{R}$ and $[\mathfrak{u}^{\mathbb{C}}_{\alpha}, \mathfrak{u}^{\mathbb{C}}_{\sigma(\alpha)}] = \mathfrak{u}^{\mathbb{C}}_{\alpha+\sigma(\alpha)}$. We note that, for any $X \in \mathfrak{u}^{\mathbb{C}}_{\alpha+\sigma(\alpha)}$, we have $\sigma(X) \in \mathfrak{u}^{\mathbb{C}}_{\alpha+\sigma(\alpha)}$ since $\mathfrak{t}$ is σ-invariant. Because $\dim_{\mathbb{C}} \mathfrak{u}^{\mathbb{C}}_{\alpha+\sigma(\alpha)} = 1$ and $\sigma^2 = Id$, it follows that $\sigma(X) = X$ or $\sigma(X) = -X$. Suppose $\sigma(X) = -X$. Since $[H, X] = \alpha(H)X + \sigma(\alpha)(H)X = 0$ for any $H \in \mathfrak{a} \subset \mathfrak{p}$, we see $X \in \mathfrak{a}^{\mathbb{C}}$, and this implies $X = 0$ because $\mathfrak{a}^{\mathbb{C}} \cap \mathfrak{u}^{\mathbb{C}}_{\alpha+\sigma(\alpha)} = \{0\}$. Therefore, we have $\sigma(X) = X$, and hence, $\mathfrak{u}^{\mathbb{C}}_{\alpha+\sigma(\alpha)} \subset \mathfrak{k}^{\mathbb{C}}$. However, for each $X_\alpha \in \mathfrak{u}^{\mathbb{C}}_{\alpha}$, we see $\sigma[X_\alpha, \sigma(X_\alpha)] = [\sigma(X_\alpha), X_\alpha] = -[X_\alpha, \sigma(X_\alpha)]$, namely, $[X_\alpha, \sigma(X_\alpha)] \in \mathfrak{p}^{\mathbb{C}}$. This contradicts $\mathfrak{k}^{\mathbb{C}} \cap \mathfrak{p}^{\mathbb{C}} = \{0\}$. Therefore, $\alpha - \beta \notin \tilde{R}$.

Suppose $\alpha + \beta (= \alpha - \sigma(\alpha)) \in \tilde{R}$. Then $2\lambda = \overline{\alpha + \beta} \in R$ and $-\sigma(\alpha + \beta) = \alpha + \beta$. However, this induces a contradiction by a similar argument as above. Therefore, we obtain $\langle \alpha, \beta \rangle = 0$.

CLAIM 2. *$\tilde{R}_0 = \{\phi\}$, namely, $\overline{\alpha} \neq 0$ for all $\alpha \in \tilde{R}$.*

We take $\gamma \in \tilde{R}_0$. If $\langle \alpha, \gamma \rangle < 0$, we have $\alpha + \gamma \in \tilde{R}$ and $\overline{\alpha + \gamma} = \overline{\alpha} = \lambda$. Then, $\alpha + \gamma = \beta$, and hence $\gamma = \beta - \alpha \in \tilde{R}$ which is a contradiction by Claim 1. Similarly, the case $\langle\alpha,\gamma\rangle > 0$ is impossible. Therefore, we have $\langle\alpha,\gamma\rangle = 0$ and hence,

$$\tilde{R} = \tilde{R}_0 \cup (\tilde{R} \setminus \tilde{R}_0) \tag{3.26}$$

is an orthogonal decomposition of the root system $\tilde{R}$. Then the ideal of $\mathfrak{g}$ which corresponds to the root system $\tilde{R}_0$ is contained in $\mathfrak{k}$ (Proposition 3.3 in [58], Chapter VI). Thus, it must be zero since $(\mathfrak{u},\sigma)$ is effective (namely, $\mathfrak{k}$ contains no ideal of $\mathfrak{u}$ except $\{0\}$). This implies $\tilde{R}_0 = \{\phi\}$.

We choose a basis $\tilde{B}$ of $\tilde{R}$ which consists of simple roots.

Claim 3. *$-\sigma(\tilde{B}) = \tilde{B}$. Moreover, $\tilde{B}$ is decomposable.*

By Claim 2, we have $\overline{\alpha} \neq 0$ for any $\alpha \in \tilde{R}$. Therefore $\mathrm{Ker}\overline{\alpha}$ is a subspace of $\mathfrak{a}$ which does not coincide with $\mathfrak{a}$. Since $\tilde{R}$ is a finite set, we can choose a non-zero vector $H \in \mathfrak{a} \setminus \bigcup_{\alpha\in\tilde{R}} \mathrm{Ker}\overline{\alpha} \neq \phi$. Then $\overline{\alpha}(H) = \alpha(H) \neq 0$ for all $\alpha \in \tilde{R}$, namely, H is a regular element in $\mathfrak{t}$. Put $\tilde{R}^+(H) := \{\alpha \in \tilde{R};\ \alpha(H) > 0\}$. Choose simple roots $\tilde{B}(H) := \{\alpha_1,\dots,\alpha_l\}$ in $\tilde{R}^+(H)$. Then $\tilde{B}(H)$ forms a basis of $\tilde{R}$.

For any $\alpha_i \in \tilde{B}(H)$, $-\sigma(\alpha_i)$ is also a root in $\tilde{R}$. Therefore, we may write $-\sigma(\alpha_i) = \sum_{j=1}^{l} a_j\alpha_j$ for $a_1,\dots,a_l \in \mathbb{Z}$. Then we have $\alpha_i = -\sum_{j=1}^{l} a_j\sigma(\alpha_j)$. Since, $-\sigma(\alpha_j)(H) = -\alpha_j(\sigma(H)) = \alpha_j(H) > 0$ as $H \in \mathfrak{a} \subset \mathfrak{p}$, we have $-\sigma(\alpha_j) \in \tilde{R}^+(H)$ for any $j = 1,\dots,l$. Because α_i is a simple root and $a_j \in \mathbb{Z}$ for all j, we obtain $\alpha_i = -\sigma(\alpha_k)$ for some k. This implies $-\sigma(\tilde{B}(H)) = \tilde{B}(H)$.

We put $\tilde{B} := \tilde{B}(H)$. Suppose that $\tilde{B}$ is indecomposable, or equivalently, any two elements of $\tilde{B}$ can be joined by a chain (see Lemma A.2.2). Take $\alpha \in \tilde{B}$. Since $-\sigma(\alpha)$ also belongs to $\tilde{B}$, there exists a chain $\{\alpha_0,\dots,\alpha_{r+1}\}$ such that $\alpha_0 = \alpha$ and $\alpha_{r+1} = -\sigma(\alpha)$. Take the smallest $i \geq 1$ such that $-\sigma(\alpha_i) = \alpha_j \in \{\alpha_0,\dots,\alpha_r\}$. Then $\{\alpha_{j+1},\dots,\alpha_{r+1},-\sigma(\alpha_1),\dots,-\sigma(\alpha_i)\}$ is a cycle in $\tilde{B}$. However, by Lemma A.2.2, it follows $j = r$ and $i = 1$, or equivalently, $-\sigma(\alpha_1) = \alpha_r$. Moreover, $\{\alpha_1,\dots,\alpha_r\}$ is a chain connecting α_1 and α_r. By an induction, we end up with a root β such that $-\sigma(\beta) = \beta$ or $-\sigma(\beta) = \gamma$ with $\langle\beta,\gamma\rangle \neq 0$. However, each case contradicts Claim 1. Therefore, $\tilde{B}$ is decomposable.

Since a basis $\tilde{B}$ of $\tilde{R}$ is decomposable, $\mathfrak{u}$ is not a simple Lie algebra. This proves (iii).

(iii) $\Rightarrow$ *(iv)*. Assume that $\mathfrak{u}$ is not simple. Since $(\mathfrak{u},\sigma)$ is effective, there are no element of $\mathfrak{k}$ except 0 which commutes with all elements of $\mathfrak{p}$. In particular, we see K acts on $\mathfrak{p}$ effectively. Moreover, by this assumption, we have

$$[\mathfrak{p},\mathfrak{p}] = \mathfrak{k}. \tag{3.27}$$

First, we show that $\mathfrak{u}$ contains no ideal invariant under σ other than $\{0\}$ and $\mathfrak{u}$. Suppose there exist an ideal $\mathfrak{u}'$ invariant under σ. Putting $\mathfrak{k}' := \mathfrak{u}' \cap \mathfrak{k}$ and $\mathfrak{p}' = \mathfrak{u}' \cap \mathfrak{p}$, we have the direct sum $\mathfrak{u}' = \mathfrak{k}' + \mathfrak{p}'$ and

$$[\mathfrak{k}, \mathfrak{p}'] \subset [\mathfrak{k}, \mathfrak{u}'] \cap [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{u}' \cap \mathfrak{p} = \mathfrak{p}'$$

since $\mathfrak{u}'$ is an ideal of $\mathfrak{u}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. This implies $\mathfrak{p}'$ is an invariant subspace under $\mathrm{ad}(\mathfrak{k})$. Since $(\mathfrak{u}, \sigma)$ is irreducible, $\mathrm{ad}(\mathfrak{k})$ acts on $\mathfrak{p}$ irreducibly, and hence, we have $\mathfrak{p}' = \{0\}$ or $\mathfrak{p}' = \mathfrak{p}$. If $\mathfrak{p}' = \mathfrak{p}$, then $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}] = [\mathfrak{p}', \mathfrak{p}'] \subset \mathfrak{k}'$ by (3.27). Therefore, $\mathfrak{k}' = \mathfrak{k}$ and we conclude $\mathfrak{u}' = \mathfrak{k} + \mathfrak{p} = \mathfrak{u}$. If $\mathfrak{p}' = \{0\}$, then $\mathfrak{k}' = \mathfrak{u}'$ and $[\mathfrak{k}', \mathfrak{p}] = [\mathfrak{u}', \mathfrak{p}] \subset \mathfrak{u}' = \mathfrak{k}'$. On the other hand, we have $[\mathfrak{k}', \mathfrak{p}] \subset \mathfrak{p}$. Since $\mathfrak{k}' \cap \mathfrak{p} = \{0\}$, we obtain $[\mathfrak{k}', \mathfrak{p}] = \{0\}$, namely, $\mathfrak{k}'$ commutes with $\mathfrak{p}$. Because K acts on $\mathfrak{p}$ irreducibly, we conclude $\mathfrak{k}' = \{0\}$. Then $\mathfrak{u}' = \{0\}$. Therefore, $\mathfrak{u}$ contains no ideal invariant σ except $\{0\}$ and $\mathfrak{u}$.

By the assumption that the semi-simple Lie algebra $\mathfrak{u}$ is not simple, there exist a simple ideal $\mathfrak{g}$ in $\mathfrak{u}$. Since σ is an automorphism on $\mathfrak{u}$, $\sigma(\mathfrak{g})$ is also a simple ideal of $\mathfrak{u}$ and isomorphic to $\mathfrak{g}$. Then the sum $\mathfrak{g} + \sigma(\mathfrak{g})$ is an ideal in $\mathfrak{u}$ invariant under σ, and it follows that

$$\mathfrak{u} = \mathfrak{g} + \sigma(\mathfrak{g}). \tag{3.28}$$

Since the intersection $\mathfrak{g} \cap \sigma(\mathfrak{g})$ is also an ideal of the simple ideal $\mathfrak{g}$, we have $\mathfrak{g} \cap \sigma(\mathfrak{g}) = \{0\}$ or $\mathfrak{g} \cap \sigma(\mathfrak{g}) = \mathfrak{g}$. If $\mathfrak{g} \cap \sigma(\mathfrak{g}) = \mathfrak{g}$, $\mathfrak{g}$ is invariant by σ and we have $\mathfrak{g} = \mathfrak{u}$ which contradicts the assumption. Therefore $\mathfrak{g} \cap \sigma(\mathfrak{g}) = \{0\}$ and the sum (3.28) is a direct sum. This shows that (u, σ) is of Type II.

We give more details of $(\mathfrak{u}, \sigma)$. Consider a map defined by

$$\phi : \mathfrak{g} \to \mathfrak{k}, \ \phi(X) := X + \sigma(X) \tag{3.29}$$

Then it is easily verified that ϕ is an isomorphism between $\mathfrak{g}$ and $\mathfrak{k}$ as a Lie algebra. In particular, $\mathfrak{k}$ is a compact, simple Lie algebra. Furthermore, (3.28) implies $\mathfrak{u}$ is isomorphic to the direct sum $\mathfrak{k} + \mathfrak{k}$. More precisely, this isomorphism is given by $\phi \times \phi \circ \sigma^{-1} : \mathfrak{g} + \sigma(\mathfrak{g}) \to \mathfrak{k} + \mathfrak{k}$. Note that $\sigma^{-1} = \sigma$ since σ is an involution. Then, under this identification, one can easily check that σ is equivalent to $\sigma_0 : \mathfrak{k} + \mathfrak{k} \to \mathfrak{k} + \mathfrak{k}, (X, Y) \mapsto (Y, X)$.

$(iv) \Rightarrow (i)$. We have already shown in the proof of Theorem 3.3.1.

$(ii) \Rightarrow (iv)$. Assume $\mathfrak{u}^*$ admits a complex structure J. We denote the Killing-Cartan form of the Lie algebra $\mathfrak{u}^*$ by β, namely, β is defined by $\beta(X, Y) := \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$ for $X, Y \in \mathfrak{u}^*$. Since J is a complex structure on $\mathfrak{u}^*$, we have $\beta(X, JY) = \beta(JX, Y)$ for $X, Y \in \mathfrak{u}^*$. In particular, we obtain $\beta(JX, JY) = -\beta(X, Y)$. Take an element $X = JY \in \mathfrak{k} \cap J\mathfrak{k}$, then we have

$$\beta(X, X) = \beta(JY, JY) = -\beta(Y, Y). \tag{3.30}$$

Since $\mathfrak{u}$ is compact and semi-simple, the Killing-Cartan form β' on $\mathfrak{u}$ is negative definite on $\mathfrak{u}$. Moreover, the restriction of β' to $\mathfrak{k}$ coincides with $\beta|_{\mathfrak{k}}$. Therefore, β is negative definite on $\mathfrak{k}$, and hence, (3.30) implies $X = 0$. This means that $\mathfrak{k} \cap J\mathfrak{k} = \{0\}$. Thus, we obtain the direct sum decomposition $\mathfrak{u}^* = \mathfrak{k} \oplus J\mathfrak{k}$ because $\dim_{\mathbb{R}}\mathfrak{k} = \dim_{\mathbb{R}}\mathfrak{p}^*$. Moreover, since $[\mathfrak{k}, J\mathfrak{k}] \subset J\mathfrak{k}$ and $[J\mathfrak{k}, J\mathfrak{k}] \subset \mathfrak{k}$, we have $\mathrm{ad}(JX)\mathrm{ad}(Y)(\mathfrak{k}) \subset J\mathfrak{k}$ and $\mathrm{ad}(JX)\mathrm{ad}(Y)(J\mathfrak{k}) \subset \mathfrak{k}$ for $X, Y \in \mathfrak{k}$. It follows that $\beta(JX, Y) = \mathrm{tr}(\mathrm{ad}(JX)\mathrm{ad}(Y)) = 0$. Thus, $\mathfrak{k}$ is orthogonal to $J\mathfrak{k}$ with respect to β. On the other hand, the direct sum decomposition $\mathfrak{u}^* = \mathfrak{k} + \mathfrak{p}^*$ is also an orthogonal decomposition relative to β, and hence, we obtain $J\mathfrak{k} = \mathfrak{p}^*$.

Next, β is decomposed as follows (see [58] I, p.140):

$$\beta(X, X) = \beta''(X, X) + \mathrm{tr}_{\mathfrak{p}^*}(\mathrm{ad}(X)|_{\mathfrak{p}^*})^2 \tag{3.31}$$

for $X \in \mathfrak{k}$, where β'' denotes the Killing-Cartan form on $\mathfrak{k}$. From $\mathfrak{p}^* = J\mathfrak{k}$, it follows

$$\mathrm{tr}_{\mathfrak{p}^*}(\mathrm{ad}(X)|_{\mathfrak{p}^*})^2 = \mathrm{tr}_{J\mathfrak{k}}(\mathrm{ad}(X)|_{J\mathfrak{k}})^2 = \mathrm{tr}_{\mathfrak{k}}(\mathrm{ad}(X)|_{\mathfrak{k}})^2 = \beta''(X, X),$$

since J is a complex structure on $\mathfrak{u}^*$. Combining this with (3.31), we obtain $\beta(X, X) = 2\beta''(X, X)$ for $X \in \mathfrak{k}$. In particular, β'' is negative definite on $\mathfrak{k}$ since so is $\beta|_{\mathfrak{k}}$. Therefore, $\mathfrak{k}$ is semi-simple and compact.

If $\mathfrak{l}$ is an ideal of $\mathfrak{k}$, then $\sqrt{-1}J\mathfrak{l} \subset \sqrt{-1}J\mathfrak{k} = \mathfrak{p}$ and $\mathrm{ad}(\mathfrak{k})(\sqrt{-1}J\mathfrak{l}) = [\mathfrak{k}, \sqrt{-1}J\mathfrak{l}] = \sqrt{-1}J[\mathfrak{k}, \mathfrak{l}] \subset \sqrt{-1}J\mathfrak{l}$. Thus, $\sqrt{-1}J\mathfrak{l}$ is an invariant subspace in $\mathfrak{p}$ with respect to the action $\mathrm{ad}(\mathfrak{k})$. However, by the assumption of $(\mathfrak{u}, \sigma)$, $\mathrm{ad}(\mathfrak{k})$ acts on $\mathfrak{p}$ irreducibly, and hence, $\mathfrak{l} = \{0\}$ or $\mathfrak{k}$. Thus, $\mathfrak{k}$ is simple. Moreover, since $\mathfrak{u}^* = \mathfrak{k} + J\mathfrak{k}$, we can define an isomorphism (as a vector space) $\phi : \mathfrak{u} = \mathfrak{k} + \sqrt{-1}J\mathfrak{k} \to \mathfrak{k} + \mathfrak{k}$ by $\phi(X + \sqrt{-1}JY) = (X + Y, X - Y)$ for $X, Y \in \mathfrak{k}$. Under this isomorphism, it is easily verified that the involution σ is equivalent to σ_0. Therefore, $(\mathfrak{u}, \sigma)$ is isomorphic to one of Type II.

(iv) $\Rightarrow$ *(ii)*. By the assumption, we identify $(\mathfrak{u}, \sigma)$ with $(\mathfrak{k} + \mathfrak{k}, \sigma_0)$. Then the ± 1-eigenspaces decomposition is given by

$$\mathfrak{u} = \mathfrak{k} + \mathfrak{p} = \{(X, X);\ X \in \mathfrak{k}\} + \{(X, -X);\ X \in \mathfrak{k}\}.$$

Define an automorphism J on $\mathfrak{u}^*$ by $J(X, X) := \sqrt{-1}(X, -X)$ and $J(\sqrt{-1}(X, -X)) := -(X, X)$ for $X \in \mathfrak{k}$. Then, it is obvious that $J^2 = -Id_{\mathfrak{u}^*}$. Moreover, for $X, Y \in \mathfrak{k}$, we have

$$\begin{aligned}
J[(X, X), (Y, Y)] &= J([X, Y], [X, Y]) = \sqrt{-1}([X, Y], -[X, Y]) \\
&= [(X, X), \sqrt{-1}(Y, -Y)] = [(X, X), J(Y, Y)]. \\
J[(X, X), \sqrt{-1}(Y, -Y)] &= J(\sqrt{-1}([X, Y], -[X, Y])) = -([X, Y], [X, Y]) \\
&= -[(X, X), (Y, Y)] = [(X, X), J(\sqrt{-1}(Y, -Y))]. \\
J[\sqrt{-1}(X, -X), \sqrt{-1}(Y, -Y)] &= -J([X, Y], [X, Y]) = -\sqrt{-1}([X, Y], -[X, Y]) \\
&= [\sqrt{-1}(X, -X), -(Y, Y)] \\
&= [\sqrt{-1}(X, -X), J(\sqrt{-1}(Y, -Y))].
\end{aligned}$$

Thus, J is a complex structure on $\mathfrak{u}^*$. $\square$

Proof of Theorem 3.3.2. Let N be a full, irreducible isoparametric submanifold in $\mathbb{R}^{n+k}$. Suppose that the normal bundle νN is H-minimal in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$. Then, by virtue of Lemma 3.2.6, we may assume that N is a principal orbit of an s-representation of an irreducible, simply connected Riemannian symmetric space U/K of compact type. Take an effective and irreducible orthogonal symmetric Lie algebra which corresponds to U/K. Then, for the restricted root system R of $(\mathfrak{u}, \sigma)$, we have $l_\lambda = m_\lambda + m_{2\lambda} = 2$ for all $\lambda \in r_+$ by Proposition 3.2.12. If there exist a root $\lambda \in r_+$ such that $2\lambda \in R_+$, then it follows $m_\lambda = m_{2\lambda} = 1$. However, by Proposition 2.3 in [5], $2\lambda \notin R$ whenever $\lambda \in R$ has an odd multiplicity and this is a contradiction. Thus, the restricted root system R is reduced and $m_\lambda = l_\lambda = 2$ for all $\lambda \in R_+ = r_+$. Then by Proposition 3.3.3, we see that $(\mathfrak{u}, \sigma)$ is isomorphic to the one of Type II and the associated globally symmetric space is a compact, connected simple Lie group G. This proves the theorem. □

3.4 Non-complete examples

Let N^n be a submanifold in the unit sphere $S^{n+k-1}(1) \subset \mathbb{R}^{n+k}$. We define the *twisted normal cone* over N by $\mathcal{CN} := \{(tp, s\nu) \in \mathbb{R}^{n+k} \oplus \mathbb{R}^{n+k};\ (p, \nu) \in \nu_1 N,\ t, s \in \mathbb{R}\}$, where $\nu_1 N$ is the unit normal bundle of N in S^{n+k-1}. This notion was introduced by Harvey-Lawson [31] in the construction of special Lagrangian varieties in $\mathbb{C}^{n+k}$ from submanifolds in the unit sphere. The twisted normal cone $\mathcal{CN}$ is regarded as the cone over

$$\mathcal{LN} := \{(\cos\theta p, \sin\theta \nu) \in \mathbb{R}^{n+k} \oplus \mathbb{R}^{n+k};\ (p, \nu) \in \nu_1 N, \theta \in S^1\}.$$

We remark that $\mathcal{LN}$ is a Legendrian variety in the unit sphere $S^{2(n+k)-1}$ with some singular points. We note that if the normal bundle over the non-singular submanifold $C(N)^* := C(N) \setminus \{0\}$ in $\mathbb{R}^{n+k}$ is H-minimal in $\mathbb{C}^{n+k}$, then so is $\mathcal{CN}$.

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_pN such that $\overline{A}^u(e_i) = \kappa_i(p, u)e_i$ for $i = 1, \cdots, n$, where $\overline{A}$ is the shape operator of N in S^{n+k-1}. We define tangent vectors $\{e_1(tp), \cdots, e_n(tp)\}$ of $T_{tp}C(N)^*$ by the parallel transport of the vectors $\{e_1, \cdots, e_n\}$ along the geodesic $\gamma(t) := tp$ in $\mathbb{R}^{n+k}$. Then $\{e_1(tp), \cdots, e_n(tp), e_{n+1}(tp) := \dot{\gamma}(t)\}$ is an orthonormal basis of $T_{tp}C(N)$. Moreover, for the parallel transport $u(tp)$ of a normal vector $u \in \nu_p N$, it is shown that $A^{u(tp)}(e_i(tp)) = (1/t)\kappa_i(p, u)e_i(tp)$ for $i = 1, \cdots, n$, and $A^{u(tp)}(e_{n+1}(tp)) = 0$, where A is the shape operator of $C(N)^*$ in $\mathbb{R}^{n+k}$. Namely, the principal curvatures of $C(N)^*$ at tp with respect to the normal direction $u(tp)$ are given by

$$\frac{1}{t}\kappa_1(p, u), \cdots, \frac{1}{t}\kappa_n(p, u) \text{ and } 0. \tag{3.32}$$

In particular, $C(N)^*$ is austere in $\mathbb{R}^{n+k}$ if and only if N is austere in S^{n+k-1}.

LEMMA 3.4.1. *Let N^n be an isoparametric hypersurface in the unit sphere $S^{n+1}(1) \subset \mathbb{R}^{n+2}$. Then the mean curvature vector H of $\nu C(N)^*$ satisfies*

$$\delta\alpha_H(tp, s\nu) = \sum_{i=1}^{n} \frac{2\kappa_i(1-\kappa_i^2)}{(t^2+s^2\kappa_i^2)^2} st - \Big(\sum_{i=1}^{n} \frac{\kappa_i}{t^2+s^2\kappa_i^2}\Big)\Big(\sum_{j=1}^{n} \frac{1-\kappa_j^2}{t^2+s^2\kappa_j^2}\Big) st.$$

A proof is similar to the proof of Lemma 3.2.1. Thus we omit it.

It is well known that there exist a unique austere hypersurface in the family of parallel hypersurfaces of an isoparametric hypersurface N when the multiplicities of the principal curvatures are the same. Moreover, the austere orbits of s-representations are classified in [40]. Thus, these twisted normal cones give examples of special Lagrangian varieties.

THEOREM 3.4.2. *Let N^n be an isoparametric hypersurface in the unit sphere $S^{n+1}(1) \subset \mathbb{R}^{n+2}$. Then the twisted normal cone $\mathcal{CN}$ is a non-minimal, H-minimal Lagrangian variety in $\mathbb{C}^{n+2}$ if and only if N is locally congruent to one of the following:*

(i) $S^2(r)$ *in* $S^3(1)$*, where* $0 < r < 1$.

(ii) $S^n(1/\sqrt{2})$*, where* $n \geq 1$.

(iii) $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$*, where* m_1 *and* m_2 *are positive integers with* $m_1 + m_2 = n$ *and* $m_1 \neq m_2$.

PROOF. We use the same notation as in the subsection 3.2.3.

(1) The case $g = 1$. We have from Lemma 3.4.1,

$$\delta\alpha_H(tp, s\nu) = \frac{2n\kappa(1-\kappa^2)}{(t^2+s^2\kappa^2)^2} st - \frac{n^2\kappa(1-\kappa^2)}{(t^2+s^2\kappa^2)^2} st = \frac{n(2-n)\kappa(1-\kappa^2)}{(t^2+s^2\kappa^2)^2} st.$$

Thus $\delta\alpha_H = 0$ if and only if one of the cases $n = 2$, $\kappa = 0$ and ± 1 occurs. When $n = 2$, N is congruent to $S^2(r)$ in $S^3(1)$ for $0 < r < 1$. When $\kappa = \pm 1$, N is congruent to the sphere of radius $1/\sqrt{2}$.

(2) The case $g = 2$. In this case, the principal curvatures are given by $\kappa_1 = \cot\theta_1$, and $\kappa_2 = -1/\cot\theta_1 = -1/\kappa_1$ for $0 < \theta_1 < \pi/2$. We denote the multiplicities of κ_1 and κ_2 by m_1 and m_2, respectively. Then by Lemma 3.4.1, we have

$$\begin{aligned} \delta\alpha_H(tp, s\nu) &= \frac{m_1(2-m_1)\kappa_1(1-\kappa_1^2)}{(t^2+s^2\kappa_1^2)^2} st + \frac{m_2(2-m_2)\kappa_2(1-\kappa_2^2)}{(t^2+s^2\kappa_2^2)^2} st \\ &\quad - \frac{m_1 m_2(\kappa_1+\kappa_2)(1-\kappa_1\kappa_2)}{(t^2+s^2\kappa_1^2)(t^2+s^2\kappa_2^2)} st. \end{aligned}$$

Now, we set $\kappa := \kappa_1$. Then it is shown that $\delta\alpha_H(tp, s\nu) = 0$ for any $s, t \in \mathbb{R} \setminus \{0\}$ if and only if κ satisfies the following equations:

$$\begin{cases} \{m_1(2-m_1)\kappa^4 + 2m_1m_2\kappa^2 + m_2(2-m_2)\}(1-\kappa^2) = 0, \\ \{m_1m_2\kappa^4 + (m_1(2-m_1) + m_2(2-m_2))\kappa^2 + m_1m_2\}(1-\kappa^2) = 0, \\ \{m_1(2-m_1) + 2m_1m_2\kappa^2 + m_2(2-m_2)\kappa^4\}(1-\kappa^2) = 0. \end{cases} \tag{3.33}$$

Therefore, if $\kappa = \cot\theta_1 = 1$ (we recall $0 < \theta_1 < \pi/2$, and hence, $\kappa > 0$), then the equations (3.33) are satisfied. In this case, N is isometric to $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$. We note that this case includes the austere submanifolds, namely, the case of same multiplicities. We consider the case $\kappa \neq 1$. Then, by taking the difference of the first and the third equations in (3.33), we obtain $m_1 = m_2$. However, one can check that this case does not have any real solution κ of (3.33).

(3) The case $g = 3$. In this case, the principal curvatures are given by $\kappa_i = \cot(\theta_1 + (i-1)\pi/3)$ for $i = 1, 2, 3$, and $0 < \theta_1 < \pi/3$, or equivalently, these are expressed by

$$\kappa_1 =: \kappa,\ \kappa_2 = \frac{\kappa - \sqrt{3}}{1 + \sqrt{3}\kappa},\ \kappa_3 = \frac{\kappa + \sqrt{3}}{1 - \sqrt{3}\kappa},\ \text{where } \kappa > \frac{1}{\sqrt{3}}, \tag{3.34}$$

and all multiplicities of these principal curvatures are the same. We denote the multiplicity by m. Then, we have from Lemma 3.4.1,

$$\delta\alpha_H(tp, s\nu) = m\sum_{i=1}^{3} \frac{2\kappa_i(1-\kappa_i^2)}{(t^2+s^2\kappa_i^2)^2} st - m^2\Big(\sum_{i=1}^{3} \frac{\kappa_i}{t^2+s^2\kappa_i^2}\Big)\Big(\sum_{j=1}^{3} \frac{1-\kappa_j^2}{t^2+s^2\kappa_j^2}\Big) st. \tag{3.35}$$

Now, we assume that $\delta\alpha_H = 0$ on $\nu C(N)^*$. Since $s \in \mathbb{R}$ is arbitrary, we put $s = 1$. Then, by canceling the denominators in the right hand side of (3.35), we have an identity for a polynomial with respect to the parameter t. Since $t \in \mathbb{R}$ is arbitrary, all coefficients in the polynomial vanish. In particular, the coefficient of the highest order term satisfies

$$m\sum_{i=1}^{3} 2\kappa_i(1-\kappa_i^2) - m^2\Big(\sum_{i=1}^{3}\kappa_i\Big)\Big(\sum_{j=1}^{3} 1-\kappa_j^2\Big) = 0. \tag{3.36}$$

Substituting (3.34) into (3.36), we see that κ satisfies the following equation:

$$3m\kappa(\kappa^2-3)\{2(9\kappa^6+9\kappa^4+39\kappa^2+7) - m(9\kappa^6-27\kappa^4+63\kappa^2+3)\} = 0. \tag{3.37}$$

This is always satisfied when $\kappa = \sqrt{3}$ which is the case of austere submanifold. Since $m \in \{1, 2, 4, 8\}$, one can see that if $m \in \{1, 2, 8\}$, there are no other real solutions. When $m = 4$, there are two solutions $\kappa > 1/\sqrt{3}$ and $\kappa \neq \sqrt{3}$, however, one can check that this is not the case.

(4) The case $g = 4$. In this case, the principal curvatures are given by $\kappa_i = \cot(\theta_1 + (i-1)\pi/4)$ $(i = 1, \cdots, 4)$, where $0 < \theta_1 < \pi/4$, and these are expressed by

$$\kappa_1 =: \kappa,\ \kappa_2 = \frac{\kappa-1}{\kappa+1},\ \kappa_3 = -\frac{1}{\kappa},\ \kappa_4 = -\frac{\kappa+1}{\kappa-1},\ \text{where } \kappa > 1.$$

The multiplicities satisfy the relation $m_1 = m_3$, and $m_2 = m_4$. First, we show that if $\delta\alpha_H = 0$ on $\nu C(N)^*$, then $m_1 = m_2$. Assume $\delta\alpha_H = 0$. When $t = s = 1$, we have from Lemma 3.4.1,

$$0 = \sum_{i=1}^{n} \frac{2\kappa_i(1-\kappa_i^2)}{(1+\kappa_i^2)^2} - \Big(\sum_{i=1}^{n} \frac{\kappa_i}{1+\kappa_i^2}\Big)\Big(\sum_{j=1}^{n} \frac{1-\kappa_j^2}{1+\kappa_j^2}\Big). \tag{3.38}$$

Since $\kappa_3 = -1/\kappa_1$ and $\kappa_4 = -1/\kappa_2$, we have the relations

$$\frac{\kappa_{i+2}(1-\kappa_{i+2}^2)}{(1+\kappa_{i+2}^2)^2} = \frac{\kappa_i(1-\kappa_i^2)}{(1+\kappa_i^2)^2}, \tag{3.39}$$

$$\frac{\kappa_{i+2}}{1+\kappa_{i+2}^2} = -\frac{\kappa_i}{1+\kappa_i^2}, \tag{3.40}$$

$$\frac{1-\kappa_{i+2}^2}{1+\kappa_{i+2}^2} = -\frac{1-\kappa_i^2}{1+\kappa_i^2} \tag{3.41}$$

for $i = 1, 2$. Therefore, by substituting (3.39) thorough (3.41) into (3.38), we obtain

$$0 = \frac{4m_1\kappa_1(1-\kappa_1^2)}{(1+\kappa_1^2)^2} + \frac{4m_2\kappa_2(1-\kappa_2^2)}{(1+\kappa_2^2)^2} = 4(m_1 - m_2)\frac{\kappa(1-\kappa^2)}{(1+\kappa^2)^2}.$$

Since $\kappa > 1$, this implies $m_1 = m_2$. Hence, if $\delta\alpha_H = 0$, then all multiplicities are the same. Then, as in the case $g = 3$, we can deduce a necessary condition of κ. The details are left to the reader. We only give the corresponding equality of (3.37):

$$\begin{aligned} &m\kappa(1-\kappa^2)(\kappa^2 - 2\kappa - 1)(\kappa^2 + 2\kappa - 1) \\ &\times\{2(\kappa^8 + 2\kappa^6 + 10\kappa^4 + 2\kappa^2 + 1) - m(\kappa^8 - 4\kappa^6 + 22\kappa^4 - 4\kappa^2 + 1)\} = 0, \end{aligned} \tag{3.42}$$

where $m = m_1 = m_2$. In the case $\kappa = \sqrt{2} + 1$, the equality (3.42) is satisfied, and this is the case of austere submanifold. On the other hand, by the result of Abresch [1], we know $m \in \{1, 2\}$, and hence, one checks that there are no real solutions of (3.42) which satisfies the conditions $\kappa \neq \sqrt{2} + 1$ and $\kappa > 1$.

(5) The case $g = 6$. In this case, the principal curvatures are given by $\kappa_i = \cot(\theta_1 + (i-1)\pi/6)$ $(i = 1, \cdots, 6)$, where $0 < \theta_1 < \pi/6$, and these are expressed by

$$\kappa_1 =: \kappa,\ \kappa_2 = \frac{\sqrt{3}\kappa - 1}{\kappa + \sqrt{3}},\ \kappa_3 = \frac{\kappa - \sqrt{3}}{\sqrt{3}\kappa + 1},\ \kappa_4 = -\frac{1}{\kappa},\ \kappa_5 = -\frac{\kappa + \sqrt{3}}{\sqrt{3}\kappa - 1},\ \kappa_6 = -\frac{\sqrt{3}\kappa + 1}{\kappa - \sqrt{3}},$$

where $\kappa > \sqrt{3}$. All multiplicities are same, and we denote it by m. Therefore, by the same argument as in the case $g = 3$, we obtain a necessary condition of κ corresponding to (3.37). The equality is given by

$$\begin{aligned} &3m(\kappa^2 - 1)(\kappa^2 - 4\kappa + 1)(\kappa^2 + 4\kappa + 1) \\ &\times\{2(9\kappa^{12} + 36\kappa^{10} + 255\kappa^8 - 56\kappa^6 + 255\kappa^4 + 36\kappa^2 + 9) \\ &-3m(3\kappa^{12} - 18\kappa^{10} + 285\kappa^8 - 412\kappa^6 + 285\kappa^4 - 18\kappa^2 + 3)\} = 0. \end{aligned} \tag{3.43}$$

When $\kappa = 2 + \sqrt{3}$, it gives an austere submanifold. On the other hand, since $m \in \{1, 2\}$, one can check that there are no real solutions which satisfy the equation (3.43) with the conditions $\kappa \neq 2 + \sqrt{3}$ and $\kappa > \sqrt{3}$. $\square$

Remark 3.4.3. In [88], Sakaki classified all surfaces in $\mathbb{R}^3$ with H-minimal normal bundles in $\mathbb{C}^3$. His classification result asserts that $S^2(r)$ with $r > 0$ and the cone over $S^1(1/\sqrt{2})$ are the only examples of non-austere surfaces in $\mathbb{R}^3$ with H-minimal normal bundles.

Remark 3.4.4. By the result of Ichiyama-Inoguchi-Urakawa in [35], it is known that $S^n(1/\sqrt{2})$ and $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$ with $m_1 \neq m_2$ are the only examples of non-minimal, *bi-harmonic* isoparametric hypersurfaces in the unit sphere.

Chapter 4

On the minimality of normal bundles and austere submanifolds

4.1 Preliminaries

Up to now, we consider a submanifold in the Euclidean space $\mathbb{R}^m$. In the following, we generalize $\mathbb{R}^n$ into a Riemannian manifold M.

4.1.1 Tangent bundles and the Sasaki metric

Let $(M, \langle , \rangle)$ be an m-dimensional Riemannian manifold, TM the tangent bundle over M, and $\pi : TM \to M$ the natural projection. For a vector field X on M, we define two vector fields on TM, the *horizontal lift* X^h and the *vertical lift* X^v by $X^h\alpha := \overline{\nabla}_X\alpha$, and $X^v\alpha = \alpha(X) \circ \pi$, respectively, where $\overline{\nabla}$ is the Levi-Civita connection with respect to the Riemannian metric $\langle , \rangle$, and α is a 1-form on M which is regarded as a function on TM. If we choose a local coordinate $(x^1, \cdots, x^m)$ of M, then we can choose a local coordinate $(p^1, \cdots, p^m, q^1, \cdots, q^m)$ of TM, where $x^i = p^i \circ \pi$, and $(q^1, \cdots, q^m)$ is the fiber coordinate. With respect to this coordinate, a local expression of X^h and X^v is given by

$$X^h = X^i \frac{\partial}{\partial p^i} - X^i q^j \Gamma^k_{ij} \frac{\partial}{\partial q^k}, \quad X^v = X^i \frac{\partial}{\partial q^i}, \tag{4.1}$$

where $X = X^i \frac{\partial}{\partial x^i}$, Γ^k_{ij}'s are the Christoffel symbols of $\overline{\nabla}$, and we use the Einstein convention. The *connection map* K is a bundle map $K : TTM \to TM$ defined as follows. Let $\exp_p : V' \to V$ be the local diffeomorphism of the exponential map from a open neighborhood $V' \subset T_pM$ to $V := \exp_p(V') \subset M$, and $\tau : \pi^{-1}(V) \to T_pM$ be the smooth map which translates every $Y \in \pi^{-1}(V)$ from $q = \pi(Y)$ to p in a parallel manner along the unique geodesic curve in V. For $u \in T_pM$, the map $R_{-u} : T_pM \to T_pM$ is defined by $R_{-u}(X) := X - u$ for $X \in T_pM$. Then we define $K_{(p,u)} := d(\exp_p \circ R_{-u} \circ \tau)$.

Let $\widetilde{c}$ be a smooth section in TM with $\widetilde{c}(0) = z, \dot{\widetilde{c}}(0) = \tilde{V} \in T_zTM$. Put $c := \pi \circ \widetilde{c}$. Then the connection map satisfies

$$K_z\tilde{V} = \overline{\nabla}_{\dot{c}}\widetilde{c}|_0. \tag{4.2}$$

For $z = (p, u) \in TM$, we define vector subspaces of T_zTM by $\mathcal{H}_z := \mathrm{Ker}K_z$ and $\mathcal{V}_z := \mathrm{Ker}d\pi_z$. Then the tangent space of TM has a direct sum decomposition $T_zTM = \mathcal{H}_z \oplus \mathcal{V}_z$. Since the horizontal and vertical lifts are characterized by the following properties

$$\pi_*(X^h)_z = X_p,\ \pi_*(X^v)_z = 0,\ K(X^h)_z = 0,\ \text{and } K(X^v)_z = X_p$$

for any vector field X on M, we have vector space isomorphisms $\pi_* : \mathcal{H}_z \tilde{\to} T_pM$ and $K : \mathcal{V}_z \tilde{\to} T_pM$. Thus, every tangent vector $\tilde{V}_z \in T_zTM$ can be decomposed into $\tilde{V}_z := (X_p)^h_z + (Y_p)^v_z$, where $X_p, Y_p \in T_pM$ are uniquely determined by $X_p := \pi_*(\tilde{V}_z)$ and $Y_p := K(\tilde{V}_z)$. We call $(X_p)^h_z$ and $(Y_p)^v_z$ the *tangential component* and the *vertical component* of the tangent vector V_z, respectively.

The tangent bundle TM admits an almost complex structure J defined by $JX^h = X^v$ and $JX^v = -X^h$ for any vector field X on M. The *Sasaki metric* $\tilde{g}$ is a Riemannian metric on TM defined by

$$\tilde{g}(\tilde{X}, \tilde{Y})_z := \langle \pi_*\tilde{X}, \pi_*\tilde{Y}\rangle_p + \langle K\tilde{X}, K\tilde{Y}\rangle_p$$

for $\tilde{X}, \tilde{Y} \in T_zTM$. By the definition, the splitting $T_zTM = \mathcal{H}_z \oplus \mathcal{V}_z$ is orthogonal with respect to $\tilde{g}$.

The Riemannian metric $\langle , \rangle$ on M defines the standard identification between the tangent bundle TM and the cotangent bundle T^*M, namely, $\iota : TM \tilde{\to} T^*M$ via $X_p \mapsto \langle X_p, \cdot\rangle$. The Liouville form $\gamma \in \Omega^1(T^*M)$ is the 1-form defined by $\gamma_{(p,\chi)}(\tilde{V}) := \chi_p(\pi_*(\tilde{V}))$, where $\tilde{V}$ is a tangent vector of T^*M. The canonical symplectic structure on T^*M is defined by $\omega^* := -d\gamma$. Then we can induce a symplectic structure on TM by $\omega := \iota^*\omega^*$. It is easily shown that the almost complex structure J and $\tilde{g}$ are associated with each other, i.e., it gives an almost Hermitian structure on TM, and ω is the associated 2-form, i.e., $\omega = \tilde{g}(J\cdot, \cdot)$ (see [13]). Since ω is closed, this almost Hermitian structure defines the almost Kähler structure on TM. We remark that this almost Kähler structure is Kähler if and only if $(M, \langle , \rangle)$ is flat.

The Levi-Civita connection with respect to the Sasaki metric satisfies the following relation for the horizontal and vertical lifts:

$$(\tilde{\nabla}_{X^h}Y^h)_z = (\overline{\nabla}_XY)^h_z - \frac{1}{2}(\overline{R}(X, Y)u)^v_z, \tag{4.3}$$

$$(\tilde{\nabla}_{X^h}Y^v)_z = \frac{1}{2}(\overline{R}(u, Y)X)^h_z + (\overline{\nabla}_XY)^v_z, \tag{4.4}$$

$$(\tilde{\nabla}_{X^v}Y^h)_z = \frac{1}{2}(\overline{R}(u, X)Y)^h_z, \tag{4.5}$$

$$\tilde{\nabla}_{X^v}Y^v = 0, \tag{4.6}$$

where $X, Y \in \Gamma(TM)$, and $\overline{R}$ denotes the curvature tensor of $\overline{\nabla}$.

4.1.2 Lemmas in the general setting

Let N be an n-dimensional submanifold of M. We denote the normal bundle of N by νN, i.e.,

$$\nu N := \{z = (p, u) \in TM |\ p \in N \text{ and } u \perp T_pN\} \subset TM.$$

Then νN is a submanifold in TM. Fix an arbitrary point $z_0 = (p_0, u_0) \in \nu N$. We can choose a local field of orthonormal frames $\{e_1, \cdots, e_n\}$ around p_0 in N such that $A^{u_0}(e_i)(p_0) = \kappa_i(z_0)e_i(p_0)$ for $i = 1, \cdots, n$. In other words, the shape operator A^{u_0} is diagonalized by the basis at p_0, and $\kappa_i(z_0)$ is the eigenvalue of A^{u_0}. Choose a local field of orthonormal frames $\{\nu_{n+1}, \cdots, \nu_m\}$ of the normal space of N around p_0.

Index convention : Throughout this chapter, we use the following indices:

$$\begin{aligned}
&i, j, k, \ldots \in \{1, \ldots, n\}.\\
&\alpha, \beta, \gamma, \ldots \in \{n+1, \ldots, m\}.\\
&\lambda, \mu, \nu, \ldots \in \{1, \ldots, m\}.
\end{aligned}$$

Lemma 4.1.1. *Let $\{e_1, \cdots, e_n\}$ and $\{\nu_{n+1}, \cdots, \nu_m\}$ be a local field of tangent frames and normal frames of N respectively defined as above. Then we have a local field of frames of νN:*

$$\begin{aligned}
E_i(z) : &= (e_i)^h_z - (A^u(e_i))^v_z,\\
E_\alpha(z) : &= (\nu_\alpha)^v_z,
\end{aligned}$$

where A^u denotes the shape operator at $p \in N$ in M with respect to the normal vector u.

Proof. First, we show that $E_i(z)$ is a tangent vector of νN. Fix a point $z = (p, u) \in \nu N$. Let $c_i : I \to N$ be a smooth curve with $c_i(0) = p$ and $\dot{c}_i = e_i(p)$ where I is an interval containing 0. We can take a curve $\tilde{c}_i(t) := (c_i(t), \nu(c_i(t)))$ in $\nu N \subset TM$ such that $\nu(c_i(t))$ is a parallel transport of the normal vector u along the curve c_i with respect to the normal connection $\nabla^\perp$. Then we have

$$\begin{aligned}
\pi_*(\dot{\tilde{c}}_i(0)) &= \frac{d}{dt}\Big|_{t=0} \pi \circ \tilde{c}_i(t) = \dot{c}_i(0) = e_i,\\
K(\dot{\tilde{c}}_i(0)) &= \frac{d}{dt}\Big|_{t=0} K \circ \tilde{c}_i(t) = \overline{\nabla}_{e_i}\nu(c_i(t))|_{t=0} = -A^u(e_i).
\end{aligned}$$

Thus $E_i(z) = \dot{\tilde{c}}_i(0) = (e_i)^h_z - (A^u(e_i))^v_z$ is a tangent vector of νN.

Next, we show $E_\alpha(z)$ is a tangent vector of νN. We take a curve $\tilde{c}_\alpha(t) := (p, t\nu_\alpha + u)$ in $\nu N \subset TM$. Obviously, $\pi_*(\dot{\tilde{c}}_\alpha(0)) = 0$. On the other hand,

$$\begin{aligned} K(\dot{\tilde{c}}_\alpha(0)) &= \frac{d}{dt}\Big|_{t=0} K \circ \tilde{c}_\alpha(t) \\ &= \frac{d}{dt}\Big|_{t=0} \exp_p(\tau(\tilde{c}_\alpha(t)) - u) \\ &= \frac{d}{dt}\Big|_{t=0} \exp_p(t\nu_\alpha) = \nu_\alpha. \end{aligned}$$

Thus, $E_\alpha(z) = \dot{\tilde{c}}_\alpha(0) = (\nu_\alpha)^v_z$ is a tangent vector of νN.

For the point $z_0 = (p_0, u_0)$, we have

$$\begin{aligned} \tilde{g}(E_i, E_j)_{z_0} &= \langle e_i, e_j \rangle_{p_0} + \langle A^u(e_i), A^u(e_j) \rangle_{p_0} = (1 + \kappa_i^2(z_0))\delta_{ij}, \\ \tilde{g}(E_i, E_\alpha)_{z_0} &= 0, \\ \tilde{g}(E_\alpha, E_\beta)_{z_0} &= \langle \nu_\alpha, \nu_\beta \rangle_{p_0} = \delta_{\alpha\beta}. \end{aligned}$$

These imply that $\{E_i, E_\alpha\}$ are linearly independent around z_0. Thus this gives a local frame of νN. □

Note that by the proof of Lemma 4.1.1, we have an orthonormal basis of $T_{z_0}\nu N$ by

$$E_i'(z_0) := \frac{1}{\sqrt{1 + \kappa_i^2(z_0)}} E_i(z_0), \ E_\alpha'(z_0) := E_\alpha(z_0). \tag{4.7}$$

Recall that the symplectic structure on $(TM, \tilde{g})$ is given by $\omega := \tilde{g}(J\cdot, \cdot)$. Then $\omega(E_i, E_j)_{z_0} = \kappa_i(z_0)\delta_{ij} - \kappa_j(z_0)\delta_{ij} = 0$, and $\omega(E_i, E_\alpha)_{z_0} = \omega(E_\alpha, E_\beta)_{z_0} = 0$ for any $i, j = 1, \cdots, n$ and $\alpha, \beta = n+1, \cdots, m$. This means that *the normal bundle* νN *is a Lagrangian submanifold in* (TM, ω).

By a direct computation using the formula (4.3) through (4.6), we have the following formula at the point $z_0 = (p_0, u_0)$.

$$\begin{aligned} (\tilde{\nabla}_{E_i} E_j)(z_0) &= \Big\{ \overline{\nabla}_{e_i} e_j - \frac{1}{2}\kappa_i(z_0)\overline{R}(u_0, e_i)e_j - \frac{1}{2}\kappa_j(z_0)\overline{R}(u_0, e_j)e_i \Big\}^h_{z_0} \\ &- \Big\{ \kappa_j(z_0)\nabla_{e_i} e_j + \frac{1}{2}\overline{R}(e_i, e_j)u_0 + \sum_{l=1}^n E_i \langle A^u(e_j), e_l \rangle e_l \Big\}^v_{z_0}, \end{aligned} \tag{4.8}$$

$$(\tilde{\nabla}_{E_i} E_\alpha)(z_0) = \Big\{ \frac{1}{2}\overline{R}(u_0, \nu_\alpha)e_i \Big\}^h_{z_0} + (\overline{\nabla}_{e_i}\nu_\alpha)^v_{z_0}, \tag{4.9}$$

$$(\tilde{\nabla}_{E_\alpha} E_i)(z_0) = \Big\{ \frac{1}{2}\overline{R}(u_0, \nu_\alpha)e_i \Big\}^h_{z_0} - \Big(\sum_{l=1}^n E_\alpha \langle A^u(e_i), e_l \rangle e_l \Big)^v_z, \tag{4.10}$$

$$(\tilde{\nabla}_{E_\alpha} E_\beta)(z_0) = 0, \tag{4.11}$$

where all the tensor fields and the differentials take values at p_0.

In general, the almost complex structure J is not integrable. The following equalities follows from a direct computation:

$$\begin{aligned}
(\tilde{\nabla}_{E_i}J)E_j(z_0) &= \frac{1}{2}\{\overline{R}(u_0,e_j)e_i - \kappa_i(z_0)\kappa_j(z_0)\overline{R}(u_0,e_i)e_j - \overline{R}(e_i,e_j)u\}^h_{z_0} \qquad (4.12)\\
&+ \frac{1}{2}\{\kappa_j(z_0)\overline{R}(u_0,e_j)e_i + \kappa_i(z_0)\overline{R}(u_0,e_i)e_j - \kappa_j(z_0)\overline{R}(e_i,e_j)u_0\}^v_{z_0},\\
(\tilde{\nabla}_{E_i}J)E_\alpha(z_0) &= \frac{1}{2}\{\kappa_i(z_0)\overline{R}(u_0,e_i)\nu_\alpha\}^h_{z_0} - \frac{1}{2}\{\overline{R}(u_0,e_i)\nu_\alpha\}^v_{z_0}, \qquad (4.13)\\
(\tilde{\nabla}_{E_\alpha}J)E_i(z_0) &= \frac{1}{2}\{\kappa_i(z_0)\overline{R}(u_0,\nu_\alpha)e_i\}^h_{z_0} - \frac{1}{2}\{\overline{R}(u_0,\nu_\alpha)e_i\}^v_{z_0}, \qquad (4.14)\\
(\tilde{\nabla}_{E_\alpha}J)E_\beta(z_0) &= -\frac{1}{2}\{\overline{R}(u_0,\nu_\alpha)\nu_\beta\}^h_{z_0}. \qquad (4.15)
\end{aligned}$$

We denote the second fundamental form of νN by $\tilde{B}$. Then the mean curvature vector of νN in TM is defined by $\tilde{H} := \mathrm{tr}\tilde{B}$. We define

$$S_{\lambda\mu\nu} := \tilde{g}(J\tilde{B}(E_\lambda, E_\mu), E_\nu).$$

Then, by using (4.3) thorough (4.8), we have the following .

$$\begin{aligned}
S_{ijk}(z_0) &= E_i\langle A^u(e_j), e_k\rangle(z_0) + (\kappa_j(z_0) - \kappa_k(z_0))\langle\overline{\nabla}_{e_i}e_j, e_k\rangle \qquad (4.16)\\
&+ \frac{1}{2}\langle\overline{R}(u_0,e_k)e_i, e_j\rangle + \frac{1}{2}\kappa_i(z_0)\kappa_k(z_0)\langle\overline{R}(u_0,e_i)e_j, e_k\rangle\\
&- \frac{1}{2}\kappa_j(z_0)\kappa_k(z_0)\langle\overline{R}(u_0,e_j)e_k, e_i\rangle,\\
S_{ij\alpha}(z_0) &= \langle A^{\nu_\alpha}(e_i), e_j\rangle - \frac{1}{2}\kappa_i(z_0)\langle\overline{R}(u_0,e_i)e_j, \nu_\alpha\rangle \qquad (4.17)\\
&- \frac{1}{2}\kappa_j(z_0)\langle\overline{R}(u_0,e_j)e_i, \nu_\alpha\rangle,\\
S_{i\alpha j}(z_0) &= \langle A^{\nu_\alpha}(e_i), e_j\rangle - \frac{1}{2}\kappa_j(z_0)\langle\overline{R}(u_0,\nu_\alpha)e_i, e_j\rangle, \qquad (4.18)\\
S_{i\alpha\beta}(z_0) &= \frac{1}{2}\langle\overline{R}(u_0,\nu_\alpha)e_i, \nu_\beta\rangle, \qquad (4.19)\\
S_{\alpha\beta\lambda}(z_0) &= 0, \qquad (4.20)
\end{aligned}$$

where we use the Bianchi identity to derive the equality (4.16). We note that $S_{\lambda\mu\nu} = S_{\mu\lambda\nu}$ by definition. Moreover, it is notable that the equality

$$S_{\lambda\mu\nu} - S_{\lambda\nu\mu} = -\tilde{g}((\tilde{\nabla}_{E_\lambda}J)E_\mu, E_\nu) \qquad (4.21)$$

holds since νN is Lagrangian. Therefore, S is symmetric in three indices when J is integrable.

Lemma 4.1.2. *For the local frame $\{E_i, E_\alpha\}$, we have the following formula at $z_0 = (p_0, u_0)$.*

$$\begin{aligned}
S_{iij}(z_0) &= E_j\langle A^u(e_i), e_i\rangle(z_0) + (\kappa_i(z_0)\kappa_j(z_0) - 1)\langle\overline{R}(u_0,e_i)e_i, e_j\rangle,\\
S_{ii\alpha}(z_0) &= E_\alpha\langle A^u(e_i), e_i\rangle(z_0) - \kappa_i(z_0)\langle\overline{R}(u_0,e_i)e_i, \nu_\alpha\rangle.
\end{aligned}$$

PROOF. By (4.12) we have

$$(\tilde{\nabla}_{E_i} J)E_i(z_0) = -\frac{1}{2}(\kappa_i^2(z_0)-1)\{\overline{R}(u_0,e_i)e_i\}^h_{z_0} + \kappa_i(z_0)\{\overline{R}(u_0,e_i)e_i\}^v_{z_0}.$$

Combining this with (4.21), we obtain

$$\begin{aligned} S_{iij} - S_{jii} &= \left\{\frac{1}{2}(\kappa_i^2(z_0)-1) + \kappa_i(z_0)\kappa_j(z_0)\right\}\langle \overline{R}(u_0,e_i)e_i, e_j\rangle, \\ S_{ii\alpha} - S_{\alpha ii} &= -\kappa_i(z_0)\langle \overline{R}(u_0,e_i)e_i, \nu_\alpha\rangle. \end{aligned}$$

On the other hand, by (4.16) and (4.18), we have

$$\begin{aligned} S_{jii} &= E_j\langle A^u(e_i), e_i\rangle - \frac{1}{2}\left\{(1+\kappa_i^2(z_0))\langle \overline{R}(u_0,e_i)e_i, e_j\rangle\right\} \\ S_{\alpha ii} &= E_\alpha\langle A^u(e_i), e_i\rangle \end{aligned}$$

From these, we get the required formula. □

The main purpose of this section is to investigate the extrinsic properties of Lagrangian normal bundles in $(TM, \tilde{g})$. First, we observe the simplest case, namely, totally geodesic case:

PROPOSITION 4.1.3. *Let N^n be a totally geodesic submanifold in a Riemannian manifold $(M^m, \langle,\rangle)$. Then the normal bundle νN is a minimal Lagrangian submanifold in $(TM, \tilde{g})$.*

PROOF. Suppose N^n is a totally geodesic submanifold in $(M^m, \langle,\rangle)$. For i, j, k, we have from (4.16)

$$\begin{aligned} S_{ijk}(z_0) &= \frac{1}{2}\langle R(e_i,e_j)u_0, e_k\rangle = -\frac{1}{2}\langle \overline{R}(e_i,e_j)e_k, u_0\rangle \\ &= -\frac{1}{2}\langle (\nabla^\perp_{e_i}\beta)(e_j,e_k) - (\nabla^\perp_{e_j}\beta)(e_i,e_k), u_0\rangle = 0, \end{aligned}$$

where $\beta(\equiv 0)$ is the second fundamental form of N in M, and we use the Codazzi equation. Moreover, we see $S_{iij} = S_{ii\alpha} = S_{\alpha\alpha i} = S_{\alpha\alpha\beta} = 0$ at any point z_0 by (4.16) through (4.20). This implies $\mathrm{tr}\overline{B}(z_0) = 0$. Thus, we obtain the proposition. □

The following proposition shows an obstruction for a normal bundle to be totally geodesic.

LEMMA 4.1.4. *Let N^n be a submanifold in a Riemannian manifold $(M^m, \langle,\rangle)$. The normal bundle νN is totally geodesic in $(TM, \tilde{g})$ if and only if N is totally geodesic and any normal space $\nu_p N$ for $p \in N$ is curvature invariant, namely, $\overline{R}(u,v)w \in \nu_p N$ for any $u, v, w \in \nu_p N$.*

Proof. Assume νN is totally geodesic, namely, $S_{\lambda\mu\nu} \equiv 0$ on νN. Fix a point $z_0 = (p_0, u_0) \in \nu N$, and take the orthonormal basis $\{e_i\}_{i=1}^n$ as above. For any $t \in \mathbb{R}$, each e_i is the principal direction with respect to the point (p_0, tu_0), i.e., $A^{tu_0}(e_i) = t\kappa_i(z_0)e_i$. Set a function of $t \in \mathbb{R}$ by $f_{\lambda\mu\nu}^{z_0}(t) := S_{\lambda\mu\nu}(p_0, tu_0)$. Then by (4.16), we see that $f_{\lambda\mu\nu}^{z_0}$ is a polynomial in t, and by the assumption, this polynomial is trivial. For instance, $f_{ij\alpha}^{z_0} \equiv 0$ implies $\langle A^{\nu_\alpha}(e_i), e_j\rangle = 0$ for any i, j, α. This means that N is totally geodesic. In particular, we have $\kappa_i(z_0) = 0$ for any i. Moreover, N must satisfy

$$\langle \overline{R}(u_0, e_k)e_i, e_j\rangle = 0, \text{ and} \tag{4.22}$$

$$\langle \overline{R}(u_0, \nu_\alpha)\nu_\beta, e_i\rangle = 0 \tag{4.23}$$

by (4.16) and (4.19) for any i, j, k, α, β. Here, (4.22) is automatically satisfied since N is totally geodesic, and we use the Codazzi equation. (4.23) implies $(\overline{R}(u_0, v)w)^\top = 0$ for any $v, w \in \nu_{p_0}N$. Because z_0 is arbitrary, N satisfies the required properties. The converse obviously follows. □

Suppose $M \simeq U/K$ is a Riemannian symmetric space with the canonical decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{m}$. Then, it follows from the result of E. Cartan (cf. [10], Theorem 8.3.1 or Corollary 9.1.1), a subspace $V \subset T_pM$ is curvature invariant if and only if there exist a totally geodesic submanifold $N^\perp$ of M such that $T_pN^\perp = V$ and $p \in N^\perp$, or equivalently, V is a Lie triple system in $\mathfrak{m}$, namely, $[[V, V], V] \subset V$.

Let N be a connected component of the fixed point set of an involutive isometry σ_N on a complete Riemannian manifold M. We call N a *reflective* submanifold, and the involution σ_N is called the *reflection* of N. Then, N is automatically totally geodesic. However, the converse is not true in general. For instance, a submanifold N in the complex space form $\mathbb{C}P^m$ (resp. $\mathbb{C}H^m$) is reflective if and only if N is a totally geodesic complex submanifold $\mathbb{C}P^n$ (resp. $\mathbb{C}H^n$) where $n = 1, \cdots, m-1$, or the totally geodesic Lagrangian submanifold $\mathbb{R}P^m$ (resp. $\mathbb{R}H^m$).

Let N be a reflective submanifold in a Riemannian symmetric space $M = U/K$. Denote the reflection of N by σ_N and the geodesic symmetry of M at p by σ_p. Then, the isometry $\sigma_M \circ \sigma_p$ is involutive, and the connected component $N^\perp$ of the fixed point set of $\sigma_M \circ \sigma_p$ is a totally geodesic submanifold such that $p \in N^\perp$ and $T_pN^\perp = \nu_pN$. Therefore, ν_pN is curvature invariant for each $p \in N$ by Cartan's theorem. Conversely, if M is simply-connected, and V is a curvature invariant subspace in T_pM satisfying $V^\perp \subset T_pM$ is also curvature invariant, then there exists a reflective submanifold N of M with $p \in N$ and $T_pN = V$ (see Proposition 9.1.6 in [10]). Combining this with Lemma 4.1.4, we obtain:

Theorem 4.1.5. *Let N be a connected submanifold (possibly, a point) in a simply* *connected Riemannian symmetric space M. Then the normal bundle νN is totally geodesic in $(TM, \tilde{g})$ if and only if N is a reflective submanifold.*

Next, we define the mean curvature form of a normal bundle νN by $\alpha_{\tilde{H}} := \omega(\tilde{H}, \cdot)|_{T\nu N} = \tilde{g}(J\tilde{H}, \cdot)|_{T\nu N}$, where $\tilde{H}$ is the mean curvature vector of νN.

LEMMA 4.1.6. *Let N^n be a submanifold in a Riemannian manifold $(M^m, \langle , \rangle)$. Denote the eigenvalue of the shape operator A of N at a point $z = (p, u) \in \nu N$ by $\{\kappa_i(z)\}_{i=1}^n$. Then the mean curvature form of the Lagrangian normal bundle νN in $(TM, \tilde{g})$ is given by*

$$\alpha_{\tilde{H}} = d\theta + \mathcal{R},$$

where

$$\theta(z) \quad := \quad \sum_{i=1}^{n} \arctan \kappa_i(z) \tag{4.24}$$

and $\mathcal{R}$ is an 1-form which is expressed by

$$\begin{aligned} \mathcal{R}(z_0) \quad := \quad & \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \frac{\kappa_i(z_0)\kappa_j(z_0) - 1}{1 + \kappa_i^2(z_0)} \langle \overline{R}(u, e_i)e_i, e_j \rangle \Big) E_j^* \\ - \quad & \sum_{\alpha=n+1}^{m} \Big(\sum_{i=1}^{n} \frac{\kappa_i(z_0)}{1 + \kappa_i^2(z_0)} \langle \overline{R}(u, e_i)e_i, \nu_\alpha \rangle \Big) E_\alpha^* \end{aligned}$$

at $z_0 \in \nu N$, by using the local frame $\{E_j, E_\alpha\}$ around z_0 given in Lemma 4.1.1.

PROOF. Since $B(E_\alpha, E_\alpha) = 0$, we have

$$\alpha_{\tilde{H}}(z_0) = \sum_{i=1}^{n} \frac{1}{1 + \kappa_i^2(z_0)} \tilde{g}(J\tilde{B}(E_i, E_i), \cdot).$$

Here, by Lemma 4.1.2, we obtain

$$\begin{aligned} \alpha_{\tilde{H}}(E_j)(z_0) \quad &= \quad \sum_{i=1}^{n} \frac{1}{1 + \kappa_i^2(z_0)} S_{iij} \\ &= \quad E_j \Big(\sum_{i=1}^{n} \arctan \langle A^u(e_i), e_i \rangle \Big)(z_0) + \sum_{i=1}^{n} \frac{\kappa_i(z_0)\kappa_j(z_0) - 1}{1 + \kappa_i^2(z_0)} \langle \overline{R}(u_0, e_i)e_i, e_j \rangle, \\ \alpha_{\tilde{H}}(E_\alpha)(z_0) \quad &= \quad \sum_{i=1}^{n} \frac{1}{1 + \kappa_i^2(z_0)} S_{ii\alpha} \\ &= \quad E_\alpha \Big(\sum_{i=1}^{n} \arctan \langle A^u(e_i), e_i \rangle \Big)(z_0) - \sum_{i=1}^{n} \frac{\kappa_i(z_0)}{1 + \kappa_i^2(z_0)} \langle \overline{R}(u_0, e_i)e_i, \nu_\alpha \rangle, \end{aligned}$$

and hence,

$$\alpha_{\tilde{H}}(z_0) = d\Big(\sum_{i=1}^{n} \arctan\langle A^u(e_i), e_i\rangle\Big)(z_0) + \mathcal{R}(z_0).$$

However, one can see that the first term coincides with $d\theta$ (see the proof of Lemma 3.2.1). $\square$

COROLLARY 4.1.7. *Let N^n be a submanifold in a Riemannian manifold $(M^m, \langle,\rangle)$. If the normal bundle νN is minimal in $(TM, \tilde{g})$, then N is a minimal submanifold in $(M, \langle,\rangle)$.*

PROOF. Suppose $\alpha_{\tilde{H}} = 0$. Then, we have from Lemma 4.1.6, $d\theta = -\mathcal{R}$. Substituting the canonical vertical vector $U := u^v_z$, we obtain the equality $U(\theta) = -\mathcal{R}(U)$, and this implies

$$\sum_{i=1}^{n} \frac{t\kappa_i(z_0)}{1+t^2\kappa_i^2(z_0)} = \sum_{i=1}^{n} \frac{t^3\kappa_i(z_0)}{1+t^2\kappa_i^2(z_0)} \langle \overline{R}(u_0, e_i)e_i, u_0\rangle \tag{4.25}$$

for a point $z_0 = (p_0, u_0) \in \nu N$ and $t \in \mathbb{R}$, since we can choose the same eigenvectors $\{e_i\}_{i=1}^n$ for each (p_0, tu_0). Taking $t \to 0$ as $t \neq 0$, we have $\sum_{i=1}^n \kappa_i(z_0) = 0$ from (4.25). Thus, N is a minimal submanifold in M. $\square$

We note that the converse of this corollary does not hold in general. If the ambient manifold M is flat, namely, $\overline{R} = 0$, Lemma 4.1.6 implies that the normal bundle νN is minimal if and only if $d\theta = 0$. However, the latter condition is equivalent to $\theta \equiv 0 \pmod{\pi}$ by the same argument as in the proof of Proposition 3.1.3. A submanifold N with $\theta \equiv 0$ is called *austere* of which notion is stronger than minimal.

LEMMA 4.1.8. *For a connected submanifold N in a Riemannian manifold $(M, \langle,\rangle)$, the following conditions are equivalent:*

(i) $\theta := \sum_{i=1}^{n} \arctan \kappa_i \equiv 0 \pmod{\pi}$ *on* νN.

(ii) $d\theta = 0$ *on* νN.

(iii) $U(\theta) := d\theta(U) = 0$ *on* νN, *where* $U := u^v_z$ *is the canonical vertical vector field.*

(iv) $P_{2k+1}(z_0) := \sum_{i=1}^{n} \kappa_i^{2k+1}(z_0) = 0$ *for any* $z_0 \in \nu N$ *and* $k \in \mathbb{Z}_{\geq 0}$.

(v) $S_{2k+1}(z_0) = 0$ *for any* $z_0 \in \nu N$ *and* $k = 1, \ldots, [(n-1)/2]$, *where* $S_l(z_0)$ *is the* l*-th elementary symmetric polynomial in* $\{\kappa_i(z_0)\}_{i=1}^n$.

(vi) The set $\{\kappa_i(z_0)\}_{i=1}^n$ is invariant under the multiplication by -1 for any $z_0 \in \nu N$.

PROOF. We have already shown that $(i) \Leftrightarrow (ii) \Leftrightarrow (v)$ in Proposition 3.1.3. $(i) \Rightarrow (iii)$ and $(vi) \Rightarrow (i)$ are obvious.

$(iii) \Rightarrow (iv)$: Suppose $U(\theta) = 0$ on νN. This is equivalent to

$$U(\theta)(p_0, tu_0) = \frac{d}{d\tau}\Big|_{\tau=0} \sum_{i=1}^n \arctan \kappa_i(p_0, (\tau+1)tu_0) = \sum_{i=1}^n \frac{t\kappa_i(z_0)}{1+t^2\kappa_i^2(z_0)} = 0$$

for any $z_0 = (p_0, u_0) \in \nu_1 N$ and $t \in \mathbb{R}$. Then the Taylor expansion at $t = 0$ implies that

$$\sum_{k=0}^{\infty} (-1)^k \Big(\sum_{i=1}^n \kappa_i^{2k+1}(z_0) \Big) t^{2k+1} = 0$$

on a small neighborhood of $t = 0$. Thus, we have (iv).

$(iv) \Rightarrow (v)$: By the Newton's formula, we have

$$(2k+1) S_{2k+1}(z_0) = \sum_{j=1}^{2k+1} (-1)^{j-1} S_{2k+1-j}(z_0) P_j(z_0)$$

for $k \geq 1$, where $P_j(z_0) := \sum_{i=1}^n \kappa_i^j(z_0)$. Suppose $P_{2k+1}(z_0) = 0$ for $k \geq 0$. Then, by an inductive argument, we obtain $S_{2k+1}(z_0) = 0$ for any $k \geq 1$.

$(v) \Rightarrow (vi)$: It obviously follows from the relation

$$\prod_{i=1}^n (x - \kappa_i(z_0)) = \sum_{k=0}^n (-1)^k S_{n-k}(z_0) x^k.$$

$\square$

DEFINITION 4.1.9. We call a submanifold N in a Riemannian manifold $(M, \langle , \rangle)$ *austere* if it satisfies one of the conditions in Lemma 4.1.8.

It is obvious that an austere submanifold is automatically a minimal submanifold.

EXAMPLE 4.1.10. (1) A surface in a Riemannian manifold is austere if and only if it is a minimal submanifold.

(2) Any complex submanifold in a Kähler manifold is austere.

(3) All austere orbits of s-representations in a sphere S^{n+1} are classified by Ikawa-Sakai-Tasaki [40]. For instance, for an isoparametric hypersurface in S^{n+1} having the principal curvatures with the same multiplicity, there exist an unique austere hypersurface in the parallel hypersurfaces.

(4) Any focal variety of isoparametric hypersurfaces in a sphere is austere [41]. As a generalization of this fact, Ge and Tang proved that any focal variety of an isoparametric hypersurface with constant principal curvatures in a Riemannian manifold is austere [28].

(5) Any singular orbit of cohomogeneity one action on a Riemannian manifold is austere [40]. In fact, these singular orbits are contained in the class of *weakly reflective submanifolds*. In general, a submanifold N in a Riemannian manifold M is called weakly reflective if, for each normal vector $u \in \nu_p N$ and $p \in N$, there exist an isometry σ_u on M such that

$$\sigma_u(p) = p, \quad (d\sigma_u)_p(u) = -u, \quad \sigma_u(N) = N.$$

This σ_u is called the *reflection* of N with respect to u. A reflective submanifold is also weakly reflective. A weakly reflective submanifold is automatically austere submanifold (see Proposition 2.5 in [40]). In [40], they classified all the weakly reflective orbits of s-representations.

(6) If N is an austere submanifold in the sphere $S^{n+1}(1)$, then the cone $C(N) := \{tp \in \mathbb{R}^{n+2};\ p \in N,\ r \in \mathbb{R}_{>0}\}$ of N is austere in $\mathbb{R}^{n+2}$. Furthermore, suppose N is contained in an odd-dimensional sphere $S^{2n+1}(1) \subset \mathbb{C}^{n+1}$ and S^1-invariant under the Hopf action. Let $\overline{N}$ be the projection of N via the Hopf fibration. In general, $\overline{N}$ is *not* an austere submanifold in $\mathbb{C}P^n$, but if N is weakly reflective and the reflections are unitary transformations, then $\overline{N}$ is weakly reflective in $\mathbb{C}P^n$ [40]. For instance, we obtain examples of austere submanifold in $\mathbb{C}P^n$ from weakly reflective orbits of the s-representation of an irreducible compact Hermitian symmetric pair.

(7) Example given in [85] (Thanks to Dr. M. Domínguez-Vázquez's information). Consider the complex hyperbolic space $\mathbb{C}H^n = G/K = SU(1,n)/S(U(1) \times U(n))$ of holomorphic sectional curvature -1 with $n \geq 2$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebra of G and K, respectively, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Since $\mathbb{C}H^n$ is a rank 1 symmetric space, $\dim_{\mathbb{R}}\mathfrak{a} = 1$. Then, we have a root space decomposition $\mathfrak{g} = \mathfrak{g}_{-2\alpha} + \mathfrak{g}_{-\alpha} + \mathfrak{g}_0 + \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$. Here, $\dim_{\mathbb{R}}\mathfrak{g}_{\pm\alpha} = 2(n-1)$ and $\dim_{\mathbb{R}}\mathfrak{g}_{\pm 2\alpha} = 1$ (see [97]). We take an ordering on $\mathfrak{a}$ so that the root α is a positive root. Let $\mathfrak{n} := \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$. Then, $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is the Iwasawa decomposition of $\mathfrak{g}$. Put $\mathfrak{s} := \mathfrak{a} + \mathfrak{n}$. Then, $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$, and the Lie subgroup S which corresponds to $\mathfrak{s}$ is a solvable Lie group. Moreover, S acts on $\mathbb{C}H^n$ simply transitively, and the isotropy subgroup at the origin o is trivial. Thus, we obtain the diffeomorphism $\mathbb{C}H^n \simeq S \cdot o \simeq S$. Under the identification,

we introduce a Kähler structure on S from $\mathbb{C}H^n$, and we identify S with $\mathbb{C}H^n$ as a Kähler manifold (see [97] for more details).

Let $\mathfrak{w}$ be a subspace of $\mathfrak{g}_\alpha$ and $\mathfrak{w}^\perp$ the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$. Then, $\mathfrak{s}_\mathfrak{w} := \mathfrak{a} + \mathfrak{w} + \mathfrak{g}_{2\alpha}$ is a solvable Lie subalgebra of $\mathfrak{s}$. Let $S_\mathfrak{w}$ be the corresponding connected Lie subgroup of S whose Lie algebra is $\mathfrak{s}_\mathfrak{w}$. We definite a submanifold $W_\mathfrak{w}$ as the orbit $S_\mathfrak{w} \cdot o$. Then $W_\mathfrak{w}$ is a homogeneous submanifold of $\mathbb{C}H^n$ with codimention $k := \dim_\mathbb{R} \mathfrak{w}^\perp$, and $W_\mathfrak{w}$ is an austere submanifold. Moreover, a tube M^r of radius $r > 0$ around $W_\mathfrak{w}$ is an isoparametric hypersurface in $\mathbb{C}H^n$ which has, in general, non-constant principal curvatures.

We refer the reader to [37] and [41] for other known examples of austere submanifolds.

In surface case, the minimality is equivalent to the *austere* condition. Corollary 4.1.7 implies that the austere condition is necessary for the minimality of normal bundle of any *surface* in a Riemannian manifold. Conversely, the next proposition gives an obstruction of the minimality of normal bundles of minimal surfaces.

Lemma 4.1.11. *Let N^2 be a minimal surface in a Riemannian manifold $(M^m, \langle , \rangle)$. Then, the normal bundle of νN is minimal in $(TM, \tilde{g})$ if and only if N^2 satisfies one of the following conditions:*

1. *N^2 is totally geodesic, or*
2. *$\overline{R}(u, e_1)e_1 = \overline{R}(u, e_2)e_2$ for any $u \in \nu_p N$ and $p \in N$, where $\{e_1, e_2\}$ is the eigenvectors of the shape operator A^u.*

Proof. Since N is austere, we have $\theta = 0$ on νN. Moreover, the 1-form $\mathcal{R}$ becomes

$$\begin{aligned}\mathcal{R}(z_0) &= -\langle \overline{R}(u, e_2)e_2, e_1\rangle E_1^* - \langle \overline{R}(u, e_1)e_1, e_2\rangle E_2^* \\ &\quad + \sum_{\alpha=3}^{m} \frac{\kappa_1(z_0)}{1+\kappa_1^2(z_0)} \Big(\langle \overline{R}(u, e_1)e_1, \nu_\alpha\rangle - \langle \overline{R}(u, e_2)e_2, \nu_\alpha\rangle \Big) E_\alpha^*.\end{aligned}$$

When N is totally geodesic, $\mathcal{R}$ identically vanishes (cf. Proposition 3.1.3). Assume N is not totally geodesic, i.e., $\kappa_1 \neq 0$. If νN is minimal, then $\mathcal{R} \equiv 0$ on νN by Lemma 4.1.6. Since $\{E_i, E_\alpha\}$ are lineally independent, this is equivalent to

$$\langle \overline{R}(u, e_2)e_2, e_1\rangle = \langle \overline{R}(u, e_1)e_1, e_2\rangle = 0, \text{ and} \tag{4.26}$$

$$\langle \overline{R}(u, e_1)e_1, \nu_\alpha\rangle = \langle \overline{R}(u, e_2)e_2, \nu_\alpha\rangle \text{ for } \alpha = 3, \dots, m. \tag{4.27}$$

(4.26) implies that $\overline{R}(u, e_i)e_i$ $(i = 1, 2)$ is a normal vector of N. Hence, by (4.27), we conclude $\overline{R}(u, e_1)e_1 = \overline{R}(u, e_2)e_2$. □

We see that the second obstruction in Lemma 4.1.11 always vanishes when the ambient space is the real space form. Thus, any minimal surface in the real space form has a minimal normal bundle in $(TM, \tilde{g})$. This fact is generalized in the next section. However, we see in Section 4.4, there exist a minimal surface in the non-flat complex space form which does not have minimal normal bundle.

4.2 On the minimality of normal bundles in tangent bundles of the real space forms

In this section, we assume that *the ambient space M is the real space form $M(c)$*, that is, a complete Riemannian manifold with constant sectional curvature c. It is well-known that if M is simply connected and $c > 0$, then $M(c) = S^m(c)$ the sphere, if $c = 0$, then $M(c) = \mathbb{R}^m$ Euclidean space, and if $c < 0$, then $M(c) = \mathbb{H}^m(c)$ Hypabolic space. The curvature tensor of the space form $M(c)$ is given by

$$\overline{R}(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y).$$

PROPOSITION 4.2.1. *Let N be a submanifold in the real space form $M = M(c)$. Then the mean curvature form of νN in $(TM, \tilde{g})$ is given by*

$$\alpha_{\tilde{H}} = d\theta - cU(\theta)U^*$$

where θ is defined by (4.24), U is the canonical vertical vector field which is defined by $U_z := u_z^v$, and $U(\theta) := d\theta(U)$.

PROOF. Since M is a real space form, we have

$$\begin{aligned} \langle \overline{R}(u,e_i)e_i, e_j\rangle &= c\{\langle e_i,e_i\rangle\langle u,e_j\rangle - \langle u,e_i\rangle\langle e_i,e_j\rangle\} = 0, \\ \langle \overline{R}(u,e_i)e_i, \nu_\alpha\rangle &= c\{\langle e_i,e_i\rangle\langle u,\nu_\alpha\rangle - \langle ue_i\rangle\langle e_i,\nu_\alpha\rangle\} = c\langle u,\nu_\alpha\rangle \end{aligned}$$

and hence,

$$\mathcal{R}(z_0) = -\sum_{\alpha=n+1}^{m}\Big(\sum_{i=1}^{n}\frac{\kappa_i(z_0)}{1+\kappa_i^2(z_0)}c\langle u,\nu_\alpha\rangle\Big)E_\alpha^* = -cU(\theta)U^*(z_0).$$

Therefore, by Lemma 4.1.6, we obtain the required equality. □

As a generalization of Proposition 3.1.3 (or Theorem 3.11 in [31]), the following proposition was first obtained by Cintract-Morvan [23]. We give another proof using the mean curvature formula.

THEOREM 4.2.2. *Let N be a connected submanifold in the real space form $M = M(c)$. Then the normal bundle νN is a minimal Lagrangian submanifold in $(TM, \tilde{g})$ if and only if N is austere in M.*

PROOF. By Proposition 4.2.1, the normal bundle νN is minimal in $(TM, \tilde{g})$ if and only if the following equality holds:

$$d\theta = cU(\theta)U^*, \tag{4.28}$$

on νN, where $\theta := \sum_{i=1}^n \mathrm{arctan}\kappa_i$. If N is austere, then $\theta = 0$ and the equality (4.28) holds. Conversely, when (4.28) holds, substituting the canonical vertical vector U to (4.28), we have $U(\theta) = cU(\theta)|u|^2$ for each point $z = (p, u) \in \nu N$. If $c = 0$ or $c \neq 0$ and $|u|^2 \neq 1/c$, this equality implies $U(\theta) = 0$. However, $U(\theta)$ is continuous on νN, the function $U(\theta)$ vanishes on νN. Thus N is austere by Lemma 4.1.8. □

4.3 Unit normal bundles

Mean curvature formula

In this section, we investigate the minimality of unit normal bundles in the unit tangent bundle over the space form. The unit tangent bundle is a hypersurface in the tangent bundle TM which is defined by $T_1M := \{z = (p, u) \in TM \mid \langle u, u\rangle = 1\}$. The *canonical vertical vector* U is a vector field on TM which is defined by $U_z := u^v_z$. For each $z \in T_1M$, the canonical vertical vector U_z gives an unit normal vector of T_1M in $(TM, \tilde{g})$. The almost Kähler structure on TM indues a contact metric structure (ϕ, ξ, η, g_1) on T_1M as follows (we refer to [13]): The Reeb vector field $\xi := -2JU$, the contact 1-form $\eta := \frac{1}{4}\tilde{g}(\xi, \cdot)|_{T_1M}$, the $(1,1)$-tensor $\phi := J - 2\eta \otimes U$, and the Riemannian metric $g_1 := \frac{1}{4}\tilde{g}$. Note that this contact metric strucure becomes K-contact (or Sasaki) if and only if $(M, \langle , \rangle)$ has constant curvature 1 (see [13]).

Let N be a submanifold in M. The *unit normal bundle* is a submanifold in $\nu_1 N$ defined by $\nu_1 N := \{z = (p, u) \in \nu N \mid \tilde{g}(u, u) = 1\}$. By a similar argument as in the previous section, one can show that $\nu_1 N$ is a Legendrian submanifold in $(T_1M, \phi, \xi, \eta, g_1)$ i.e., $\eta|_{\nu_1 N} = 0$. Using the Legendrian condition, we have the following isomorphism:

$$\begin{array}{ccc} \Gamma(\nu(\nu_1 N)) & \tilde{\rightarrow} & \Omega^1(\nu_1 N) \oplus C^\infty(\nu_1 N) \\ V & \mapsto & \Big(-\dfrac{1}{8}V \rfloor d\eta, \dfrac{1}{2}\eta(V)\Big), \end{array}$$

where $\rfloor$ denotes the inner product. Note that $-\frac{1}{2}V \rfloor d\eta = g_1(\phi V, \cdot)|_{\nu_1 N}$ in our notation. The following mean curvature formula is a generalization of Theorem 2.3.8 (see also the next subsection).

Proposition 4.3.1. *Let N be a submanifold in the real space form $M(c)$. Then the mean curvature vector H_1 of $\nu_1 N \to (T_1 M, g_1)$ corresponds to*

$$\Big(d(\theta|_{\nu_1 N}), (1-c)U(\theta)|_{\nu_1 N}\Big).$$

Proof. We have the following diagram:

$$\begin{array}{ccc} \nu N & \longrightarrow & TM \\ \uparrow & \nearrow & \uparrow \\ \nu_1 N & \longrightarrow & T_1 M \end{array}$$

We assume all manifolds in the above diagram admit the induced metric $\tilde{g}$. Denote the mean curvature vector of the immersions $\nu_1 N \to (T_1 M, \tilde{g})$ and $\nu_1 N \to (TM, \tilde{g})$ by H_1' and H_1'' respectively. By using the above diagram, one can easily show that $H_1'' = H + fU = H_1' + fU$ on $\nu_1 N$ for some function $f \in C^\infty(\nu_1 N)$. Hence, we have $H_1' = H|_{\nu_1 N}$.

By Theorem 4.2.1, we have $H = -J\nabla\theta + cU(\theta)JU$, where $\theta := \sum_{i=1}^n \arctan \kappa_i$. Since the canonical vertical vector field U is tangent to νN and is the unit normal vector of $\nu_1 N$ in νN, we have $\nabla\theta|_{\nu_1 N} = \nabla'\theta' + U(\theta)|_{\nu_1 N} U$ where $\theta' := \theta|_{\nu_1 N}$, and $\nabla'\theta'$ is the gradient of θ' on $(\nu_1 N, \tilde{g})$. Moreover, since νN is Lagrangian, we have $2\eta(\nabla\theta) = -\tilde{g}(J\nabla\theta, U) = 0$ on $\nu_1 N$. Thus, we obtain

$$H_1' = H|_{\nu_1 N} = -\phi(\nabla'\theta') + (c-1)U(\theta)|_{\nu_1 N} JU. \tag{4.29}$$

If we take the metric on $\nu_1 N$ by $g_1 = \frac{1}{4}\tilde{g}$, the relation between the mean curvature vectors is given by $H_1 = 4H_1'$. Hence, we have from (4.29) that $H_1 = -4\phi(\nabla'\theta') + 2(1-c)U(\theta)|_{\nu_1 N}\xi$. This implies the required correspondence. □

Corollary 4.3.2. *Let N be an austere submanifold in the real space form $M = M(c)$. Then the unit normal bundle $\nu_1 N$ is a minimal Legendrian submanifold in $(T_1 M, g_1)$.*

We remark that the converse of Corollary 4.3.2 does not hold in general. Such an example may be found below.

The case $M = S^m(1)$

We consider the case $M = S^m(1)$. In this case, the contact metric structure (ϕ, ξ, η, g_1) on the unit tangent bundle $T_1 S^m$ becomes a Sasaki strucure [13]. This standard metric is not Einstein, but η-Einstein. In fact, the Ricci tensor of g_1 is given by

$$\mathrm{Ric} = 2(2m-3)g_1 + 2(2-m)\eta \otimes \eta.$$

Thus, if we take a D-homothetic deformation

$$\eta' := \alpha\eta, \quad \xi' := \frac{1}{\alpha}\xi, \quad g' := \alpha g_1 + (\alpha^2 - \alpha)\eta \otimes \eta$$

with $\alpha := 2(m-1)/m$, then (ϕ, ξ', η', g') is a $SO(m+1)$-invariant Sasaki-Einstein structure on T_1S^m with Einstein constant $2(m-1)$ (cf. [100]). Since the mean curvature vector of a Legendrian submanifold is D-homothetic invariant (see Appendix A.3), by Corollary 4.1, we get minimal Legendrian submanifolds in the Sasaki-Einstein manifold T_1S^m from austere submanifolds in S^m. Moreover, T_1S^m is diffeomorphic to the Stiefel manifold $V_2(\mathbb{R}^{m+1}) \simeq SO(m+1)/SO(m-1)$, which is the principal S^1 bundle over the oriented 2-plane Grasmann manifold $\tilde{Gr}_2(\mathbb{R}^{m+1}) \simeq SO(m+1)/SO(2) \times SO(m-1)$. Denote the standard projection by $\overline{\pi} : V_2(\mathbb{R}^{m+1}) \to \tilde{Gr}_2(\mathbb{R}^{m+1})$. Then the following three are equivalent (cf. [74]):

(i) *L is a minimal Legendrian submanifold in the Sasaki-Einstein manifold $T_1S^m \simeq V_2(\mathbb{R}^{m+1})$.*

(ii) *The cone $C(L)$ of L is a special Lagrangian submanifold of some phase in the Ricci-flat Kähler cone $(C(V_2(\mathbb{R}^{m+1})), G) = (V_2(\mathbb{R}^{m+1}) \times \mathbb{R}_+, r^2 g + dr^2)$.*

(iii) *The projected image $\overline{\pi}(L)$ is a minimal Lagrangian submanifold in $\tilde{Gr}_2(\mathbb{R}^{m+1})$.*

By the mean curvature formula and (ii) in the above, we summarize

Corollary 4.3.3. *Let N be a submanifold in the unit sphere $S^m(1)$. If N is austere, or more generally, a submanifold such that $\theta|_{\nu_1 N}$ is constant, then the cone of the unit normal bundle $\nu_1 N$ is a special Lagrangian submanifold of some phase in $C(T_1S^m)$.*

We explain the geometrical interpretation of (iii). For $t > 0$, define a map ψ_t as follows:

$$\begin{array}{rcl} \psi_t : \nu_1 N & \to & S^m(1) \\ (p, u) & \mapsto & \exp_p(tu) \end{array}$$

If t is small enough, the map ψ_t is an immersion. Thus, the unit normal bundle $\nu_1 N$ is realized as the *tubular hypersurface* of N in S^m. For an oriented hypersurface P in S^m, the *Gauss map* is the map: $\mathcal{G} : P \to \tilde{Gr}_2(\mathbb{R}^{m+1})$, $p \mapsto \mathbf{p} \wedge \mathbf{n}$ where $\mathbf{p}$ is the position vector of p, and $\mathbf{n}$ is the unit normal vector at p of P in S^m. Then we have $\mathcal{G} = \overline{\pi} \circ \iota$:

$$\begin{array}{ccccc} & \nu N & \xrightarrow{\text{min.}} & TS^m & \\ & \cup & & \cup & \\ & \nu_1 N & \xrightarrow{\text{min.}} & T_1S^m \simeq V_2(\mathbb{R}^{m+1}) & \\ \text{exp} \swarrow & \downarrow & \searrow^{\text{min.}} & \downarrow \overline{\pi} & \\ S^m & \underset{\text{aust.}}{\leftarrow} N & \mathcal{G} & \tilde{Gr}_2(\mathbb{R}^{m+1}) & \end{array}$$

In [84], Palmer considers which hypersurfaces in S^m have minimal Gauss maps. The following corollary gives an example of minimal Gauss map including his result.

Corollary 4.3.4. *Let N be an oriented submanifold in $S^m(1)$ with the property given in Corollary 4.3.3 . Then the tubular hypersurface around N in S^m has a minimal Gauss map into the complex hyperquadric $Q_{m-1}(\mathbb{C}) \simeq \tilde{Gr}_2(\mathbb{R}^{m+1})$, and hence it gives a minimal Lagrangian immersion.*

If we take N as an oriented connected *hypersurface* in $S^m(1)$, then one of the connected component of $\nu_1 N$ is diffeomorphic to N. In this case, the tubular hypersurface is nothing but the parallel hypersurface of N. Moreover, in this case, the mean curvature formula in Proposition 4.3.1 coincides with the Palmer's formula given in Theorem 2.3.8. Thus Proposition 4.3.1 is a generalization of the formula. One can also prove Corollary 4.3.4 by computing the principal curvatures of a tubular hypersurface, and using the Palmer's formula.

The typical and large class of examples of Corollary 4.3.4 is the *isoparametric hypersurfaces.* It is known that any isoparametric hypersurfaces in a Riemannian manifold are tubular hypersurfaces of some focal manifolds. Moreover, the focal manifolds of any isoparametric hypersurfaces are austere (cf. Example 4.1.8). As a special case of this fact, the minimality of the Gauss maps of isoparametric hypersurfaces in S^m (Corollary 2.3.10) comes from the austere condition of the focal manifolds.

When an isoparametric hypersurface in $S^m(1)$ has the same multiplicity, then the (unique) minimal prallel hypersurface is austere. For these austere hypersurface, we obtain the following:

Proposition 4.3.5. *Let N^n be an austere isoparametric hypersurface with g distinct principal curvatures of the same multiplicity m in the unit sphere $S^{n+1}(1)$. Define a function on the unit normal bundle $\nu_1 N$ of N by $f_a(p,\nu) := \langle a, \nu_p \rangle$ for a constant vector $a \in \mathbb{R}^{n+2}$. Then we have $\Delta f_a = c_{g,m} f_a$, where Δ is the Laplace-Beltrami operator acting on $C^\infty(\nu_1 N)$ with respect to the induced metric from $(T_1 S^m, g_1)$, and $c_{g,m}$ is a constant which depends on g and m.*

Proof. First, we note that $\nu_1 N$ is diffeormorphic to N since N is a hypersurface, and sometimes, we may identify the function f_a as a function on N. We can choose a local orthonormal tangent frame $\{e_1, \ldots, e_n\}$ of N so that $A^\nu(e_i)(p) = \kappa_i(p) e_i$ around a point $p_0 \in N$. Then, $\{E_1, \ldots, E_n\}$ given in Lemma 4.1.1 is a local orthogonal frame of $\nu_1 N$. By using this frame and the isoparametric condition, we have

$$\Delta f_a = -\sum_{i=1}^{n} \frac{4}{1+\kappa_i^2} \{E_i^2(f_a) - \tilde{\nabla}'_{E_i} E_i(f_a)\}, \tag{4.30}$$

where $\tilde{\nabla}'$ denotes the Levi-Civita connection on $\nu_1 N$. Here, since N is an isoparametric hypersurface, we have

$$\begin{aligned} E_i(f_a) &= e_i(f_a) = -\kappa_i\langle a, e_i\rangle & (4.31) \\ E_i^2(f_a) &= e_i^2(f_a) = -\kappa_i\langle a, \overline{\nabla}_{e_i}e_i\rangle = -\kappa_i\langle a, \nabla_{e_i}e_i + \kappa_i\nu - \mathbf{p}\rangle \\ &= -\kappa_i^2 f_a + \kappa_i\langle a, \mathbf{p}\rangle - \kappa_i\sum_{j=1}^n\langle\nabla_{e_i}e_i, e_j\rangle\langle a, e_j\rangle, & (4.32) \end{aligned}$$

where ∇ denotes the Levi-Civita connection on N, and $\mathbf{p}$ is a position vector at $p \in N$ in $\mathbb{R}^{n+2}$. On the other hand, by (4.8), we see

$$\tilde{\nabla}'_{E_i}E_i = \sum_{j=1}^n \frac{1}{1+\kappa_j^2}\tilde{g}(\tilde{\nabla}_{E_i}E_i, E_j)E_j = \sum_{j=1}^n \frac{1+\kappa_i\kappa_j}{1+\kappa_j^2}\langle\nabla_{e_i}e_i, e_j\rangle E_j. \quad (4.33)$$

Substituting (4.31) through (4.33) into (4.30), we obtain

$$\begin{aligned} \frac{1}{4}\Delta f_a &= \Big(\sum_{i=1}^n \frac{\kappa_i^2}{1+\kappa_i^2}\Big)f_a - \Big(\sum_{i=1}^n \frac{\kappa_i}{1+\kappa_i^2}\Big)\langle a, \mathbf{p}\rangle \\ &\quad + \sum_{i,j=1}^n \frac{\kappa_i - \kappa_j}{(1+\kappa_i^2)(1+\kappa_j^2)}\langle\nabla_{e_i}e_i, e_j\rangle\langle a, e_j\rangle & (4.34) \end{aligned}$$

Suppose $\kappa_j \neq 0$. Then, we have

$$\begin{aligned} \langle\nabla_{e_i}e_j, e_k\rangle &= \frac{1}{\kappa_j}\langle\overline{\nabla}_{e_i}(A^\nu(e_j)), e_k\rangle \\ &= \frac{1}{\kappa_j}\{\langle(\overline{\nabla}_{e_i}A^\nu)(e_j), e_k\rangle + \langle A^\nu(\overline{\nabla}_{e_i}e_j), e_k\rangle\} \\ &= \frac{1}{\kappa_j}\{\langle(\overline{\nabla}_{e_i}A^\nu)(e_j), e_k\rangle + \kappa_k\langle\nabla_{e_i}e_j, e_k\rangle\} \end{aligned}$$

and hence,

$$(\kappa_j - \kappa_k)\langle\nabla_{e_i}e_j, e_k\rangle = \langle(\overline{\nabla}_{e_i}A^\nu)(e_j), e_k\rangle. \quad (4.35)$$

We note this equality holds even if $\kappa_j = 0$. Moreover, by using the Codazzi equation and since A^ν is a symmetric operator, we have

$$\langle(\overline{\nabla}_{e_i}A^\nu)(e_j), e_k\rangle = \langle(\overline{\nabla}_{e_k}A^\nu)(e_i), e_j\rangle.$$

Therefore, combining this with (4.35), we obtain

$$(\kappa_j - \kappa_k)\langle\nabla_{e_i}e_j, e_k\rangle = (\kappa_i - \kappa_j)\langle\nabla_{e_k}e_i, e_j\rangle.$$

This implies

$$\sum_{i,j=1}^n \frac{\kappa_i - \kappa_j}{(1+\kappa_i^2)(1+\kappa_j^2)}\langle\nabla_{e_i}e_i, e_j\rangle\langle a, e_j\rangle = 0$$

in (4.34). Therefore, for the unit normal bundle $\nu_1 N$ of an isoparametric hypersurface N, we obtain

$$\frac{1}{4}\Delta f_a = \Big(\sum_{i=1}^{n}\frac{\kappa_i^2}{1+\kappa_i^2}\Big) f_a - \Big(\sum_{i=1}^{n}\frac{\kappa_i}{1+\kappa_i^2}\Big)\langle a, \mathbf{p}\rangle.$$

Here, if we assume N is austere, the second term vanishes, and hence, f_a is an eigenfunction of Δ since $\kappa_1, \ldots, \kappa_n$ are constant. Moreover, by Münzner's formula (3.11), we see

$$\sum_{i=1}^{n}\frac{\kappa_i^2}{1+\kappa_i^2} = m\sum_{i=1}^{g}\cos^2\left(\theta_1 + \frac{i-1}{g}\pi\right) = mc_g,$$

where $c_g = 0, 1, 3/2, 2, 3$ when $g = 1, 2, 3, 4, 6$, respectively. $\square$

REMARK 4.3.6. The austere condition is essential in this proposition. We also remark that the eigenvalue $c_{g,m}$ of f_a is smaller than $4n$ which is the η-Einstein constant of $(T_1S^{n+1}, \phi, \xi, \eta, g_1)$. This implies that *the compact minimal Legendrian embedding $\nu_1 N \simeq N$ into T_1S^{n+1} is Legendrian unstable* in the sense of [80] (see also [46]), except the case $g = 1$. Since Proposition 4.3.5 is a local argument, we see that f_a is also an eigenfunction of the Gauss map of an austere isoparametric hypersurface N into $\tilde{Gr}_2(\mathbb{R}^{n+2})$. Then, the Legendrian instability corresponds to the Hamiltonian instability of the Gauss map (see Theorem 2.3.14).

4.4 On the minimality of normal bundles in the tangent bundle of the complex space form

From now on, we assume that the ambient space is the complex space form $M = M^m(4c)$ of the holomorphic sectional curvature $4c$ with complex dimension m. Then the curvature tensor of the complex space form $M(4c)$ is given by

$$\begin{aligned} \overline{R}(X,Y)Z &= c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle J_0Y, Z\rangle J_0X \\ &\quad -\langle J_0X, Z\rangle J_0Y + 2\langle X, J_0Y\rangle J_0Z\}, \end{aligned} \tag{4.36}$$

for $X, Y, Z \in \Gamma(TM)$, where J_0 denotes the almost complex structure of M.

Complex submanifolds

THEOREM 4.4.1. *Let N be a complex submanifold in the complex space form $M = M^m(4c)$. Then the normal bundle νN is a minimal Lagrangian submanifold in $(TM, \tilde{g})$.*

PROOF. We choose a local frame $\{E_i, E_\alpha\}$ around $z_0 \in \nu N$ given in Section 4.2. Recall that this frame gives an orthonormal basis $\{E'_i, E'_\alpha\}$ of $T_{z_0}\nu N$. We calculate the mean curvature at z_0. First, for a complex submanifold N in the complex space form, we have

$$\langle \overline{R}(u_0, e_i)e_i, e_j\rangle = 0, \ \langle \overline{R}(u_0, e_i)e_i, \nu_\alpha\rangle = c\langle u_0, \nu_\alpha\rangle, \tag{4.37}$$

since the tangent space of N is J_0-invariant, and M satisfies the curvature condition (4.36). Then by Lemma 4.1.6 and the same calculation as in the proof of Proposition 4.2.1, we obtain

$$\alpha_{\tilde{H}} = d\theta - cU(\theta)U^*. \tag{4.38}$$

On the other hand, any complex submanifold N in the complex space form is austere, we have $\theta = 0$ on νN. Combining this with (4.38), we obtain $\alpha_{\tilde{H}} = 0$. $\square$

REMARK 4.4.2. In the case $c = 0$, the statement of Theorem 4.4.1 has already appeared in [31].

Hopf hypersurfaces

A real hypersurface N^{2m-1} in a non-flat complex space form $M = M^m(4c)$ is called the *Hopf hypersurface* if the characteristic vector field $\xi := -J_0\nu$ is a principal direction of N (where J_0 is the complex structure of M and ν is a local normal vector of N in M). A real hypersurface in the complex space form M admits a contact structure induced from the Kähler structure of M. More precisely, we define the contact 1-form η by $\eta := \langle \xi, \cdot\rangle|_N$. Then Ker$\eta$ is a $2m-2$ dimensional distribution.

For the Hopf hypersurface N, we can take the local orthonormal tangent frame $\{e_1, \cdots, e_{2m-1}\}$ so that $e_{2m-1} = \xi$, and Kerη is spanned by $\{e_1, \cdots, e_{2m-2}\}$ around p_0 and $A(e_i)(p_0) = \kappa_i(z_0)e_i(p_0)$ for $i = 1, \cdots, 2m-1$. Then, by the curvature condition (4.30), we have

$$\begin{aligned} \langle \overline{R}(u_0, e_i)e_i, e_j\rangle &= 0, \text{ for } i, j = 1, \cdots, 2m-1, \\ \langle \overline{R}(u_0, e_i)e_i, \nu\rangle &= c\langle u_0, \nu\rangle, \text{ for } i = 1, \cdots, 2m-2, \\ \langle \overline{R}(u_0, \xi)\xi, \nu\rangle &= 4c\langle u_0, \nu\rangle. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \mathcal{R}(z_0) &= -c\Big(\sum_{i=1}^{2m-2} \frac{\kappa_i(z_0)}{1+\kappa_i^2(z_0)} + \frac{4\kappa_{2m-1}(z_0)}{1+\kappa_{2m-1}^2(z_0)}\Big)\langle u_0, \nu\rangle E^*_\nu \\ &= -c\Big\{U(\theta) + \frac{3\kappa_{2m-1}(z_0)}{1+\kappa_{2m-1}^2(z_0)}\Big\}U^* \end{aligned}$$

and hence, it follows

$$\alpha_{\tilde{H}} = d\theta - c\Big\{U(\theta) + \frac{3\kappa_{2m-1}}{1+\kappa_{2m-1}^2}\Big\}U^*. \tag{4.39}$$

THEOREM 4.4.3. *Let N^{2m-1} be a Hopf hypersurface with constant principal curvatures in the non-flat complex space form $M = M^m(4c)$. Then*

1. *if $c > 0$, νN is a minimal Lagrangian submanifold in $(TM, \tilde{g})$ if and only if m is odd and N is the tube of radius $\pi/(4\sqrt{c})$ over the totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^{2k+1}(4c)$.*
2. *if $c < 0$, there are no Hopf hypersurface with constant principal curvatures which gives a minimal normal bundle in $(TM, \tilde{g})$.*

PROOF. In this proof, we assume that the Hopf hypersurface N has constant principal curvatures.

(1) *The case $c > 0$.* By (4.39), the normal bundle νN is minimal in $(TM, \tilde{g})$ if and only if the following equality holds:

$$d\theta = c\{U(\theta) + 3\alpha\}U^*, \tag{4.40}$$

where $\theta := \sum_{i=1}^{2m-1} \tan^{-1}\kappa_i$, and $\alpha := \kappa_{2m-1}/(1+\kappa_{2m-1}^2)$. If N is the tube of radius $\pi/(4\sqrt{c})$ over totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^{2k+1}(4c)$, then we have $\theta = \kappa_{2m-1} = 0$ on νN, and hence, the equality (4.40) holds. Conversely, assume the equality (4.40) holds. Substituting the canonical vertical vector U into (4.40), we have

$$(1 - c|u|^2)U(\theta) = 3c\alpha|u|^2, \tag{4.41}$$

for any normal vector u. Since $c > 0$, we can take a normal vector u_0 so that $|u_0|^2 = 1/c$. Then we have $\alpha(u_0) = 0$, and hence $\kappa_{2m-1}(p_0, u_0) = 0$. This implies that $\kappa_{2m-1}(p_0, u) = 0$ for any $u \in \nu_{p_0}N$. Since p_0 is any, we have $\kappa_{2m-1} = 0$ on νN. By [2], such a Hopf hypersurface is locally congruent to the tube of radius $\pi/(4\sqrt{c})$ over a complex submanifold in $M = \mathbb{C}P^m(4c)$. From this information, and by using the classification theorem of Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}P^m$ (cf. [6]), one can easily check that only the tube of radius $\pi/(4\sqrt{c})$ over $\mathbb{C}P^k$ in $\mathbb{C}P^{2k+1}$ satisfies the equality (4.40).

(2) *The case $c < 0$.* Assume the equality (4.41) holds. Then we have,

$$\sum_{i=1}^{2m-1} \frac{\kappa_i(z_0)}{1+\kappa_i^2(z_0)} = \frac{3c}{1-c|u|^2} \cdot \frac{\kappa_{2m-1}(z_0)}{1+\kappa_{2m-1}^2(z_0)}. \tag{4.42}$$

By the classification theorem (cf. [6]), we can show that all principal curvatures have the same sign. Therefore, since $c < 0$, the equality (4.42) holds if and only if both side of (4.42) are equal to zero. By the classification theorem, one can easily check that this situation dose not occur. □

Remark 4.4.4. From Theorem 4.4.3, we can see that the minimality of the submanifold N in M does not imply the minimality of the normal bundle in general. However, one can show that the minimality of the normal bundle of a Hopf hypersurface is equivalent to the austere condition of the hypersurface under the assumption $\kappa_{2m-1} = 0$. In fact, the tube of radius $\pi/(4\sqrt{c})$ over $\mathbb{C}P^k$ in $\mathbb{C}P^{2k+1}(4c)$ is the only austere Hopf hypersurface with constant principal curvatures and $\kappa_{2m-1} = 0$.

Minimal surfaces

Let N^2 be a minimal surface in the non-flat complex space form $M = M^m(4c)$. Let $\{e_1, e_2\}$ be an eigenvector of the shape operator A^u at a point $p \in N$ with respect to a normal vector $u \in \nu_p N$. By the curvature condition (4.36), we have

$$\begin{aligned}\overline{R}(u, e_i)e_i &= c\{\langle e_i, e_i\rangle u - \langle u, e_i\rangle e_i + \langle J_0 e_i, e_i\rangle J_0 u \\ &\quad -\langle J_0 u, e_i\rangle J_0 e_i + 2\langle u, J_0 e_i\rangle J_0 e_i\} \\ &= c\{u - 3\langle J_0 u, e_i\rangle J_0 e_i\}\end{aligned}$$

for $i = 1, 2$. Suppose N is not totally geodesic. Then, Lemma 4.1.11 implies that νN is minimal if and only if $\langle J_0 u, e_1\rangle e_1 = \langle J_0 u, e_2\rangle e_2$. Since e_1 and e_2 are linearly independent, this is equivalent to

$$\langle J_0 u, e_1\rangle = \langle J_0 u, e_2\rangle = 0.$$

Since $\{e_1, e_2\}$ spans the tangent space $T_p N$, this implies $J_0 u \in \nu_p N$. Because $u \in T_p N$ is arbitrary, we have $J_0(\nu_p N) \subset \nu_p N$ or equivalently, $J_0(T_p N) \subset T_p N$ for any $p \in N$. Therefore, combining this with Corollary 4.1.7, Lemma 4.1.11 and Theorem 4.4.1, we obtain the following:

Proposition 4.4.5. *Let N^2 be a surface in the non-flat complex space form $M = M^m(4c)$. Then νN is minimal in $(TM, \tilde{g})$ if and only if N is totally geodesic or a complex curve.*

We recall that, for a surface N, the minimality of N is equivalent to the austere condition. In the contrast to the case of the real space forms (Theorem 4.2.2), this theorem shows that the austerity is *not* a sufficient condition for the minimality of normal bundles in general.

From the above results, it is natural to ask the following questions: *Which submanifolds in the non-flat complex space form $M = M^m(4c)$ have minimal normal bundles in $(TM, \tilde{g})$?* Examples with the required property obtained in this chapter are all austere.

Appendix A

A.1 Special Lagrangian submanifolds

First, we recall the notion of calibrated submanifolds.

Definition A.1.1 ([31]). Let (M, g) be a Riemannian manifold. A closed k-form ϕ on M is called a *calibration* if $\phi|_V \leq \mathrm{vol_V}$ holds for any oriented k-plane $V \subset T_pM$ at $p \in M$. A k-dimensional submanifold N of M is said to be *calibrated* by a calibration ϕ if $\phi|_{T_pM} = \mathrm{vol}_{T_pM}$ for all $p \in M$.

A remarkable property of a compact calibrated submanifold is that it is volume minimizing in its homology class (When a submanifold is non-compact, it is *locally* volume-minimizing as well). A typical examples of calibrated submanifolds are complex submanifolds in a Kähler manifold (M^{2n}, ω, J). These are calibrated by $\omega^k/k!$ for $(1 \leq k \leq n)$, and, in this case, the volume miniming property follows from *Wirtinger's inequality.*

A special Lagrangian submanifold is a calibrated submanifold in a Calabi-Yau manifold. There exist several definition of Calabi-Yau manifolds. Here, we use the following definition:

Definition A.1.2 (cf. [32], [44]). A Kähler manifold (M, ω, J) with $\dim_{\mathbb{C}} M = n$ $(n \geq 2)$ is called an *almost Calabi-Yau* manifold if there exist a non-vanishing holomorphic $(n, 0)$-form Ω on M. In addition, if Kähler form ω and Ω satisfy the equality

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2}\Big(\frac{\sqrt{-1}}{2}\Big)^n \Omega \wedge \overline{\Omega},$$

then we call (M, ω, J, Ω) a *Calabi-Yau* n-fold.

When (M, ω, J, Ω) is Calabi-Yau, $\mathrm{Re}(e^{\sqrt{-1}\theta}\Omega)$ is a calibration for any $\theta \in \mathbb{R}$.

Definition A.1.3. A submanifold L in a Calabi-Yau n-fold (M, ω, J, Ω) is called *special Lagrangian* with phase $e^{\sqrt{-1}\theta}$ if it is calibrated by the calibration $\mathrm{Re}(e^{-\sqrt{-1}\theta}\Omega)$.

A.2 Proof of Lemmas in Chapter 3

A.2.1 Proof of Lemma 3.2.9

We refer to [91] for the general fact of the root system.

Let V be a finite dimensional vector space and r a root system on V. We choose an inner product $\langle , \rangle$ on V such that the Weyl group is a subgroup of $O(V)$. Denote the reflection of $\gamma \in r$ by s_γ. Suppose that the root system r is reduced. Choose a basis on V and denote the set of all positive roots by r_+. Set

$$\mathcal{R} := \{(\lambda, \mu) \in r_+ \times r_+;\ \lambda \neq \mu\}.$$

We define the angle of $(\lambda, \mu) \in \mathcal{R}$ by the relation $\langle \lambda, \mu \rangle = |\lambda||\mu| \cos \angle(\lambda, \mu)$ with $0 < \angle(\lambda, \mu) < \pi$. Define subsets of $\mathcal{R}$ as follows (see also Figure A.1):

(i) If $\angle(\lambda, \mu) = \pi/2$ and $|\lambda| = |\mu|$, then we set

$$O(\lambda; \mu) := \{(\lambda, \mu)\}.$$

We call a subset of this type the *Type O*.

(ii) If $\angle(\lambda, \mu) = 2\pi/3$ and $|\lambda| = |\mu|$, then we set

$$A(\lambda; \mu) := \{(\lambda, \mu), (\mu, \lambda + \mu), (\lambda + \mu, \lambda)\}.$$

We call a subset of this type the *Type A*.

(iii) If $\angle(\lambda, \mu) = 3\pi/4$ and $|\lambda| : |\mu| = 1 : \sqrt{2}$, then we set

$$\begin{aligned} B_1(\lambda; \mu) &:= \{(\lambda, \mu), (\mu, \lambda + \mu), (\lambda + \mu, 2\lambda + \mu), (2\lambda + \mu, \lambda)\}, \\ B_2(\lambda; \mu) &:= \{(\mu, \lambda), (\lambda, 2\lambda + \mu), (2\lambda + \mu, \lambda + \mu), (\lambda + \mu, \mu)\}. \end{aligned}$$

We call a subset of this type the *Type* B_i for $i = 1, 2$.

(iv) If $\angle(\lambda, \mu) = 5\pi/6$ and $|\lambda| : |\mu| = 1 : \sqrt{3}$ then we set

$$\begin{aligned} G_1(\lambda; \mu) &:= \{(\lambda, \mu), (\mu, \lambda + \mu), (\lambda + \mu, 3\lambda + 2\mu), (3\lambda + 2\mu, 2\lambda + \mu), \\ &\qquad (2\lambda + \mu, 3\lambda + \mu), (3\lambda + \mu, \lambda)\}. \\ G_2(\lambda; \mu) &:= \{(\mu, \lambda), (\lambda, 3\lambda + \mu), (3\lambda + \mu, 2\lambda + \mu), (2\lambda + \mu, 3\lambda + 2\mu), \\ &\qquad (3\lambda + 2\mu, \lambda + \mu), (\lambda + \mu, \mu)\}. \end{aligned}$$

We call a subset of this type the *Type* G_i for $i = 1, 2$.

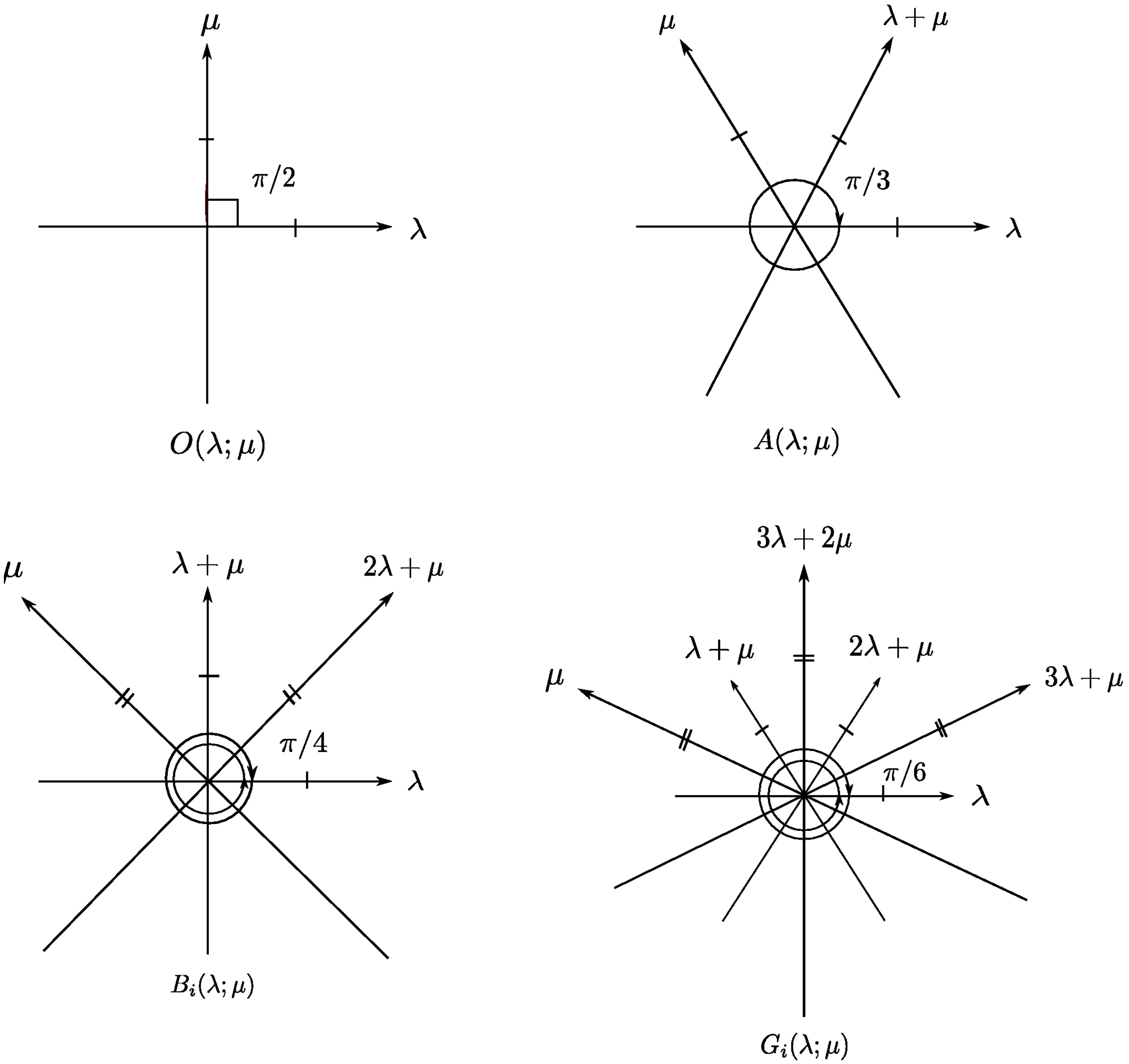
μ
$\pi/2$
λ
$O(\lambda;\mu)$
μ
$\lambda+\mu$
$\pi/3$
λ
$A(\lambda;\mu)$
μ
$\lambda+\mu$
$2\lambda+\mu$
$\pi/4$
λ
$B_i(\lambda;\mu)$
$3\lambda+2\mu$
$\lambda+\mu$
$2\lambda+\mu$
μ
$3\lambda+\mu$
$\pi/6$
λ
$G_i(\lambda;\mu)$

Figure A.1

We call these subsets the *cyclic subsets* of $\mathcal{R}$. For any cyclic subset $X(\lambda;\mu)$, we have a graph which corresponds to $X(\lambda;\mu)$ given in Figure A.1. We denote the graph by $gr(X(\lambda;\mu))$.

We note that if λ,μ are positive roots, $a\lambda+b\mu$ with $a,b\in\mathbb{Z}_+$ is also a positive root whenever $a\lambda+b\mu\in r$. If $(\lambda,\mu)\in\mathcal{R}$ satisfies one of (i) – (iv), and $\langle\lambda,\mu\rangle<0$, then $\lambda+\mu\in r_+$. Hence, $A(\lambda;\mu)$ is well-defined. When $(\lambda,\mu)\in$ (iii), then $s_\lambda(\mu)=\mu-2\sqrt{2}(-1/\sqrt{2})\lambda=2\lambda+\mu\in r_+$ since the reflection acts on r (see Figure B.1). Thus $B_i(\lambda;\mu)$ $(i=1,2)$ are well-defined, namely, any element is a pair of positive roots. By a similar argument, we see that $G_i(\lambda;\mu)(i=1,2)$ are well-defined. We remark that the set $\mathcal{R}$ contains the cyclic subsets $G_i(\lambda;\mu)(i=1,2)$ if and only if the reduced root system r is isomorphic to the system of the exceptional simple Lie group G_2.

We assert the following.

Lemma A.2.1. *$\mathcal{R}$ is a disjoint union of cyclic subsets.*

Proof. First, we show that the set $\mathcal{R}$ coincides with the union of all cyclic subsets. For $(\alpha,\beta)\in\mathcal{R}$, one can easily check that the following properties (see Figure B.1):

(o) If $(\alpha,\beta)\in O(\lambda;\mu)$, then $\angle(\alpha,\beta)=\pi/2$ and $|\alpha|=|\beta|$.

(a) If $(\alpha,\beta)\in A(\lambda;\mu)$, then $\angle(\alpha,\beta)=\pi/3$ or $2\pi/3$ and $|\alpha|=|\beta|$.

(b) If $(\alpha,\beta)\in B_i(\lambda;\mu)$ $(i=1,2)$, then $\angle(\alpha,\beta)=\pi/4$ or $3\pi/4$ and $|\alpha|:|\beta|=1:\sqrt{2}$ or $\sqrt{2}:1$.

(g) If $(\alpha,\beta)\in G_i(\lambda;\mu)$ $(i=1,2)$, then $\angle(\alpha,\beta)=\pi/6$ or $5\pi/6$ and $|\alpha|:|\beta|=1:\sqrt{3}$ or $\sqrt{3}:1$.

Moreover, for each case, the angle $\angle(\alpha,\beta)$ is equal to the maximal angle if and only if $(\alpha,\beta)=(\lambda,\mu)$. We show the converse of theses properties.

The case (o) and the case when $\angle(\alpha,\beta)$ is maximal in each case are obvious. In the case (g), since the root system r is isomorphic to one of G_2, one can easily check the converse. In the following, without loss of generality, we may assume $\alpha<\beta$.

If $\angle(\alpha,\beta)=\pi/3$ and $|\alpha|=|\beta|$, then $\beta-\alpha\in r_+$. Furthermore, we see that $(\alpha,\beta)\in A(\beta-\alpha,\alpha)$.

If $\angle(\alpha,\beta)=\pi/4$ and $|\alpha|:|\beta|=1:\sqrt{2}$, then $\beta-\alpha\in r_+$ and $s_\alpha(\beta)=\beta-2\sqrt{2}(1/\sqrt{2})\alpha=\beta-2\alpha\in r$. We note $|\beta-\alpha|=|\alpha|$, $|\beta-2\alpha|=\sqrt{2}|\alpha|$ and $\angle(\beta-2\alpha,\alpha)=\angle(\beta-\alpha,-\beta+2\alpha)=3\pi/4$. There are two possibilities, namely, $\beta-2\alpha\in r_+$ or $-\beta+2\alpha\in r_+$. If $\beta-2\alpha\in r_+$, then $(\alpha,\beta)\in B_2(\alpha;\beta-2\alpha)$. If $-\beta+2\alpha\in r_+$, then $(\alpha,\beta)\in B_1(\beta-\alpha;-\beta+2\alpha)$.

If $\angle(\alpha, \beta) = \pi/4$ and $|\alpha| : |\beta| = \sqrt{2} : 1$, then $\beta - \alpha \in r_+$, $|\beta - \alpha| = (1/\sqrt{2})|\alpha|$ and $\angle(\beta - \alpha, \alpha) = 3\pi/4$. Moreover, we see that $(\alpha, \beta) \in B_1(\beta - \alpha; \alpha)$.

Therefore, the converse of the properties (o) through (g) are valid. This implies that the set $\mathcal{R}$ coincides with the union of all cyclic subsets. Moreover, we see that two cyclic subsets with different types are disjoint. Thus, it is sufficient to show that two cyclic subsets $X(\lambda; \mu), X(\lambda'; \mu')$ of the same type are disjoint whenever $(\lambda, \mu) \neq (\lambda', \mu')$, where $X = O, A, B_i$ and G_i.

Suppose $X(\lambda; \mu) \cap X(\lambda'; \mu') \neq \{\phi\}$ and choose an element $(\alpha, \beta) \in X(\lambda; \mu) \cap X(\lambda'; \mu')$. Then the vectors α and β belong to the subspace $V_1 \cap V_2$, where $V_1 := \mathrm{span}_{\mathbb{R}}\{\lambda, \mu\}$ and $V_2 := \mathrm{span}_{\mathbb{R}}\{\lambda', \mu'\}$. If $\dim(V_1 \cap V_2) = 1$, α is proportional to β, a contradiction. Therefore, we have $V_1 = V_2$. Since α, β belong to the graphs of the same type, these two graphs coincide on $V_1 = V_2$. Then one can easily check that the case $(\lambda, \mu) \neq (\lambda', \mu')$ is impossible, since otherwise some positive root and negative root coincide, a contradiction. Thus we have $(\lambda, \mu) = (\lambda', \mu')$. This completes the proof. $\square$

A.2.2 A Lemma for a basis of a root system

Let r be a root system on V as before. A basis of r is a subset B in r such that the following properties hold.

(i) B is a basis of the vector space V.

(ii) Every $\beta \in r$ can be written as $\beta = \sum_{\alpha \in B} n_\alpha \alpha$, where n_α are integers of the same sign.

There is a one-to-one correspondence between basis of r and the Wely chambers of r (see [58], Theorem 2.2). In particular, there exist a basis.

A root system r is called *decomposable* (or *reducible*) if it decompose into a disjoint union of two orthogonal subsets r_1 and r_2. Otherwise, it is said to be *indecomposable* (or *irreducible*). If V_i $(i = 1, 2)$ are subspaces in V spanned by r_i $(i = 1, 2)$, then $V = V_1 \oplus V_2$, where r_i is a root system on V_i $(i = 1, 2)$. A root system r is decomposable if and only if a basis B of r is decomposable (see Proposition 4.1 in [58]).

If r is isomorphic to a root system of a simple Lie algebra, r is indecomposable. When r is isomorphic to a root system of a semi-simple Lie algebra $\mathfrak{g}$, r is decomposed into indecomposable root systems which correspond to each simple ideal of $\mathfrak{g}$. Therefore, the reduced root system r is indecomposable if and only if it is isomorphic to a root system of a simple Lie algebra.

We give a characterization of an indecomposable root system. Let B be a basis of the root system r. A *chain* in B is a subset $\{\alpha_1, \ldots, \alpha_k\}$ such that $\langle \alpha_i, \alpha_{i+1} \rangle \neq 0$ for any $i = 1, \ldots, k$. A chain is called a *cycle* if $\langle \alpha_l, \alpha_1 \rangle \neq 0$ for $l > 2$.

Lemma A.2.2 ([58], Lemma 4.5). *(1) B is indecomposable if and only if any two elements of B can be joined by a chain. (2) B does not contain cycles.*

Proof. (1) It is obvious from the definition.

(2) Let $\{\alpha_1, \ldots, \alpha_k\}$ be a chain in B and $\beta_i := \alpha_i/|\alpha_i|$ for $i = 1, \ldots, k$. Then

$$(2\langle\beta_i, \beta_j\rangle)^2 = 2\frac{\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_i, \alpha_i\rangle} \times 2\frac{\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_j, \alpha_j\rangle} = 4\cos^2\theta,$$

where $\langle\alpha_i, \alpha_j\rangle = |\alpha_i||\alpha_j|\cos\theta$. Moreover, we see $4\cos^2\theta \in \mathbb{Z}$ by the definition of the root system. In particular, we have $(2\langle\beta_i, \beta_j\rangle)^2 = 4\cos^2\theta = 0, 1, 2, 3$, or 4. When $4\cos^2\theta = 4$, α_i is proportional to α_j, however, this is impossible since $\alpha_i, \alpha_j \in B$. If $\langle\beta_i, \beta_j\rangle > 0$, then $\beta_i - \beta_j = \alpha_i/|\alpha_i| - \alpha_j/|\alpha_j| \in r$, and this contradicts (ii) above. Therefore, we obtain $\langle\beta_i, \beta_j\rangle \leq -1$ whenever $\langle\beta_i, \beta_j\rangle = 0$.

On the other hand, we have

$$0 < \langle\sum_{i=1}^{k} \beta_i, \sum_{j=1}^{k} \beta_j\rangle = k + 2\sum_{i\neq j}\langle\beta_i, \beta_j\rangle. \tag{A.1}$$

If $\{\alpha_1, \ldots, \alpha_k\}$ is a cycle, then $k \geq 3$ and $\langle\beta_l, \beta_1\rangle \leq -1$ for $l = 2, \ldots, k$. Then we have

$$k + 2\sum_{i\neq j}\langle\beta_i, \beta_j\rangle \leq k - 2(k-1) = -k + 2 < 0.$$

However, this contradicts (A.1). Therefore, B does not contain cycles. □

A.3 Some properties of D-homothetic deformations

Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a Sasaki η-Einstein manifold. The *D-homothetic deformation* is the structure $({}^*\phi, {}^*\xi, {}^*\eta, {}^*g)$ on M such that

$${}^*\phi = \phi, \quad {}^*\eta = \alpha\eta, \quad ,{}^*\xi = \frac{1}{\alpha}\xi, \quad {}^*g = \alpha g + \alpha(\alpha-1)\eta\otimes\eta$$

for some positive constant α. We denote ${}^*M := (M, {}^*\phi, {}^*\xi, {}^*\eta, {}^*g)$. This preserves the distribution $\mathcal{D} := \mathrm{Ker}\ \eta = \mathrm{Ker}\ {}^*\eta$. Moreover, ${}^*M$ is a contact metric structure. Since M is K-contact, the relation of the Christoffel symbols is given by

$${}^*\Gamma^i_{jk} = \Gamma^i_{jk} - (\alpha-1)(\phi^i_j\eta_k + \eta_j\phi^i_k), \tag{A.2}$$

where ${}^*\Gamma$ and Γ denotes the Christoffel symbols of ${}^*M$ and M, respectively (see (2.13)′ in [99]). Combining the normal condition of M with (2.14) in [100], we have a relation

$$\begin{aligned} {}^*R^i_{jkl} &= R^i_{jkl} + (\alpha-1)(2\phi^i_j\phi_{kl} + \phi^i_k\phi_{jl} - \phi^i_l\phi_{jk}) \\ &\quad +(\alpha-1)\{g_{ik}\xi^i(\eta_l - \eta_j) - g_{il}\xi^i(\eta_k - \eta_j)\} + (\alpha^2-1)(\delta^i_l\eta_j\eta_k - \delta^i_k\eta_j\eta_l). \end{aligned} \tag{A.3}$$

Recall that a contact metric structure is Sasaki if and only if the curvature tensor satisfies (see Proposition 7.6 in [13])

$$\xi^j R^i_{jkl} = \delta^i_k \eta_l - \delta^i_l \eta_k. \tag{A.4}$$

By the relation (A.3), we obtain

$$\begin{aligned} {}^*\xi^{i*} R^i_{jkl} &= \frac{1}{\alpha}\{\xi^j R^i_{jkl} + (\alpha^2 - 1)(\delta^i_l \eta_k - \delta^i_k \eta_l)\} \\ &= \frac{1}{\alpha} \times \alpha^2 (\delta^i_l \eta_k - \delta^i_k \eta_l) \quad (\because\ M \text{ satisfies (A.4)}) \\ &= \delta^{i*}_l \eta_k - \delta^{i*}_k \eta_l. \end{aligned}$$

Therefore, ${}^*M$ is also a Sasaki manifold. Moreover, by contracting (A.3), we have

$${}^*\mathrm{Ric} = \mathrm{Ric} - 2(\alpha - 1)g + (\alpha - 1)\{(2m+1)(\alpha+1) - (\alpha - 1)\}\eta \otimes \eta. \tag{A.5}$$

Here, we suppose that M is η-Einstein with η-Einstein constant $a > -2$, namely, there exist a constant a such that $\mathrm{Ric} = ag + (2n - a)\eta \otimes \eta$. Then choosing the constant $\alpha = (a+2)/(2m+2)$, we see that

$${}^*\mathrm{Ric} = \Big(\frac{a + 2 - 2\alpha}{\alpha}\Big)(\alpha g + (\alpha^2 - \alpha)\eta \otimes \eta) = \Big(\frac{a + 2 - 2\alpha}{\alpha}\Big){}^* g.$$

Thus, we obtain the following:

Lemma A.3.1 (cf. [100]). *Let $(M^{2m+1}, \phi, \xi, \eta, g)$ be a Sasaki η-Einstein manifold with η-Eisntein constant $a > -2$. Then the D-homothetic structure ${}^*M = (M, {}^*\phi, {}^*\xi, {}^*\eta, {}^*g)$ with $\alpha := (a+2)/(2m+2)$ is a Sasaki-Einstein manifold with Einstein constant $(a + 2 - 2\alpha)/\alpha$.*

Consider an *integral immersion* $\iota : L^n \to M^{2m-1}$, namely, $\iota^{-1}\eta = 0$, where ι^{-1} denotes the pull-back. Since a D-homothetic deformation preserves $\mathcal{D}$, an immersion ${}^*\iota : L \to {}^*M$ is also integral. Moreover, the induced metric via ${}^*\iota$ is given by $({}^*\iota)^{-1*}g = \alpha({}^*\iota)^{-1}g = \alpha\iota^{-1}g$. Hence, ι and ${}^*\iota$ are conformal equivalent (even though M and ${}^*M$ are *not* conformal equivalent). In particular, the Levi-Civita connections ∇ and ${}^*\nabla$ of the induced metrics coincide. Let $\{e_1, \ldots, e_n\}$ be a local frames of L. It is obvious that $\iota_* e_i = {}^*\iota_* e_i$ for any $i = 1, \ldots, n$, and so, we omit ι_* and ${}^*\iota_*$. We take a local chart $\{x_1, \ldots, x_{2m+1}\}$ of M. Set

$e_i = \sum_{A=1}^{2m+1} a_{iA}\partial_A$, where $\partial_A = \partial/\partial x_A$. Then by (A.2), we see that

$$\begin{aligned} {}^*\overline{\nabla}_{e_i}e_j - \overline{\nabla}_{e_i}e_j &= \sum_{A,B} a_{iA}a_{iB}({}^*\overline{\nabla}_{\partial_A}\partial_B - \overline{\nabla}_{\partial_A}\partial_B) \\ &= \sum_{A,B,C} a_{iA}a_{iB}({}^*\Gamma^C_{AB} - \Gamma^C_{AB})\partial_C \\ &= \sum_{A,B,C} a_{iA}a_{iB}\{-(\alpha-1)(\phi^C_A\eta_B + \eta_A\phi^C_B)\}\partial_C \\ &= -(\alpha-1)\{\eta(e_j)\phi(e_i) + \eta(e_i)\phi(e_j)\} \\ &= 0 \end{aligned}$$

since L is integrable. Combining this with $\nabla = {}^*\nabla$, we obtain the following:

Proposition A.3.2. *Let B and ${}^*B$ be the second fundamental form of the integral immersion $\iota : L \to M$ and ${}^*\iota : L \to {}^*M$, respectively. Then $B = {}^*B$ as a tensor field on L. In particular, the mean curvature vectors H and ${}^*H$ satisfy the relation $H = \alpha^*H$.*

Corollary A.3.3. *An integral immersion $\iota : L \to M$ is minimal if and only if ${}^*\iota : L \to {}^*M$ is minimal. Moreover, suppose ι is Legendrian, then ι is L-minimal if and only if ${}^*\iota$ is L-minimal.*

Appendix B

Tables

Table B.1: [Takagi's list [93], [50]] Hopf hypersurfaces N with constant principal curvatures (or equivalently, homogeneous real hypersurfaces) in $\mathbb{C}P^n(4c)$ (cf. [69]). N is a tube of a focal variety of radius r.

Type	Focal variety	Radius	Principal curvatures	Multiplicities
A_1	$\mathbb{C}P^{n-1}$	$0 < r < \pi/2$	$\lambda_1 = \frac{1}{\sqrt{c}} \cot r$, $a = \frac{2}{\sqrt{c}} \cot 2r$	$2n-2, 1$
A_2	$\mathbb{C}P^k$ $(1 \leq k \leq n-2)$	$0 < r < \pi/2$	$\lambda_1 = \frac{-1}{\sqrt{c}} \tan r$, $\lambda_2 = \frac{1}{\sqrt{c}} \cot r$, $a = \frac{2}{\sqrt{c}} \cot 2r$	$2p, 2q, 1$ $(p+q = n-1)$
B	$\mathbb{Q}^{n-1}$	$0 < r < \pi/4$	$\lambda_1 = \frac{-1}{\sqrt{c}} \cot r$, $\lambda_2 = \frac{1}{\sqrt{c}} \tan r$, $a = \frac{2}{\sqrt{c}} \tan 2r$	$n-1, n-1,$ 1
C	$\mathbb{C}P^1 \times \mathbb{C}P^{(n-1)/2}$ ($n \geq 5$, odd)	$0 < r < \pi/4$	$\lambda_1 = \frac{-1}{\sqrt{c}} \cot r$, $\lambda_2 = \frac{1}{\sqrt{c}} \cot(\frac{\pi}{4} - r)$, $\lambda_3 = \frac{1}{\sqrt{c}} \cot(\frac{\pi}{2} - r)$, $\lambda_4 = \frac{1}{\sqrt{c}} \cot(\frac{3\pi}{4} - r)$, $a = \frac{-2}{\sqrt{c}} \tan 2r$	$n-3, 2,$ $n-3, 2,$ 1
D	$Gr_2(\mathbb{C}^5)$ $(n = 9)$	$0 < r < \pi/4$	$\lambda_1 = \frac{-1}{\sqrt{c}} \cot r$, $\lambda_2 = \frac{1}{\sqrt{c}} \cot(\frac{\pi}{4} - r)$, $\lambda_3 = \frac{1}{\sqrt{c}} \cot(\frac{\pi}{2} - r)$, $\lambda_4 = \frac{1}{\sqrt{c}} \cot(\frac{3\pi}{4} - r)$, $a = \frac{-2}{\sqrt{c}} \tan 2r$	$4, 4,$ $4, 4,$ 1
E	$SO(10)/U(5)$ $(n = 15)$	$0 < r < \pi/4$	$\lambda_1 = \frac{-1}{\sqrt{c}} \cot r$, $\lambda_2 = \frac{1}{\sqrt{c}} \cot(\frac{\pi}{4} - r)$, $\lambda_3 = \frac{1}{\sqrt{c}} \cot(\frac{\pi}{2} - r)$, $\lambda_4 = \frac{1}{\sqrt{c}} \cot(\frac{3\pi}{4} - r)$, $a = \frac{-2}{\sqrt{c}} \tan 2r$	$8, 6,$ $8, 6,$ 1

Table B.2: [Montiel's list] Hopf hypersurfaces N with constant principal curvatures (cf. [69]). N is the holosphere or a tube of radius r of a focal variety in the following.

Type	Focal variety	Radius	Principal curvatures	Multiplicities
A_0 (horosphere)	–	–	$\lambda_1 = \frac{1}{\sqrt{c}}$, $a = \frac{2}{\sqrt{c}}$	$2n-2, 1$
A_1 (geodesic sphere)	$\mathbb{C}H^k$ $(k = 0, n-2)$	$0 < r < \infty$	$\lambda_1 = \frac{1}{\sqrt{c}} \tanh r$, $a = \frac{2}{\sqrt{c}} \coth 2r$	$2n-2$, 1
A_2	$\mathbb{C}H^k$ $(1 \leq k \leq n-2)$	$0 < r < \infty$	$\lambda_1 = \frac{1}{\sqrt{c}} \tanh r$, $\lambda_2 = \frac{1}{\sqrt{c}} \coth r$, $a = \frac{2}{\sqrt{c}} \coth 2r$	$2p, 2q, 1$ $(p+q = n-1)$
B	$\mathbb{R}H^n$	$0 < r < \infty$	$\lambda_1 = \frac{1}{\sqrt{c}} \coth r$, $\lambda_2 = \frac{1}{\sqrt{c}} \tanh r$, $a = \frac{2}{\sqrt{c}} \tanh 2r$	$n-1$, $n-1$ 1

Table B.3: Homogeneous isoparametric hypersurfaces N in spheres (cf. [59], [98]) and focal manifolds. N is a principal orbit of the isotropy representation of (U, K). $N_\pm$ denotes the focal manifolds.

g	(U, K)	dim$N$	(m_1, m_2)	$N = K/K_0$	N_+	N_-
1	$(S^1 \times SO(n+2), SO(n+1)), (n \geq 1)$	n	n	S^n	$\{pt\}$	$\{pt\}$
2	$(SO(p+2) \times SO(n+2-p), SO(p+1) \times SO(n+1-p)), (1 \leq p \leq n-1)$	n	$p, n-p$	$S^p \times S^{n-p}$	S^p	S^{n-p}
3	$(SU(3), SO(3))$	3	$1, 1$	$\frac{SO(3)}{\mathbb{Z}_2+\mathbb{Z}_2}$	$\frac{SO(3)}{S(O(2)\times O(1))}$	$\frac{SO(3)}{S(O(1)\times O(2))}$
3	$(SU(3) \times SU(3), SU(3)),$	6	$2, 2$	$\frac{SU(3)}{T^2}$	$\frac{SU(3)}{S(U(2)\times U(1))}$	$\frac{SU(3)}{S(U(2)\times U(1))}$
3	$(SU(6), Sp(3))$	12	$4, 4$	$\frac{Sp(3)}{Sp(1)^3}$	$\frac{Sp(3)}{Sp(2)\times Sp(1))}$	$\frac{Sp(3)}{Sp(1)\times Sp(2))}$
3	(E_6, F_4)	24	$8, 8$	$\frac{F_4}{Spin(8)}$	$\frac{F_4}{Spin(9)}$	$\frac{F_4}{Spin(9)}$
4	$(SO(5) \times SO(5), SO(5))$	8	$2, 2$	$\frac{SO(5)}{T^2}$	$\frac{SO(5)}{SO(2)\times SO(3)}$	$\frac{SO(5)}{U(2)}$
4	$(SU(m+2), S(U(m) \times U(2)), (m \geq 2)$	$4m-2$	$2, 2m-3$	$\frac{S(U(m)\times U(2))}{SU(m-2)\times T^2}$	$\frac{S(U(m)\times U(2))}{SU(m-2)\times U(2)}$	$\frac{S(U(m)\times U(2))}{SU(m-1)\times T^2}$
4	$(SO(m+2), SO(m) \times SO(2)), (m \geq 3)$	$2m-2$	$1, m-2$	$\frac{SO(m)\times SO(2)}{SO(m-2)\times \mathbb{Z}^2}$	$\frac{SO(m)\times SO(2)}{SO(m-2)\times SO(2)}$	$\frac{SO(m)\times SO(2)}{O(m-1)}$
4	$(Sp(m+2), Sp(m) \times Sp(2)), (m \geq 2)$	$8m-2$	$4, 4m-5$	$\frac{Sp(m)\times Sp(2)}{Sp(m-2)\times Sp(1)^2}$	$\frac{Sp(m)\times Sp(2)}{Sp(m-2)\times Sp(2)}$	$\frac{Sp(m)\times Sp(2)}{Sp(m-1)\times Sp(1)^2}$
4	$(SO(10), U(5))$	18	$4, 5$	$\frac{U(5)}{SU(2)^2\times T^1}$	$\frac{U(5)}{Sp(2)\times U(1)\times T^1}$	$\frac{U(5)}{S(U(2)\times U(3))}$
4	$(E(6), Spin(10) \cdot T)$	30	$6, 9$	$\frac{Spin(10)\cdot T}{SU(4)\cdot T}$	$\frac{Spin(10)\cdot T}{U(1)\cdot Spin(7)}$	$\frac{Spin(10)\cdot T}{S^1\cdot SU(5)}$
6	$(G_2 \times G_2, G_2),$	12	$2, 2$	$\frac{G_2)}{T^2}$	$\frac{G_2}{U(2)}$	$\frac{G_2}{U(2)}$
6	$(G_2, SO(4)),$	6	$1, 1$	$\frac{SO(4))}{\mathbb{Z}_2+\mathbb{Z}_2}$	$\frac{SO(4)}{O(2)}$	$\frac{SO(4)}{O(2)}$

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